## Bachelor's thesis

# The anomalous magnetic moment of the muon in the scotogenic model 

Jan Wissmann

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First examiner: Prof. Dr. Michael Klasen
Second examiner: PD Dr. Karol Kovařík

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## 1 Introduction

The Standard Model (SM) is the greatest achievement of particle physics. It explains experimental results very accurately and was able to predict a sizeable number of particles, e.g. the top quark, the W and Z bosons and the Higgs boson. But there are also a number of phenomena which can not be explained by the Standard Model. Notable examples are gravity, dark matter (DM), and neutrino masses. To see where exactly the Standard Model fails, physicists around the world try to find experimental results which contradict the Standard Model. One of these experimental values is the muon anomalous magnetic moment $a_{\mu}=\left(g_{\mu}-2\right) / 2$, which is the topic of this thesis. It describes the deviation of the $g$-factor $g_{\mu}$ from the result predicted by the Dirac equation, namely $g_{\mu}=2$. The $g_{\mu}$-factor describes a proportionality constant between the magnetic moment $\vec{\mu}$ of the muon and its spin $\vec{S}$, and is therefore defined by:

$$
\vec{\mu}=g_{\mu} \frac{e}{2 m_{\mu}} \vec{S}
$$

Recent experimental results from the Fermilab National Accelerator Laboratory (FNAL) show a combined average of $a_{\mu}^{\exp }=(116592061 \pm 41) \cdot 10^{-11}[\mathrm{Abi}+21]$, which deviates from the value predicted by the Standard Model, $a_{\mu}^{\mathrm{SM}}=(116591810 \pm 43) \cdot 10^{-11}$ [Aoy +20 ], by $\Delta a_{\mu}=a_{\mu}^{\exp }-a_{\mu}^{\text {SM }}=(251 \pm 59) \cdot 10^{-11}$ or $4.2 \sigma$. This is just short of $5 \sigma$, above which a deviation is considered a discovery in particle physics. The theoretical Standard Model value is calculated considering electroweak and hadronic contributions, and contributions due to Quantum Electrodynamics (QED). From these values it can be seen that both theoretical and experimental results can be determined with similar precision, which makes the anomalous magnetic moment a promising test for new physics.

Beyond the Standard Model (BSM) minimal extensions of the Standard Model can be used to explain neutrino masses and yield a candidate for dark matter. An example for this is the scotogenic model [Ma06], which extends the Standard Model by an exact $\mathbb{Z}_{2}$-symmetry and a set of particles odd under $\mathbb{Z}_{2}$. This enables radiative corrections to the neutrino masses and yields dark matter candidates from the new particles. In this thesis an extension to the scotogenic model, presented in [CN19], is analyzed with respect to the anomalous magnetic moment. Experimental constraints regarding relic density, lepton flavor violation (LFV) and the Large Electron-Positron Collider (LEP) charged particle mass limit are imposed.

First the anomalous magnetic moment is calculated analytically both in the SM and in the extended scotogenic model by expressing the invariant amplitude in different form factors and identifying the form factors in the explicit calculation. The anomalous magnetic moment can then be obtained from one of the form factors in
the non-relativistic limit. After that, the analytical result is compared to the one numerically computed by SPheno [Por03; PS12]. At last, the whole parameter space is scanned using SPheno and micrOMEGAs [Bél+18]. The necessary files for this were generated using SARAH [Sta08] and minimal-lagrangians [May21].

## 2 Dark matter

The $\Lambda$ CDM model is a cosmological model to describe evolution of the universe, starting from the Big Bang. It includes the so called cold dark matter (CDM), which is a type of matter that makes up $26.4 \%$ of the energy density of the universe or $84.4 \%$ of the matter density $[\mathrm{Zyl}+20]$. The " $\Lambda$ " in $\Lambda \mathrm{CDM}$ stands for the cosmological constant $\Lambda$, which describes the energy density of the vacuum and can be associated with dark energy in the universe. This accounts for the biggest part of the energy density present in the universe and is hypothesized to be the cause of its accelerating expansion. The "coldness" of this dark matter means that it is moving slowly compared to the speed of light.

### 2.1 Evidence for the existence of dark matter

There are several sources of evidence for the existence of dark matter. The first one are the rotation curves of spiral galaxies: Equating the gravitational and the centripetal force yields the rotation velocity $v$ dependent on the distance $r$ from the origin of the galaxy [GD11]:

$$
\begin{equation*}
v(r)=\sqrt{\frac{G M(r)}{r}} \tag{2.1}
\end{equation*}
$$

Here $M(r)$ is the mass enclosed within a sphere of radius $r$. For large distances $r$ from the galaxy's origin, $M(r)$ can be assumed to be constant in eq. (2.1), such that $v(r) \sim \frac{1}{\sqrt{r}}$, i.e. the rotation velocity $v(r)$ should decrease with increasing radius $r$. However, the observed results deviate from this prediction: Measured rotation curves stay constant at large distances, as shown in fig. 1. This means that $M(r)$ can not stay


Figure 1: Measured velocity distribution of the spiral galaxy NGC 6503. The labeled curves show the contributions by the observed disk, gas, and the dark matter halo. Source: [JKG96, p. 207]
constant but has to increase, even for large $r$. It follows that the matter contained in a spiral galaxy is not concentrated at the center, but it has a halo of electromagnetically non-interacting, or dark, matter.

Another type of evidence is provided by gravitational lensing around heavy objects, e.g. galaxies or clusters of galaxies. It describes the property of light to take the shortest path in space, which can be curved, according to the theory of general relativity. Around large gravitational wells, like the aforementioned galaxies, this effect enables light to travel around an object completely. The amount of curvature is predicted by general relativity, so that images of gravitational lensing can be used to measure the mass distribution of a distant object. This works even if the mass is non-luminous matter, i.e. does not interact electromagnetically and can thus not be detected by optical telescopes. With this method the Bullet Cluster (fig. 2) can be examined: It consists of two colliding clusters of galaxies, which show a discrepancy between the concentration of luminous matter, and dark matter. In fig. 2 the luminous matter (hot gas), observed by the X-ray Chandra telescope and shown in blue, is separated from most of the dark matter, shown in red. The latter was observed by measurements of gravitational lensing [Opt06]. During the collision, a drag effect slowed down the hot gas and separated it from the dark matter, which in turn did not undergo this effect, since it only interacts very weakly.

At last, another source of evidence is the cosmic microwave background (CMB). From peaks in its power spectrum the contents of the universe can be determined.


Figure 2: The Bullet Cluster (1E 0657-56). Luminous matter is shown in red, and dark matter, observed by gravitational lensing, is shown in blue. Source: [Opt06]

### 2.2 WIMPs as a candidate for dark matter

One of the most promising dark matter candidates are weakly interacting massive particles (WIMPs). These are generally a type of particle with an interaction strength
on the scale of the electroweak interaction or weaker, but nonetheless with a nonzero strength. WIMPs are said to have been produced during a period of thermal equilibrium after the Big Bang, until the temperature of the universe decreased due to expansion and the WIMPs began to only annihilate. This led to an exponential decrease of their number density, up to a point after which the annihilation would stop completely and leave behind a WIMP relic density. This is called the freeze-out mechanism and is shown in fig. 3. Here the comoving number density $Y=\frac{n}{s}$ is plotted logarithmically against $x=\frac{m}{T}$. $n$ is the dark matter number density and $s$ is the entropy density, such that $Y$ is the number of particles in a comoving volume, i.e. in a volume unaffected by the expansion of the universe. Because the temperature $T$ decreases with time due to the expansion of the universe, the $x$-axis also shows increasing time. The number density $Y_{\mathrm{EQ}}$ in thermal equilibrium decreases exponentially, up until when the freeze-out happens at a decoupling temperature $T_{\text {dec }}$, which is defined by the point in time where the interaction rate $\Gamma=\sigma v n$ of the annihilation is equal to the Hubble parameter $H(t)$. Here $\sigma$ is the WIMP annihilation cross section and $v$ the velocity. At this decoupling temperature the actual number density starts to diverge from the one in thermal equilibrium and remains constant. Additionally, a dependence on $\langle\sigma v\rangle$ is shown in fig. 3, which is the thermal average of the WIMP annihilation cross section $\sigma$ times the velocity $v$. For an increasing cross section $\sigma$ the annihilation would of course continue up to a later point in time, so that the number density after the freeze-out is smaller. The time evolution of the number density $n(t)$ can be described by the Boltzmann equation [GD11],


Figure 3: The freeze-out mechanism of WIMPs, where $Y$ is the comoving number density, dependent on the mass $m$ divided by temperature $T$. The solid line shows its exponential decrease at thermal equilibrium, the dashed lines show its actual trend, dependent on $\langle\sigma v\rangle$. Source: [KT90, p. 126]

$$
\begin{equation*}
\frac{\mathrm{d} n}{\mathrm{~d} t}=-3 H(t) n(t)-\langle\sigma v\rangle\left(n(t)^{2}-n_{\mathrm{eq}}(t)^{2}\right) \tag{2.2}
\end{equation*}
$$

where $H(t)$ is the Hubble parameter at time $t$ and $n_{\text {eq }}(t)$ is the number density at thermal equilibrium.

The dark matter relic density $\Omega h^{2}$ describes how much dark matter is present in the universe. It is determined by fitting Cosmic Microwave Background (CMB) radiation spectra, which was done by the Planck collaboration and results in a value of [Agh+20]

$$
\begin{equation*}
\Omega h^{2}=0.1200 \pm 0.0012 \tag{2.3}
\end{equation*}
$$

The density parameter $\Omega$ is defined as the dark matter mass density divided by the critical density $\rho_{c}$, which is the matter density in a universe with no curvature, and $h$ is the Hubble constant $H_{0}$ at present time divided by $100 \frac{\mathrm{~km}}{\mathrm{smpc}}$.

## $3 a_{\mu}$ in the Standard Model

### 3.1 From the Dirac equation to the Pauli equation

A great success of the Dirac equation was the theoretical result of $g_{\mu}=2$, which is already very close to the experimental result. It is obtained by taking the non-relativistic limit of the Dirac equation, the Pauli equation, which was formulated by Wolfgang Pauli in 1927. The Pauli equation describes a charged spin- $1 / 2$ particle in an external electromagnetic field, moving at a speed much less than the speed of light. The derivation of the Pauli equation and the value $g_{\mu}=2$ shall be demonstrated in this section. It is based on [Kuh16, pp. 45-46].

The Dirac equation in Hamiltonian form can be written as

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \psi=H_{\mathrm{D}} \psi \tag{3.1}
\end{equation*}
$$

where $\psi$ is a 4-component Dirac spinor and $H_{\mathrm{D}}$ is the Hamilton operator corresponding to the Dirac equation:

$$
\begin{equation*}
H_{\mathrm{D}}=-\mathrm{i} \vec{\alpha} \cdot \vec{\nabla}+\beta m \tag{3.2}
\end{equation*}
$$

Here $\vec{\nabla}$ is the nabla operator with respect to position space and $m$ is the mass of $\psi$. The Dirac representation of the $4 \times 4$ matrices $\alpha^{i}(i \in\{1,2,3\})$ and $\beta$ is

$$
\alpha^{i}=\left(\begin{array}{cc}
0 & \sigma_{i}  \tag{3.3}\\
\sigma_{i} & 0
\end{array}\right) \quad \text { and } \quad \beta=\left(\begin{array}{cc}
\mathbb{1}_{2} & 0 \\
0 & -\mathbb{1}_{2}
\end{array}\right)
$$

$\sigma_{i}$ denotes the $i$-th Pauli matrix:

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1  \tag{3.4}\\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

To obtain the Dirac equation for a particle with charge $-e$ in an electromagnetic field the replacement $p^{\mu} \rightarrow p^{\mu}+e A^{\mu}$, where

$$
p^{\mu}=\mathrm{i} \partial^{\mu}=\mathrm{i}\left(\frac{\partial}{\partial t},-\vec{\nabla}\right),
$$

can be made. Here $\left(A^{\mu}\right)=(\phi, \vec{A})$ is the Lorentz contravariant electromagnetic potential, containing the scalar electric potential $\phi$ and the vector potential $\vec{A}$. With this the Dirac equation becomes

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \psi=[\vec{\alpha} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A})+\beta m-e \phi] \psi \tag{3.5}
\end{equation*}
$$

Now the following ansatz can be used:

$$
\begin{equation*}
\psi=\binom{\varphi}{\chi} e^{-\mathrm{i} m t} \tag{3.6}
\end{equation*}
$$

where the rapid time dependence caused by the rest mass is factored out. $\varphi$ and $\chi$ are two-component spinors. Plugging eq. (3.6) into eq. (3.5) results in:

$$
\mathrm{i} \frac{\partial}{\partial t}\binom{\varphi}{\chi}+m\binom{\varphi}{\chi}=\left[\left(\begin{array}{cc}
0 & \sigma_{i} \\
\sigma_{i} & 0
\end{array}\right)\left(-\mathrm{i} \frac{\partial}{\partial x_{i}}+e A_{i}\right)\binom{\varphi}{\chi}+m\left(\begin{array}{cc}
\mathbb{1}_{2} & 0 \\
0 & -\mathbb{1}_{2}
\end{array}\right)\binom{\varphi}{\chi}-e \phi\binom{\varphi}{\chi}\right] .
$$

Carrying out the matrix products yields two coupled differential equations:

$$
\begin{align*}
& \mathrm{i} \frac{\partial}{\partial t} \varphi+m \varphi=\vec{\sigma} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A}) \chi+(m-e \phi) \varphi  \tag{3.7}\\
& \mathrm{i} \frac{\partial}{\partial t} \chi+m \chi=\vec{\sigma} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A}) \varphi+(-m-e \phi) \chi \tag{3.8}
\end{align*}
$$

These can be simplified using the non-relativistic limit, i.e. the change in time of $\chi$ and the electric potential can be neglected in comparison to the mass,

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \chi \ll m \chi \quad \text { and } \quad e \phi \chi \ll m \chi \tag{3.9}
\end{equation*}
$$

such that eq. (3.8) becomes

$$
\chi=\frac{1}{2 m} \vec{\sigma} \cdot(-i \vec{\nabla}+e \vec{A}) \varphi .
$$

Plugging this into eq. (3.7) yields

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \varphi=\vec{\sigma} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A}) \chi-e \phi \varphi=\frac{1}{2 m}[\vec{\sigma} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A})]^{2} \varphi-e \phi \varphi . \tag{3.10}
\end{equation*}
$$

The first term can be simplified as follows (using the identity $\vec{\nabla} \times(\vec{A} \varphi)=(\vec{\nabla} \times \vec{A}) \varphi+$ $(\vec{\nabla} \varphi) \times \vec{A}):$

$$
\begin{align*}
{[\vec{\sigma} \cdot(-\mathrm{i} \vec{\nabla}+e \vec{A})]^{2} \varphi } & =\sigma_{i} \sigma_{j}\left(-\mathrm{i} \frac{\partial}{\partial x_{i}}+e A_{i}\right)\left(-\mathrm{i} \frac{\partial}{\partial x_{j}}+e A_{j}\right) \varphi \\
& =\left(\delta_{i j}+\mathrm{i} \varepsilon_{i j k} \sigma_{k}\right)\left(-\mathrm{i} \frac{\partial}{\partial x_{i}}+e A_{i}\right)\left(-\mathrm{i} \frac{\partial}{\partial x_{j}}+e A_{j}\right) \varphi \\
& =(-\mathrm{i} \vec{\nabla}+e \vec{A})^{2} \varphi+\mathrm{i} \vec{\sigma} \cdot[(-\mathrm{i} \vec{\nabla}+e \vec{A}) \times(-\mathrm{i} \vec{\nabla}+e \vec{A})] \varphi \\
& =(-\mathrm{i} \vec{\nabla}+e \vec{A})^{2} \varphi+e \vec{\sigma} \cdot[\vec{\nabla} \times(\vec{A} \varphi)+\vec{A} \times(\vec{\nabla} \varphi)] \\
& =(-\mathrm{i} \vec{\nabla}+e \vec{A})^{2} \varphi+e \vec{\sigma} \cdot(\vec{\nabla} \times \vec{A}) \varphi . \tag{3.11}
\end{align*}
$$

By plugging eq. (3.11) into eq. (3.10) together with using $\vec{B}=\vec{\nabla} \times \vec{A}$ and $\vec{S}=\frac{1}{2} \vec{\sigma}$, the following can be obtained:

$$
\begin{equation*}
\mathrm{i} \frac{\partial}{\partial t} \varphi=\left[\frac{1}{2 m}(-\mathrm{i} \vec{\nabla}+e \vec{A})^{2}+\frac{e}{m} \vec{S} \cdot \vec{B}-e \phi\right] \varphi \tag{3.12}
\end{equation*}
$$

which is the Pauli equation in its general form. From this the magnetic moment, i.e. the coupling of the spin to the magnetic field, can be obtained. For the muon it is defined as:

$$
\begin{equation*}
\vec{\mu}=g_{\mu} \frac{e}{2 m_{\mu}} \vec{S} \tag{3.13}
\end{equation*}
$$

which is also the definition of the proportionality constant, the $g$-factor. From eq. (3.12) it can be seen that $g_{\mu}=2$, which is of course also true for the electron and the tau. To quantify the deviation of an experimentally measured or theoretically predicted value from $g_{\mu}=2$, a new variable is introduced, namely the anomalous magnetic moment $a_{\mu}$. It is defined via

$$
\begin{equation*}
a_{\mu}=\frac{g_{\mu}-2}{2} . \tag{3.14}
\end{equation*}
$$

where the factor $1 / 2$ arises due to the definition of the magnetic form factor, which will be introduced in the next section. The reason for these deviations are loop corrections, which are not considered by only looking at the Dirac equation in the non-relativistic limit.

### 3.2 The electromagnetic form factors

The relevant process from which $a_{\mu}$ can be calculated is shown in fig. 4. It describes the interaction of a muon with an electromagnetic field, which is why the incoming photon does not have to be on-shell. The blob in the middle in principle represents all possible loops, which introduce corrections to $g_{\mu}=2$.

In the beginning it is sensible to think about the structure of the invariant amplitude $\mathcal{M}=\epsilon_{\mu} \mathcal{N}^{\mu}$, where $\epsilon_{\mu}$ is the polarization vector of the incoming photon. Because $\mathcal{M}^{\mu}$ transforms like a Lorentz vector, we can parametrize it by the possible Lorentz vectors that can appear in the calculation of the blob in fig. 4, similar to the calculation shown in [Jeg17, pp. 202-203]:

$$
\begin{align*}
\mathrm{i} \mathcal{M}^{\mu}=\bar{u}\left(q_{2}\right) & {\left[f_{1} \gamma^{\mu}\right.} \\
& +f_{2} p^{\mu}+f_{3} q_{1}^{\mu}+f_{4} q_{2}^{\mu}  \tag{3.15}\\
& \left.+f_{5} \gamma^{\mu} \gamma^{5}+f_{6} p^{\mu} \gamma^{5}+f_{7} q_{1}^{\mu} \gamma^{5}+f_{8} q_{2}^{\mu} \gamma^{5}\right] u\left(q_{1}\right)
\end{align*}
$$

Only $f_{1}$ to $f_{4}$ can appear in quantum electrodynamics (QED), as $f_{5}$ to $f_{8}$ are combined with $\gamma^{5}$, but QED is parity invariant. Because the photon can not be assumed to be on-shell, $p^{2} \neq 0$. Furthermore, momentum conservation requires $p=q_{2}-q_{1}$.


Figure 4: The relevant process in the calculation of the anomalous magnetic moment.

Because they are Lorentz scalars, the $f_{i}$ can only depend on other Lorentz scalars, such as scalar products of two momenta ( $p^{2}, q_{1 / 2}^{2}, q_{1} \cdot q_{2}, p \cdot q_{1 / 2}$ ) or slashed momenta ( $p, q_{1 / 2}$ ). This simplifies eq. (3.15) by using the following considerations:

1. Because the incoming and outgoing muon are on their mass shells, $q_{1 / 2}^{2}=m_{\mu}^{2}$. Therefore $q_{1} \cdot q_{2}$ can be rewritten as

$$
\begin{equation*}
q_{1} \cdot q_{2}=-\frac{1}{2}\left[\left(q_{2}-q_{1}\right)^{2}-q_{2}^{2}-q_{1}^{2}\right]=m_{\mu}^{2}-\frac{1}{2} p^{2} \tag{3.16}
\end{equation*}
$$

and $p \cdot q_{1 / 2}$ can be expressed as

$$
\begin{equation*}
p \cdot q_{1 / 2}= \pm \frac{1}{2}\left[\left(q_{1 / 2} \pm p\right)^{2}-p^{2}-q_{1 / 2}^{2}\right]= \pm \frac{1}{2}\left[q_{2 / 1}^{2}-p^{2}-m_{\mu}^{2}\right]=\mp \frac{1}{2} p^{2} . \tag{3.17}
\end{equation*}
$$

2. The slashed momenta $q_{1 / 2}$ and $p=q_{2}-q_{1}$ can be converted into terms dependent on $m_{\mu}$ by using the Dirac equation (A.1) and its adjoint equation (A.2).

Terms with $\gamma^{5}$ can be brought into the right order to apply the Dirac equation via the anticommutation relation (A.8) of $\gamma^{5}$ and a Dirac-matrix $\gamma^{\mu}$.

So it is apparent that the $f_{i}$ only depend on $p^{2}$ and $m_{\mu}$ (and on other scalar parameters such as particle masses (e.g. of particles in loops) or the fine structure constant $\alpha=$ $e^{2} /(4 \pi)$ ).

Momentum conservation $p=q_{2}-q_{1}$ allows to substitute $p^{\mu}$ by $q_{1}^{\mu}$ and $q_{2}^{\mu}$ in eq. (3.15), which is equivalent to setting $f_{2}=0$ and $f_{6}=0$. The Ward identity $p_{\mu} \mathcal{M}^{\mu}=$ 0 can be used to relate some of the remaining $f_{i}$ :

$$
\begin{aligned}
& p_{\mu} \mathcal{N}^{\mu}=0 \\
& =p_{\mu} \bar{u}\left(q_{2}\right)\left[f_{1} \gamma^{\mu}+f_{3} q_{1}^{\mu}+f_{4} q_{2}^{\mu}+f_{5} \gamma^{\mu} \gamma^{5}+f_{7} q_{1}^{\mu} \gamma^{5}+f_{8} q_{2}^{\mu} \gamma^{5}\right] u\left(q_{1}\right) \\
& =\bar{u}\left(q_{2}\right)\left[f_{1} p p+f_{3} p \cdot q_{1}+f_{4} p \cdot q_{2}+f_{5} p \gamma^{5}+f_{7} p \cdot q_{1} \gamma^{5}+f_{8} p \cdot q_{2} \gamma^{5}\right] u\left(q_{1}\right) .
\end{aligned}
$$

The first term becomes

$$
\begin{aligned}
& f_{1} \bar{u}\left(q_{2}\right) p u\left(q_{1}\right)= \\
& \quad \stackrel{f_{1}}{ } \bar{u}\left(q_{2}\right)\left[q_{2}-q_{1}\right] u\left(q_{1}\right) \\
& \quad \stackrel{\text { A.1),(A.2) }}{=} f_{1} \bar{u}\left(q_{2}\right)\left[m_{\mu}-m_{\mu}\right] u\left(q_{1}\right)=0,
\end{aligned}
$$

and the fourth term can be written as

$$
\begin{aligned}
& f_{5} \bar{u}\left(q_{2}\right) p \gamma^{5} u\left(q_{1}\right)=f_{5} \bar{u}\left(q_{2}\right)\left[q_{2}-q_{1}\right] \gamma^{5} u\left(q_{1}\right) \\
& \stackrel{(\text { A. } 8)}{=} f_{5} \bar{u}\left(q_{2}\right)\left[q_{2} \gamma^{5}+\gamma^{5} q_{1}\right] u\left(q_{1}\right) \\
& \stackrel{(\text { A.1) (A. (A) })}{=} 2 m_{\mu} f_{5} \bar{u}\left(q_{2}\right) \gamma^{5} u\left(q_{1}\right) .
\end{aligned}
$$

So the Ward identity simplifies to

$$
\begin{aligned}
& \bar{u}\left(q_{2}\right)\left[f_{3} p \cdot q_{1}+f_{4} p \cdot q_{2}+2 m_{\mu} f_{5} \gamma^{5}+f_{7} p \cdot q_{1} \gamma^{5}+f_{8} p \cdot q_{2} \gamma^{5}\right] u\left(q_{1}\right) \\
& \stackrel{(3.17)}{=} \bar{u}\left(q_{2}\right)\left[-\frac{1}{2} p^{2}\left(f_{3}-f_{4}\right) \mathbb{1}_{4}+2 m_{\mu} f_{5} \gamma^{5}-\frac{1}{2} p^{2}\left(f_{7}-f_{8}\right) \gamma^{5}\right] u\left(q_{1}\right)=0
\end{aligned}
$$

Because $\mathbb{1}_{4}$ and $\gamma^{5}$ are linearly independent, their factors must vanish separately. Equating the coefficients yields

$$
\begin{align*}
f_{3} & =f_{4}  \tag{3.18}\\
f_{8}-f_{7} & =-\frac{4 m_{\mu}}{p^{2}} f_{5} \tag{3.19}
\end{align*}
$$

This result can now be plugged into eq. (3.15):

$$
\begin{aligned}
& \mathrm{i} \mathcal{M}^{\mu}=\bar{u}\left(q_{2}\right)\left[f_{1} \gamma^{\mu}+f_{3} q_{1}^{\mu}+f_{4} q_{2}^{\mu}\right. \\
& \left.+f_{5} \gamma^{\mu} \gamma^{5}+f_{7} q_{1}^{\mu} \gamma^{5}+f_{8} q_{2}^{\mu} \gamma^{5}\right] u\left(q_{1}\right) \\
& =\bar{u}\left(q_{2}\right)\left[f_{1} \gamma^{\mu}+f_{3} q_{1}^{\mu}+f_{4} q_{2}^{\mu}\right. \\
& \left.+f_{5} \gamma^{\mu} \gamma^{5}+\frac{1}{2}\left(f_{8}-f_{7}\right)\left(q_{2}^{\mu}-q_{1}^{\mu}\right) \gamma^{5}+\frac{1}{2}\left(f_{7}+f_{8}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) \gamma^{5}\right] u\left(q_{1}\right) \\
& \stackrel{(3.18),(3.19)}{=} \bar{u}\left(q_{2}\right)\left[f_{1} \gamma^{\mu}+f_{3}\left(q_{1}^{\mu}+q_{2}^{\mu}\right)\right. \\
& \left.+f_{5}\left(\gamma^{\mu}-\frac{2 m_{\mu}}{p^{2}} p^{\mu}\right) \gamma^{5}+\frac{1}{2}\left(f_{7}+f_{8}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) \gamma^{5}\right] u\left(q_{1}\right) .
\end{aligned}
$$

Now the Gordon decomposition, eq. (A.15), and its counterpart including $\gamma^{5}$, eq. (A.16), can be used to express the $q_{1}^{\mu}+q_{2}^{\mu}$ occurrences in terms of the commutator $\sigma^{\mu \nu}$ of the gamma matrices (defined by eq. (A.4)):

$$
\begin{aligned}
\mathrm{i} \mathcal{M}^{\mu}=\bar{u}\left(q_{2}\right) & {\left[\left(f_{1}+2 m_{\mu} f_{3}\right) \gamma^{\mu}-\mathrm{i} f_{3} \sigma^{\mu \nu} p_{\nu}\right.} \\
& \left.+f_{5}\left(\gamma^{\mu}-\frac{2 m_{\mu}}{p^{2}} p^{\mu}\right) \gamma^{5}-\frac{\mathrm{i}}{2}\left(f_{7}+f_{8}\right) \sigma^{\mu \nu} p_{\nu} \gamma^{5}\right] u\left(q_{1}\right)
\end{aligned}
$$

These coefficients are now defined as four different form factors [Jeg17, p. 203]:
The electric charge form factor $F_{\mathrm{E}}\left(p^{2}\right)=\frac{\mathrm{i}}{e}\left[f_{1}\left(q^{2}\right)+2 m_{\mu} f_{3}\left(p^{2}\right)\right]$, the magnetic form factor $F_{\mathrm{M}}\left(p^{2}\right)=\frac{\mathrm{i}}{e}\left[-2 m_{\mu} f_{3}\left(p^{2}\right)\right]$,

$$
\begin{equation*}
\text { the anapole moment } F_{\mathrm{A}}\left(p^{2}\right)=\frac{\mathrm{i}}{e} f_{5}\left(p^{2}\right), \tag{3.20}
\end{equation*}
$$

and the electric dipole moment $F_{\mathrm{D}}\left(p^{2}\right)=\frac{\mathrm{i}}{e}\left[-\mathrm{i} m_{\mu}\left(f_{7}+f_{8}\right)\right]$.
Using these, the invariant amplitude can be written in the following form:

$$
\begin{align*}
\mathrm{i} \mathcal{M}^{\mu}=-\mathrm{i} e \bar{u}\left(q_{2}\right) & {\left[\gamma^{\mu} F_{\mathrm{E}}\left(p^{2}\right)+\frac{\mathrm{i}}{2 m_{\mu}} \sigma^{\mu \nu} p_{\nu} F_{\mathrm{M}}\left(p^{2}\right)\right.}  \tag{3.21}\\
& \left.+\left(\gamma^{\mu}-\frac{2 m_{\mu}}{p^{2}} p^{\mu}\right) \gamma^{5} F_{\mathrm{A}}\left(p^{2}\right)+\frac{1}{2 m_{\mu}} \sigma^{\mu \nu} p_{\nu} \gamma^{5} F_{\mathrm{D}}\left(p^{2}\right)\right] u\left(q_{1}\right)
\end{align*}
$$

At tree level, the "blob" in the middle of fig. 4 is just a vertex. This results in the invariant amplitude

$$
\begin{equation*}
\mathrm{i} \mathcal{M}_{\mathrm{tl}}^{\mu}=-\mathrm{i} e \bar{u}\left(q_{2}\right) \gamma^{\mu} u\left(q_{1}\right) \tag{3.22}
\end{equation*}
$$

for which the $g$-factor is $g_{\mu}=2$. Corrections to this result thus have to come from Feynman diagrams including loops. This also shows that at tree level all form factors are 0 , except for $F_{\mathrm{E}}\left(p^{2}\right)=1$.

The magnetic form factor yields the anomalous magnetic moment in the nonrelativistic limit, i.e. for $p=0$, because the magnetic moment is measured at nonrelativistic energies:

$$
\begin{equation*}
a_{\mu}=\frac{g_{\mu}-2}{2}=F_{\mathrm{M}}(0) \tag{3.23}
\end{equation*}
$$

This means that a term proportional to $\sigma^{\mu \nu}$ (and not proportional to $\gamma^{5}$ ) has to be identified in the result of the calculation.

### 3.3 Calculation of the anomalous magnetic moment in the Standard Model

In the Standard Model, the mainly contributing one-loop diagram takes the form of fig. 5. The blob of fig. 4 is thus replaced by the loop in fig. 5. In this chapter the anomalous magnetic moment $a_{\mu}$ will be obtained by first explicitly carrying out the calculation of the invariant amplitude and then identifying the magnetic form factor of eq. (3.21). The calculation of this chapter is based on [Sch14, pp. 318-320].


Figure 5: The Feynman diagram from which the first-order loop corrections to the $g$-factor can be calculated in the Standard Model.

Using the Feynman rules in appendix A.7, the diagram 5 leads to the invariant amplitude

$$
\begin{align*}
\mathrm{i} \mathcal{M}_{\mathrm{SM}}= & \varepsilon_{\mu} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{-\mathrm{i} g_{\nu \rho}}{\left(k-q_{1}\right)^{2}+\mathrm{i} \varepsilon} \bar{u}\left(q_{2}\right)\left(-\mathrm{i} e \gamma^{\nu}\right) \\
& \cdot \frac{\mathrm{i}\left(k+\not p+m_{\mu}\right)}{(k+p)^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon}\left(-\mathrm{i} e \gamma^{\mu}\right) \frac{\mathrm{i}\left(k+m_{\mu}\right)}{k^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon}\left(-\mathrm{i} e \gamma^{\rho}\right) u\left(q_{1}\right) \\
= & -e^{3} \varepsilon_{\mu} \bar{u}\left(q_{2}\right) \int \frac{\mathrm{d}^{4} k}{(2 \pi)^{4}} \frac{\gamma^{\nu}\left(k+\not p+m_{\mu}\right)^{\mu}\left(k+m_{\mu}\right) \gamma_{\nu}}{\left[\left(k-q_{1}\right)^{2}+\mathrm{i} \varepsilon\right]\left[(k+p)^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon\right]\left[k^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon\right]} u\left(q_{1}\right) . \tag{3.24}
\end{align*}
$$

The mass $m_{\mu}$ always stands for the muon mass and therefore has an upright index $\mu$, whereas a $\mu$ in italics acts as a Lorentz index and therefore can have the values $\mu \in\{0,1,2,3\}$.

To consider the numerator and denominator separately, the following abbreviations are introduced:

$$
\begin{equation*}
\mathrm{i} \mathcal{M}_{\mathrm{SM}}=-e^{3} \varepsilon_{\mu} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{N^{\mu}}{A B C}, \tag{3.25}
\end{equation*}
$$

where

$$
\begin{align*}
N^{\mu} & =\bar{u}\left(q_{2}\right) \gamma^{\nu}\left(k+\not p+m_{\mu}\right) \gamma^{\mu}\left(k+m_{\mu}\right) \gamma_{\nu} u\left(q_{1}\right), \\
A & =k^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon,  \tag{3.26}\\
B & =(k+p)^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon, \\
C & =\left(k-q_{1}\right)^{2}+\mathrm{i} \varepsilon .
\end{align*}
$$

To aid with the evaluation of the integral in the end, the following identity (see appendix A.5) is used:

$$
\begin{equation*}
\frac{1}{A B C}=2 \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{1}{(x A+y B+z C)^{3}} . \tag{3.27}
\end{equation*}
$$

Because of the $\delta$-function the identity

$$
\begin{equation*}
x+y+z=1 \tag{3.28}
\end{equation*}
$$

can be used, momentum conservation gives the relation

$$
\begin{equation*}
p=q_{2}-q_{1}, \tag{3.29}
\end{equation*}
$$

and the incoming and outgoing muon are on their mass shells:

$$
\begin{equation*}
q_{1}^{2}=q_{2}^{2}=m_{\mu}^{2} . \tag{3.30}
\end{equation*}
$$

The denominator of eq. (3.27) can be written as

$$
\begin{aligned}
x A+y B+z C=x\left(k^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon\right)+y\left(k^{2}+2 k \cdot p\right. & \left.+p^{2}-m_{\mu}^{2}+\mathrm{i} \varepsilon\right) \\
& +z\left(k^{2}-2 k \cdot q_{1}+q_{1}^{2}+\mathrm{i} \varepsilon\right) \\
& \begin{aligned}
& \stackrel{(3.28)}{=} k^{2}+2 k \cdot\left(y p-z q_{1}\right)-(1-z) m_{\mu}^{2}+y p^{2}+z q_{1}^{2}+\mathrm{i} \varepsilon \\
& \stackrel{(3.31)}{=}\left(k+y p-z q_{1}\right)^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \\
&-(1-z) m_{\mu}^{2}+y p^{2}+z q_{1}^{2}+\mathrm{i} \varepsilon
\end{aligned}
\end{aligned}
$$

$$
=:\left(k+y p-z q_{1}\right)^{2}-\Delta+\mathrm{i} \varepsilon
$$

where the following was used in the third line:

$$
\begin{align*}
\left(k+y p-z q_{1}\right)^{2} & =k^{2}+y^{2} p^{2}+z^{2} q_{1}^{2}+2 y k \cdot p-2 z k \cdot q_{1}-2 y z p \cdot q_{1} \\
\Leftrightarrow \quad k^{2}+2 k \cdot\left(y p-z q_{1}\right) & =\left(k+y p-z q_{1}\right)^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \cdot \tag{3.31}
\end{align*}
$$

Also the variable $\Delta$ was defined. It can be simplified further:

$$
\begin{aligned}
& -\Delta=-(1-z) m_{\mu}^{2}+y p^{2}+z q_{1}^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \\
& \stackrel{(3.30)}{=}-(1-z) m_{\mu}^{2}+z(1-z) m_{\mu}^{2}+y(1-y) p^{2}+2 y z p \cdot q_{1} \\
& \stackrel{(3.28)}{=}-(1-z)^{2} m_{\mu}^{2}+x y p^{2}+y z\left[p^{2}+2 p \cdot q_{1}\right] \\
& \quad=-(1-z)^{2} m_{\mu}^{2}+x y p^{2}+y z[\underbrace{p+q_{1}}_{\stackrel{(3+29)}{=} q_{2}^{2}}-q_{1}^{2}] \\
& \stackrel{(3.28)}{=}-(1-z)^{2} m_{\mu}^{2}+x y p^{2} .
\end{aligned}
$$

So in conclusion the basis in the denominator of (3.25) simplifies to

$$
\begin{align*}
x A+y B+z C & =\left(k+y p-z q_{1}\right)^{2}-\Delta+\mathrm{i} \varepsilon  \tag{3.32}\\
\text { with } \quad \Delta & =-x y p^{2}+(1-z)^{2} m_{\mu}^{2} .
\end{align*}
$$

If an integral substitution $k \mapsto k-y p+z q_{1}$ is carried out, this result changes to

$$
\begin{equation*}
x A+y B+z C=k^{2}-\Delta+\mathrm{i} \varepsilon . \tag{3.33}
\end{equation*}
$$

Now the numerator $N^{\mu}$ can be evaluated. By using various $\gamma$-matrix identities (see eq. (A.7)), it can be written as

$$
\begin{aligned}
N^{\mu}= & \bar{u}\left(q_{2}\right) \gamma^{\nu}\left(k+p p+m_{\mu}\right) \gamma^{\mu}\left(k+m_{\mu}\right) \gamma_{\nu} u\left(q_{1}\right) \\
= & \bar{u}\left(q_{2}\right)\left[\gamma^{\nu}(k+p p) \gamma^{\mu} k k \gamma_{\nu}+m_{\mu} \gamma^{\nu}(k+p p) \gamma^{\mu} \gamma_{\nu}\right. \\
& \left.\quad+m_{\mu} \gamma^{\nu} \gamma^{\mu} k \gamma_{\nu}+m_{\mu}^{2} \gamma^{\nu} \gamma^{\mu} \gamma_{\nu}\right] u\left(q_{1}\right) \\
\stackrel{(\text { A. } 7)}{=} & \bar{u}\left(q_{2}\right)\left[-2 k \gamma^{\mu}(k+p p)+4 m_{\mu}(p+k)^{\mu}+4 m_{\mu} k^{\mu}-2 m_{\mu}^{2} \gamma^{\mu}\right] u\left(q_{1}\right) \\
= & -2 \bar{u}\left(q_{2}\right)\left[k \gamma^{\mu} p p+k \gamma^{\mu} k k+m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}\left(2 k^{\mu}+p^{\mu}\right)\right] u\left(q_{1}\right) .
\end{aligned}
$$

The aforementioned integral substitution changes this in the following way:

$$
\begin{gathered}
-\frac{1}{2} N^{\mu} \stackrel{\text { subst. }}{\mapsto} \bar{u}\left(q_{2}\right)\left[\left(k k-y p p+z q_{1}\right) \gamma^{\mu} p p+\left(k k-y p+z q_{1}\right) \gamma^{\mu}\left(k z-y p+z q_{1}\right)\right. \\
\left.+m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}\left(2 k^{\mu}-2 y p^{\mu}+2 z q_{1}^{\mu}+p^{\mu}\right)\right] u\left(q_{1}\right)
\end{gathered}
$$

$$
\begin{aligned}
=\bar{u}\left(q_{2}\right) & {\left[k \gamma^{\mu} p+\left(-y p+z q_{1}\right) \gamma^{\mu} p\right.} \\
& +k \gamma^{\mu} k+\underline{k \gamma^{\mu}\left(-y p+z q_{1}\right)} \\
& \left.+\underline{\left(-y p+z q_{1}\right) \gamma^{\mu} k+(-y p}+z q_{1}\right) \gamma^{\mu}\left(-y p+z q_{1}\right) \\
& \left.+m_{\mu}^{2} \gamma^{\mu}-\underline{m_{\mu} k^{\mu}}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu}\right] u\left(q_{1}\right) .
\end{aligned}
$$

The underlined terms are linear, i.e. antisymmetric, in one component of $k$, respectively. Therefore integrals with these integrands vanish, as explained in appendix A.8. This is equivalent to just leaving them out of $N^{\mu}$ :

$$
\begin{aligned}
&-\frac{1}{2} N^{\mu}=\bar{u}\left(q_{2}\right) {\left[\left(-y p+z q_{1}\right) \gamma^{\mu} p+k \gamma^{\mu} k k+\left(-y p+z q_{1}\right) \gamma^{\mu}\left(-y p+z q_{1}\right)\right.} \\
&\left.+m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu}\right] u\left(q_{1}\right) \\
& \stackrel{(3.29)}{=} \bar{u}\left(q_{2}\right) {\left[k \gamma^{\mu} k+\left(-(y+z) p+z q_{2}\right) \gamma^{\mu}\left((1-y) p+z q_{1}\right)\right.} \\
&\left.+m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu}\right] u\left(q_{1}\right) \\
& \stackrel{(\text { A.1),(A.2) }}{=} \bar{u}\left(q_{2}\right)\left[k \gamma^{\mu} k+\left(-(y+z) p+z m_{\mu}\right) \gamma^{\mu}\left((1-y) p+z m_{\mu}\right)\right. \\
&\left.+m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu}\right] u\left(q_{1}\right) \\
&=\bar{u}\left(q_{2}\right)\left[k \gamma^{\mu} k-(1-x)(1-y) p \gamma^{\mu} p\right. \\
&(1-x) z m_{\mu} p \gamma^{\mu}+(1-y) z m_{\mu} \gamma^{\mu} p+z^{2} m_{\mu}^{2} \gamma^{\mu} \\
&+\left.m_{\mu}^{2} \gamma^{\mu}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu}\right] u\left(q_{1}\right) .
\end{aligned}
$$

Some terms can be examined individually:

$$
\begin{align*}
& k \gamma^{\mu} k=k^{\nu} k^{\rho} \gamma_{\nu} \gamma^{\mu} \gamma_{\rho} \stackrel{(\mathrm{A} .26)}{=} \frac{1}{4} k^{2} g^{\nu \rho} \gamma_{\nu} \gamma^{\mu} \gamma_{\rho} \stackrel{(\mathrm{A} .7)}{=}-\frac{1}{2} k^{2} \gamma^{\mu}  \tag{3.34}\\
& p \gamma^{\mu} p=p_{\nu} p_{\rho} \gamma^{\nu} \gamma^{\mu} \gamma^{\rho} \stackrel{(\text { A.3) }}{=} p_{\nu} p_{\rho} \gamma^{\nu}\left(2 g^{\mu \rho}-\gamma^{\rho} \gamma^{\mu}\right) \\
& =p^{\mu} p-p p \gamma^{\mu}=p^{\mu} p-p^{2} \gamma^{\mu} \\
& \Rightarrow \quad \bar{u}\left(q_{2}\right) p \gamma^{\mu} p u\left(q_{1}\right) \stackrel{(3.29)}{=} \bar{u}\left(q_{2}\right)\left[p^{\mu}\left(q_{2}-q_{1}\right)-p^{2} \gamma^{\mu}\right] u\left(q_{1}\right) \\
& \stackrel{(\mathrm{A} .1)(\mathrm{A} .2)}{=} \bar{u}\left(q_{2}\right)\left[-p^{2} \gamma^{\mu}\right] u\left(q_{1}\right)  \tag{3.35}\\
& \bar{u}\left(q_{2}\right) p \gamma^{\mu} u\left(q_{1}\right) \stackrel{(3.29),(\mathrm{A.2)}}{=} \bar{u}\left(q_{2}\right)\left(m_{\mu}-q_{1}\right) \gamma^{\mu} u\left(q_{1}\right) \\
& \stackrel{(\text { A.B) }}{=} \bar{u}\left(q_{2}\right)\left(m_{\mu} \gamma^{\mu}-2 q_{1}^{\mu}+\gamma^{\mu} q_{1}\right) u\left(q_{1}\right) \\
& \stackrel{(\mathrm{A} .1)}{=} 2 \bar{u}\left(q_{2}\right)\left(m_{\mu} \gamma^{\mu}-q_{1}^{\mu}\right) u\left(q_{1}\right)  \tag{3.36}\\
& \bar{u}\left(q_{2}\right) \gamma^{\mu} p u\left(q_{1}\right) \stackrel{(3.29),(\text { A. } 1)}{=} \bar{u}\left(q_{2}\right) \gamma^{\mu}\left(q_{2}-m_{\mu}\right) u\left(q_{1}\right) \\
& \stackrel{(\text { A.3) }}{=} \bar{u}\left(q_{2}\right)\left(2 q_{2}^{\mu}-q_{2} \gamma^{\mu}-m_{\mu} \gamma^{\mu}\right) u\left(q_{1}\right) \\
& \stackrel{(\text { A.1) }}{=} 2 \bar{u}\left(q_{2}\right)\left(q_{2}^{\mu}-m_{\mu} \gamma^{\mu}\right) u\left(q_{1}\right) \text {. } \tag{3.37}
\end{align*}
$$

Using this, the numerator becomes

$$
\begin{aligned}
-\frac{1}{2} N^{\mu}= & \bar{u}\left(q_{2}\right)[
\end{aligned} \quad-\frac{1}{2} k^{2} \gamma^{\mu}+(1-x)(1-y) p^{2} \gamma^{\mu} .
$$

The Gordon identities proved in appendix A. 6 can be used to find the desired form of the numerator. To apply them, the required term $q_{1}^{\mu}+q_{2}^{\mu}$ has to be obtained from the following terms occurring in the numerator $N^{\mu}$ :

$$
\begin{aligned}
& 2(1-x) z m_{\mu} q_{1}^{\mu}+2(1-y) z m_{\mu} q_{2}^{\mu}-2 m_{\mu}(1-2 y) p^{\mu}-4 m_{\mu} z q_{1}^{\mu} \\
&=-2(1+x) z m_{\mu} q_{1}^{\mu}+2(1-y) z m_{\mu} q_{2}^{\mu}-2 m_{\mu}(1-2 y) p^{\mu} \\
& \stackrel{(3.29)}{=}-2(x+y) z m_{\mu} q_{1}^{\mu}-2(1-2 y-(1-y) z) m_{\mu} p^{\mu} \\
&=-(1-z) z m_{\mu}\left(q_{1}^{\mu}+q_{2}^{\mu}-p^{\mu}\right)-2(1-2 y-(1-y) z) m_{\mu} p^{\mu} \\
&=-(1-z) z m_{\mu}\left(q_{1}^{\mu}+q_{2}^{\mu}\right)+\left(-2+4 y+2 z-2 y z+z-z^{2}\right) m_{\mu} p^{\mu} \\
& \stackrel{(3.28)}{=}-(1-z) z m_{\mu}\left(q_{1}^{\mu}+q_{2}^{\mu}\right)+(z-2)(x-y) m_{\mu} p^{\mu} .
\end{aligned}
$$

Now the Gordon identity can be used:

$$
\begin{aligned}
&-\frac{1}{2} N^{\mu}=\bar{u}\left(q_{2}\right) {\left[-\frac{1}{2} k^{2} \gamma^{\mu}+(1-x)(1-y) p^{2} \gamma^{\mu}+\left(1-2 z-z^{2}\right) m_{\mu}^{2} \gamma^{\mu}\right.} \\
&\left.\quad(1-z) z m_{\mu}\left(q_{1}^{\mu}+q_{2}^{\mu}\right)+(z-2)(x-y) m_{\mu} p^{\mu}\right] u\left(q_{1}\right) \\
& \stackrel{(\mathrm{A} .15)}{=} \bar{u}\left(q_{2}\right)\left[-\frac{1}{2} k^{2} \gamma^{\mu}+(1-x)(1-y) p^{2} \gamma^{\mu}+\left(1-2 z-z^{2}\right) m_{\mu}^{2} \gamma^{\mu}\right. \\
&\left.-2(1-z) z m_{\mu}^{2} \gamma^{\mu}+\mathrm{i} z(1-z) m_{\mu} \sigma^{\mu \nu} p_{\nu}+(z-2)(x-y) m_{\mu} p^{\mu}\right] u\left(q_{1}\right) \\
&= \bar{u}\left(q_{2}\right)\left[-\frac{1}{2} k^{2} \gamma^{\mu}+(1-x)(1-y) p^{2} \gamma^{\mu}+\left(1-4 z+z^{2}\right) m_{\mu}^{2} \gamma^{\mu}\right. \\
&\left.+\mathrm{i} z(1-z) m_{\mu} \sigma^{\mu \nu} p_{\nu}+(z-2)(x-y) m_{\mu} p^{\mu}\right] u\left(q_{1}\right) .
\end{aligned}
$$

Following eq. (3.21), there should be a term proportional to $\gamma^{\mu}$ and one proportional to $\sigma^{\mu \nu} p_{\nu}$. Because fig. 5 is a QED diagram, $\gamma^{5}$ can not be present in the invariant amplitude, such that $F_{\mathrm{A}}\left(p^{2}\right)=F_{\mathrm{D}}\left(p^{2}\right)=0$ in eq. (3.21). This is exactly what can be seen here, except for an extra term proportional to $p^{\mu}$. However, this term does actually vanish, because currently the integral can be written as follows:

$$
\mathrm{i} \mathcal{M}_{\mathrm{SM}}=-2 e^{3} \varepsilon_{\mu} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{N^{\mu}}{\left(k^{2}-\Delta+\mathrm{i} \varepsilon\right)^{3}} .
$$

Except for $N^{\mu}$, the integral is symmetric in exchanging $x \leftrightarrow y$, because the respective integrals have the same bounds, the $\delta$-function is symmetric in $x \leftrightarrow y$, and $\Delta$ is too. In $N^{\mu}$, the term proportional to $p^{\mu}$ is antisymmetric in swapping $x \leftrightarrow y$. That means the integral over this term is equal to its negative, after doing the substitution $x \leftrightarrow y$, which can only be true if it is zero.

By eq. (3.21) the relevant contribution now comes from the term proportional to $\sigma^{\mu \nu} p_{\nu}$, which will be called $N_{\mathrm{M}}^{\mu}$ :

$$
\begin{equation*}
N_{\mathrm{M}}^{\mu}=-2 z(1-z) m_{\mu} \bar{u}\left(q_{2}\right) \mathrm{i} \sigma^{\mu \nu} p_{\nu} u\left(q_{1}\right) \tag{3.38}
\end{equation*}
$$

It is apparent that $N_{\mathrm{M}}^{\mu}$ does not depend on $k$. With that knowledge the $k$-integration can be carried out by using the following identity:

$$
\begin{equation*}
\int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{\left(2 \pi^{4}\right)} \frac{1}{\left(k^{2}-\Delta+\mathrm{i} \varepsilon\right)^{3}}=\frac{-\mathrm{i}}{32 \pi^{2} \Delta} \tag{3.39}
\end{equation*}
$$

This can be proven by utilizing Wick rotations in the complex plane, as shown in [Sch14, pp. 823-824]. Thus the invariant amplitude can be simplified to

$$
\begin{align*}
\mathrm{i} \mathcal{M}_{\mathrm{M}} & =-2 e^{3} \varepsilon_{\mu} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{N_{\mathrm{M}}^{\mu}}{\left(k^{2}-\Delta+\mathrm{i} \varepsilon\right)^{3}} \\
& =-2 e^{3} \varepsilon_{\mu} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{-\mathrm{i}}{32 \pi^{2} \Delta} N_{\mathrm{M}}^{\mu} \tag{3.40}
\end{align*}
$$

where again $\mathcal{M}_{\mathrm{M}}$ is the contribution proportional to the magnetic form factor to the invariant amplitude. Equating i $\mathcal{M}_{\mathrm{M}}$ from eq. (3.21) with eq. (3.40) results in

$$
\varepsilon_{\mu}(-\mathrm{i} e) \bar{u}\left(q_{2}\right) \frac{\mathrm{i}}{2 m_{\mu}} \sigma^{\mu \nu} p_{\nu} F_{\mathrm{M}}\left(p^{2}\right) u\left(q_{1}\right)=-2 e^{3} \varepsilon_{\mu} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{-\mathrm{i}}{32 \pi^{2} \Delta} N_{\mathrm{M}}^{\mu},
$$

which yields the following for the magnetic form factor:

$$
\begin{equation*}
F_{\mathrm{M}}\left(p^{2}\right) \stackrel{(3.32),(3.38)}{=} \frac{\alpha}{\pi} m_{\mu}^{2} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{z(1-z)}{-x y p^{2}+(1-z)^{2} m_{\mu}^{2}} \tag{3.41}
\end{equation*}
$$

where $\alpha=e^{2} / 4 \pi \approx 1 / 137$ is the fine-structure constant. To get the anomalous magnetic moment, the non-relativistic limit $p=0$ has to be taken (see eq. (3.23)). This leads to the following anomalous magnetic moment:

$$
\begin{aligned}
& a_{\mu}=F_{\mathrm{M}}(0) \stackrel{(3.41)}{=} \frac{\alpha}{\pi} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{z}{1-z} \\
&=\frac{\alpha}{\pi} \int_{0}^{1} \mathrm{~d} z \int_{0}^{1-z} \mathrm{~d} x \frac{z}{1-z}=\frac{\alpha}{\pi} \int_{0}^{1} z \mathrm{~d} z
\end{aligned}
$$

$$
\begin{equation*}
=\frac{\alpha}{2 \pi} \tag{3.42}
\end{equation*}
$$

The limits on the $x$-integral in the second line are caused by the $\delta$-function: Because $y=1-x-z$, and $0 \leq y \leq 1$, the inequalities

$$
\begin{gather*}
0 \leq 1-x-z \\
\Rightarrow \quad x \leq 1-z, \tag{3.43}
\end{gather*}
$$

and

$$
\begin{gather*}
1-x-z \leq 1  \tag{3.44}\\
\Rightarrow \quad-z \leq 0 \leq x
\end{gather*}
$$

yield the integral limits $0 \leq x \leq 1-z$. The end result is thus

$$
\begin{align*}
& a_{\mu}=\frac{\alpha}{2 \pi} \approx 0.001161 \\
\Leftrightarrow \quad & g_{\mu}=2+\frac{\alpha}{\pi} \approx 2.002323 . \tag{3.45}
\end{align*}
$$

The first one to calculate this result was Schwinger in 1948. The current loop corrected theoretical value includes pure QED, electroweak and hadronic contributions [Aoy+20]. The QED contribution is calculated by using a perturbative expansion in $\alpha$, including all terms up to $\mathcal{O}\left(\alpha^{5}\right)$, i.e. Feynman diagrams with more and more loops. These contributions are the largest of the three, and have negligible numerical uncertainties, as do the very small electroweak contributions. The latter are calculated from Feynman diagrams including the W bosons, the Z boson or the Higgs boson. The most difficult to calculate are the hadronic contributions, i.e. Feynman diagrams including loop quarks and gluons, and additionally they yield the main part of the numerical uncertainties. It is currently tried to calculate them using either a datadriven, or a lattice-QCD approach. Reducing the uncertainty of the theoretical side is very important to keep pace with current experiments, as the muon anomalous magnetic moment is a very promising hint of BSM physics.

## $4 a_{\mu}$ in an extended scotogenic model

### 4.1 Introduction to the model

The scotogenic model is a radiative seesaw model which was first introduced by Ernest Ma in [Ma06]. It is a minimal extension of the Standard Model, which yields small neutrino masses via radiative corrections and also a candidate for dark matter. This is achieved by extending the SM by an exact $\mathbb{Z}_{2}$ symmetry (see appendix A.1). Now three heavy singlet Majorana neutrinos $\mathrm{N}_{i}, i \in\{1,2,3\}$ and a scalar doublet $\left(\eta^{+}, \eta^{0}\right)$ are
added to the Standard Model. These have the properties listed for N and $\eta$ in table 1. The $\mathrm{SU}(2)$ column shows the dimension of the representation (i.e. if it is an $\mathrm{SU}(2)$ singlet or doublet) of the respective particle and the $U(1)$ one shows its weak hypercharge $Y=2\left(Q-T_{3}\right)$, where $Q$ is the electric charge and $T_{3}$ is the third component of the weak isospin.

The exactness of the $\mathbb{Z}_{2}$ symmetry prevents the $\eta$ field from obtaining a vacuum expectation value (VEV). It is notable that the new particles all have a $\mathbb{Z}_{2}$ charge of -1 , while the Standard Model particles all have a $\mathbb{Z}_{2}$ charge of 1 . This has the effect that every possible vertex has to consist of an even number of new particles to preserve $\mathbb{Z}_{2}$ charge conservation. Firstly, the $\mathbb{Z}_{2}$ symmetry is thus the reason why no new particles can decay into just Standard Model fields, which makes the lightest $\mathbb{Z}_{2}$-odd particle a suitable dark matter candidate, if it has a neutral electric charge. Secondly, the neutrino masses can be created only via loop processes, which keeps them sufficiently small and does not have to involve particles with exorbitant masses.

In this thesis an extension to the scotogenic model, featured in [CN19], is considered. Here the vector-like fermion doublets $X_{1}=\left(\chi_{1}^{+}, \chi_{1}^{0}\right)$ and $X_{2}=\left(\chi_{2}^{0}, \chi_{2}^{-}\right)$are included in addition to the fields already present in the scotogenic model. The properties of these are also listed in table 1.

Table 1: The field content of the scotogenic model (SM plus $N$ and $\eta$ ) and its extension (SM plus $\mathrm{N}, \eta$ and $\mathrm{X}_{1 / 2}$ ).

| Field | Generations | Spin | $\mathrm{SU}(3)_{C}$ | $\mathrm{SU}(2)_{L}$ | $\mathrm{U}(1)_{Y}$ | $\mathbb{Z}_{2}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Standard Model fields |  |  |  |  |  |  |  |
| N | 3 | $1 / 2$ | 1 | 1 | 0 | -1 |  |
| $\eta$ | 1 | 0 | 1 | 2 | 1 | -1 |  |
| $\mathrm{X}_{1}$ | 1 | $1 / 2$ | 1 | 2 | 1 | -1 |  |
| $\mathrm{X}_{2}$ | 1 | $1 / 2$ | 1 | 2 | -1 | -1 |  |

The BSM Lagrangian $\mathcal{L}$ of this model can be written as follows:

$$
\begin{align*}
\mathcal{L}=- & \frac{1}{2} \lambda_{2}\left(\eta^{\dagger} \eta\right)^{2}-\lambda_{3} \mathrm{H}^{\dagger} \mathrm{H} \eta^{\dagger} \eta-\lambda_{4} \mathrm{H}^{\dagger} \eta \eta^{\dagger} \mathrm{H}-m_{\eta}^{2} \eta^{\dagger} \eta \\
+ & {\left[-\frac{1}{2} \lambda_{5}\left(\eta^{\dagger} \mathrm{H}\right)^{2}-m_{\mathrm{X}} \mathrm{X}_{1} \mathrm{X}_{2}-\frac{1}{2} m_{\mathrm{N}_{k}} \mathrm{~N}_{k} \mathrm{~N}_{k}\right.}  \tag{4.1}\\
& \left.-y_{1} \mathrm{H}^{\dagger} \mathrm{X}_{1} \mathrm{~N}-y_{2} \eta^{\dagger} \mathrm{X}_{2} \mathrm{e}_{\mathrm{R}}^{\mathrm{c}}-y_{3} \mathrm{HX} \mathrm{X}_{2} \mathrm{~N}-y_{4} \mathrm{~L} \eta \mathrm{~N}+\mathrm{H} . \mathrm{c.} .\right]
\end{align*}
$$

where $H$ is the SM Higgs doublet, $L$ is the SM left-handed lepton doublet, $e_{R}^{c}$ is the SM right-handed lepton singlet, $\mathrm{N}=\left(\mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{~N}_{3}\right)$ and $k \in\{1,2,3\}$. Implicitly the convention of $\operatorname{SU}(2)$-invariant products, presented in [May18, pp. 57-74], is used here. Because of the multiple generations of N , e and $\mathrm{L}, y_{1}, y_{2}$ and $y_{3}$ are 3-component vectors, and $y_{4}$ is a $3 \times 3$-matrix. For the parameters $\lambda_{i}$ the notation from [Ma06]
is followed. Because in [CN19] the Yukawa coupling $y_{1}$ is set to 0 , this will be done here too. The mixed states of the neutral fermions $\left(\chi_{1}^{0}, \chi_{2}^{0}, \mathrm{~N}_{1}, \mathrm{~N}_{2}, \mathrm{~N}_{3}\right)$ will be called $\psi_{i}$, $i \in\{1,2,3,4,5\}$, where their masses increase with increasing $i$, i.e. $\psi_{1}$ is the fermion with the smallest mass and will therefore always be the dark matter candidate in this thesis.

### 4.2 Calculation of the anomalous magnetic moment in the extended scotogenic model

In the extended scotogenic model the main contribution to the anomalous magnetic moment arises from the diagram in fig. 6 [CN19]. Its invariant amplitude $\mathcal{M}$ and the resulting anomalous magnetic moment will be calculated in this chapter.


Figure 6: The relevant Feynman diagram for the anomalous magnetic moment.

The arrows show the direction of charge flow. The bottom loop propagator shows one of the mixed states $\psi_{i}, i \in\{1,2,3,4,5\}$, here called $\psi$ for simplicity. The contributions of the different fermions will be summed in the end.

The invariant amplitude is again compiled by using the Feynman rules established in appendix A.7:

$$
\begin{align*}
& \mathrm{i} \mathcal{M}= \varepsilon_{\mu} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \underbrace{(-\mathrm{i} e)\left(2 k^{\mu}-p^{\mu}\right)}_{\gamma, \eta \text {-vertex }} \underbrace{\frac{\mathrm{i}}{k^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon} \frac{\mathrm{i}}{\eta-p)^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon}} \\
&=-\bar{u}\left(q_{2}\right) \underbrace{\left(d+d^{\prime} \gamma^{5}\right)}_{\text {left } \mu, \eta, \psi \text {-vertex }} \underbrace{\left(d\left(q_{1}\right)\right.}_{\begin{array}{c}
\psi \text {-propagators } \\
\frac{\mathrm{i}\left(k+q_{1}+m_{\psi}\right)}{\left(k+q_{1}\right)^{2}-m_{\psi}^{2}+\mathrm{i} \varepsilon} \\
\text { right } \mu, \eta, \psi-\text {-vertex }
\end{array}\left(f+f^{\prime} \gamma^{5}\right)} \\
&=\int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \bar{u}\left(q_{2}\right) \frac{\left(2 k^{\mu}-p^{\mu}\right)\left(d+d^{\prime} \gamma^{5}\right)}{\left[k^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon\right]\left[(k-p)^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon\right]}  \tag{4.2}\\
& \cdot \frac{\left(k+q_{1}+m_{\psi}\right)\left(f+f^{\prime} \gamma^{5}\right)}{\left[\left(k+q_{1}\right)^{2}-m_{\psi}^{2}+\mathrm{i} \varepsilon\right]} u\left(q_{1}\right) .
\end{align*}
$$

First the same abbreviations as in section 3.3 can be introduced again:

$$
\begin{equation*}
\mathrm{i} \mathcal{M}=-\varepsilon_{\mu} e \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{N^{\mu}}{A B C} \tag{4.3}
\end{equation*}
$$

such that

$$
\begin{align*}
N^{\mu} & =\bar{u}\left(q_{2}\right)\left(2 k^{\mu}-p^{\mu}\right)\left(d+d^{\prime} \gamma^{5}\right)\left(k+q_{1}+m_{\psi}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
A & =k^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon  \tag{4.4}\\
B & =(k-p)^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon \\
C & =\left(k+q_{1}\right)^{2}-m_{\psi}^{2}+\mathrm{i} \varepsilon
\end{align*}
$$

Again, eqs. (3.28) to (3.30) can be used to simplify the denominator of eq. (3.27):

$$
\begin{aligned}
& x A+y B+z C= x\left(k^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon\right)+y\left(k^{2}-\right. \\
&\left.2 k \cdot p+p^{2}-m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon\right) \\
&+z\left(k^{2}+2 k \cdot q_{1}+q_{1}^{2}-m_{\psi}^{2}+\mathrm{i} \varepsilon\right) \\
& \stackrel{(3.28)}{=} k^{2}+2 k \cdot\left(z q_{1}-y p\right)+z q_{1}^{2}+y p^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon \\
& \stackrel{(4.5)}{=}\left(k-y p+z q_{1}\right)^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \\
&+z q_{1}^{2}+y p^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2}+\mathrm{i} \varepsilon
\end{aligned}
$$

$$
=:\left(k-y p+z q_{1}\right)^{2}-\Delta+\mathrm{i} \varepsilon,
$$

where the following was used in the third line:

$$
\begin{align*}
\left(k-y p+z q_{1}\right)^{2} & =k^{2}+y^{2} p^{2}+z^{2} q_{1}^{2}-2 y k \cdot p+2 z k \cdot q_{1}-2 y z p \cdot q_{1} \\
\Leftrightarrow \quad k^{2}+2 k \cdot\left(z q_{1}-y p\right) & =\left(k-y p+z q_{1}\right)^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \tag{4.5}
\end{align*}
$$

Analogous to the SM calculation the variable $\Delta$ was defined:

$$
\begin{aligned}
-\Delta & =z q_{1}^{2}+y p^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2}-y^{2} p^{2}-z^{2} q_{1}^{2}+2 y z p \cdot q_{1} \\
& =y(1-y) p^{2}+2 y z p \cdot q_{1}+z(1-z) q_{1}^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2} \\
& \stackrel{(3.28)}{=}(x y+y z) p^{2}+2 y z p \cdot q_{1}+(x z+y z) q_{1}^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2} \\
& =y z\left(p+q_{1}\right)^{2}+x y p^{2}+x z q_{1}^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2} \\
& \stackrel{(3.29)}{=} y z q_{2}^{2}+x y p^{2}+x z q_{1}^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2} \\
& \stackrel{(3.30),(3.28)}{=} x y p^{2}+z(1-z) m_{\mu}^{2}-z m_{\psi}^{2}-(1-z) m_{\eta^{ \pm}}^{2} .
\end{aligned}
$$

Using this, the basis in the denominator of eq. (4.3) simplifies to

$$
\begin{align*}
x A+y B+z C & =\left(k-y p+z q_{1}\right)^{2}-\Delta+\mathrm{i} \varepsilon \\
\text { with } \quad \Delta & =-x y p^{2}-z(1-z) m_{\mu}^{2}+z m_{\psi}^{2}+(1-z) m_{\eta^{ \pm}}^{2} . \tag{4.6}
\end{align*}
$$

Now the integral substitution $k \mapsto k+y p-z q_{1}$ can be carried out again. The overall sign differs from the substitution used in the SM calculation, because in this section the direction of the variable $k$ was reversed in fig. 6 with respect to fig. 5. The substitution again gets rid of the terms added to $k$ :

$$
\begin{equation*}
x A+y B+z C=k^{2}-\Delta+\mathrm{i} \varepsilon \tag{4.7}
\end{equation*}
$$

The numerator $N^{\mu}$ takes the following form:

$$
\begin{aligned}
& N^{\mu}=\bar{u}\left(q_{2}\right)\left(2 k^{\mu}-p^{\mu}\right)\left(d+d^{\prime} \gamma^{5}\right)\left(\not k+q_{1}+m_{\psi}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
& \qquad \begin{aligned}
\text { subst. } \\
\mapsto \\
u
\end{aligned}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left(2 k^{\mu}+(2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right)
\end{aligned} \quad \begin{aligned}
& \cdot\left(\not k+y p p+(1-z) q_{1}+m_{\psi}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
=\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right) & {\left[2 k^{\mu} \not k+2 k^{\mu}\left(y p+(1-z) q_{1}+m_{\psi}\right)\right.} \\
& +\frac{\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) k}{\left.\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right)\left(y p+(1-z) q_{1}+m_{\psi}\right)\right]\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right)}
\end{aligned}
$$

The underlined terms are linear in the components of $k$ again, so the integrals result in zero (see appendix A.8). Additionally, to simplify the first term in the brackets, eq. (A.26) can be used:

$$
\begin{equation*}
2 k^{\mu} k=2 k^{\mu} k^{\nu} \gamma_{\nu} \stackrel{(\mathrm{A} .26)}{=} \frac{1}{2} g^{\mu \nu} k^{2} \gamma_{\nu}=\frac{1}{2} k^{2} \gamma^{\mu} . \tag{4.8}
\end{equation*}
$$

This leads to

$$
\begin{aligned}
N^{\mu}=\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left[\frac{1}{2} k^{2} \gamma^{\mu}\right. & +\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) \\
\cdot & \left.\left(y p+(1-z) q_{1}+m_{\psi}\right)\right]\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right)
\end{aligned}
$$

To make use of the Dirac equation (A.1) and its adjoint (A.2), momentum conservation (eq. (3.29)) can be used to get rid of $p$ :

$$
\begin{aligned}
& N^{\mu} \stackrel{(3.28),(3.29)}{=} \bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left[\frac{1}{2} k^{2} \gamma^{\mu}+\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right)\right. \\
& \left.\quad \cdot\left(y q_{2}+x q_{1}+m_{\psi}\right)\right]\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
& \stackrel{(\mathrm{A} .8)}{=} \\
& \quad \bar{u}\left(q_{2}\right)\left[\frac{1}{2} k^{2} \gamma^{\mu}+\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) y q_{2}\right]\left(d-d^{\prime} \gamma^{5}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
& \quad+\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left(f-f^{\prime} \gamma^{5}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) x q_{1} u\left(q_{1}\right) \\
& \quad+\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) m_{\psi}\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\stackrel{(\mathrm{A} .1),(\mathrm{A} .2)}{=} & \bar{u}\left(q_{2}\right)\left[\frac{1}{2} k^{2} \gamma^{\mu}+\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) y m_{\mu}\right]\left(d-d^{\prime} \gamma^{5}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) \\
& +\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left(f-f^{\prime} \gamma^{5}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) x m_{\mu} u\left(q_{1}\right) \\
& +\bar{u}\left(q_{2}\right)\left(d+d^{\prime} \gamma^{5}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) m_{\psi}\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right) .
\end{aligned}
$$

Now the part proportional to $\gamma^{5}$ can be separated:

$$
\begin{aligned}
N^{\mu}= & \frac{1}{2} \bar{u}\left(q_{2}\right) k^{2} \gamma^{\mu}\left(d-d^{\prime} \gamma^{5}\right)\left(f+f^{\prime} \gamma^{5}\right) u\left(q_{1}\right)
\end{aligned} \quad \begin{aligned}
& \quad+\bar{u}\left(q_{2}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right)\left[\left(d+d^{\prime} \gamma^{5}\right)\left(f-f^{\prime} \gamma^{5}\right) x m_{\mu}\right. \\
& \left.\quad+\left(d-d^{\prime} \gamma^{5}\right)\left(f+f^{\prime} \gamma^{5}\right) y m_{\mu}+\left(d+d^{\prime} \gamma^{5}\right)\left(f+f^{\prime} \gamma^{5}\right) m_{\psi}\right] u\left(q_{1}\right) \\
& \stackrel{(\mathrm{A} .9)}{=} \frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f-d^{\prime} f^{\prime}\right) k^{2} \gamma^{\mu} u\left(q_{1}\right)+\frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f^{\prime}-d^{\prime} f\right) k^{2} \gamma^{\mu} \gamma^{5} u\left(q_{1}\right) \\
& \\
& +\bar{u}\left(q_{2}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right)\left[\left(d f-d^{\prime} f^{\prime}\right)(x+y) m_{\mu}\right. \\
& \left.\quad+\left(d f+d^{\prime} f^{\prime}\right) m_{\psi}\right] u\left(q_{1}\right)
\end{aligned} \quad \begin{aligned}
& \quad+\bar{u}\left(q_{2}\right)\left((2 y-1) p^{\mu}-2 z q_{1}^{\mu}\right) \gamma^{5}\left[\left(d^{\prime} f-d f^{\prime}\right)(x-y) m_{\mu}\right. \\
& \\
& \left.\quad+\left(d^{\prime} f+d f^{\prime}\right) m_{\psi}\right] u\left(q_{1}\right) .
\end{aligned}
$$

The Gordon identities proved in appendix A. 6 can be used to find the desired form of the numerator. To apply them, the required $q_{1}^{\mu}+q_{2}^{\mu}$ is obtained by using

$$
\begin{equation*}
2 q_{1}^{\mu} \stackrel{(3.29)}{=} q_{1}^{\mu}+q_{2}^{\mu}-p^{\mu} \tag{4.9}
\end{equation*}
$$

This leads to

$$
\begin{aligned}
\begin{aligned}
& N^{\mu} \stackrel{(3.28),(4.9)}{=} \frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f-d^{\prime} f^{\prime}\right) k^{2} \gamma^{\mu} u\left(q_{1}\right)+\frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f^{\prime}-d^{\prime} f\right) k^{2} \gamma^{\mu} \gamma^{5} u\left(q_{1}\right) \\
&-\bar{u}\left(q_{2}\right)\left((x-y) p^{\mu}+z\left(q_{1}^{\mu}+q_{2}^{\mu}\right)\right)\left[\left(d f-d^{\prime} f^{\prime}\right)(x+y) m_{\mu}\right. \\
&\left.+\left(d f+d^{\prime} f^{\prime}\right) m_{\psi}\right] u\left(q_{1}\right)
\end{aligned} \\
-\bar{u}\left(q_{2}\right)\left((x-y) p^{\mu}+z\left(q_{1}^{\mu}+q_{2}^{\mu}\right)\right) \gamma^{5}\left[\left(d^{\prime} f-d f^{\prime}\right)(x-y) m_{\mu}\right. \\
\left.+\left(d^{\prime} f+d f^{\prime}\right) m_{\psi}\right] u\left(q_{1}\right)
\end{aligned} \quad \begin{array}{r}
(\mathrm{A} .15),(\mathrm{A} .16) \\
= \\
\frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f-d^{\prime} f^{\prime}\right) k^{2} \gamma^{\mu} u\left(q_{1}\right)+\frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f^{\prime}-d^{\prime} f\right) k^{2} \gamma^{\mu} \gamma^{5} u\left(q_{1}\right) \\
-\bar{u}\left(q_{2}\right)\left(\underline{(x-y) p^{\mu}}+\underline{\left.z\left(2 m_{\mu} \gamma^{\mu}+\mathrm{i} \sigma^{\mu \nu}\left(q_{1 \nu}-q_{2 v}\right)\right)\right)}\right) \\
\cdot
\end{array}
$$

The terms underlined in red are antisymmetric in substituting $x \leftrightarrow y$, the blue ones are symmetric in $x \leftrightarrow y$, so products which combine red and blue are antisymmetric. The denominator of the integral is symmetric in $x \leftrightarrow y$, see eq. (4.6), so these whole integrals are antisymmetric in $x \leftrightarrow y$, which means that the integrals must vanish. This is again equivalent to leaving the terms out of the numerator $N^{\mu}$ altogether. Also momentum conservation can be used to replace $q_{1 \nu}-q_{2 \nu}$ by $-p_{\nu}$ :

$$
\begin{aligned}
N^{\mu}= & \frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f-d^{\prime} f^{\prime}\right) k^{2} \gamma^{\mu} u\left(q_{1}\right)+\frac{1}{2} \bar{u}\left(q_{2}\right)\left(d f^{\prime}-d^{\prime} f\right) k^{2} \gamma^{\mu} \gamma^{5} u\left(q_{1}\right) \\
& -\bar{u}\left(q_{2}\right) z\left(2 m_{\mu} \gamma^{\mu}-\mathrm{i} \sigma^{\mu \nu} p_{\nu}\right) \\
& \cdot\left[\left(d f-d^{\prime} f^{\prime}\right)(x+y) m_{\mu}+\left(d f+d^{\prime} f^{\prime}\right) m_{\psi}\right] u\left(q_{1}\right) \\
& -\bar{u}\left(q_{2}\right)(x-y)^{2}\left(d^{\prime} f-d f^{\prime}\right) m_{\mu} p^{\mu} \gamma^{5} u\left(q_{1}\right) \\
& +\bar{u}\left(q_{2}\right) \mathrm{i} z \sigma^{\mu \nu} p_{\nu}\left(d^{\prime} f+d f^{\prime}\right) m_{\psi} \gamma^{5} u\left(q_{1}\right) .
\end{aligned}
$$

According to eq. (3.21) the relevant contribution now comes from the term proportional to $\sigma^{\mu \nu} p_{\nu}$ (but not proportional to $\gamma^{5}$ ), which will be called $N_{\mathrm{M}}^{\mu}$ :

$$
\begin{equation*}
N_{\mathrm{M}}^{\mu}=\bar{u}\left(q_{2}\right) \mathrm{i} \sigma^{\mu \nu} p_{\nu} z\left[\left((1-z) m_{\mu}+m_{\psi}\right) d f-\left((1-z) m_{\mu}-m_{\psi}\right) d^{\prime} f^{\prime}\right] u\left(q_{1}\right) . \tag{4.10}
\end{equation*}
$$

Here too it is apparent that $N_{\mathrm{M}}^{\mu}$ does not depend on $k$. Therefore eq. (3.39) can again be applied directly:

$$
\begin{align*}
\mathrm{i} \mathcal{M}_{\mathrm{M}} & =-2 \varepsilon_{\mu} e \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{N_{\mathrm{M}}^{\mu}}{\left(k^{2}-\Delta+\mathrm{i} \varepsilon\right)^{3}} \\
& =-2 \varepsilon_{\mu} e \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{-\mathrm{i}}{32 \pi^{2} \Delta} N_{\mathrm{M}}^{\mu}, \tag{4.11}
\end{align*}
$$

where again $\mathcal{M}_{\mathrm{M}}$ is the contribution proportional to the magnetic form factor. Equating $\mathrm{i} \mathcal{M}_{\mathrm{M}}$ from eqs. (3.21) and (4.11),
$\varepsilon_{\mu}(-\mathrm{i} e) \bar{u}\left(q_{2}\right) \frac{\mathrm{i}}{2 m_{\mu}} \sigma^{\mu \nu} p_{\nu} F_{\mathrm{M}}\left(p^{2}\right) u\left(q_{1}\right)=-2 \varepsilon_{\mu} e \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{-\mathrm{i}}{32 \pi^{2} \Delta} N_{\mathrm{M}}^{\mu}$,
yields the magnetic form factor in a preliminary integral form by rearranging and division by $\varepsilon_{\mu} \bar{u}\left(q_{2}\right) \sigma^{\mu \nu} p_{\nu} u\left(q_{1}\right)$ :

$$
\begin{align*}
F_{\mathrm{M}}\left(p^{2}\right) \stackrel{(4.6),(4.10)}{=}-\frac{1}{8 \pi^{2}} m_{\mu} & \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \\
& \cdot \frac{z\left((1-z) m_{\mu}+m_{\psi}\right) d f-z\left((1-z) m_{\mu}-m_{\psi}\right) d^{\prime} f^{\prime}}{-x y p^{2}-z(1-z) m_{\mu}^{2}+z m_{\psi}^{2}+(1-z) m_{\eta^{ \pm}}^{2}} \tag{4.12}
\end{align*}
$$

Here the non-relativistic limit can be employed again (see eq. (3.23)), i.e. $p^{2}$ can be set to 0 . For convenience the occurring integral is defined as $I_{\psi}^{ \pm}$. Its index $\psi$ should remind of the different masses of the propagator fermions in fig. 6 , which means that the integral yields a different contribution for every fermion.

$$
\begin{aligned}
& F_{\mathrm{M}}(0)=-\frac{1}{8 \pi^{2}} m_{\mu} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \\
& \cdot \frac{z\left((1-z) m_{\mu}+m_{\psi}\right) d f-z\left((1-z) m_{\mu}-m_{\psi}\right) d^{\prime} f^{\prime}}{-z(1-z) m_{\mu}^{2}+z m_{\psi}^{2}+(1-z) m_{\eta^{ \pm}}^{2}} \\
&=:-\frac{1}{8 \pi^{2}} \frac{m_{\mu}^{2}}{m_{\eta^{ \pm}}^{2}}\left(I_{\mathrm{N}}^{+} d f-I_{\mathrm{N}}^{-} d^{\prime} f^{\prime}\right),
\end{aligned}
$$

which includes the definition

$$
\begin{align*}
I_{\psi}^{ \pm} & :=\frac{m_{\eta^{ \pm}}^{2}}{m_{\mu}} \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{z\left((1-z) m_{\mu} \pm m_{\psi}\right)}{-z(1-z) m_{\mu}^{2}+z m_{\psi}^{2}+(1-z) m_{\eta^{ \pm}}^{2}} \\
& =\int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} z \delta(x+y+z-1) \frac{z\left(1-z \pm \varepsilon_{\psi}\right)}{-z(1-z) \lambda^{2}+z\left(\varepsilon_{\psi} \lambda\right)^{2}+1-z} \tag{4.13}
\end{align*}
$$

where the abbreviations

$$
\begin{equation*}
\varepsilon_{\psi}=\frac{m_{\psi}}{m_{\mu}} \quad \text { and } \quad \lambda=\frac{m_{\mu}}{m_{\eta^{ \pm}}} \tag{4.14}
\end{equation*}
$$

were used. Evaluation of the $\delta$-function and the substitution $z^{\prime}=1-z$ change $I_{\psi}^{ \pm}$to

$$
\begin{align*}
I_{\psi}^{ \pm} & \stackrel{\text { eval. } \delta}{=} \int_{0}^{1} \mathrm{~d} z \int_{0}^{1-z} \mathrm{~d} y \frac{z\left(1-z \pm \varepsilon_{\psi}\right)}{-z(1-z) \lambda^{2}+z\left(\varepsilon_{\psi} \lambda\right)^{2}+1-z} \\
& =\int_{0}^{1} \mathrm{~d} z \frac{z(1-z)\left(1-z \pm \varepsilon_{\psi}\right)}{-z(1-z) \lambda^{2}+z\left(\varepsilon_{\psi} \lambda\right)^{2}+1-z} \\
& \stackrel{\text { sub. }}{=} \int_{0}^{1} \mathrm{~d} z^{\prime} \frac{z^{\prime}\left(1-z^{\prime}\right)\left(z^{\prime} \pm \varepsilon_{\psi}\right)}{-z^{\prime}\left(1-z^{\prime}\right) \lambda^{2}+\left(1-z^{\prime}\right)\left(\varepsilon_{\psi} \lambda\right)^{2}+z^{\prime}} \\
& =\int_{0}^{1} \mathrm{~d} z^{\prime} \frac{z^{\prime}\left(1-z^{\prime}\right)\left(z^{\prime} \pm \varepsilon_{\psi}\right)}{\left(\varepsilon_{\psi} \lambda\right)^{2}\left(1-z^{\prime}\right)\left(1-\varepsilon_{\psi}^{-2} z^{\prime}\right)+z^{\prime}} . \tag{4.15}
\end{align*}
$$

The limits for the $y$-integration after the evaluation of the $\delta$-function can be obtained by the same considerations made for the SM calculation, namely eq.(3.43) and eq.(3.44).

The couplings $f$ and $f^{\prime}$ can be expressed in terms of $d$ and $d^{\prime}$. To see why, the relevant interaction terms in the Lagrangian after electroweak symmetry breaking are looked at [LPQ18, p. 25]:

$$
\mathcal{L}_{\mathrm{int}}=\tilde{d} \eta^{+} \bar{\psi} \mu^{-}+\tilde{d}^{\prime} \eta^{+} \bar{\psi} \gamma^{5} \mu^{-}+\text {H. c. }
$$

The coefficients are equal to the couplings up to a constant phase. To get the couplings on the other side of fig. 6 , the hermitian conjugate is examined:

$$
\begin{aligned}
\text { H. c. } & =\tilde{d}^{*}\left(\eta^{+} \psi^{\dagger} \gamma^{0} \mu^{-}\right)^{\dagger}+\tilde{d}^{\prime *}\left(\eta^{+} \psi^{\dagger} \gamma^{0} \gamma^{5} \mu^{-}\right)^{\dagger} \\
& =\tilde{d}^{*}\left(\mu^{-}\right)^{\dagger} \gamma^{0} \psi\left(\eta^{+}\right)^{\dagger}+\tilde{d}^{\prime *}\left(\mu^{-}\right)^{\dagger} \gamma^{5} \gamma^{0} \psi\left(\eta^{+}\right)^{\dagger} \\
& \stackrel{(\text { A. } 8)}{=} \tilde{d}^{*}\left(\mu^{-}\right)^{\dagger} \gamma^{0} \psi\left(\eta^{+}\right)^{\dagger}-\tilde{d}^{\prime *} \bar{\mu}^{-} \gamma^{5} \psi\left(\eta^{+}\right)^{\dagger} \\
& \stackrel{!}{=} \tilde{f}\left(\mu^{-}\right)^{\dagger} \gamma^{0} \psi\left(\eta^{+}\right)^{\dagger}+\tilde{f}^{\prime} \bar{\mu}^{-} \gamma^{5} \psi\left(\eta^{+}\right)^{\dagger} .
\end{aligned}
$$

Thus the result is $f=d^{*}$ and $f^{\prime}=-d^{*}$. The couplings of course also depend on the fermion in the propagator of fig. 6 , so they will get an index in the following summation over the fermions. The summation index $\psi$ will count the mass eigenstates of the mixed fermions. Finally, the anomalous magnetic moment in integral form can be written as

$$
\begin{align*}
\Delta a_{\mu}=F_{\mathrm{M}}(0) & =-\frac{1}{8 \pi^{2}} \frac{m_{\mu}^{2}}{m_{\eta^{ \pm}}^{2}} \sum_{\text {fermion } \psi}\left(I_{\psi}^{+}\left|d_{\psi}\right|^{2}+I_{\psi}^{-}\left|d_{\psi}^{\prime}\right|^{2}\right) \\
& =-\frac{1}{8 \pi^{2}} \frac{m_{\mu}^{2}}{m_{\eta^{ \pm}}^{2}} \sum_{\text {fermion } \psi} \int_{0}^{1} \mathrm{~d} x \frac{x(1-x)\left(x+\varepsilon_{\psi}\right)\left|d_{\psi}\right|^{2}+x(1-x)\left(x-\varepsilon_{\psi}\right)\left|d_{\psi}^{\prime}\right|^{2}}{\left(\varepsilon_{\psi} \lambda\right)^{2}(1-x)\left(1-\varepsilon_{\psi}^{-2} x\right)+x} \tag{4.16}
\end{align*}
$$

This result is also featured in [LPQ18, p. 26] in eqs. (26a) and (26b), and in [JN09, p. 101] in eq. (264). In the latter the polynomials for the scalar and pseudoscalar coupling directly under eq. (264) have to be plugged in to agree with the result in [LPQ18] and the one calculated here.

Equation (4.16) is suited for actual numerical evaluation with a method to calculate integrals numerically, because $I_{\psi}^{ \pm}$consists only of polynomials of up to third order, and the limits of integration are not prone to be affected by numerical errors. Therefore this result is used in section 5.3.

Further analytical evaluation of $I_{\psi}^{ \pm}$is still possible. For that, at first the square in the denominator is completed:

$$
\begin{aligned}
\left(\varepsilon_{\psi} \lambda\right)^{2}(1-x)\left(1-\varepsilon_{\psi}^{-2} x\right)+x & =\lambda^{2} x^{2}+\left(1-\left(\varepsilon_{\psi} \lambda\right)^{2}-\lambda^{2}\right) x+\left(\varepsilon_{\psi} \lambda\right)^{2} \\
& =\lambda^{2}(x^{2}+\underbrace{\left(\lambda^{-2}-\varepsilon_{\psi}^{2}-1\right)}_{=: 2 m} x)+\left(\varepsilon_{\psi} \lambda\right)^{2} \\
& =\lambda^{2}(x+m)^{2}-\lambda^{2} m^{2}+\left(\varepsilon_{\psi} \lambda\right)^{2} .
\end{aligned}
$$

where $m$ is a dimensionless abbreviation. Using the substitution $y=x+m$, the
integral transforms into

$$
I_{f}^{ \pm}=\frac{1}{\lambda^{2}} \int_{m}^{m+1} \mathrm{~d} y \frac{(y-m)(1-y+m)\left(y-m \pm \varepsilon_{\psi}\right)}{y^{2} \underbrace{-m^{2}+\varepsilon_{\psi}^{2}}_{:=-a^{2}}}
$$

with a real constant $a$. The numerator can then be rewritten as a polynomial in $y$ :

$$
\begin{aligned}
(y-m)(1-y+m)(y-m & \left. \pm \varepsilon_{\psi}\right)=-y^{3}+\left(1+3 m \mp \varepsilon_{\psi}\right) y^{2} \\
& +\left(-3 m^{2}-2 m \pm(2 m+1) \varepsilon_{\psi}\right) y-m(m+1)\left(-m \pm \varepsilon_{\psi}\right)
\end{aligned}
$$

Now for the different powers of $y$ the integrals shown in appendix A. 9 can be used, which leads to the rather lengthy equation

$$
\begin{aligned}
I_{f}^{ \pm}= & -\frac{1}{2 \lambda^{2}}\left[2 m+1+a^{2} \ln \left|\frac{(m+1)^{2}-a^{2}}{m^{2}-a^{2}}\right|\right] \\
& +\frac{1}{\lambda^{2}}\left(1+3 m \mp \varepsilon_{\psi}\right)\left[1+\frac{a}{2} \ln \left|\frac{m+1-a}{m+1+a}\right|-\frac{a}{2} \ln \left|\frac{m-a}{m+a}\right|\right] \\
& +\frac{1}{2 \lambda^{2}}\left(-3 m^{2}-2 m \pm(2 m+1) \varepsilon_{\psi}\right) \ln \left|\frac{(m+1)^{2}-a^{2}}{m^{2}-a^{2}}\right| \\
& -\frac{1}{2 a \lambda^{2}} m(m+1)\left(-m \pm \varepsilon_{\psi}\right)\left[\ln \left|\frac{m+1-a}{m+1+a}\right|-\ln \left|\frac{m-a}{m+a}\right|\right] \\
= & \frac{1}{2 \lambda^{2}}\left[-a^{2}-3 m^{2}-2 m \pm(2 m+1) \varepsilon_{\psi}\right] \ln \left|\frac{(m+1)^{2}-a^{2}}{m^{2}-a^{2}}\right| \\
& +\frac{1}{2 \lambda^{2}}\left[\left(1+3 m \mp \varepsilon_{\psi}\right) a-\frac{m}{a}(m+1)\left(-m \pm \varepsilon_{\psi}\right)\right]\left(\ln \left|\frac{m+1-a}{m+1+a}\right|-\ln \left|\frac{m-a}{m+a}\right|\right) \\
& +\frac{1}{\lambda^{2}}\left[2 m+\frac{1}{2} \mp \varepsilon_{\psi}\right]
\end{aligned}
$$

## 5 Numerical calculations

### 5.1 The coupling of the $\mu-\psi-\eta$ vertex

The couplings $d$ and $d^{\prime}$ from section 4.2 still have to be determined, which is done here for the extended scotogenic model. This was achieved by using the Mathematica extension SARAH [Sta08]. It analytically calculates various quantities for a model, e.g. mass matrices or vertex couplings. The SARAH output is obtained as


$$
\begin{aligned}
& \stackrel{(5.2)}{=}-\mathrm{i} \sum_{n=1}^{3}\left(U_{\psi}\right)_{j 2}^{*}\left(y_{2}\right)_{i} P_{\mathrm{L}} \\
& \\
& \quad+\mathrm{i} \sum_{n=1}^{3}\left(y_{4}\right)_{i n}\left(U_{\psi}\right)_{\mathrm{j}, n+2} P_{\mathrm{R}}
\end{aligned}
$$

where $V_{\mathrm{e}}$ is the mixing matrix of the left-handed SM leptons, $U_{\mathrm{e}}$ is its right-handed counterpart, and $U_{\psi}$ is the mixing matrix of the BSM fermions. $y_{2}$ and $y_{4}$ are Yukawa couplings defined in eq. (4.1). At last, $P_{\mathrm{L}}$ and $P_{\mathrm{R}}$ are the left-handed and right-handed projectors (see appendix A.4). Because there is no mixing between the leptons and the antileptons, $V_{\mathrm{e}}$ and $U_{\mathrm{e}}$ are the identity matrices, i.e.

$$
\begin{equation*}
\left(V_{\mathrm{e}}\right)_{i m}=\left(U_{\mathrm{e}}\right)_{i m}=\delta_{i m} . \tag{5.2}
\end{equation*}
$$

### 5.2 Introduction to the used toolchain

In this section the toolchain for analyzing the parameter space is presented briefly. The model files for the extended scotogenic model were created using the Python module "minimal-lagrangians" [May21]. These then were used by SARAH [Sta08] to build the files for SPheno [Por03; PS12] and micrOMEGAS [Bél+18]. Using these tools, a Python toolchain was used to input the parameters into the SPheno and micrOMEGAS, either sampling one parameter continuously or a set of values randomly, and reading out the output values, e.g. $\Delta a_{\mu}$. The former is used in section 5.3 and the latter in section 5.4.

### 5.3 Comparison of SPheno and the analytical result

In this section the analytical result is compared to the anomalous magnetic moment calculated by SPheno. For this, eq. (4.16) is used to calculate $\Delta a_{\mu}$, where the couplings $d_{\psi}$ and $d_{\psi}^{\prime}$ are taken as the term proportional to $\mathbb{1}_{4}$ and $\gamma^{5}$, respectively, from eq. (5.1). The mixing matrix $U_{\psi}$ for the analytical result was also determined by SPheno. Using the aforementioned toolchain, $m_{\chi}, y_{2}, y_{3}$ and $y_{4}$ were probed.

The results are shown in fig. 7 and the parameters and ranges are listed in table 2. It should be noted that the lightest of the mixed $\psi$ fermions is always taken as the dark matter candidate, which is always $\psi_{1}$, such that $m_{\mathrm{DM}}=m_{\psi_{1}}$. In fig. $7 \mathrm{a} m_{\mathrm{X}}$ was sampled, for which the analytical results are in excellent agreement with SPheno. The assumption that the diagram in fig. 6 yields the main contribution to $\Delta a_{\mu}$ thus seems to be correct (in this range of parameters). The shape of the curve also makes sense, because with increasing masses of the loop particles, i.e. $m_{\mathrm{X}}$, the contribution of the relevant diagram should get smaller. This can be seen in fig. 7a. In fig. 7b the vector Yukawa couplings $y_{2}$ and $y_{3}$ were sampled with every component the same,
and the matrix Yukawa coupling $y_{4}$ was sampled as being proportional to $\mathbb{1}_{4}$ (see table 2). It is apparent that the results agree with SPheno above a certain value of the different Yukawa couplings, which seems to vary between them. The discrepancy could stem from other contributing diagrams, which SPheno considers but are not calculated in this thesis, or from SPheno making approximations to increase the performance of the calculations. Either way, this is not a problem, because mainly the order $\Delta a_{\mu} \sim \mathcal{O}\left(10^{-9}\right)$ is of interest, in which the analytical results agree with SPheno. The downward spikes shown in fig. 7 b are caused by a change of $\operatorname{sign}$ in $\Delta a_{\mu}$, i.e. $\Delta a_{\mu}$ becomes negative for decreasing values of the Yukawa couplings, which could not be plotted here because of the logarithmic scale. At last, it is apparent from fig. 7 that a value of $\Delta a_{\mu}$ can (for now) be reached in this model.


Figure 7: Comparison of analytical results and results calculated by SPheno. In (a) $m_{\mathrm{X}}$ is varied, in (b) the Yukawa couplings. Spikes due to logarithmic scaling show where the sign of $\Delta a_{\mu}$ changes. The parameters and variation ranges can be found in table 2.

### 5.4 Random sampling of the parameter space

Now the random sampling of the parameter space is presented. Similar to the previous section, the varied parameters and their limits can be found in table 3 in appendix A.10. To ensure correct neutrino masses, the Casas-Ibarra parametrization was used to determine the Yukawa coupling $y_{4}$. Instead of sampling $y_{3}$ directly, the 3component vector $\xi$ (introduced in [CN19, p. 12]) where $\xi=y_{4} y_{3}$, i.e. $\xi_{i}=\left(y_{4}\right)_{i j}\left(y_{3}\right)_{j}$, was sampled around a benchmark point presented in [CN19, p. 16]. y $y_{3}$ was thus determined by $y_{3}=\left(y_{4}\right)^{-1} \xi$, which was achieved by solving the corresponding system of equations numerically.

After the probing, the following limits were imposed on the parameter space:

1. relic density: $\Omega h^{2}=0.12 \pm 0.07$ (see eq. (2.3))
2. lepton flavor violation: $\operatorname{BR}(\mu \rightarrow \mathrm{e} \gamma)<4.2 \cdot 10^{-13}, \operatorname{BR}(\tau \rightarrow \mathrm{e} \gamma)<3.3 \cdot 10^{-8}$ and $\operatorname{BR}(\mu \rightarrow 3 \mathrm{e})<1 \cdot 10^{-12}[\mathrm{Zyl}+20]$
3. LEP (Large Electron-Positron Collider): $m_{\chi_{1}^{+}}>102 \mathrm{GeV}[\mathrm{Abb}+03]$

Note that the actual experimental uncertainty on the relic density $\Omega h^{2}$ (see eq. (2.3)) is much smaller than the interval used here. The problem is that the probability of reaching this interval is very small if most of the free parameters are sampled over multiple orders of magnitude. Many points would then be discarded, even if they have a relic density very close to the experimental value. Another reason are numerical errors produced by micrOMEGAS, which introduce another theoretical uncertainty.

The parameter points, constrained by the above limits, are shown in fig. 8. In total, 1033580 data points were sampled, from which 840 survive all constraints. The most points are excluded by the relic density and LFV constraints. Because the lepton flavor violating $\mu \rightarrow \mathrm{e} \gamma$ process has essentially the same diagram as the anomalous magnetic moment contribution, LFV excludes the greater $\Delta a_{\mu}$ values more prominently, as seen in fig. 8 b . In fig. 8 c the excluded values are plotted in front of the allowed values, because many more values are allowed than excluded. The hard cut at $\approx 100 \mathrm{GeV}$ in fig. 8c can be explained by the fact that the dark matter candidate is taken as the lightest neutral fermion, i.e. $m_{\mathrm{DM}}=m_{\psi_{1}}$ : Thus the masses of the mass eigenstates $m_{\chi_{1}^{+}}$have to be bigger than $m_{\psi_{1}}$. The reason for this is that $m_{\chi_{1}^{+}}$is charged, so it can not be a dark matter candidate. At $m_{\mathrm{DM}}>102 \mathrm{GeV}$ this means that $m_{\chi_{1}^{+}}$is automatically bigger than 102 GeV , so the LEP constraint is always fulfilled.

In fig. 8 b it can be seen that the limiting factor seems to be LFV, although it seems to be possible to achieve $\Delta a_{\mu} \sim \mathcal{O}\left(10^{-9}\right)$. Because the probability of fulfilling the LFV constraint and getting a sizable $\Delta a_{\mu}$ is so small, none of the points with all constraints get a $\Delta a_{\mu}$ of the right size. This is also shown in fig. 9. The upper left quadrant here is the region where parameter points which fulfill both constraints would show up.

It can be seen that there are some points in this region, although with all constraints imposed there are none left. Conclusively, the right size of the anomalous magnetic moment could still be reached, but getting the right parameter points is very improbable in random sampling the parameter space. Maybe finding the right combinations of parameters can better be achieved with more sophisticated methods of probing.

Also the relationship between the branching rations of the three lepton flavor violating processes used as limits can be evaluated. This is done in fig. 10. Here all sampled points are plotted, where by the LFV constraints, shown as lines, it can be seen that the process $\mu \rightarrow \mathrm{e} \gamma$ imposes the strictest constraint.

At last the direct detection spin-independent cross section $\sigma_{p}(\mathrm{SI})$ was examined. This is shown in fig. 11. If a parameter point has a main annihilation channel (i.e. one with a branching ration greater than $50 \%$ ), it is colored accordingly. As expected, the main annihilation channel changes with the dark matter mass. With an increase of mass, more annihilations get kinematically allowed, which can be seen e.g. for $\psi_{1} \psi_{1} \rightarrow \mathrm{t} \overline{\mathrm{t}}$ : Its mass is approximately 173 GeV , and this is the smallest dark matter mass which shows this annihilation channel. The downward spikes at about 45.5 GeV and 62.5 GeV are the $\mathrm{Z}^{0}$ (Z-boson) and h (Higgs boson) resonances. These can be explained as follows: The $\psi_{1} \psi_{1}$ annihilation happens with either a $Z^{0}$ or a has the propagator, so the invariant amplitude of this process gets terms $\frac{1}{p^{2}-m_{Z^{0}}^{2}}$ and $\frac{1}{p^{2}-m_{\mathrm{h}}}$, where $p$ is the total four-momentum of the incoming $\psi_{1}$. These become very large for $p^{2} \approx m_{\mathrm{Z}^{0} / \mathrm{h}}$, so in turn the cross section of this process also gets bigger. That means less scattering with the direct detection atoms happen and such $\sigma_{p}$ gets smaller. The process $\psi_{1} \psi_{1} \rightarrow \mathrm{~W}^{+} \mathrm{W}^{-}$is favored in such a way over the elastic scattering with the proton that the cross section of the latter becomes several orders of magnitude smaller. The XENON1T limit excludes many parameter points, which makes finding points with the correct anomalous magnetic moment even more improbable. A solution for this could be to use an algorithmic determination of the parameters, i.e. moving the parameter steadily into regions with output values agreeing with the experimental limits, instead of random sampling the points. The disadvantage here would be that for this a certain smaller region of parameters has to be chosen. Also the Yukawa coupling $y_{1}$, which was set to 0 from the start, as it is done in [CN19], could influence the anomalous magnetic moment if it has a nonzero value.


Figure 8: The sampled values, shown as $\Delta a_{\mu}$ against $m_{\text {DM }}=m_{\psi_{1}}$. In (a), (b) and (c) the individual constraints are imposed upon the parameter points, (d) shows all the constraints at once.


Figure 9: Correlation of the anomalous magnetic moment with LFV. The upper left quadrant shows parameter points that are both allowed by the LFV constraint and also have a $\Delta a_{\mu}$ of the right magnitude.


Figure 10: The different branching ratios which where used as limiting constraints in this section, plotted against each other. A correlation between the three processes is visible. The process $\mu \rightarrow \mathrm{e} \gamma$ imposes the strictest constraint.


Figure 11: The spin-independent direct detection cross section against the dark matter mass. The XENON1T limit is taken from [Apr+18]. The parameter points shown here are the ones fulfilling all imposed constraints.

## 6 Conclusion and outlook

In this thesis, the anomalous magnetic moment of the muon was examined in an extended scotogenic model, i.e. a radiative seesaw model. For this, it was first shown how $g_{\mu}=2$ follows from the Dirac equation in the non-relativistic limit. Then the invariant amplitude of the process which yields the anomalous magnetic moment was considered. First it was written in terms of the electromagnetic form factors, and after that it was calculated in the Standard Model, leading to the expected result of a correction of $\frac{\alpha}{\pi}$ to $g_{\mu}$. Then the extension of the scotogenic model was introduced, in which the one-loop contribution to $g_{\mu}$ was calculated again. The result could be found to be in agreement with the literature. For the numerical calculations a toolchain of different tools was presented, in which the analytical calculation was compared to the numerical result of the anomalous magnetic moment. In the relevant parameter regions both results could be found to be in good agreement. At last a random scan was conducted, where it was found that the model can indeed yield parameter points with an anomalous magnetic moment in the right order of magnitude. But these are severely constrained by LFV processes, as both are connected by similar Feynman diagrams, so only a few points satisfying both these constraints remained. The imposed constraint on the relic density then discarded all of these points, so that in conclusion the experimental constraints lead to an anomalous magnetic moment which is too small. In principle the method of scanning randomly could be at fault, where the probability of reaching a specific suitable parameter configuration gets smaller with the addition of every free parameter. A solution could be a more sophisticated probing method which gradually moves the parameter points to regions where the experimental constraints can be fulfilled.

The anomalous magnetic moment of the muon continues to pose a promising hint on BSM particle physics. It can therefore be hoped that the improvements in determining its experimental and theoretical value lead to a bigger discrepancy in the next years.

## A Appendix

## A. 1 The group $\mathbb{Z}_{\mathbf{2}}$

The group $\mathbb{Z}_{2}$ is interpretable as two isomorphic groups: The quotient group $(\mathbb{Z} / 2 \mathbb{Z},+)$, where

$$
\mathbb{Z} / 2 \mathbb{Z}=\{0,1\} \quad \text { and } \quad+: \mathbb{Z} / 2 \mathbb{Z} \rightarrow \mathbb{Z} / 2 \mathbb{Z}, \quad(a, b) \mapsto a+b \quad \bmod 2,
$$

or the cyclic group of order $2,\left(\mathbb{Z}_{2}, \cdot\right)$, where

$$
\mathbb{Z}_{2}=\{-1,+1\} \quad \text { and } \quad \cdot: \mathbb{Z}_{2} \rightarrow \mathbb{Z}_{2}, \quad(a, b) \mapsto a \cdot b
$$

Here it is more meaningful to use the second interpretation, because then, like it is done in [Ma06] and [CN19], the terms "odd" and "even" and the abbreviations "-" and " + " can be used for the elements of $\mathbb{Z}_{2},-1$ and +1 .

## A. 2 The Dirac equation

The Dirac equation regarding a spin $1 / 2$ particle with momentum $q$ in momentum space reads

$$
\begin{equation*}
q u(q)=m u(q), \tag{A.1}
\end{equation*}
$$

where $m$ is the mass of said particle and $u(q)$ its momentum space spinor. Here the Feynman slash notation $q=q_{\mu} \gamma^{\mu}$ is used. The adjoint Dirac equation can then be obtained as

$$
\begin{equation*}
\bar{u}(q) q=m \bar{u}(q), \tag{A.2}
\end{equation*}
$$

with the adjoint spinor $\bar{u}(q)=u(q)^{\dagger} \gamma^{0}$.

## A. 3 Relations regarding the Dirac gamma matrices

The gamma matrices $\gamma^{\mu}$ fulfill an anticommutator relation:

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}=2 g^{\mu \nu} \mathbb{1}_{4}, \tag{A.3}
\end{equation*}
$$

where $\mathbb{1}_{4}$ is the four-dimensional identity matrix.
The commutator of the gamma matrices $\gamma^{\mu}$ is defined as

$$
\begin{equation*}
\sigma^{\mu \nu}:=\frac{\mathrm{i}}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] . \tag{A.4}
\end{equation*}
$$

Useful for the proof of the Gordon identities, the following can be observed from
eq. (A.4):

$$
\begin{equation*}
\mathrm{i} \sigma^{\mu \nu} \stackrel{(\mathrm{A} .3)}{=} \mathrm{g}^{\mu \nu}-\gamma^{\mu} \gamma^{\nu}, \tag{A.5}
\end{equation*}
$$

and also substituting $\gamma^{\mu} \gamma^{\nu}$ in eq. (A.5) by using eq. (A.3) results in

$$
\begin{equation*}
\mathrm{i} \sigma^{\mu \nu}=\gamma^{\nu} \gamma^{\mu}-g^{\mu \nu} \tag{A.6}
\end{equation*}
$$

The following identities simplify handling products of the $\gamma$-matrices [Sch14, p. 820]:

$$
\begin{align*}
\gamma^{\mu} \gamma_{\mu} & =4 \mathbb{1}_{4} \\
\gamma^{\mu} \gamma^{\nu} \gamma_{\mu} & =-2 \gamma^{\nu}  \tag{A.7}\\
\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma_{\mu} & =4 g^{\nu \rho} \mathbb{1}_{4} \\
\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma} \gamma_{\mu} & =-2 \gamma^{\sigma} \gamma^{\rho} \gamma^{\nu}
\end{align*}
$$

The fifth Dirac matrix $\gamma^{5}=\mathrm{i} \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}$ anticommutes with the other four Dirac matrices:

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{5}\right\}=0 \quad \Leftrightarrow \quad \gamma^{\mu} \gamma^{5}=-\gamma^{5} \gamma^{\mu} \tag{A.8}
\end{equation*}
$$

Moreover, its square is the identity:

$$
\begin{equation*}
\left(\gamma^{5}\right)^{2}=\mathbb{1}_{4} \tag{A.9}
\end{equation*}
$$

## A. 4 The left- and right-handed projectors

The projectors $P_{\mathrm{L}}$ and $P_{\mathrm{R}}$ are defined as follows:

$$
\begin{equation*}
P_{\mathrm{L}}=\frac{1}{2}\left(1-\gamma^{5}\right) \quad \text { and } \quad P_{\mathrm{R}}=\frac{1}{2}\left(1+\gamma^{5}\right) . \tag{A.10}
\end{equation*}
$$

They can be used to get the left- and right-handed Weyl spinors out of a Dirac spinor. In the Weyl-basis, the Dirac spinor $\psi$ can be written as a doublet of the two-component Weyl spinors:

$$
\psi=\binom{\psi_{\mathrm{L}}}{\psi_{\mathrm{R}}} .
$$

$\gamma^{5}$ is represented as follows in the Weyl basis (where free space is to be interpreted as $0)$ :

$$
\gamma^{5}=\left(\begin{array}{cc}
-\mathbb{1}_{2} & \\
& \mathbb{1}_{2}
\end{array}\right) .
$$

Therefore the projection operators take a very simple form in the Weyl representation:

$$
P_{\mathrm{L}}=\left(\begin{array}{ll}
\mathbb{1}_{2} & \\
& 0
\end{array}\right) \quad \text { and } \quad P_{\mathrm{R}}=\left(\begin{array}{ll}
0 & \\
& \mathbb{1}_{2}
\end{array}\right),
$$

from which the projection property

$$
P_{\mathrm{L}} \psi=\psi_{\mathrm{L}} \quad \text { and } \quad P_{\mathrm{R}} \psi=\psi_{\mathrm{R}}
$$

can be seen.

## A. 5 Feynman parametrization

This section is based on the proof sketch presented in [PS95, pp. 189-190]. The Feynman parametrization is used to simplify denominators arising due to propagators, where the multiplied terms are converted into a sum at the cost of some extra integrals. The advantage is that the loop momentum which comes up in this sum can be shifted, such that a complete square is present in the denominator, which is the case in section 4.2. This makes the integration over the loop momentum easier.

It is convenient to consider the special case for two factors $A$ and $B$ in the denominator first. For this the following integral is evaluated (using the substitution $u=x A+(1-x) B)$ :

$$
\begin{aligned}
\int_{0}^{1} \mathrm{~d} x \frac{1}{[x A+(1-x) B]^{2}} & \stackrel{\text { sub. }}{=} \int_{B}^{A} \frac{\mathrm{~d} u}{A-B} \frac{1}{u^{2}}=\left[-\frac{1}{u}\right]_{B}^{A} \frac{1}{A-B} \\
& =-\left(\frac{1}{A}-\frac{1}{B}\right) \frac{1}{A-B}=-\frac{B-A}{A B} \frac{1}{A-B}=\frac{1}{A B}
\end{aligned}
$$

which can also be written in a form where $1-x$ is converted to a new variable $y$ by introducing a second integral:

$$
\begin{equation*}
\frac{1}{A B}=\int_{0}^{1} \mathrm{~d} x \frac{1}{[x A+(1-x) B]^{2}}=\int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \delta(x+y-1) \frac{1}{[x A+y B]^{2}} \tag{A.11}
\end{equation*}
$$

Differentiating eq. (A.11) $n-1$ times with respect to $B$ yields

$$
\begin{align*}
(-1)^{n-1}(n-1)!\frac{1}{A B^{n}} & =\int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \delta(x+y-1) \frac{(-1)^{n-1} n!y^{n-1}}{[x A+y B]^{n+1}} \\
\Leftrightarrow \frac{1}{A B^{n}} & =\int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \delta(x+y-1) \frac{n y^{n-1}}{[x A+y B]^{n+1}} . \tag{A.12}
\end{align*}
$$

Now the actual proof for $n$ factors $A_{1}, \ldots, A_{n}$ in the denominator can be carried out.

The formula which shall be proven reads

$$
\begin{equation*}
\frac{1}{A_{1} \cdots A_{n}}=\int_{0}^{1} \mathrm{~d} x_{1} \cdots \mathrm{~d} x_{n} \delta\left(\sum_{i=1}^{n} x_{i}-1\right) \frac{(n-1)!}{\left[x_{1} A_{1}+\cdots+x_{n} A_{n}\right]^{n}} . \tag{A.13}
\end{equation*}
$$

The proof is done by induction. The base case is eq. (A.11) for $n=2$, the induction hypothesis (IH) is eq. (A.13). The induction step $n \rightarrow n+1$ is then conducted as

$$
\begin{aligned}
& \frac{1}{A_{1} \cdots A_{n+1}} \stackrel{\mathrm{IH}}{=} \int_{0}^{1} \mathrm{~d} x_{1} \cdots \mathrm{~d} x_{n} \delta\left(\sum_{i=1}^{n} x_{i}-1\right) \frac{(n-1)!}{\left[x_{1} A_{1}+\cdots+x_{n} A_{n}\right]^{n}} \frac{1}{A_{n+1}} \\
& \stackrel{(\mathrm{~A} .12)}{=} \int_{0}^{1} \mathrm{~d} x_{1} \cdots \mathrm{~d} x_{n} \delta\left(\sum_{i=1}^{n} x_{i}-1\right)(n-1)! \\
& \quad \cdot \int_{0}^{1} \mathrm{~d} x \mathrm{~d} y \delta(x+y-1) \frac{n y^{n-1}}{\left[x A_{n+1}+y\left(x_{1} A_{1}+\cdots+x_{n} A_{n}\right)\right]^{n+1}} \\
&=\int_{0}^{1} \mathrm{~d} x_{1} \cdots \mathrm{~d} x_{n} \delta\left(\sum_{i=1}^{n} x_{i}-1\right) n! \\
& \quad \int_{0}^{1} \mathrm{~d} x \frac{(1-x)^{n-1}}{\left[x A_{n+1}+(1-x)\left(x_{1} A_{1}+\cdots+x_{n} A_{n}\right)\right]^{n+1}}
\end{aligned}
$$

Now the substitution $x_{i}^{\prime}=(1-x) x_{i}$, the renaming $x \rightarrow x_{n+1}^{\prime}$ and the identity $\delta(\alpha x)=$ $\delta(x) /|\alpha|$ can be used:

$$
\begin{aligned}
\frac{1}{A_{1} \cdots A_{n+1}} & \stackrel{\text { sub. }}{=} \int_{0}^{1} \mathrm{~d} x_{n+1}^{\prime} \int_{0}^{1-x_{n+1}^{\prime}} \mathrm{d} x_{1}^{\prime} \cdots \mathrm{d} x_{n}^{\prime} \delta\left(\sum_{i=1}^{n} \frac{x_{i}^{\prime}}{1-x_{n+1}^{\prime}}-1\right) \\
& \cdot \frac{n!}{1-x_{n+1}^{\prime}} \frac{1}{\left[x_{1}^{\prime} A_{1}+\cdots+x_{n+1}^{\prime} A_{n+1}\right]^{n+1}} \\
= & \int_{0}^{1} \mathrm{~d} x_{n+1}^{\prime} \int_{0}^{1-x_{n+1}^{\prime}} \mathrm{d} x_{1}^{\prime} \cdots \mathrm{d} x_{n}^{\prime} \delta\left(\sum_{i=1}^{n} x_{i}^{\prime}-1\right) \frac{n!}{\left[x_{1}^{\prime} A_{1}+\cdots+x_{n+1}^{\prime} A_{n+1}\right]^{n+1}} \\
& \stackrel{\text { (A.14) }}{=} \int_{0}^{1} \mathrm{~d} x_{1}^{\prime} \cdots \mathrm{d} x_{n+1}^{\prime} \delta\left(\sum_{i=1}^{n+1} x_{i}^{\prime}-1\right) \frac{n!}{\left[x_{1}^{\prime} A_{1}+\cdots+x_{n+1}^{\prime} A_{n+1}\right]^{n+1}} .
\end{aligned}
$$

The last equality holds because the $\delta$-function is 0 if any $x_{j}^{\prime}(j \in\{1, \ldots, n\})$ is greater than the upper integral limit:

$$
\begin{equation*}
\delta\left(\sum_{i=1}^{n+1} x_{i}^{\prime}-1\right)=0 \quad \text { if } \quad x_{j}^{\prime}>1-x_{n+1}^{\prime} . \tag{A.14}
\end{equation*}
$$

This can be seen by considering

$$
\begin{aligned}
\sum_{i=1}^{n+1} x_{i}^{\prime}-1 & =x_{1}^{\prime}+\ldots+x_{j-1}^{\prime}+x_{j+1}^{\prime}+\ldots+x_{n}^{\prime}+\underbrace{x_{j}^{\prime}+x_{n+1}^{\prime}-1}_{>0 \text { if } x_{j}^{\prime}>1-x_{n+1}^{\prime}} \\
& >x_{1}^{\prime}+\ldots+x_{j-1}^{\prime}+x_{j+1}^{\prime}+\ldots+x_{n}^{\prime}>0
\end{aligned}
$$

but the $\delta$-function is only nonzero if its argument is 0 .
With this the proof of eq. (A.13) is complete. Equation (A.13) with $n=3$ is used in section 3.3 and section 4.2.

## A. 6 Gordon identity

The Gordon identity can be used to express the sum of two four vector momenta, sandwiched between the corresponding spinors, in terms of gamma matrices $\gamma^{\mu}$ and their commutator $\sigma^{\mu \nu}$. For the Standard Model calculation the identity shown in [Sch14, p. 316] can be used:

$$
\begin{equation*}
\bar{u}\left(q_{2}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) u\left(q_{1}\right)=2 m_{\mu} \bar{u}\left(q_{2}\right) \gamma^{\mu} u\left(q_{1}\right)+i \bar{u}\left(q_{2}\right) \sigma^{\mu \nu}\left(q_{1 v}-q_{2 v}\right) u\left(q_{1}\right) \tag{A.15}
\end{equation*}
$$

Here $u\left(q_{1}\right)$ and $u\left(q_{2}\right)$ are two on-shell spinors in momentum space with momenta $q_{1}$ and $q_{2}$, respectively, and mass $m_{\mu}$. It can be proven by using the Dirac equation (A.1) and its adjoint equation (A.2):

$$
\begin{aligned}
i \bar{u}\left(q_{2}\right) \sigma^{\mu \nu}\left(q_{1 \nu}-q_{2 \nu}\right) u\left(q_{1}\right) & \stackrel{(\mathrm{A} .5),(\mathrm{A} .6)}{=} \bar{u}\left(q_{2}\right)\left[\left(g^{\mu \nu}-\gamma^{\mu} \gamma^{\nu}\right) q_{1 \nu}-\left(\gamma^{\nu} \gamma^{\mu}-g^{\mu \nu}\right) q_{2 \nu}\right] u\left(q_{1}\right) \\
& =\bar{u}\left(q_{2}\right)\left[-q_{2} \gamma^{\mu}-\gamma^{\mu} q_{1}+q_{1}^{\mu}+q_{2}^{\mu}\right] u\left(q_{1}\right) \\
& \stackrel{\text { A. } .1)(\mathrm{A} .2)}{=} \bar{u}\left(q_{2}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) u\left(q_{1}\right)-2 m_{\mu} \bar{u}\left(q_{2}\right) \gamma^{\mu} u\left(q_{1}\right)
\end{aligned}
$$

which is equivalent to eq. (A.15).
A similar identity holds if $\gamma^{5}$ is present:

$$
\begin{equation*}
\bar{u}\left(q_{2}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) \gamma^{5} u\left(q_{1}\right)=i \bar{u}\left(q_{2}\right) \sigma^{\mu \nu}\left(q_{1 \nu}-q_{2 \nu}\right) \gamma^{5} u\left(q_{1}\right) \tag{A.16}
\end{equation*}
$$

This can be proven in the same way:

$$
\begin{gathered}
i \bar{u}\left(q_{2}\right) \sigma^{\mu \nu}\left(q_{1 \nu}-q_{2 \nu}\right) \gamma^{5} u\left(q_{1}\right) \stackrel{(\mathrm{A.S}),(\mathrm{A} .6)}{=} \bar{u}\left(q_{2}\right)\left[\left(g^{\mu \nu}-\gamma^{\mu} \gamma^{\nu}\right) q_{1 \nu}-\left(\gamma^{\nu} \gamma^{\mu}-g^{\mu \nu}\right) q_{2 \nu}\right] \gamma^{5} u\left(q_{1}\right) \\
\stackrel{(\mathrm{A} .8)}{=} \bar{u}\left(q_{2}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) \gamma^{5} u\left(q_{1}\right)+\bar{u}\left(q_{2}\right)\left(\gamma^{\mu} \gamma^{5} q_{1}-q_{2} \gamma^{\mu} \gamma^{5}\right) u\left(q_{1}\right) \\
\stackrel{\text { (A.1),(A.2) }}{=} \bar{u}\left(q_{2}\right)\left(q_{1}^{\mu}+q_{2}^{\mu}\right) \gamma^{5} u\left(q_{1}\right) .
\end{gathered}
$$

## A. 7 Feynman rules

The following Feynman rules are used in the calculation in section 3.3 and section 4.2. Those also occurring in the Standard model are taken from [HM84].


In section 4.2 there is also the specific coupling $\mu^{-}-\eta^{+}-\psi$. The coupling consists of a linear combination of the projectors $P_{\mathrm{L}}$ and $P_{\mathrm{R}}$, which is why it includes a scalar and a pseudoscalar part.


Here $d$ and $d^{\prime}$ are two scalars, that still depend on e.g. mixing matrices. The concrete coupling for the extended scotogenic model is determined in section 5.1.

## A. 8 Loop momentum integrals

In section 3.3 and section 4.2 integrals linear in $k^{\mu}$ arise:

$$
\begin{equation*}
D^{\mu}(\Delta):=\int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{k^{\mu}}{\left(k^{2}-\Delta\right)^{3}}=0 . \tag{A.24}
\end{equation*}
$$

The right equality holds because the integrand is antisymmetric in $k^{\mu}$, and the integral has symmetric improper bounds, so the integral with integration variable $k^{\mu}$ vanishes, which results in the whole integral vanishing.

Furthermore, integrals with terms consisting of $k^{\mu} k^{\nu}$ appear:

$$
\begin{align*}
D^{\mu \nu}(\Delta) & :=\int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{k^{\mu} k^{\nu}}{\left(k^{2}-\Delta\right)^{3}}  \tag{A.25}\\
& \stackrel{(\mathrm{~A} .266}{=} \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{\mathrm{~g}^{\mu \nu} k^{2}}{4} \frac{1}{\left(k^{2}-\Delta\right)^{3}} .
\end{align*}
$$

The second equality in eq. (A.25) means that

$$
\begin{equation*}
k^{\mu} k^{\nu}=\frac{1}{4} g^{\mu \nu} k^{2} \tag{A.26}
\end{equation*}
$$

in the context of the integral. The proof is featured in [Sch14, p. 380]. Due to $D^{\mu \nu}$ being a tensor, it must be proportional to the only other occurring tensor in this calculation, which is $g^{\mu \nu}$. To match the dimensions, $D^{\mu \nu}$ must also include the factor $k^{2}$, which results in the proportionality

$$
\int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} \frac{k^{\mu} k^{\nu}}{\left(k^{2}-\Delta\right)^{3}} \sim \int_{\mathbb{R}^{4}} \frac{\mathrm{~d}^{4} k}{(2 \pi)^{4}} g^{\mu \nu} k^{2} \frac{1}{\left(k^{2}-\Delta\right)^{3}} .
$$

The proportionality constant $c$ can be obtained by calculating $g_{\mu \nu} D^{\mu \nu}$ :

$$
\begin{aligned}
g_{\mu \nu} \int_{\mathbb{R}^{4}} \frac{d^{4} k}{(2 \pi)^{4}} \frac{k^{\mu} k^{\nu}}{\left(k^{2}-\Delta\right)^{3}} & =g_{\mu \nu} \int_{\mathbb{R}^{4}} \frac{d^{4} k}{(2 \pi)^{4}} c g^{\mu \nu} \frac{k^{2}}{\left(k^{2}-\Delta\right)^{3}} \\
\Leftrightarrow \quad \int_{\mathbb{R}^{4}} \frac{d^{4} k}{(2 \pi)^{4}} \frac{k^{2}}{\left(k^{2}-\Delta\right)^{3}} & =4 c \int_{\mathbb{R}^{4}} \frac{d^{4} k}{(2 \pi)^{4}} \frac{k^{2}}{\left(k^{2}-\Delta\right)^{3}} \\
\Leftrightarrow \quad c & =\frac{1}{4} .
\end{aligned}
$$

Together this results in eq. (A.26)

## A. 9 Integrals needed for the analytical calculation of $\Delta \mathbf{a}_{\mu}$

The needed integrals are

$$
\begin{equation*}
\int \mathrm{d} x \frac{x^{n}}{x^{2}-a^{2}} \tag{A.27}
\end{equation*}
$$

where $n \in\{0,1,2,3\}$. The first one $(n=0)$ can be calculated using partial fraction decomposition:

$$
\frac{1}{x^{2}-a^{2}}=\frac{1}{(x+a)(x-a)}=\frac{A}{x+a}+\frac{B}{x-a},
$$

where $A$ and $B$ are the constants that have to be found. Multiplying by $(x+a)(x-a)$ gets

$$
1=A(x-a)+B(x+a) \quad \Leftrightarrow \quad 1=a(B-A)+(A+B) x
$$

and equating the coefficients results in

$$
A=-\frac{1}{2 a} \quad \text { and } \quad B=\frac{1}{2 a} .
$$

Now the first integral can be calculated:

$$
\begin{align*}
\int \mathrm{d} x \frac{1}{x^{2}-a^{2}} & =\frac{1}{2 a} \int \mathrm{~d} x \frac{1}{x-a}-\frac{1}{2 a} \int \mathrm{~d} x \frac{1}{x+a} \\
& =\frac{1}{2 a} \ln |x-a|-\frac{1}{2 a} \ln |x+a| \\
& =\frac{1}{2 a} \ln \left|\frac{x-a}{x+a}\right| \tag{A.28}
\end{align*}
$$

The second integral can be proven by using the substitution $u=x^{2}-a^{2}$ :

$$
\begin{align*}
\int \mathrm{d} x \frac{x}{x^{2}-a^{2}} & =\int \mathrm{d} u \frac{1}{2 u} \\
& =\frac{1}{2} \ln \left|x^{2}-a^{2}\right| \tag{A.29}
\end{align*}
$$

The calculation of the third integral uses the first integral:

$$
\begin{gather*}
\int \mathrm{d} x \frac{x^{2}}{x^{2}-a^{2}}=\int \mathrm{d} x+\int \mathrm{d} x \frac{a^{2}}{x^{2}-a^{2}} \\
\stackrel{(\text { A.28) }}{=} x+\frac{a}{2} \ln \left|\frac{x-a}{x+a}\right| \tag{A.30}
\end{gather*}
$$

Analogously, the fourth integral makes use of the second one:

$$
\begin{align*}
& \int \mathrm{d} x \frac{x^{3}}{x^{2}-a^{2}}=\int \mathrm{d} x x+\int \mathrm{d} x \frac{x a^{2}}{x^{2}-a^{2}} \\
& \stackrel{\text { (A.29) }}{=} \frac{x^{2}}{2}+\frac{a^{2}}{2} \ln \left|x^{2}-a^{2}\right| \tag{A.31}
\end{align*}
$$

## A. 10 Parameters used for the numerical calculations

Table 2: The parameters used for fig. 7.
(a) The parameters that were left constant throughout the probing.

| $\lambda$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ | $m_{\eta}[\mathrm{GeV}]$ | $m_{\mathrm{N}}[\mathrm{GeV}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.26 | 0.5 | 6 | -5 | $2 \cdot 10^{-9}$ | 1050 | $1000 \cdot \mathbb{1}_{3}$ |

(b) The variable parameters during the probing: if one parameter was varied, the others were left constant. The ranges are shown as intervals.

| parameter | value if constant | value if varied |
| :---: | :---: | :---: |
| $m_{\mathrm{X}}$ in GeV | 950 | [ $\left.10^{2}, 10^{4}\right]$ |
| $y_{2}$ | $\left(\begin{array}{c}5 \cdot 10^{-5} \\ 1 \\ 0.5\end{array}\right)$ | $\left[10^{-5}, 10\right] \cdot\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ |
| $y_{3}$ | $\left(\begin{array}{c}-1.40 \\ -1.81 \\ 1.99\end{array}\right)$ | $\left[10^{-5}, 10\right] \cdot\left(\begin{array}{l}-1 \\ -1 \\ -1\end{array}\right)$ |
| $y_{4}$ | $\left(\begin{array}{ccc}1.3 \cdot 10^{-2} & 5.0 \cdot 10^{-4} & 8.9 \cdot 10^{-3} \\ 1.0 \cdot 10^{-2} & 2.4 \cdot 10^{-2} & 4.1 \cdot 10^{-3} \\ -2.8 \cdot 10^{-3} & 2.0 \cdot 10^{-2} & 2.7 \cdot 10^{-2}\end{array}\right)$ | $\left[10^{-5}, 10\right] \cdot \mathbb{1}_{3}$ |

Table 3: The parameters used for the random sampling in section 5.4. If a general component of a vector or a matrix is shown, then all those components were sampled independently. If the sampling interval includes several orders of magnitude, the variable was sampled uniformly on a logarithmic scale. The neutrino mass is an input parameter for the Casas-Ibarra parametrization.
$\left.\begin{array}{cc}\hline \text { parameter } & \text { value } \\ \hline \lambda & 0.26 \\ \lambda_{2} & 0.5 \\ \lambda_{3} & {\left[-\sqrt{\lambda \cdot \lambda_{2}}, 4 \pi\right]} \\ \lambda_{4} & {\left[\max \left(-4 \pi,-\sqrt{\lambda \cdot \lambda_{2}}-\lambda_{3}+\left|\lambda_{5}\right|\right), 4 \pi\right]} \\ \lambda_{5} & \pm\left[10^{-15}, 10\right] \\ \left(y_{2}\right)_{i} & \pm\left[10^{-5}, 10\right] \\ \xi & \left( \pm\left[10^{-8}, 10^{-3}\right]\right. \\ \xi & \pm\left[10^{-4}, 1\right] \\ & \pm\left[10^{-4}, 1\right]\end{array}\right)$

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