Deep Inelastic Scattering with Massive Quarks at Next-to-leading Order in QCD

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1. Introduction

The history of the Standard Model is strongly related to an increasing understanding of the structure of hadrons. What started with the famous experiments of Ernest Rutherford in the beginning of the 20th century forms nowadays the basis of new discoveries at the Large Hadron Collider at CERN. The realization of Deep Inelastic Scattering (DIS) was perhaps one of the most fruitful steps in this whole development, which has spanned over more than the last hundred years. A very comprehensive overview can be found in the three Nobel prize lectures of 1990, [Friedman et al., 1991]. Starting in the 1960's at the Stanford Linear Accelerator, this research led to the experimental validation of quarks and set the ground for the theory which is today known as Quantum Chromodynamics (QCD). These two components form our modern understanding of hadrons, which is formulated with the help of the parton model.

This thesis focuses on adding a certain aspect to the basic (or naive) parton model, namely the implementation of quark masses. Although this was a subject since the early 1980's, it was solved in a satisfactory way only more than ten years later by the ACOT (Aivazis, Collins, Olness and Tung) formalism, which was established within a series consisting of two papers [Aivazis et al., 1994a] and [Aivazis et al., 1994b]. There, the authors focused on heavy quark production in DIS, but their ideas extend to any desired process involving hadrons. It is the main object of this thesis to reproduce and explain these publications as detailed as possible.

In order to do so, we will first review some basic aspects of Quantum Field Theory and the Standard Model of particle physics, and, after that, focus on the parton model, using DIS as an example. In that chapter, we also make efforts to establish conventions in such a way that they can be conveniently generalized to non-vanishing masses as well as retained again when changing back to the massless case. Then, we turn to the ACOT formalism. Another preliminary chapter is needed to explain special theoretical background. Finally, the publications themselves can be reviewed in the last two chapters. The first one will deal with the general formulation of the problem in terms of the helicity formalism as well as some initial leading order calculations. The second one turns to the topic of next-to-leading order calculations, entailing additional issues such as finding a proper renormalization and subtraction scheme for the quantities of the parton model. However, the core of this thesis is what follows afterwards: The appendix. At various points in the main text, calculations are outsourced to these sections. Here, the interested reader may find much deeper and more detailed discussions than would have been possible in the original publications. In particular, many formulae are only stated in the ACOT papers, while their derivations are by no means trivial. The different appendices aim to close this gap and, additionally, add some further notes and calculations for the sake of a broader understanding. Moreover, the last appendix contains points in the original ACOT publications that, at least from the author's perspective, needed some clarifications or even corrections.

All diagrams in this work were created with the help of JaxoDraw, see [Binosi and Theußl, 2004].

2. Theoretical overview

This chapter shall provide a broad overview over the underlying theory and some special techniques that are used throughout the whole work. There is no claim of completeness; the interested reader's attention is rather drawn to the mentioned literature in front of every section.

2.1. Lie groups

A good introduction to group theory in general and Lie groups in particular can be found in [Tung, 1985], a more application oriented introduction for the Standard Model is given in [Schwartz, 2014], sections 10.1 and 25.1. This short section will only introduce the most important terms in a more physical way, ignoring many mathematical formalities.

2.1.1. A general overview

A group G is a set of elements $U \in G$ with an operation $*$ combining two of these elements that fulfills the group axioms:

$$\begin{aligned} \text{Completeness:} \quad & \forall U, V \in G : U * V \in G \\ \text{Associativity:} \quad & \forall U, V, W \in G : (U * V) * W = U * (V * W) \\ \text{Neutral element:} \quad & \forall U \in G \; \exists \, U^{-1} \in G : U * U^{-1} = E, \end{aligned}$$

where E is the neutral element of the group, i.e.

$$U * E = U. \tag{2.1}$$

A Lie group is, in addition, a differentiable manifold[1]. In physics, Lie groups are often introduced with their property that they are continuously connected to the identity so the group elements can be displaced via the exponential map:

$$U = \exp(-i\alpha^a T^a), \tag{2.2}$$

where $\alpha^a \in \mathbb{C}$, where $a \in \mathbb{N}$ are coefficients specifying the element g and T^a the so-called generators of the group. As it can be deduced directly from (2.2), the group generators are the first differential of the elements,

$$i \frac{dU}{d\alpha^a}\bigg|_{\alpha^a=0} = T^a. \tag{2.3}$$

[1] This is a differential geometric object, cf. e.g. [Lee, 2012] for a proper introduction.

This at least gives a hint to the fact that generators form the tangent space of G at 0.

Generators form a Lie algebra[2]:

$$[T^a, T^b] = if^{abc}T^c, \tag{2.4}$$

where $[\cdot, \cdot]$ is a Lie bracket, i.e. the commutator of T^a and T^b, and f^{abc} are the antisymmetric structure constants of the algebra. The connection to the Lie brackets implies that these constants also obey the Jacobi identity

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0. \tag{2.5}$$

We call a Lie group simple, if it is irreducible[3], and semi-simple, if the group decomposes into a direct product of simple Lie groups.

Up until now, group elements and generators were not further specified and introduced as abstract objects. In fact they are, but one can find homomorphisms that "transport" this group and algebra structure to e.g. a matrix space. There are infinitely many homomorphisms, each corresponding to a so-called representation in matrix shape. Let us deeper examine some concrete examples of Lie groups, namely the $SU(N)$.

2.1.2. The Lie group $SU(N)$

The set of all N-dimensional unitary[4] matrices with $\det(U) = 1$ form the group $SU(N)$, describing rotations in an N dimensional complex vector space. The unitarity of $SU(N)$ implies that all generators of the according Lie algebra, denoted by $\mathfrak{su}(N)$, must be hermitian (cf. (2.2)). Two representations are of great importance in the further explanations:

At first the fundamental representation, which is the representation with the smallest possible dimension, taking trivial representations like

$$T^a_{\text{trivial}} = 0 \;\forall a \tag{2.6}$$

aside. In the case of $SU(N)$ this dimension is $d(F) = N$ and the generators ($N \times N$ matrices) will be denoted as $T^a_F \equiv T^a$, suppressing the subscript F.

The second one is called adjoint representation. In this representation the matrix elements of the generators are formed by the structure constants,

$$(T^a_A)_{lm} = -if^a_{lm}. \tag{2.7}$$

In the case of $SU(N)$ there are $N^2 - 1$ generators, so the T^a_A are $(N^2 - 1) \times (N^2 - 1)$ matrices and $d(A) = N^2 - 1$.

There are several constants that characterize different representations. First of all, it is often useful to find elements of G that commute with all others, called Casimir

[2] If not stated otherwise, here and in all following calculations we use the Einstein summation convention.

[3] I.e. we cannot find a closed subgroup.

[4] I.e. the complex conjugate of U is equal to its transverse, $U^* = U^{-1}$.

operators or simply Casimirs. Schur's Lemma[5] implies that elements of this form need to be proportional to $\mathbb{1}$ in an irreducible representation. The most important one is the quadratic Casimir formed by the sum of all generators squared:

$$T_R^a T_R^a = C_2(R)\mathbb{1}. \tag{2.8}$$

The constant of proportionality $C_2(R)$ in the expression above depends on the arbitrary representation R.

Generators are normalized in the sense of

$$\mathrm{Tr}\left(T_R^a T_R^b\right) = T(R)\delta^{ab}. \tag{2.9}$$

The index of the representation is closely linked to the normalization of the structure constants. The convention mostly used in physics is

$$f^{acd} f^{bcd} = N\delta^{ab}, \tag{2.10}$$

which directly implies

$$T_F \equiv T(F) = \frac{1}{2} \tag{2.11}$$

and

$$T_A \equiv T(A) = N. \tag{2.12}$$

(2.9) also lets us specify $C_2(R)$. By exploring the special case $b = a$, which implies a summation over a, we end with

$$C_2(R) = T(R)\frac{N^2 - 1}{d(R)}. \tag{2.13}$$

For the fundamental and adjoint representation we obtain

$$C_F \equiv C_2(F) = \frac{N^2 - 1}{2N} \tag{2.14}$$

and

$$C_A \equiv C_2(A) = N. \tag{2.15}$$

2.1.2.1. $U(1)$, $SU(2)$ and $SU(3)$

The special cases $N = 1, 2, 3$ are by far the most important ones, since they represent the gauge symmetries of fundamental Lagrangians in the Standard Model (cf. sections 2.6.1 and 2.6.2).

$U(1)$ has a very simple structure. There is only one generator, the identity itself. Therefore it is an abelian group, which is a rare exception. For the same reason the structure of the Lie algebra is trivial, too.

[5] Actually, this is only a special case of Schur's Lemma applied on matrices.

$SU(2)$ and $SU(3)$ are non-abelian groups. The generators of $SU(2)$ are the well-known Pauli spin matrices (in fundamental representation) up to a prefactor,

$$T^a = \frac{1}{2}\sigma^a. \tag{2.16}$$

The Casimir in the fundamental representation is (using (2.14))

$$C_F = \frac{3}{4}, \tag{2.17}$$

all other group constants in the fundamental and adjoint representation are trivial (see above). The structure constants are the well-known Levi-Civita symbols,

$$f^{abc} = \epsilon^{abc}. \tag{2.18}$$

It is worth noting that $SU(2)$ and $SO(3)$[6] are isomorph, which means that they have the same Lie algebra. This general Lie algebra is denoted by

$$[J_l, J_m] = i\epsilon_{lmn}J_n, \tag{2.19}$$

where the J_1, J_2 and J_3 are the angular momentum operators, well-known from Quantum mechanics. A more convenient choice in many cases are linear combinations of J_1 and J_2,

$$J_\pm = J_1 \pm iJ_2, \tag{2.20}$$

which together with J_3 form an equivalent algebra. One can simultaneously find eigenstates of the quadratic Casimir

$$J^2 \equiv J_l J_l \tag{2.21}$$

and one J_l, typically J_3. These states, labeled by $|j, j_3\rangle$, have eigenvalues[7]

$$J^2 |j, j_3\rangle = j(j+1) |j, j_3\rangle \tag{2.22}$$

$$J_3 |j, j_3\rangle = j_3 |j, j_3\rangle, \tag{2.23}$$

where

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \tag{2.24}$$

$$j_3 = -j, -j+1, -j+2, ..., j-1, j. \tag{2.25}$$

The eigenvalue j can be used to label the representation, for example $j = \frac{1}{2}$ is the fundamental representation describing particles with spin $\frac{1}{2}$. The effects of $J_\pm$ on $|j, j_3\rangle$ are

$$J_\pm |j, j_3\rangle = \sqrt{j(j+1) - j_3(j_3 \pm 1)}\, |j, j_3 \pm 1\rangle. \tag{2.26}$$

[6] $SO(3)$ is the group of rotations in a three dimensional real vector space. For real matrices unitarity is replaced by orthogonality, i.e. $O^\dagger = O^{-1}$.

[7] We can recognize by the first formula that the quadratic Casimir of $\mathfrak{su}(2)$ is proportional to $j(j+1)$, so the choice of a specific j comes along with a choice of a representation.

For $SU(3)$, the generators in fundamental representation are formed by the so-called Gell-Mann-matrices λ^a,

$$T^a = \frac{1}{2}\lambda^a, \tag{2.27}$$

cf. e.g. [Schwartz, 2014], p. 485. The Casimir is (again using (2.14))

$$C_F = \frac{4}{3}. \tag{2.28}$$

2.1.2.2. Irreducible tensors and the Wigner-Eckart theorem

In the context of the helicity formalism we encounter tensors in the $|j, j_3\rangle$ basis which are not angular momentum operators. A proper introduction to this topic is e.g. given in [Sakurai, 1985], section 3.10.

In general, one can distinguish between vector and spherical operators. Both types are introduced in the section above: An example for vector operators is the angular momentum $\boldsymbol{J} = (J_1, J_2, J_3)$. The components of a general vector operator $\boldsymbol{V}$ transform like a three-vector in a Euclidean vector-space, i.e.

$$V'_i = R_{ij} V_j, \tag{2.29}$$

where R_{ij} are components of a rotational matrix $R \in SO(3)$ created by some combination of angular momentum operators J_i. An infinitesimal rotation implies the commutation relation

$$[V_l, J_m] = i\epsilon_{lmn} V_n, \tag{2.30}$$

which can also be used as a definition and is trivially fulfilled by $\boldsymbol{J}$ itself. $\boldsymbol{x}$ and $\boldsymbol{p}$ are other well-known examples. These considerations can be generalized to tensors of arbitrary rank $T_{m_1 m_2 \ldots}$ by multiplying a R_{ij} for every component in a transformation:

$$T'_{m_1 m_2 \ldots} = R_{m_1 l_1} R_{m_2 l_2} \cdots T_{l_1 l_2 \ldots}. \tag{2.31}$$

Tensors with this transformation property are called cartesian tensors.

These tensors are reducible with respect to different representations of $\mathfrak{su}(2)$. Hence, we can find a combination of irreducible tensors[8] transforming in different representations that is equal to the original cartesian tensor. Tensors that transform in one specific representation of $\mathfrak{su}(2)$ labeled with j are called irreducible tensors. A possible notation for an irreducible tensor is T^k_q, where k stands for the representation the tensor transforms in and $q = -k, -k+1, ..., k$ is the according equivalent to j_3. The simplest non-trivial example is a rank-1 tensor V_i in the representation $j = k = 1$ (which implies $q = -1, 0, 1 \equiv -, 0, +$), where one defines[9]

$$V_0 \equiv V_3 \tag{2.32}$$

$$\text{and} \quad V_\pm \equiv V_1 \pm i V_2. \tag{2.33}$$

[8] From now on, "irreducible" means "irreducible with respect to representations of $\mathfrak{su}(2)$".

[9] These definitions differ in the literature by conventional prefactors.

Such defined tensors meet the commutation relations for spherical tensors, which, just as the relation (2.30) in the case of vector tensors, can be used as a definition of spherical tensors.

The matrix elements of irreducible tensors in the angular momentum basis can be calculated by using the Wigner-Eckart theorem[10],

$$\langle j' j_3' | \, T_q^k \, | j j_3 \rangle \propto T_{jj'}^k \langle j' j_3' (k, j) q j_3 \rangle. \tag{2.34}$$

The strength of this theorem lies in the fact that an arbitrary complicated matrix element $\langle j' j_3' | \, T_q^k \, | j j_3 \rangle$ can be expressed through well-known Clebsch-Gordan coefficients $\langle jk; j_3 q | jk; j' j_3' \rangle$ and so-called reduced matrix elements $T_{jj'}^k$. They simply take the role of a proportionality constants, since for fixed j and j' the only factor which makes two matrix elements distinguishable is the Clebsch-Gordan coefficient. There are two selection rules for $\langle j' j_3' | \, T_q^k \, | j j_3 \rangle$ to be non-zero:

$$j_3' = j_3 + q \tag{2.35}$$

$$\text{and} \quad |j - k| \leq j' \leq j + k, \tag{2.36}$$

both arising from well-known properties of the addition of two angular momenta [11].

2.1.3. The Lorentz group

One of the most important outer or global symmetries of Standard Model Lagrangians is the Lorentz symmetry, well-known from special relativity. This symmetry is in many cases a very obvious one, since Lorentz-invariant objects are likely formulated in a covariant way using 4-vectors in Minkovski space and the Minkovski metric[12]

$$g^{\mu\nu} = \text{diag}(1, -1, -1, -1). \tag{2.37}$$

Elements of the Lorentz group $\Lambda^{\mu\nu}$ are transformations that leave $g^{\mu\nu}$ invariant, i.e.

$$\Lambda_\mu^\lambda g^{\mu\nu} \Lambda_\nu^\sigma = g^{\lambda\sigma}. \tag{2.38}$$

The Lorentz group is often denoted as $SO(1,3)$, referring to the number of dimensions with positive (1) and negative (3) sign in the metric[13].

It can be shown that the generators of the Lorentz group are J_j and K_j, where $j = 1, 2, 3$. J_i are angular momentum operators generating the $SO(3)$ subgroup and

[10] For a proof cf. e.g. [Sakurai, 1985], p. 239 and 240. For the Clebsch-Gordan coefficients, we use the convention of [Tung, 1985].

[11] Cf. e.g. [Sakurai, 1985], section 3.7, which can also serve as an introduction to Clebsch-Gordan coefficients.

[12] The reader's knowledge about the underlying covariant formalism used in special relativity and also relativistic Quantum Field Theory will be assumed in this work. For a complete introduction cf. e.g. [Sakurai, 1985].

[13] One distinguishes between $O(1,3)$ and $SO(1,3)$, the latter called the proper Lorentz group. The S again refers to $\det(\Lambda) = 1$, which, in this case, excludes parity and time reversal operations.

K_j are the boost generators. For example, a boost along the z- or x_3-axis can be written as[14]

$$\Lambda_{z\text{-boost}}(\psi) = \exp(-i\psi K_3) = \begin{pmatrix} \cosh(\psi) & 0 & 0 & -\sinh(\psi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\psi) & 0 & 0 & \cosh(\psi) \end{pmatrix}. \tag{2.39}$$

In this notation, the close relation between rotations and boosts becomes obvious: Boosts can be seen as hyperbolic rotations, containing hyperbolic sines and cosines instead of trigonometric ones. If we apply the boost on a particle with vanishing three-momentum, the hyperbolic angle is called rapidity. Then,

$$\psi \equiv \operatorname{artanh}(|\boldsymbol{v}|) \tag{2.40}$$

holds, with $\boldsymbol{v} = \frac{\boldsymbol{p}}{E}$ being the velocity of a particle after the boost.

The complete Lie algebra generating the Lorentz group is

$$[J_l, J_m] = i\epsilon_{lmn} J_n \tag{2.41}$$
$$[J_l, K_m] = i\epsilon_{lmn} K_n \tag{2.42}$$
$$[K_l, K_m] = -i\epsilon_{lmn} K_n. \tag{2.43}$$

Hence, an arbitrary Lorentz transformation can be written as

$$\Lambda = \exp\Big[i(\theta_l J_l + \psi_m K_m)\Big], \tag{2.44}$$

where θ_j denotes the rotation angles around all three axes, also known as the Euler angles, and ψ_m are rapidities along the three axes, respectively.

We can recombine J_l and K_l to $J_l^{\pm}$ via

$$J_l^{\pm} = \frac{1}{2}\Big(J_l \pm iK_l\Big), \tag{2.45}$$

which greatly simplifies the Algebra structure:

$$[J_l^{\pm}, J_m^{\pm}] = i\epsilon_{lmn} J_n^{\pm} \tag{2.46}$$
$$[J_l^{+}, J_l^{-}] = 0. \tag{2.47}$$

Now there is no mixing between both generator families, which means that we can write the Lorentz Lie algebra as a direct sum of two angular momentum algebras:

$$\mathfrak{so}(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{2.48}$$

Consequently, $SO(1,3)$ is a direct product of $SU(2)$ groups,

$$SO(1,3) = SU(2) \otimes SU(2), \tag{2.49}$$

[14] An explicit calculation of this boost matrix can be found in appendix B.

and therefore a semi-simple Lie group. This opens the door for a very practical way to label Lorentz group representations, namely by the quantum number j of both $SU(2)$ groups, i.e. (j_a, j_b). The two most important ones are $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, describing fermions and antifermions via four dimensional spinors, and $(\frac{1}{2}, \frac{1}{2})$, describing four-vector-like spin-1 particles as the gauge bosons in the Standard Model[15]. Note that in this situation we deal with (space dependent) fields, i.e. infinite dimensional Lorentz group representations. The problem behind all finite dimensional representations is that they are non-unitary and so do not conserve probability. This can be traced back to the non-compact nature of the Lorentz group. Actually, in the case of finite dimensional representations, the generators are anti-hermitian, which makes the group elements anti-unitary. This can nicely be seen at the relation between generators and group elements (2.2). However, there are unitary subgroups, the so-called little groups. They are such defined that they leave a specific momentum unchanged. An example is the rotation group in three dimensions $SO(3)$, which does not change the momentum if its spacial components fulfill $\boldsymbol{p} = \boldsymbol{0}$.

[15] To be more precise: The matrices of the according Lorentz group representations act on these spinors and vectors.

2.2. Spinors and the Weyl representation

Whenever we want to describe fermions, we have to deal with so-called spinors, denoted by $u(p)$. Hence, here we will give a short introduction on them. A more detailed discussion can be found in every textbook about QFT, such as [Schwartz, 2014] or [Peskin and Schroeder, 1995].

Solving the Dirac equation

$$(\gamma_\mu p^\mu - m)u(p) = 0 \tag{2.50}$$

in the Weyl representation[16] yields the following form of the $u_s(p)$[17]:

$$\xi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \xi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.51}$$

where σ^μ and $\overline{\sigma}^\mu$ are vectors of Pauli matrices and the Weyl spinors ξ_s are solutions of the Dirac equation for a particle at rest. There is a freedom of choice when one specifies the two ξ, which only need to solve the massless Dirac equation[18] and fulfill

$$\xi_s^\dagger \xi_{s'} = \delta_{ss'}. \tag{2.52}$$

The ξ_s defined above are eigenstates of σ^3 and thus define a particle with spin $\pm\frac{1}{2}$. The same is then true for the $u_s(p)$. In principal, other choices are also possible. We will introduce the for this work most important of them, the helicity spinors, in section 2.3.2.

$\overline{u}_s(p)$ is defined as

$$\overline{u}_s(p) = u_s^\dagger(0)\gamma_0, \tag{2.53}$$

such that

$$\overline{u}_s(p)u_{s'}(p) = 2m\delta_{ss'} \tag{2.54}$$

is a Lorentz scalar. Of great importance is the completeness relation

$$\sum_s u_s(p)\overline{u}_s(p) = p_\mu\gamma^\mu + m \equiv \not{p} + m, \tag{2.55}$$

where

$$m^2 \equiv p^2. \tag{2.56}$$

[16] The Weyl representation for Dirac matrices is defined in appendix A.

[17] Taking the square root of a matrix M is mathematically not a uniquely defined expression. The conventional definition in this case is the following: Change to the eigenvector space via the matrix S^{-1}, take the square root of all entries, which are the eigenvalues, of the diagonalized matrix M' (This operation can be uniquely denoted by $\sqrt{M'}$, trivially fulfilling $\sqrt{M'} \cdot \sqrt{M'} = M'$) and change back to cartesian space via S. One easily checks that $\sqrt{M} \equiv S\sqrt{M'}S^{-1}$ fulfills $\sqrt{M} \cdot \sqrt{M} = S\sqrt{M'}S^{-1}S\sqrt{M'}S^{-1} = SM'S^{-1} = M$.

[18] This equation also is sometimes called Weyl equation, referring to the Weyl spinors introduced below.

When quantizing a spinor field $\psi(x)$, the spinors take the role of polarizations,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_s \Big(e^{-ipx} u_s(p) a_s(p) + e^{ipx} v_s(p) b_s^\dagger(p) \Big), \tag{2.57}$$

where the $a_s(p)$ and $b_s^\dagger(p)$ are the creation and annihilation operators for (anti-) particles and $v_s(p)$ is an anti-particle spinor. These anti-spinors obey a slightly different completeness relation, namely

$$\sum_s v_s(p)\overline{v}_s(p) = p_\mu \gamma^\mu + m \equiv \not{p} - m. \tag{2.58}$$

Throughout this work, we will stick to the Weyl notation of spinors. The reason for that is that it makes the decomposition of the Lorentz group into a product of two $SU(2)$ subgroups[19] evident, since the two upper and lower components transform independently in their own subgroups, respectively. A general spinor field in Weyl notation is often denoted as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.59}$$

where the subscripts of the Weyl spinors $\psi_{L,R}$ is called the particle's chirality (left- or right-handed). With the help of the projection operators

$$P_{L,R} = \frac{1}{2}(1 \pm \gamma^5), \tag{2.60}$$

one can obtain spinors containing only left- or right-handed Weyl spinors.

[19] Cf. section 2.1.3 for further information.

2.3. Helicity

A comprehensive introduction to calculating amplitudes of Feynman graphs with the help of helicity is given in [Olness and Tung, 1987]. A discussion of helicity itself can be found in every textbook about the Standard Model, e.g. [Schwartz, 2014], section 11.1. The eigenvalues and eigenstates in the case of spin-$\frac{1}{2}$ and spin-1 are calculated in appendix C.

2.3.1. General definition

The helicity operator h of a particle with momentum $\boldsymbol{p}$ is defined as

$$h = \frac{\boldsymbol{p} \cdot \boldsymbol{S}}{|\boldsymbol{p}|} \tag{2.61}$$

with the help of the spin operator $\boldsymbol{S}$ in an arbitrary representation. The scalar product in the numerator indicates that it operates as a projection operator for the spin on the momentum axis $\boldsymbol{p}/|\boldsymbol{p}|$. It should be stressed that since the momentum and spin are conserved, so is the helicity. We see that the helicity is closely related to angular momentum, i.e. the particle's spin. In fact, for every spin eigenvalue there is a corresponding helicity eigenvalue. In the helicity formalism, states of an arbitrary particle are not defined via the momentum p and the spin s anymore[20], but rather with p and the helicity λ: $|p, \lambda\rangle$. We define the scalar λ as the eigenvalue of the helicity operator h,

$$h\,|p, \lambda\rangle = \lambda\,|p, \lambda\rangle\,. \tag{2.62}$$

Furthermore, it can be shown that in the zero-mass limit helicity and chirality operators have the same eigenstates. To see this (in the case of spin-$\frac{1}{2}$), consider the Dirac equation (2.50) with vanishing mass:

$$i\gamma_\mu \partial^\mu \psi(x) = 0, \tag{2.63}$$

which becomes

$$\begin{pmatrix} 0 & \sigma_\mu p^\mu \\ \overline{\sigma}_\mu p^\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_L(p) \\ \psi_R(p) \end{pmatrix} = 0 \tag{2.64}$$

in Weyl representation and Fourier space. Inserting (2.61) yields

$$h\psi_{L,R}(p) = \pm \frac{E}{|\boldsymbol{p}|} \psi_{L,R}(p) = \pm \psi_{L,R}(p), \tag{2.65}$$

meaning that chirality eigenstates are simultaneously helicity eigenstates, too[21]. For example, in the presence of a purely left-handed coupling, there is only one helicity

[20] We work with these in the context of defining the leptonic tensor in the naive parton model, cf. section 3.1.

[21] A physical explanation goes as follows: Due to the particle moving with the speed of light, an observer is not able to boost to a Lorentz frame in which the particle's direction of motion flips. This means that a projection of the (Lorentz invariant) spin onto the direction of motion also does not change its sign. Therefore, the helicity is Lorentz invariant and nothing else than the chirality of the particle.

allowed for massless particles. Since in the ACOT-formalism leptonic masses will taken to be zero, this fact will become useful later.

In the following, we will have a detailed look at the eigenvalues and corresponding eigenvectors in the case of $s = \frac{1}{2}$ and $s = 1$.

2.3.2. Polarizations and spinors

2.3.2.1. Spin-1: Gauge Bosons

The role of polarizations in the context of a spin-1 field are discussed e.g. throughout chapter 6 of [Greiner and Reinhardt, 1996].

The approach of the helicity formalism is to make the degrees of freedom not manifest in form of Minkovski coordinates, but with the help of helicity eigenstates, i.e. polarizations ϵ^μ_λ in the case of bosons. Eigenvalues of h (often also simply called helicities) from now on will be denoted by λ, which takes the values 0 and ± 1 in the case of bosons with $s = 1$. These polarizations are defined with respect of a reference axis, which is taken to be the direction of another momentum taking part in the process. In addition, polarizations form an (nearly) orthonormal basis of the Minkovski space.

Of special interest for us are the polarizations of an arbitrary spacelike (i.e. having a negative norm) exchanged vector boson with momentum q^μ, which is aligned to the z-axis. Hence, let us explicitly give these polarizations corresponding to this special q^μ with reference to an arbitrary timelike (i.e. having a positive norm) momentum denoted by P^μ. E.g., when q^μ is aligned to the z-axis, the reference momentum defines the t-z plane together with q^μ. In this work, we work in a frame where P^μ also has vanishing transverse momenta, i.e. vanishing x- and y-components.

The alignment of $\boldsymbol{q}$ and the z-axis makes the transverse polarizations rather easy. We choose the Condon-Shortley convention[22]

$$\epsilon^\mu_\pm(P, q) \equiv \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0). \tag{2.66}$$

In the case of photons, these two polarizations correspond to circular polarized light.

In general, there is also a longitudinal polarization depending on P^μ and q^μ, corresponding to the $\lambda = 0$ eigenvalue:

$$\epsilon^\mu_0(P, q) \equiv \frac{Q^2 P^\mu + (P \cdot q) q^\mu}{Q\sqrt{(P \cdot q)^2 + Q^2 P^2}}. \tag{2.67}$$

Here, the reference momentum takes the role of giving the longitudinal polarization a rectangular component in the t-z plane with respect to q^μ. Since we set the x-

[22] In fact, the actual Condon-Shortley convention is $\epsilon^\mu_\pm(P, q) \equiv \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0)$ for momenta positively aligned to the z-axis (cf. appendix C.1). Since $q^3 < 0$, we have to rotate around the y-axis by π (in other words: taking the negative of the x- and z-component) to obtain the $\epsilon^\mu_\pm(P, q)$ fitting to our purposes.

and y-components of P^μ and q^μ to zero, it is easy to see that $\epsilon_0^\mu(P,q)$ is indeed orthogonal to $\epsilon_\pm^\mu(P,q)$.

As discussed in appendix C.1, the $\lambda = 0$ helicity has a two-dimensional eigenspace. Thus, there needs to be a second linear independent polarization, which is chosen such that it is orthogonal to $\epsilon_0^\mu(P,q)$[23]:

$$\epsilon_q^\mu(P,q) \equiv \frac{q^\mu}{Q}. \tag{2.68}$$

With a complete set of polarizations at hand, we can now give the completeness relation of this basis. However, it is not completely normalized because squaring some polarizations, namely $\epsilon_\pm^\mu(P,q)$ and $\epsilon_q^\mu(P,q)$, gives -1 and not 1´. Hence, we need an additional factor (compared to a generic completeness relation) of

$$v_\lambda = \begin{cases} 1 & \text{for } \lambda = 0 \\ -1 & \text{for } \lambda = \pm, q \end{cases}. \tag{2.69}$$

Using this factor, we have

$$\sum_{\lambda=\pm,0,q} v_\lambda \epsilon_\lambda^{\mu\star} \epsilon_\lambda^\nu = g^{\mu\nu}. \tag{2.70}$$

As it is the case for spinors (cf. section 2.2), the four vectors above can be used to quantize a field of an arbitrary gauge boson[24]:

$$A^\mu(x) = \int \frac{d^3q}{\sqrt{2\omega_q(2\pi)^3}} \sum_\lambda \Big(\hat{a}_\lambda(q)\epsilon_\lambda^\mu(P,q)e^{-iqx} + \hat{a}_\lambda^\dagger(q)\epsilon_\lambda^{\star\mu}(P,q)e^{iqx}\Big), \tag{2.71}$$

where $\hat{a}_\lambda^\dagger(q)$ and $\hat{a}_\lambda(q)$ are operators that create or destroy a boson with helicity λ and momentum q^μ. Which helicities λ appear in the sum above depends on whether the boson is massive or not. This will be discussed in more detail in section 2.4.

2.3.2.2. Spin-$\frac{1}{2}$: Fermions

When describing fermions, one needs to introduce helicity spinors[25] $u_\lambda(p)$, in contrast to the usual spin-dependent spinor $u_s(p)$. The reader shall be reminded of the fact that there is a freedom of choice when defining a spinor basis. In section 2.2, we chose the solutions for the Dirac equation of a particle at rest to be spin eigenstates of σ^3 with the well-known eigenvalues $\pm\frac{1}{2}$. Hence, the spinors $u_s(p)$ describe

[23]It is automatically orthogonal to $\epsilon_\pm^\mu(P,q)$, because both polarizations correspond to different eigenvalues and h is an unitary operator.

[24]In fact, this is the actual reason why we call the $\epsilon_\lambda^\mu(P,q)$ polarizations, although at first instance they are only eigenstates of the helicity operator.

[25]To put it in other words, when we change from ϵ_λ^μ to u_λ, we just change the Lorentz group representation from spin 1 to spin $\frac{1}{2}$. Both objects take the role of polarizations when quantizing vector or spinor fields.

fermions with spin up and down along the momentum axis. Of course, this choice is to some extend arbitrary. We can just as well choose helicity eigenstates fulfilling

$$h\xi_\lambda(p) = \lambda\xi_\lambda(p). \tag{2.72}$$

Note that in this work the explicit helicity indices of fermions are denoted by $\pm\frac{1}{2}$, only to distinguish them from bosonic ones. This shall of course not be confused with the actual helicity eigenvalues of fermions, ± 1. For example, an eigenstate of $\lambda = -1$ would be denoted as $\xi_{-\frac{1}{2}}(p)$.

As the equation above suggests, eigenstates will now be momentum-dependent and not constant anymore. In general, λ can take the values ± 1 with according eigenstates

$$\xi_{+\frac{1}{2}}(\boldsymbol{p}) = \frac{1}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}|+p^3)}}\begin{pmatrix}|\boldsymbol{p}|+p^3\\ p^1+ip^2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \tag{2.73}$$

$$\text{and } \xi_{-\frac{1}{2}}(\boldsymbol{p}) = \frac{1}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}|+p^3)}}\begin{pmatrix}-p^1+ip^2\\ |\boldsymbol{p}|+p^3\end{pmatrix} = \begin{pmatrix}-e^{i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}, \tag{2.74}$$

where the last expression is formulated in spherical coordinates of $\boldsymbol{p}$. These spinors form a basis and can be used to quantize spinor fields $\psi(x)$. Hence, we can also define according four dimensional spinors via

$$u_\lambda(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_\lambda(p)\\ \sqrt{p\cdot\bar{\sigma}}\,\xi_\lambda(p)\end{pmatrix}. \tag{2.75}$$

Completely analogous to (3.3), one can obtain this spinor when acting with an accordingly quantized spinor field[26] on a helicity eigenstate:

$$\psi(p)\,|p,\lambda\rangle = u_\lambda(p)\,|0\rangle\,. \tag{2.76}$$

[26] "Accordingly quantized" means that we replace the spinors $u_s(p)$ in equation (2.57) with the $u_\lambda(p)$ from above.

2.4. Gauge theories

Almost every theory of particle physics, the most famous one probably the Standard Model itself, is a gauge theory. Hence, a discussion of gauge theories is given in every QFT textbook, e.g. [Schwartz, 2014] and [Peskin and Schroeder, 1995]. However, gauge invariances are often introduced little by little throughout the whole book. For this reason, let us summarize the most important facts in this section. It is closely based on chapters 6 and 7 of [Greiner and Reinhardt, 1996], where the treatment of physical and unphysical degrees of freedom in a massive or massless spin-1 field together with different gauges is discussed in a particular constructive manner.

2.4.1. (Un-)physical polarizations of gauge bosons

2.4.1.1. The massive case

As a four-dimensional real (Lorentz) vector, A^μ a priori has four degrees of freedom. However, this does not reflect the real world.

For simplicity, let us start with the Lagrangian of a massive, uncharged (i.e. real) vector boson, which is known as the Proca theory:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu, \tag{2.77}$$

where the field strength tensor $F^{\mu\nu}$ is defined as[27]

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.78}$$

The corresponding Euler-Langrange equations are

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu + M^2 A^\mu = 0. \tag{2.79}$$

Contracting with ∂_μ gives

$$\partial_\mu A^\mu = 0, \tag{2.80}$$

which puts an additional constraint onto A^μ. This simplifies the Euler-Lagrange equation to

$$\left(\partial_\nu \partial^\nu + M^2\right) A^\mu = 0, \tag{2.81}$$

meaning that every component of A^μ obeys the Klein-Gordon equation. When substituting the quantization for A^μ (2.71) into (2.80) and switching to momentum space, we can reformulate the expression above into a condition for the polarizations which were introduced in section 2.3.2.1:

$$q_\mu \epsilon^\mu_\lambda(P, q) = 0. \tag{2.82}$$

A simple calculation yields that only $\epsilon^\mu_q(P, q)$ does not meet this condition. Hence, we have reduced the number of physical polarizations and the sum over polarizations

[27] One can find a more general definition in section 2.6.1.

in the quantization (2.71) only runs over $\lambda = 0, \pm 1$. In fact, with this condition we could separate the Spin-0 degree of freedom from the three spin-1 degrees of freedom in a general vector field. This is the reason why $\epsilon_q^\mu(P,q)$ is sometimes called scalar polarization.

For completeness, let us give the propagator of a massive spin-1 field. For our sakes it is sufficient to simply invert the kinetic part of the Proca-Lagrangian (2.77) and not bother with the more formal field theoretic approach of calculating $\langle 0| T(A^\mu(x)A^\mu(y)|0\rangle$. Both ways are described in section 7.4 and 7.5 of [Greiner and Reinhardt, 1996]. We can rewrite $\mathcal{L}_{\text{Proca}}$ into[28]

$$\mathcal{L}_{\text{Proca}} = \frac{1}{2}A^\mu\Big(\overleftarrow{\partial}_\mu\partial_\nu - g_{\mu\nu}(\overleftarrow{\partial}_\rho\partial^\rho - M^2)\Big)A^\nu \equiv \frac{1}{2}A^\mu D_{\mu\nu}A^\nu. \tag{2.83}$$

The (Feynman-) propagator $\Pi^{\mu\nu}$ is the inverted Fourier transformed kinetic operator $D_{\mu\nu}$, thus

$$\Pi^{\mu\rho}D_{\rho\nu}(q) \overset{!}{=} g^\mu_\nu. \tag{2.84}$$

The only possible tensors of rank 2 that $\Pi^{\mu\nu}$ can depend on are $g^{\mu\nu}$ and $q^\mu q^\nu$, since q^μ is the only Lorentz four-vector involved in $\mathcal{L}$. Substituting the ansatz

$$\Pi^{\mu\rho} = Ag^{\mu\rho} + Bq^\mu q^\rho \tag{2.85}$$

into (2.84) yields[29]

$$\Pi^{\mu\nu} = \frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}}{q^2 - M^2}. \tag{2.86}$$

2.4.1.2. The massless case

A massless spin-1 boson, e.g. a photon, can be described by setting $M = 0$ in the Proca-Lagrangian (2.77), leaving

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.87}$$

where $F^{\mu\nu}$ is defined as in (2.78). The Euler-Lagrange equations are simply the Maxwell equation,

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.88}$$

Further differentiating gives zero, since we contract one symmetric and one antisymmetric tensor. Thus, opposite to the massive case, there is no additional condition for A^μ arising from the equations of motion.

[28] The arrow in $\overleftarrow{\partial}_\mu$ indicates that the derivates acts on the vector field in front of the brackets. This object can be seen as the hermitian conjugate of the normal differential operator ∂^μ, so it transforms to $+iq_\mu$ when going to momentum space.

[29] A complete field theoretical approach would have also led to the proper time-ordering by adding an additional $\pm i\epsilon$ in the denominator. For our purposes, the time-ordering is irrelevant, so ϵ can be simply set to 0.

Let us try to derive the propagator as we did for the massive spin-1 field. Rewriting the Maxwell Lagrangian gives

$$\mathcal{L}_{\text{Maxwell}} = \frac{1}{2} A^\mu \left(\overleftarrow{\partial}_\mu \partial_\nu - g_{\mu\nu} \overleftarrow{\partial}_\rho \partial^\rho \right) A^\nu \equiv \frac{1}{2} A^\mu D_{\mu\nu} A^\nu. \tag{2.89}$$

However, following the approach in the section above by making a general ansatz for $\Pi^{\mu\nu}$ and contracting with $D_{\mu\nu}(q)$ does not lead any results for A and B at all. The reason for that is that $D^{\mu\nu}(q)$ is simply not invertible, which can be seen by the fact that it has an eigenvalue of 0 with eigenvector q^μ.

2.4.2. Different Gauges of a massless spin-1 field

The difficulties above can be traced back to fact that a massless vector field A^μ has one less degree of freedom compared to a massive one. The underlying reason for this is an equivalence of several solutions for A^μ under a so-called gauge-transformation. Hence, we need to establish a condition like (2.80) artificially, i.e. we need to choose a gauge. The most common one from electrodynamics is the Coulomb gauge, where we demand that the three spacial components of A^μ fulfill

$$\nabla \boldsymbol{A} = 0. \tag{2.90}$$

This has the advantage of automatically canceling the longitudinal degree of freedom, $\epsilon_0^\mu(P, q)$. On the other side, the obvious disadvantage is that it is not Lorentz invariant, meaning that in this case we (implicitly) also choose a specific frame to work in.

Coming from the Proca-theory, the most natural choice would be to simply choose

$$\partial_\mu A^\mu = 0 \tag{2.91}$$

as our gauge. Unfortunately, it can be shown that this is not possible for a massless spin-1 field (cf. section 7.2 in [Greiner and Reinhardt, 1996]). However, there is a way out: We can add a gauge-fixing term to $\mathcal{L}_{\text{Maxwell}}$,

$$\mathcal{L}'_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2, \tag{2.92}$$

which "enforces" this constraint in the fashion of a Lagrange multiplier. Depending on the literature, ξ is introduced as a parameter or as a so-called auxillary field, i.e. a field without a kinetic term in the Lagrangian.

In this way, we have introduced the class of covariant gauges. Common choices of ξ are

- $\xi = 0$, called Landau gauge,
- $\xi = 1$, called Feynman gauge,
- $\xi \to \infty$, called unitary gauge.

The resulting differential operator in $\mathcal{L}'_{\text{Maxwell}}$ can indeed be inverted and we obtain the propagator[30]

$$\Pi^{\mu\nu} = \frac{-g^{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2}}{q^2}. \tag{2.93}$$

Note that, although ξ appears in the propagator, as an artificial parameter it cannot be part of the physical amplitude $\mathcal{M}$. Thus, it drops out of any perturbative calculation when left arbitrary.

Compared to the Coulomb gauge, choosing one of the covariant gauges does not lead to an automatic cancellation of unphysical[31] degrees of freedom. Hence, for a photon, the sum in the quantization (2.71) runs over all helicities $\lambda = \pm 1, 0, q$. The cancellation of ϵ_0^μ and ϵ_q^μ can be achieved by a proper choice of a Hilbert space, a process which is called the Gupta-Bleuler method and is explained e.g. in section 7.3 of [Greiner and Reinhardt, 1996]. Hence, physical photons (e.g. in asymptotic states) can only be transversely polarized, described by $\epsilon_\pm^\mu$. In contrast, a virtual photon (e.g. in a loop) can have all four polarizations.

[30] Again, one needs to add a $\pm i\epsilon$ in the denominator for proper time-ordering.

[31] It can be shown that polarizations belonging to a helicity of $\lambda = 0$ lead to negative norms, cf. section 7.2 of [Greiner and Reinhardt, 1996].

2.5. Renormalization

The topic of renormalization is one of the key aspects of QFT and therefore extensively explained in every textbook like [Schwartz, 2014] or [Peskin and Schroeder, 1995]. In this section, we give only a short overview while we leave all the mathematical details to be explained, when needed, in later calculations of this work or in the books mentioned beforehand.

Historically, one of the most irritating aspects of quantum field theories was the divergences one encounters when calculating processes at higher orders in perturbation theory. At first sight, this behavior spoils the whole perturbation series and, even worse, causes unphysical behavior due to infinitely large observables. Renormalization is able to take care of some of this unphysical behavior, the so-called ultraviolet (UV) divergences. But before we sketch the process of Renormalization, we give a short classification of divergences.

2.5.1. Divergences

Calculating an arbitrary process at higher orders contains two things which are absent in leading order (or Born-) graphs: Loops and real emissions.

Loops introduce unobservable momenta which are integrated over when calculating the respective matrix element. These loop integrals can contain divergences when taking the loop momentum to infinity: The UV divergences. Their counterpart are the infrared (IR) divergences, which can also occur in loop calculations and, furthermore, when there are additional particles in the final state, the so-called real emissions.

There is a simple way how to characterize divergences in real emissions: Consider a process with an arbitrary number of final states. When we add an emission of another unobserved particle, i.e. a real emission, because we want to include higher orders in perturbation theory, it is guaranteed (by the Feynman rules given in appendix A.3) that we obtain an additional propagator with a denominator proportional to $(p_r - p_f)^2$. This is valid for any massless particle, whether it is a fermion or gauge boson. In fact, IR divergences only occur when all involved particles in the final state are taken to be massless, so additional to $p_i^2 = 0$ we also demand $p_r^2 = 0$. When we then perform the phase space integration over p_r^μ and p_f^μ (as a part of the integration over all final state momenta), we encounter poles created by the propagator above. These poles arise from the roots of the denominator, which is

$$(p_r - p_f)^2 \propto p_r \cdot p_i = E_r E_f (1 - \cos\theta) \tag{2.94}$$

in the massless case. θ is the angle between both three-momenta. The denominator above vanishes in two regions of phase space: Either one of the energies E_r or E_i vanishes (or becomes negligible compared to other quantities like the center-of-mass energy) or $\cos\theta \to 1$, which corresponds to collinear three-momenta. Consequently, the first case is called "soft divergence" and is a subclass of an IR divergence, while the second one is called "collinear divergence".

The famous Kinoshita-Lee-Nauenberg theorem ensures that, at a given order in perturbation theory, all the IR divergences of loop diagrams and real emissions completely cancel each other. Therefore, when taking all possible Feynman diagrams into account, these complete amplitudes will not contain any IR divergences, they are "IR safe". This is not the case for UV divergences, which is the reason why they need a special treatment: Renormalization.

2.5.2. Regulators

Before we are able to take care of possible divergences, we need to make them manifest. The most evident way to do this is to introduce an upper (lower) bound Λ instead of letting the integration limit go to infinity (zero). Then calculating the loop integral and taking the limit $\Lambda \to \infty(0)$ afterwards makes the UV (IR) divergence apparent. Λ is sometimes called a "hard cutoff". It can be shown that in renormalizable theories only logarithmic divergences $\propto \ln(\Lambda^2)$ can occur.

Although this is an intuitive way to handle divergences, it has one major downside: A hard cutoff is not Lorentz-invariant, meaning that a Lorentz transformation would also change the integration limits, i.e. Λ, if they do not cover the whole kinematic region from zero to infinity. A Lorentz-invariant way to treat divergences is changing the dimension of the integral instead of its limits. Hence, we go from 4 space-time dimensions to[32] $d = 4 - 2\epsilon$, $\epsilon > 0$, since UV divergent loop integrals become convergent when computed in less than 4 dimensions. After doing all calculations, one needs to take the limit $\epsilon \to 0$. Logarithmic divergences of hard cutoffs correspond to divergences of the form $1/\epsilon$ in dimensional regularization. It is also possible to regulate IR divergences in loop integrals when setting $\epsilon < 0$.

When changing the space-time dimension, also the dimension of the Lagrangian changes, since the action as space-time integral over the Lagrangian still needs to be dimensionless. This leads to couplings with non-vanishing mass dimension. If these couplings are part of a loop integral, we introduce a so-called renormalization scale μ and substitute the coupling with an alternative dimensionless one:

$$g_0 \to \mu^{4-d} g_0 = \mu^{2\epsilon} g_0. \tag{2.95}$$

Both μ and ϵ are artificial quantities and therefore need to drop out of every calculation of physical quantities like Greens functions or matrix elements. In addition, the metric tensor and therefore also the Dirac algebra adapt to the space-time dimension:

$$g^\mu_\mu = d \tag{2.96}$$

$$\text{and} \gamma^\mu \gamma_\mu = d, \tag{2.97}$$

which also influences more complex contractions. Since

$$\mathrm{Tr}[\mathbb{1}] = 4 \tag{2.98}$$

[32] The factor of 2 is purely conventional and only simplifies some calculations.

remains untouched, the trace theorems such as

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = g^{\mu\nu} (\text{and so on, cf. appendix A.2}) \tag{2.99}$$

stay as they are in d dimensions.

As it was explained in the section above, IR divergences in the phase space integrations of real emission graphs originate from massless particles. Similar to the hard cutoff, these divergences are also logarithmic. Hence, one can choose regulators to be non-zero masses, which are set to zero again after the calculation.

There is of course a variety of other regulators available when dealing with divergences in QFT, but for us it is sufficient to stick to the last two regulators explained above.

2.5.3. Subtraction schemes and renormalized perturbation theory

After classifying and unfolding divergences with the help of regulators, the last open question is how to treat them. As mentioned above, infinite quantities, e.g. observables, are unphysical. The basic idea of renormalization is that the quantities of the original Lagrangian, i.e. the couplings, particle masses and fields, are ill defined, meaning they are not physical. To obtain the actual, i.e. physical quantities, we need to renormalize them:

$$g_0 = Z_g g, \tag{2.100}$$

where g_0 is the bare coupling of the original Lagrangian, g is the renormalized physical coupling and Z_g is the infinite renormalization constant. The same concept is applied to masses and fields.

At tree level, where there are no divergences at all, both bare and renormalized quantities need to be equal, so we can write

$$Z_g = 1 + \delta_g. \tag{2.101}$$

The counterterms can be formally expanded in the (now renormalized) coupling (starting at g^2), which enables a perturbative approach that goes hand in hand with the actual perturbation theory in QFT. We obtain the so-called renormalized perturbation theory. Starting with the Lagrangian $\mathcal{L}_0$, which contains the bare fields and parameters, we substitute (2.100) and (2.101) and split up $\mathcal{L}_0$ into two parts: One that looks just as $\mathcal{L}_0$, but with renormalized instead of bare quantities, and one that contains all counterterms up to the desired order. These new terms introduce a new set of Feynman rules for propagators and vertices, both indicated with a cross in the Feynman diagrams, which give rise to new terms in Greens functions or matrix elements. By the choice of the infinitely large counterterms, these terms cancel the divergences that may occur in the rest of the graphs by loop diagrams.

However, there is a freedom of choice when defining counterterms, which leads us to the subtraction schemes[33]. One of the most famous ones is the minimal subtraction (MS) scheme, where the counterterms only contain the divergences themselves

[33] The name originates from the idea that we choose counterterms in a way to subtract all the UV divergences that appear at a certain order in perturbation theory.

(expressed in an arbitrary regulator) and nothing else. The modified minimal subtraction ($\overline{\text{MS}}$) scheme is applied when we use dimensional regularization. With this choice, the counterterms also contain some finite contributions proportional to $\ln(4\pi)$ and the Euler-Mascheroni constant γ_E. These constants emerge when expanding the gamma function, that appears in results of some loop integrals in d dimensions, around $\epsilon = 0$.

2.5.4. The renormalization group equation

The bare quantities remain untouched throughout the whole process of renormalization. Therefore, they need to be independent of the renormalization scale:

$$\mu \frac{d}{d\mu} g = 0. \tag{2.102}$$

By looking at (2.100), we see that the scale dependence of g is completely determined by the dependence of Z_g on g up to any desired order. This leads to the so-called running coupling,

$$\mu \frac{d\alpha}{d\mu} = \beta(\alpha), \tag{2.103}$$

where $\alpha \propto g^2$ is the well-known coupling constant and the β-function is a series in α (or g) determined by the renormalization constant Z_g. The same concept can be applied to masses, where we obtain

$$\frac{\mu}{m_R} \frac{dm_R}{d\mu} = \gamma_m(g). \tag{2.104}$$

Here, the anomalous dimension γ_m depends on the coupling g, since Z_m is, as explained in the section above, expanded in g.

2.6. A brief Introduction to the Standard Model of particle physics

The Standard Model of particle physics is one of best tested theories today's physics can offer. Throughout the last and current century it succeeded to describe almost all phenomena that occurred on the stage of elemental particles. It fills whole text-books, e.g. [Schwartz, 2014] and [Peskin and Schroeder, 1995], but we simply give an compact overview of the theory's two essential components: Quantum Chromodynamics and the electroweak theory. Furthermore, the reader is supposed to be familiar with the theoretical basis of the Standard Model, which is Quantum Field Theory. If this is not the case, both afore mentioned books give an adequate introduction concerning this topic. This work makes use of the perturbative approach of Quantum Field Theory including the well-known Feynman diagrams[34].

This approach, including all its famous cornerstones like Wick's theorem, LSZ reduction and many more, is assumed to be known to the reader as well, so that perturbative calculations do not need any form of deeper motivation or explanation.

2.6.1. Quantum chromodynamics (QCD)

The theory of Quantum chromodynamics was constructed to describe the binding between quarks, which form all kinds of hadrons, for example protons and neutrons[35]. The interaction is exchanged by gluons. These gauge bosons couple to the charge of QCD, the name-giving colors (anti-) red, (anti-) green and (anti-) blue for (anti-) particles. Since the theory predicts three charges, the complex fermionic quark fields are triplets ψ and so transform under the fundamental representation of $SU(3)$. Demanding invariance of the langrangian $\mathcal{L}_{\text{QCD}}$ under local transformations of this kind makes this group a gauge group and couples, with the help of the covariant derivative, the quark fields to the gluon field A^a_μ transforming in the adjoint representation of $SU(3)$. $a = 1...8$ labels the components in this representation. Since we deal with a non-abelian gauge group, there are also cubic and quartic interaction terms of these gluon fields contained in the square of the gluon field strength

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.105}$$

The full QCD-Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{QCD}} = &-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu_a)(\partial_\nu A^{a\nu}) + (\partial_\mu \bar{c}^a)(\delta^{ac}\partial^\mu + g f^{abc} A^{b\mu}) c^c \\ &+ \overline{\psi}_i (i\not{\partial} + g\not{A}^a T^a_{ij} - m\delta_{ij})\psi_j + \text{counterterms},\end{aligned} \tag{2.106}$$

where ψ_i is a quark field (in the fundamental representation) with the color index i. The second term is a gauge term introduced in section 2.4.2, generalized to $SU(3)$.

[34] One can find the exact conventions for Feynman rules used in this work in appendix A.3.

[35] In fact, the parton model and Deep Inelastic scattering, both discussed in chapter 3, were and are one of the most important tests and applications of QCD.

One can also find so-called Faddeev-Popov ghost fields c and $\bar{c}$ in the expression above. These fields are necessary if one wants to preserve Lorentz invariance during the quantization of gluon fields. It is easiest to see this when quantizing the theory in the functional integral formalism, as it can be found e.g. in section 25.4 of [Schwartz, 2014].

The coupling constant of QCD is

$$\alpha_s \equiv \frac{g_s^2}{4\pi}. \tag{2.107}$$

In opposition to QED, its β-function[36] is negative, which leads to high coupling strengths at low energy scales and low strengths at high scales. This phenomenon is called "asymptotic freedom". High couplings at low energy scales lead to color-neutral bound states, which is the reason why free quarks do not exist. The process of quarks forming bound states like the proton, called "confinement", is therefore a non-perturbative one. This, together with asymptotic freedom, will become one of the key aspects in the following chapter when talking about the parton model. It is also possible to give a numerical value at which pertubation theory breaks down. In a process called dimensional trasmutation we can replace the (dimensionless) starting condition in an RGE solution for the coupling constant with an energy constant Λ, the Landau pole. As the name suggests, Λ marks the pole of $\alpha_s(\mu)$ which makes a perturbative treatment impossible. For QCD, Λ depends on the number of considered quark flavor and lies in a range of a few hundred MeV.

To the present day there have been found three quark generations, each containing two quarks. They also can be sorted into up- and down types with charges

$$e_u = \frac{2}{3}e \equiv Q_u e \tag{2.108}$$

$$\text{and } e_d = -\frac{1}{3}e \equiv Q_d e, \tag{2.109}$$

where e is the charge of a positron. Their masses can be found in table 2.1.

Table 2.1.: Quark masses and charges of all flavors according to [Tanabashi, M., et. al. (Particle Data Group), 2018] in $\overline{\text{MS}}$ renormalization at $\mu = 2$GeV.

Flavor	Electric Charge	Mass	
up	Q_u	2.2	MeV
down	Q_d	4.7	MeV
strange	Q_u	95	MeV
charm	Q_d	1.275	GeV
bottom	Q_d	4.8	GeV
top	Q_u	173.0	GeV (direct measurements)

[36] cf. section 2.5.4 for further details.

2.6.2. Electroweak interactions

The remaining part of the Standard Model is described by the electroweak interaction. In this work, we will concentrate on QCD interactions and therefore omit most of the details. The interested reader shall be referred to any desired textbook on the Standard Model like [Schwartz, 2014] or [Peskin and Schroeder, 1995].

The most famous aspect of the electroweak model is surely the spontaneous symmetry breaking of a semi-simple Lie group, $SU(2) \times U(1) \to U(1)_e$, where the last Lie group stands for the gauge group of QED (see below). This gives rise to massive $W^{\pm}$- and Z-bosons, with masses[37]

$$M_W = 80.38\,\text{GeV} \tag{2.110}$$

$$\text{and } M_Z = 91.19\,\text{GeV}, \tag{2.111}$$

and the Higgs particle as the Goldstone boson.

Other than QCD and QED, chiral symmetry is not conserved under electroweak interactions. Of special interest for this work are the different gauge couplings for left- and right-handed particles $g_{R,L}$ as well as neutral (Z-boson and photon) and charged ($W^{\pm}$-bosons) currents. Both can be found in table 2.2. Following the notation in the ACOT papers [Aivazis et al., 1994a] and [Aivazis et al., 1994b], we choose the following form for a general vertex Feynman rule[38]:

$$ig_B \Gamma^\mu \equiv ig_B \Big(g_R(1+\gamma^5) + g_L(1-\gamma^5) \Big) \gamma^\mu. \tag{2.112}$$

These general gauge couplings g_B are also given in table 2.2.

Table 2.2.: General and chiral couplings of all electroweak gauge couplings. e is the electric charge of a positron, while $g = e/\sin\theta_W$ is the gauge coupling of the $SU(2)$ gauge group mentioned in the text. θ_W is the so-called Weinberg angle. As it is also done at some other places in this work, the dependences on the quark flavors i and j are kept implicit. The charge fractions Q_i are defined in (2.108) and (2.109). The V_{ij} are matrix elements of the CKM-matrix which mixes mass and interaction eigenstates of quarks. T_3^i is the eigenvalue of the weak isospin's third component represented by the $SU(2)$ gauge group.

coupling	γ	Z	$W^{\pm}$
g_B	$-e$	$\frac{g}{2\sqrt{2}}$	$\frac{g}{2\cos\theta_W}$
g_R	$\frac{Q_i}{2}$	$-Q_i \sin^2\theta_W$	0
g_L	$\frac{Q_i}{2}$	$T_L^i - Q_i \sin^2\theta_W$	V_{ij}

Although the details on electroweak Lagrangians are not important to us in this context, one remark on the choice of the gauge should be made: We choose the

[37] Cf. [Tanabashi, M., et. al. (Particle Data Group), 2018].
[38] The chiral projectors are introduced in section 2.2.

unitary gauge so that the propagator of the massive gauge bosons is the same as the one of a simple Proca Lagrangian, cf. section 2.4.1.1. More on this can be found e.g. in section 28.4 of [Schwartz, 2014].

For energies much smaller than the masses of the heavy gauge bosons, a good approximation is the theory of quantum electrodynamics (QED). In this theory, the only gauge boson is the photon A^μ, i.e. the underlying gauge-symmetry is a $U(1)$ symmetry with charge q:

$$\mathcal{L}_{\text{QED}} = \overline{\Psi}\Big(i\gamma^\mu\partial_\mu + m^2\Big)\Psi + F^{\mu\nu}F_{\mu\nu} - q\overline{\Psi}\gamma^\mu\Psi A_\mu + \text{counterterms}. \tag{2.113}$$

In this case, the symmetry is abelian and so the field strength tensor is simply

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.114}$$

Additionally, for theories with abelian gauge symmetries one can derive special Ward identities,

$$q^\mu \mathcal{M} = 0, \tag{2.115}$$

where $\mathcal{M}$ is the amplitude of an arbitrary process and q^μ the momentum of an external photon attached to it.

3. An Introduction to Deep Inelastic Scattering (DIS) and the Parton Model

Proper introductions to Deep Inelastic Scattering in general can be found in [Halzen and Martin, 1984], chapters 8 and 9, as well as in [Schwartz, 2014], chapter 32.

Probing hadrons with a lepton beam that has energies above the corresponding hadron mass and resonance peaks is called Deep Inelastic Scattering (DIS). The leading interaction, at least at low enough energies, is the photon exchange. For this introductory chapter, we will restrict ourselves to these processes.

Since the earliest applications in the 1960's, the reason behind lepton-hadron scattering is to explore and understand the inner structure of the probed hadrons. The lectures [Friedman et al., 1991] in the context of the nobel price 1990, awarded for the experimental realization of DIS, are a very comprehensive summary of the theoretical and experimental development of DIS and therefore should be consulted for a more detailed overview.

Due to the very simple production options the used leptons were electrons back then. Hypothetically, also the heavier generations, muons and tauons, as well as the respective neutrinos are an option. One of the most common examined hadrons is the proton, which can be analyzed very easily as it is the nucleus of the lightest hydrogen isotope. Its mass is $m_p = 938\,\mathrm{MeV}$[1]. In order to talk about Deep Inelastic scattering in this case, the transfered energy Q^2 as well as the mass of the products must greatly exceed the proton mass, so Q^2 needs to be at least of order $1\,\mathrm{GeV}$.

Throughout this work fermion masses m_l are set to zero. They would enter calculations via contributions such as m_l^2/Q^2. This can be neglected when $Q^2 \gg m_l^2$, which is practically always the case if the fermion is not a tauon.

[1] The most exact measurements of all masses mentioned in this work can be looked up in [Tanabashi, M., et. al. (Particle Data Group), 2018].

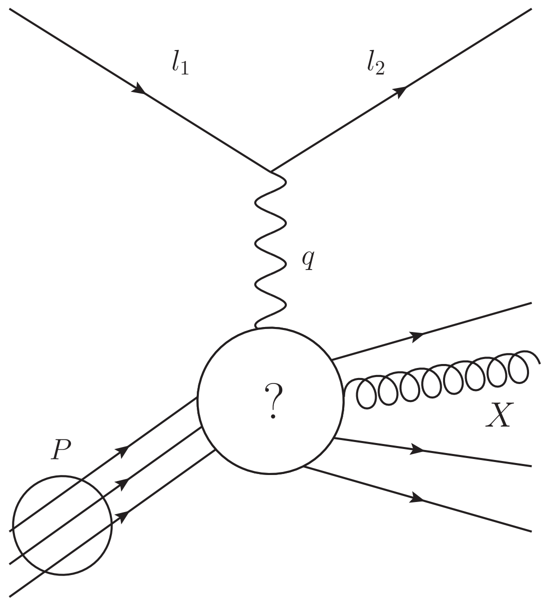


Figure 3.1.: Lepton with momentum $l_{1,2}$ scattering off a hadron with total momentum P. The exchanged gauge boson is a photon with momentum q. The hadronic vertex "?" and the products X (with total momentum P_X) are not further specified. This process, with high enough exchanged energy $Q^2 = -q^2$, is called "Deep Inelastic Scattering".

3.1. The hadronic tensor $W^{\mu\nu}$

How can one describe the scattering between an lepton and a hadron mathematically? Formally, one defines a hadronic tensor $W^{\mu\nu}$ that contributes to the averaged matrix element $\overline{|\mathcal{M}|^2}$ of the process as follows:

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{Q^4} 4\pi Q^2 L_{\mu\nu} W^{\mu\nu}, \tag{3.1}$$

where

$$Q^2 \equiv -q^2 > 0 \tag{3.2}$$

is the transferred energy, sometimes called virtuality of the photon. For this chapter, it is sufficient to only deal with QED interactions, i.e. the exchange of a photon. We see (by using the according Feynman rules, cf. appendix A.3) that the amplitude can be split up into leptonic and hadronic constituents as well as the exchanged gauge boson and vertices. The latter are reflected in the numerator of the prefactor, e^4, and the denominator Q^4 comes from the related photon propagator. $4\pi Q^2$ cancels to normalizations of both leptonic and hadronic tensors.

The leptonic tensor in the matrix element above,

$$\begin{aligned} L^{\mu\nu} &\equiv \frac{1}{Q^2} \frac{1}{2s_1+1} \sum_{s_{1,2}} \langle l_1, s_1 | J_L^{\mu}(0) | l_2, s_2 \rangle \langle l_2, s_2 | J_L^{\dagger\nu}(0) | l_1, s_1 \rangle \\ &= \frac{1}{Q^2} \frac{1}{2s_1+1} \sum_{s_{1,2}} \Big[\overline{u}_{s_1}(l_1) \gamma^\mu u_{s_2}(l_2) \Big] \Big[\overline{u}_{s_1}(l_1) \gamma^\nu u_{s_2}(l_2) \Big]^*, \end{aligned} \tag{3.3}$$

arises from the tree-level scattering of a Dirac particle, in this case the lepton.

$$J_L^\mu(x) \equiv \overline{\Psi}(x) \gamma^\mu \Psi(x) \tag{3.4}$$

is the leptonic current deduced from the Lagrangian (2.113) including the lepton field $\Psi(x)$[2]. $L^{\mu\nu}$ can be seen as the squared "upper half"-contribution to a whole t-channel[3] Feynman diagram of leading order in α, cf. e.g. figure 3.1. It contains an averaging over all initial spins s_1 and a sum over all spins $s_{1,2}$, which expresses the unpolarized nature of an incoming and outgoing lepton beam[4]. In the process considered in this chapter, the incoming lepton remains unchanged, so $s_i = s_f = \pm\frac{1}{2}$.

As it can be seen in figure 3.1, the treatment of the lower part of the t-channel scattering is in no way that trivial. In principle, one could ignore all subprocesses

[2] That the equality between the first and second line in (3.3) holds can nicely be seen if one decomposes $\Psi(x)$ into a sum of creation and annihilation operators $a_s^\dagger(\boldsymbol{p})$ and $a_s(\boldsymbol{p})$, i.e. writes down the canonical quantization of $\Psi(x)$.

[3] t-channel referes to the equality of Q^2 to the Mandelstam variable t, cf. (D.16).

[4] This is a very common approach in DIS (and in Standard Model calculations in general). However, one should always keep in mind that it is not possible to extract spin-related informations about the proton with this strategy.

inside the hadron and just describe $W^{\mu\nu}$ by a sum over all final states X[5],

$$W^{\mu\nu} \equiv \frac{1}{4\pi}\overline{\sum}_X (2\pi)^4 \delta^{(4)}(P + q - P_X) \langle P| J_H^\mu |P_X\rangle \langle P_X| J_H^{\dagger\nu} |P\rangle\,, \qquad (3.5)$$

containing, in analogy to (3.3), the hadronic current J_H^μ. The sum over all final states[6] X implicitly contains the phase space of the final hadronic products, so according modifications to the phase space $d\Pi_{\text{LIPS}}$ in (3.13) need to be made when calculating the cross section (cf. e.g. appendix D).

Equivalently, one can decompose $W^{\mu\nu}$ into a sum over all possible Lorentz-invariant tensors of rank two,

$$\begin{aligned} W^{\mu\nu} \equiv &- g^{\mu\nu} W_1 + \frac{1}{M^2} P^\mu P^\nu W_2 - \frac{i}{2M^2}\epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta W_3 + \frac{1}{M^2} q^\mu q^\nu W_4 \\ &+ \frac{1}{2M^2}(P^\mu q^\nu + q^\mu P^\nu) W_5 + \frac{1}{2M^2}(P^\mu q^\nu - q^\mu P^\nu) W_6. \end{aligned} \qquad (3.6)$$

The expression above covers all relevant tensors, since the whole process is characterized by the hadron momentum P and the photon momentum q. The outgoing momenta are implicitly integrated over and the total final hadronic momentum is fixed by $P_X^\mu = P^\mu + q^\mu$, so the only independent variables are P and q. It is convention to scale some prefactors with the hadron mass $M = \sqrt{P^2}$, so that all W_i have the same mass dimension[7]. W_3 and W_6 cover the antisymmetric part of $W^{\mu\nu}$. Both contributions vanish when multiplied with the leptonic tensor $L^{\mu\nu}$, which is symmetric when only considering QED interactions[8], and therefore can be neglected in this chapter.

As a consequence of ignoring all internal processes in the hadron, the explicit shape of the W_i is not known right from the beginning. To put it in another way, the W_i encode the inner structure of the hadron and thus are often called structure functions. The aim of the next pages will be to find the explicit shape of these functions by making specific assumptions on the composition of hadrons based on experimental results.

Using the Ward-identity[9] (2.115), namely

$$q_\mu W^{\mu\nu} = 0, \qquad (3.7)$$

one can express W_4 and W_5 in terms of W_1 and W_2. That one can withdraw two dependences out of one equation can be traced back to the independence of P and q.

[5] A word of caution: It should always be kept in mind that the normalization of both leptonic and hadronic tensor differs in the literature. In this work, $1/Q^2$ for $L^{\mu\nu}$ and $1/(4\pi)$ for $W^{\mu\nu}$ is chosen.
[6] Summing over all final states is sometimes called a total inclusive process.
[7] There are other conventions used in the literature, usually differing by factors of M.
[8] This is due to chirality conservation in QED.
[9] Here, $W^{\mu\nu}$ takes the role of the (squared) amplitude of the sub process $P + q \to P_X$.

(3.7) results in

$$W_4 = \left(\frac{P\cdot q}{Q^2}\right)^2 W_2 - \frac{M^2}{Q^2} W_1 \tag{3.8}$$

$$\text{and} \quad W_5 = \frac{2P\cdot q}{Q^2} W_2. \tag{3.9}$$

The hadronic tensor then has the following form:

$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{1}{Q^2} q^\mu q^\nu\right) W_1 + \frac{1}{M^2}\left(P^\mu + \frac{P\cdot q}{Q^2} q^\mu\right)\left(P^\nu + \frac{P\cdot q}{Q^2} q^\nu\right) W_2. \tag{3.10}$$

If the hadron would be point-like[10], or in general the scattering is elastic, the structure functions only depend on one Lorentz scalar, e.g. the transferred energy Q.

In the case of inelastic scattering, there is an additional dependence on another Lorentz-scalar. A common choice is the so called Bjorken x,

$$x \equiv \frac{Q^2}{2P\cdot q} \in (0,1], \tag{3.11}$$

so that

$$W_{1,2} = W_{1,2}(Q^2, x). \tag{3.12}$$

As an example, let us discuss the hadronic tensor and the structure functions for the scattering off a point-like hadron with charge $+e$. To find the explicit form of W_1 and W_2, we can compare the differential cross sections for $\overline{|\mathcal{M}|^2} \propto L_{\mu\nu}L^{\mu\nu}$ and $\overline{|\mathcal{M}|^2} \propto L_{\mu\nu}W^{\mu\nu}$[11] via

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{F} d\Pi_{\text{LIPS}}. \tag{3.13}$$

The Flux Factor[12]

$$F \equiv 2\Delta(s, m_l^2, M^2), \tag{3.14}$$

here defined via the Δ-function

$$\Delta(a,b,c) \equiv \sqrt{a^2 + b^2 + c^2 - 2ab - 2ac - 2bc}, \tag{3.15}$$

[10]Strictly speaking, in this case the particle would not be a hadron any more, since there is nothing left that can be bound together via strong interactions.

[11]The calculation of both products can be found in appendix D.

[12](3.13) is a general formular for the differential cross section and so is (3.14) a general expression for the flux factor of a process involving two incident particles. In our case, as mentioned before, fermion masses m_l are always set to zero, even if they appear in some expressions. Additionally, the Mandelstam variable s appears in F. For a definition, see (D.15).

and the well-known Lorentz-invariant phase space element[13] $d\Pi_{\text{LIPS}}$ (both for two particles) are obviously in both cross sections the same. The calculation of both cross sections is done in appendix D, here it is sufficient to only give the result. The first cross section reads

$$\left(\frac{d\sigma}{dE_2 d\cos\theta}\right)_L = \frac{2\pi\alpha^2}{E_1^2 \sin^4\frac{\theta}{2}} \left(\frac{Q^2}{2M^2}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)\delta\left(E_1 - E_2 - \frac{Q^2}{2M}\right), \tag{3.16}$$

where $E_1 = l_{1,0}$ and $E_2 = l_{2,0}$ are the initial and final lepton energies. As stated, the differential cross section is given in the laboratory frame for a fixed target experiment, so the particle corresponding to the second $L^{\mu\nu}$ is at rest, i.e. $P = (M, \mathbf{0})$[14]. θ denotes the corresponding scattering angle in this frame. The second result is[15]

$$\boxed{\left(\frac{d\sigma}{dE_2 d\cos\theta}\right)_W = \frac{4\pi\alpha^2}{E_1^2 \sin^4\frac{\theta}{2}} \frac{1}{M}\left(W_1 \sin^2\frac{\theta}{2} + \frac{1}{2}W_2\cos^2\frac{\theta}{2}\right).} \tag{3.17}$$

Now the key step is to demand that both expressions are equal because the hadron is taken to be point-like. A simple comparison of coefficients gives

$$W_1 = \frac{Q^2}{4M}\cdot\delta\left(E_1 - E_2 - \frac{Q^2}{2M}\right) \tag{3.18}$$

$$W_2 = M\cdot\delta\left(E_1 - E_2 - \frac{Q^2}{2M}\right). \tag{3.19}$$

If we write $E_1 - E_2$ in terms of Lorentz-scalars,

$$E_1 - E_2 = \left(\frac{P\cdot q}{M}\right)_{\text{lab}} \equiv (\nu)_{\text{lab}}, \tag{3.20}$$

we arrive at the alternative expressions

$$W_1 = \frac{1}{2}x\cdot\delta(1-x) \tag{3.21}$$

$$\nu W_2 = M\cdot\delta(1-x), \tag{3.22}$$

since

$$\frac{Q^2}{2M\nu} = x. \tag{3.23}$$

We see, as stated above, that there is indeed a dependence on only one kinematic variable. It is also worth noting that x is set to 1 in this specific situation. In the next section, when x is identified with a momentum fraction, it becomes clear that this must be the case for every point-like scattering. The results for W_1 and W_2 of course are not realistic ones, but they are a good preparation for the next section. There, we will apply the exact same steps to achieve actually meaningful results.

[13]For a definition, see (A.6).

[14]A more detailed discussion can be found in appendix D.1

[15]Note that in this and all following cross sections the coupling strength of the hadron is included in the structure functions W_i. For examples, cf. e.g. (3.21), where it is simply a 1, or (3.33), where it is a fraction of the electric charge.

3.2. The naive Parton Model

Until now not a single word was written about the actual structure of hadrons. The last section only treated the (electromagnetic) scattering process and the general framework, processes explaining the hadron itself were not mentioned. In order to find a suitable model it is helpful to look at a measurement at SLAC from the 1960's, cf. figure 3.2. It shows measured values of νW_2 at different Q^2 and fixed x. In general, the behavior could be arbitrary complex, but instead we simply see a constant line, very similar to a point-like hadron, cf. (3.22): νW_2 scales with Q^2, which is why this phenomenon is called Bjorken scaling. In other words, the description of point-like particles needs only one kinematic variable, e.g. x. If now hadronic quantities like $\nu W_1(x, Q^2)$ scale with Q^2, there actually is a dependency on only one variable. Hence, one can trace back these hadronic quantities to well-known point-like ones. That hadrons "behave" like point-like particles at high energies is one of the key insights of DIS.

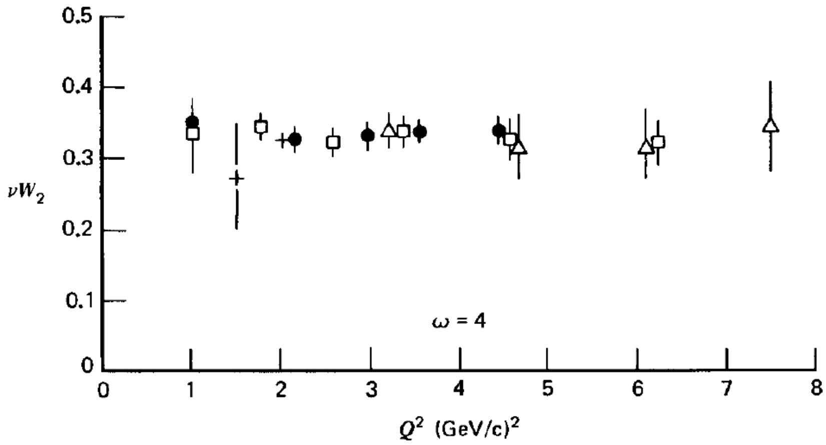


Figure 3.2.: Measurement of νW_2 at different Q^2 and $x = \frac{1}{\omega} = \frac{1}{4}$, SLAC (Stanford Linear Accelerator). The Bjorken scaling can nicely be seen. Taken from [Halzen and Martin, 1984], p. 190.

The conclusion, so natural for today's readers as revolutionary for physicists in the 1960's[16], is that hadrons consist of elementary, point-like particles, called "partons".

According to the parton model, a parton of type i carries a fraction ξ of the hadron momentum,

$$p_i^\mu = \xi P^\mu, \ \xi \in (0, 1). \tag{3.24}$$

ξ is an internal variable that cannot be measured explicitly. What is inherent to every parton type is the parton distribution function (PDF) $f_i(\xi)$ that describes the

[16] There were many competing theories aiming to describe the hadron structure back then, for a detailed view cf. e.g. [?], p. 617 et seq.

distribution of ξ on the interval $(0, 1)$. There is of course also a dependence in the hadron type, which we will keep implicit if not stated otherwise.

$f_i(\tilde{\xi})d\xi$ can be interpreted as the probability to find a parton with momentum $\tilde{\xi}P$ in the infinitesimal interval $[\tilde{\xi}, \tilde{\xi} + d\xi]$. However, it does not fulfill every condition of a conventional probability density, as it is not normalized to 1 but to the number N_i of partons of type i, i.e.

$$\int_0^1 f_i(\xi)d\xi = N_i. \tag{3.25}$$

The parton model in this plane form makes some simplifications that shall be mentioned before proceeding. First of all, it does not allow partons to have a three-momentum not parallel to the one of the hadron (cf. (3.24)). Additionally, it assumes that the partons are free inside the hadron. Lepton-hadron scattering then reduces to a sum of lepton-parton scattering[17] with free partons in initial and final states, illustrated in figure 3.3.

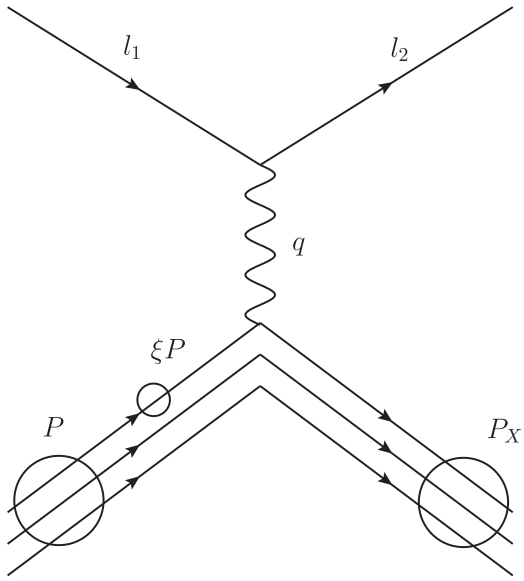


Figure 3.3.: Lepton-Hadron scattering at tree-level in the Parton model. The hadron with momentum P consists of point-like partons with momentum fraction ξ. After the scattering the products X with momentum P_X, also consisting of partons, are again not specified.

At first sight, free partons seem like a contradiction in terms. Partons were introduced as constituents of the hadron, so they should be bound together by strong interaction and not free. What "free partons" actually means is that scattering interaction happens on a much smaller timescale than binding interactions before and

[17] To be more precise: A sum over all parton types and an integration over all possible parton momenta.

after the scattering. This can be only true at sufficiently high momentum transfers Q, which means that Q is much larger than any other energy scale involved in the process. According to Heisenberg's uncertainty principle, this corresponds to a short lifetime of the virtual photon and therefore a small time scale on which the scattering happens. The ignorance of interactions is also the reason why assuming a fixed number of partons N_i is valid. As we will see below, the ignorance of interactions inside the hadron, namely strong interactions, leads to an incoherence, i.e. a "classical" multiplication of probabilities as in (3.27).

The behavior of parton masses is quite strange. Squaring the partonic momentum gives

$$m_i = \sqrt{p_i^2} = \xi M. \tag{3.26}$$

A ξ-depending mass is obviously far away from reality. This is why the naive parton model is strictly speaking only appropriate in a frame in which all masses are negligible. A frame with this condition must move with an infinite velocity, so that the masses can be ignored due to the infinite momenta. This is the so-called infinite momentum frame[18].

In the parton model, every quantity describing the hadron, above all the hadronic tensor, can now be written as a (PDF-weighted) sum over the partonic quantities, i.e.[19]

$$\begin{aligned} W^{\mu\nu}(x, Q^2) &= \sum_i \int_0^1 dz \int_0^1 d\xi f_i(\xi) \hat{W}^{\mu\nu}(z, Q^2) \delta(x - \xi z) \\ &= \sum_i \int_x^1 \frac{d\xi}{\xi} f_i(\xi) \hat{W}^{\mu\nu}\left(\frac{x}{\xi}, Q^2\right). \end{aligned} \tag{3.27}$$

In the equation above, we introduced a circumflex notation to indicate partonic quantities. The partonic hadronic tensor can be interpreted as the scattering between lepton and parton instead of the hadron. However, $\hat{W}^{\mu\nu}$ cannot depend on x, since x is defined with the help of the hadron momentum P. Instead, a partonic version of x in analogy to definition (3.11) is introduced:

$$z \equiv \frac{Q^2}{2p_i \cdot q}. \tag{3.28}$$

Because z, as a partonic variable, is neither determined nor measurable, (3.27) also contains an integration over the full range of z. But there is a relation between x, ξ and z. By recalling the definition of ξ (3.24), we easily see that

$$x = \xi z, \tag{3.29}$$

[18] A mathematical treatment of this frame as a constituent of a class of frames, the collinear frames, can be found in section 4.3.

[19] In general, the partonic tensor could also depend on the parton type, denoted by $\hat{W}_i^{\mu\nu}$. For the sake of simplicity, we will omit this index in the first instance.

which explains the δ distribution in (3.27). The limits of ξ shift, since x/ξ must lay in the z integration range, namely between 0 and 1, for the δ distribution to be non-zero.

Now the assumption of point-like partons comes into play. This implies that $\hat{W}^{\mu\nu}$ must have exactly the same shape as $L^{\mu\nu}$. Hence, the cross section for parton-lepton scattering can be simply copied from the lepton-lepton cross section (3.16) with slight changes to the mass and charge. So once again plugging $W^{\mu\nu}$, which now contains the parton sum, into the general cross section (3.13) gives

$$\frac{d\sigma}{dE_2 d\cos\theta} = \frac{2\pi\alpha^2}{E_1^2 \sin^4\frac{\theta}{2}} \sum_i Q_i^2 \int_0^1 d\xi f_i(\xi) \left(\frac{Q^2}{2m_i^2} \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} \right) \delta\left(\nu - \frac{Q^2}{2m_i} \right). \tag{3.30}$$

Q_i is the charge of parton type i. Using (3.20), we can rewrite the δ distribution to cancel the ξ-integration, i.e.

$$\delta\left(\nu - \frac{Q^2}{2m_i}\right) = \delta\left(\frac{Q^2}{2Mx} - \frac{Q^2}{2M\xi}\right) = \frac{2M}{Q^2}\delta\left(\frac{1}{x} - \frac{1}{\xi}\right) = \frac{2M}{Q^2} x^2 \delta(\xi - x). \tag{3.31}$$

This leads to

$$\frac{d\sigma}{dE_2 d\cos\theta} = \frac{2\pi\alpha^2}{E_1^2 \sin^4\frac{\theta}{2}} \sum_i Q_i^2 f_i(x) \left(\frac{1}{M} \sin^2\frac{\theta}{2} + \frac{2M}{Q^2} x^2 \cos^2\frac{\theta}{2} \right). \tag{3.32}$$

Comparing this to the general cross section containing the hadronic tensor (3.17), yields, exactly as in the end of section 3.1, expressions for the structure functions:

$$W_1(x, Q^2) = \frac{1}{2} \sum_i Q_i^2 f_i(x) \tag{3.33}$$

$$\nu W_2(x, Q^2) = Mx \sum_i Q_i^2 f_i(x) \tag{3.34}$$

(3.31) allows us to use ξ and x completely equivalent. This is a noteworthy result, since, at least at first sight, there is no obvious connection between the probabilistic quantity ξ and the kinematic variable x.

Additionally, W_1 and νW_2 are both indeed independent of Q^2 at fixed x, so we reproduced the Bjorken scaling, which was discussed in the context of figure 3.2. Another common convention is to use

$$F_1(x, Q^2) \equiv W_1(x, Q^2) \tag{3.35}$$

$$\text{and } F_2(x, Q^2) \equiv \frac{\nu}{M} W_2(x, Q^2) \tag{3.36}$$

instead of W_1 and W_2. In this case, (3.34) becomes a little simpler:

$$F_2(x, Q^2) = x \sum_i Q_i^2 f_i(x). \tag{3.37}$$

Also of great importance is the Callan-Gross relation. Substituting (3.33) into (3.34) yields a connection between $W_1(x, Q^2)$ and $W_2(x, Q^2)$, namely

$$\nu W_2(x, Q^2) = 2MxW_1(x, Q^2)$$
$$\Leftrightarrow W_1(x, Q^2) = \frac{Q^2}{4M^2x^2} W_2(x, Q^2). \tag{3.38}$$

The Callan-Gross relation gets its exact structure from the at first only assumed fermionic nature of the partons, so the experimental reproduction was a huge indicator that this assumption was right. In the context of this work, it will help us to compare later introduced parton models with this often called "naive" parton model, being the in many ways simplest way to describe hadronic structures.

After the introduction of the parton model formalism we will now fill it with physical meaning. In order to do so, we make use of the quark model, originally introduced by Gell-Mann in [Gell-Mann, 1964]. Hence, we identify the partons with quarks and gluons. In this context, one often uses the notation $q(\xi)$ and $g(\xi)$ instead of $f_i(\xi)$. The quark model differentiates between valence quarks that specify general properties like charge and spin of the hadron, and sea quarks, originating from QCD processes like quark-anti-quark production of a gluon. Therefore, a quark distribution can be split up into a valence and a sea PDF:

$$q(\xi) = q_v(\xi) + q_s(\xi). \tag{3.39}$$

As instructive as all this new knowledge about hadrons is, there is one huge limitation that was not dealt with up to this point. All these calculations build on tree-level scattering between leptons and quarks. In fact, experiments in higher energy regions, cf. e.g. figure 3.4, show a behavior of the structure functions which is not constant anymore. This motivates calculations in higher orders, namely next-to-leading order (NLO) corrections to the lepton-quark scattering. In other words, we need to improve the parton model by including QCD interactions. The arising contributions will, just as indicated by the diagram 3.4, violate the Bjorken scaling. This work contains such a NLO calculation in a more generalized parton model, see section 6.2. For now, we will turn to a more detailed treatment of PDFs and partonic tensors.

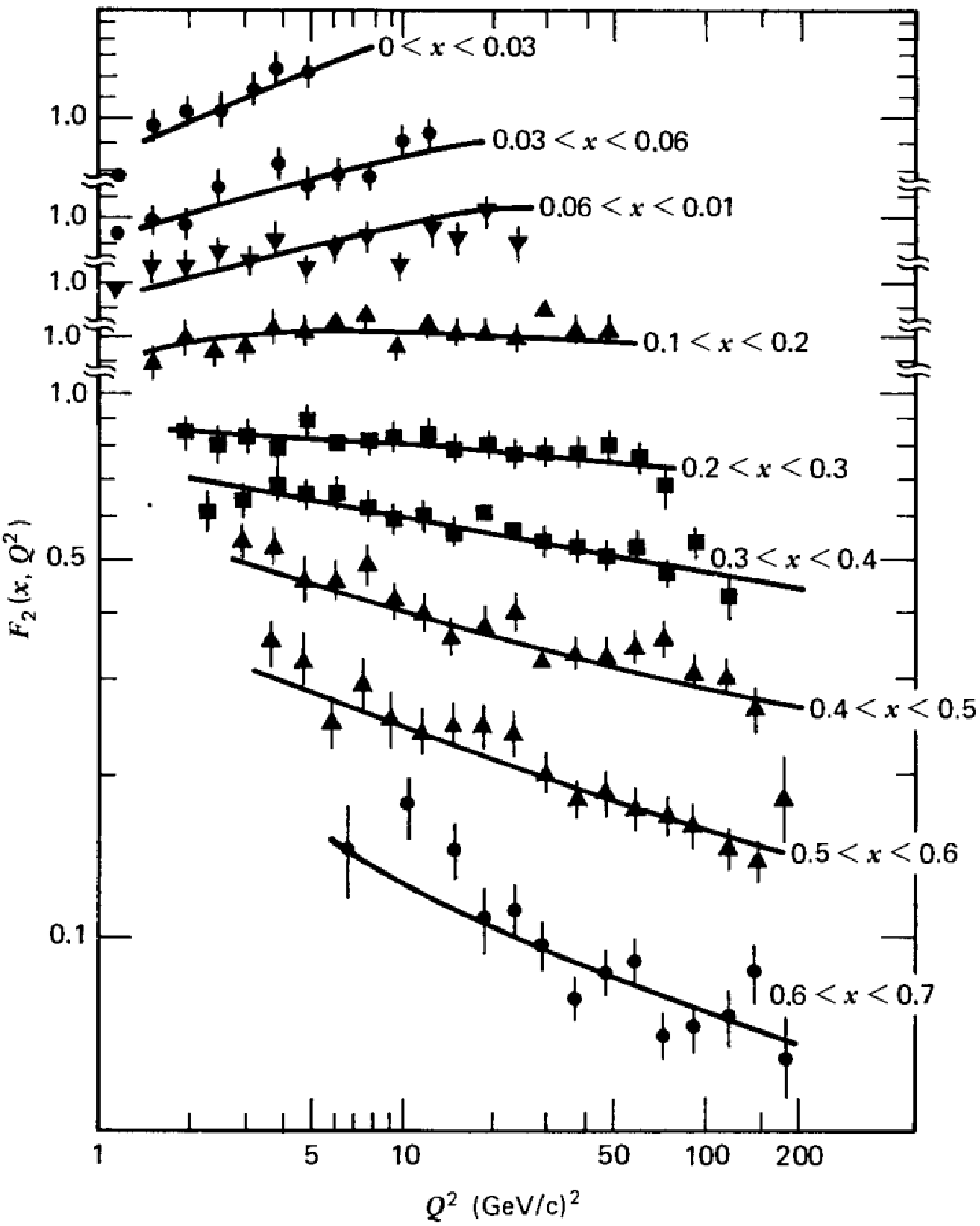


Figure 3.4.: Q^2-dependence of the structure function $F_2(x, Q^2)$ measured at different x, CDHS counter experiment at CERN. Deviations from Bjorken scaling at very large and small x can nicely be seen. Taken from [Halzen and Martin, 1984], p. 218.

3.3. A more general view on the parton model

Far more detailed introductions on factorizations and their proofs as well as PDFs on a field theoretic level can be found e.g. in [Collins, 2011] and [Collins et al., 2004].

3.3.1. Factorization

After this rather heuristic introduction to PDFs, let us make one step back and look at a general process with the only restriction that one hadron needs to be involved. In section 3.2, the hadronic tensor in the context of the naive parton model turned out to be (cf. e.g. equation (3.27))

$$W^{\mu\nu}(x, Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi} f_i(\xi) \hat{W}_i^{\mu\nu}\left(\frac{x}{\xi}, Q^2\right), \tag{3.40}$$

when we established the PDFs as probability density functions. Let us define the integration in the equation above as a generalized product $\otimes$. We can interpret it as follows: When assuming partons, one is able to factorize out the tensor for partonic interactions $\hat{W}_i^{\mu\nu}$ out of the hadronic tensor $W^{\mu\nu}$. What is left is the according PDF f_i and a summation over all parton types i. Figure 3.5 shows a diagrammatic visualization of the factorization approach.

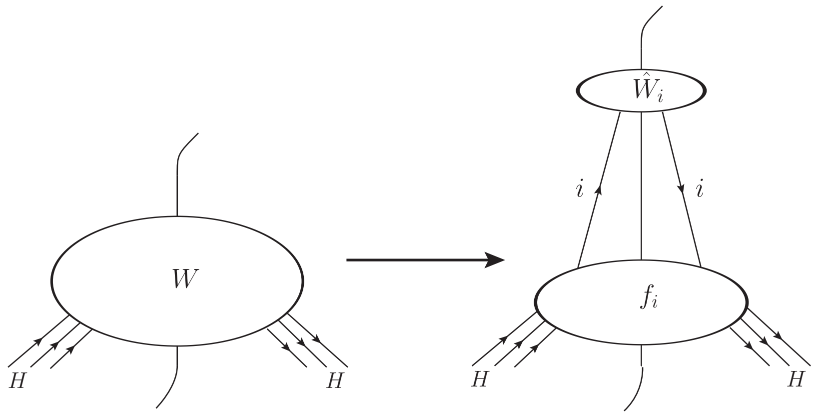


Figure 3.5.: Factorization of a general process involving a hadron. The hadronic tensor W "splits" into the partonic tensor $\hat{W}_i$ and the PDF f_i. They are connected by parton lines of type i (quark flavors or gluons), which are on-shell and collinear to the hadron H. The line going vertically through the bubbles is a final state cut which sets all crossing lines inside the bubbles on-shell. Diagrammatically, it visualizes a squared amplitude with the right side corresponding to the complex conjugate amplitude. This gives us the ability to depict the whole hadronic and partonic tensor in one diagram.

What was deduced heuristically in the parton model, cf. section 3.2, actually needs to be proven strictly. These factorization proofs[20] are far beyond the scope of this work. For DIS, operator product expansion (OPE) is applied to the hadronic tensor. OPE treats the multiplication of operators, like the leptonic and hadronic currents in (3.5), at large Q (while holding all other kinematic variables, like x, fixed). It yields a power series in Q with each term consisting of a product between a Q-dependent so-called Wilson coefficient and a Q-independent operator. A detailed explanation can be found in [Collins, 1984], chapter 10. Then, in a complex procedure, the regions of the Feynman graphs[21] which correspond to the leading term in the OPE, the so-called leading twist contribution, are identified. In the end, one finds diagrams of the form of figure 3.5. They contain one hard region with contributions of order Q, the partonic tensor (which was originally the Wilson coefficient), and one region with all momenta in them being collinear, described by the PDFs[22]. Thus, we arrive at

$$W^{\mu\nu}(x, Q^2) = \sum_i f_i \otimes \hat{W}_i^{\mu\nu} + \text{remainder}, \tag{3.41}$$

where the remainder stands for higher twist terms. These terms will be omitted in the rest of the work.

3.3.2. PDFs and the DGLAP Equation

More detailed explanations can be found in section 4 of [Collins et al., 2004].

Another advantage of this formal approach is that we obtain a field theoretic definition for PDFs[23]:

$$f(\xi) = \frac{1}{4\pi} \int dx^- e^{-i\xi P^+ x^-} \langle P| \overline{\psi}(0, x^-, \mathbf{0}) P_R \mathcal{G} \psi(0, 0, \mathbf{0}) |P\rangle , \tag{3.42}$$

with the Wilson line[24]

$$\mathcal{G} \equiv \mathcal{P} \exp\left(i g_s \int_0^{x^-} dy^- A^{c+}(0, y^-, \mathbf{0}) T^c \right). \tag{3.43}$$

[20] Other processes apart from DIS, the most prominent ones are perhaps two hadrons in the initial state like e.g. the Drell-Yan process, have other factorization formulae and therefore need different proofs.

[21] These regions belong to a greater class of regions, the so-called pinch-singular surfaces, which are explained in more detailed e.g. in chapter 5 of [Collins, 2011].

[22] A third type, the soft region, can be neglected due to Ward-identities. Further work needs to be done to show that a special part of it, the Glauber region, vanishes or is not leading. There, the Grammar-Yennie approximations, which open the possibility to use the Ward identities mentioned above, fail, which would spoil the whole factorization procedure.

[23] Here we give the PDF of a quark field ψ depending on lightcone coordinates $x_\pm \equiv (x_0 \pm x_3)/\sqrt{2}$, which will be introduced in section 4.1. The definition of a gluon PDF involving gluon fields can be done analogously.

[24] $\mathcal{P}$ denotes ordering the gluon field light-cone component A^{a+} along the path from 0 to x^-.

In the light-cone gauge, characterized by $A^+ = 0$, it is obvious that $\mathcal{G} = 1$ holds and so the PDF reduces to

$$f(\xi)d\xi = \frac{1}{(2\pi)^3}\sum_s \frac{1}{2\xi P^+}\int d\boldsymbol{p}\,\langle P|\, b^\dagger\Big(\xi P^+, \boldsymbol{p}, s\Big) b\Big(\xi P^+, \boldsymbol{p}, s\Big)\,|P\rangle\,. \tag{3.44}$$

The spin sum and the integration over transverse momenta $\boldsymbol{k}$ indicate that we work with unpolarized and collinear PDFs. We see by the particle number operator $b^\dagger(...)b(...)$ sandwiched between the hadronic state $|P\rangle$ that there is indeed a close connection between PDFs and parton numbers, as mentioned during the introduction to PDFs in section 3.2. PDFs contain UV divergences and, as they are nothing else than Greens functions, are therefore renormalized.

As a consequence of factorization, we can describe the scale dependence of the PDFs using the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation. It reads

$$\frac{df_i(x,\mu_f)}{d\ln\mu_f^2} = \frac{\alpha_s(\mu_f^2)}{2\pi}\sum_j P_{j\to i}\Big(\frac{x}{\xi}\Big)\otimes f_j(x,\mu_f), \tag{3.45}$$

where the generalized product is the same as the one introduced in the context of factorization. $\alpha_s(\mu^2)$ is the coupling constant of QCD, cf. (2.107). The so-called splitting kernels P_{ij} are universal (i.e. hadron-independent) perturbative objects and can be interpreted as the probability of the splitting of parton type j into type i. Note that the often called factorization scale μ_f is not necessarily equal to the renormalization scale μ of e.g. α_s. It can be interpreted as the scale that separates the long-distant/soft and the short-distant/hard part of the hadronic tensor.

With the help of the DGLAP equation, we can divide the PDF into a perturbative, scale dependent part and a non-perturbative, x dependent part[25] which needs to be measured by experiment. In addition, the non-perturbative part contains the whole (implicit) dependence on the hadron itself.

3.3.3. Partonic Tensors at NLO

A quite similar explanation to the one presented here, only applied on the Drell-Yan process, can be found in section 2 of [Collins et al., 2004]. Here and in the following, we adopt the notation that is introduced in this section from [Aivazis et al., 1994b].

For this section, we make the dependence on the hadron in the PDFs explicit via an upper index, i.e.

$$f_j^H(\xi,\mu) \equiv f_j(\xi,\mu). \tag{3.46}$$

The reason for this is that the most common way to calculate a partonic tensor is to use special types of PDFs, where one substitutes the arbitrary hadron H by another parton of type j, namely an (anti-) quark or a gluon. This means that in the following we will make use of PDFs $f_j^i(\xi,\mu)$, which describe the probability of finding a parton of type j inside a parton i.

[25] This part can also be seen as the starting condition for the DGLAP equation, similar to RGEs.

Additionally, we need a notation to distinguish subtracted[26] and non-subtracted parton tensors. We will stick, as introduced in the section above, to $\hat{W}_i^{\mu\nu}$ when describing the partonic tensor used in the factorization theorem, i.e. the subtracted one containing no divergences. For the unrenormalized one, we reserve the symbol $\hat{\Omega}_i^{\mu\nu}$. This is the tensor that can be directly deduced from the Feynman diagrams. It contains IR and UV divergences (before renormalization) and therefore corresponds to parton scattering at all energy scales, not only at scales of order Q (like $\hat{W}_i^{\mu\nu}$). Consequently, there is a factorization theorem for $\hat{\Omega}_i^{\mu\nu}$ using the "partonic" PDFs introduced above:

$$\hat{\Omega}_i^{\mu\nu} = \sum_j f_j^i \otimes \hat{W}_j^{\mu\nu}. \tag{3.47}$$

Our goal is now to obtain expressions for $\hat{W}_i^{\mu\nu}$ depending on $\hat{\Omega}_i^{\mu\nu}$ order by order in pertubation theory. At leading order, since there is no (QCD-) interaction allowed, a natural choice for a PDF (which holds independently of j being inside a parton i or a hadron H) is

$$f_j^{i/H(0)}(\xi, \mu) = \delta(1 - \xi). \tag{3.48}$$

Also expanding both partonic tensors yields

$$\hat{W}_i^{(0)\mu\nu} = \hat{\Omega}_i^{(0)\mu\nu} \tag{3.49}$$

at LO. Hence, as expected, there is no subtraction needed at leading order.

At NLO, a perturbative expansion[27] of (3.42) (With a partonic state instead of the hadronic one) leads to

$$f_j^{i(1)} = \frac{\alpha_s}{2\pi} P_{i\to j}^{(1)} \ln \frac{\mu_f^2}{m_i^2}, \tag{3.50}$$

where $P_{i\to j}^{(1)}$ is the leading order contribution[28] of the splitting function $P_{i\to j}$ and the factorization scale μ_f. Note that the UV divergences in $f_j^{i(1)}$ are already taken care of, meaning that here and in the following we assume that PDFs are renormalized via an $\overline{\text{MS}}$ subtraction. In anticipation of the ACOT formalism and similar to the conventions in [Aivazis et al., 1994b], we choose a non-zero mass m_i to regulate the IR divergence in $f_j^{i(1)}$. This IR divergence can be traced back to the assumption of massless partons, cf. section 2.5.1. As it is described in section 2.5.2 and can be

[26] The reason why we use the term "subtracted" instead of "renormalized" in this context is that the renormalization is already done on the level of PDFs and hard scattering, i.e. after evaluating the Feynman diagrams. What is left are IR divergences coming from long-scale physics that should not appear in the (hard) partonic tensor. It is their treatment which is explained below.

[27] This is easier said than done. At least the dependence on $P_{i\to j}^{(1)}$ could have been guessed right from the beginning since they both describe the same situation at the same order: Partons splitting into other partons. More information on the perturbative expansion of PDFs can be found in [Collins and Soper, 1982].

[28] Since there is no splitting at order α_s^0, for splitting functions "LO" means order α_s.

found in other sources[29], it is also suitable to use dimensional regularization with an $\epsilon > 0$. Hence, at NLO, (3.47) results in

$$\hat{W}_i^{(1)\mu\nu} = \hat{\Omega}_i^{(1)\mu\nu} - \sum_j f_j^{i(1)} \otimes \hat{W}_j^{(0)\mu\nu}. \tag{3.51}$$

This equation justifies our nomenclature of subtracted and non-subtracted partonic tensors. It is now evident that we need to subtract the low energy physics from $\hat{\Omega}_i^{(1)\mu\nu}$ to obtain the partonic tensor describing only the hard scattering, $\hat{W}_i^{(1)\mu\nu}$. It is also common to interpret the additional subtraction terms as the part of the hadronic tensor that deals with the double counting of overlapping parts in the hard and collinear subgraphs. To be precise, general graphs behave like subtraction terms in collinear regions. Subtracting them turns a general diagram into a hard one, i.e. a subtracted partonic tensor. A concrete calculation at the example of heavy quark production is given in section 6.3.

[29] Cf. e.g. [Collins et al., 2004], equation (52).

4. The ACOT formalism: Preparations

Before presenting the ACOT formalism in its full glory, we need to introduce some important concepts, which are crucial for understanding the further explanations. Building on an alternative set of coordinates in four-dimensional spacetime, we will establish a set of frames for lepton-hadron scattering containing the Breit frame, which will be used in the ACOT formalism. Lastly, we give a short introduction to the helicity formalism. More detailed explanations on these topics can be found in [Collins, 1997] and appendix A.2 of [Aivazis et al., 1994a].

4.1. Light-cone coordinates

In special and general relativity, it is a widespread convention to use coordinates of the form[1]

$$x^\mu = (x_0, \boldsymbol{x}) = (x_0, x_1, x_2, x_3), \tag{4.1}$$

which transform under the metric (2.37). We say that x_0 transforms lightlike and x_1 to x_3 transform spacelike. This basis is the natural choice, but is by no means physically distinguished. Just as well, we can establish partially new coordinates

$$x_L^\mu = (x^+, x^-, \boldsymbol{x}_T) \equiv \left(\frac{x_0 + x_3}{\sqrt{2}}, \frac{x_0 - x_3}{\sqrt{2}}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right). \tag{4.2}$$

One easily sees that this new basis is orthonormalized, too. Additionally, note that changing to light-cone coordinates is a linear transformation, so an addition of two vectors can still be performed coordinate-wise. The according back-transformation is

$$x^\mu = \left(\frac{x^+ + x^-}{\sqrt{2}}, x_1, x_2, \frac{x^+ - x^-}{\sqrt{2}} \right). \tag{4.3}$$

Linear changes of basis are not allowed to alter the scalar product, which yields

$$\begin{aligned} x_L \cdot x_L &\stackrel{!}{=} x \cdot x = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0^2 - x_3^2 + x_0 x_3 - x_0 x_3 - x_1^2 - x_2^2 \\ &= (x_0 + x_3)(x_0 - x_3) - x_1^2 - x_2^2 = 2x^+ x^- - \boldsymbol{x}_T^2. \end{aligned} \tag{4.4}$$

[1] x^μ stands for an arbitrary 4-vector and is not only restricted to space-time coordinates.

We see that there are now two timelike Light-cone coordinates, x^+ and x^-. A scalar product between arbitrary four-dimensional vectors x^μ and y^μ in light-cone coordinates is therefore

$$x_L \cdot y_L = x^+ y^- + y^+ x^- - \boldsymbol{x}_T \cdot \boldsymbol{y}_T, \tag{4.5}$$

where the scalar product of the two transverse vectors $\boldsymbol{x}_T$ and $\boldsymbol{y}_T$ is the usual Euclidean one.

The light-cone coordinates will be the in many cases more natural approach when parametrizing four-momenta. Hence, we will use both sets of coordinates throughout the next chapters. To avoid confusion, we will always try to clarify which coordinates are used in a specific situation. Moreover, since in this work we always deal with vanishing transverse momenta, a $\mathbf{0}_T$ as the last two coordinates always indicates the use of light-cone coordinates.

4.2. Boosts in light-cone coordinates, the collinear frames and rapidity

Light-cone coordinates play out their full strength in Lorentz boosts, introduced in section 2.1.3. Let us start with a particle in its rest frame, meaning that its rest energy is only the mass M, while its three-momentum vanishes:

$$P_R^\mu = (M, \mathbf{0}). \tag{4.6}$$

If we now apply a boost along the z-axis, denoted by $\Lambda_{z\text{-boost}}(\psi)$ [2], with arbitrary rapidity ψ[3], the four-momentum becomes

$$P^\mu = \Lambda^\mu_{z\text{-boost},\nu}(\psi) P_R^\nu = \Big(M\cosh(\psi), 0, 0, M\sinh(\psi)\Big). \tag{4.7}$$

Transforming this vector into light-cone coordinates leads to[4]

$$P^\mu = \left(M\frac{\cosh(\psi)+\sinh(\psi)}{\sqrt{2}}, M\frac{\cosh(\psi)-\sinh(\psi)}{\sqrt{2}}, \mathbf{0}_T\right), \tag{4.8}$$

which can be rewritten as

$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_T\right) \tag{4.9}$$

by using the hyperbolic identies

$$\sinh(\psi) = \frac{e^\psi - e^{-\psi}}{2} \tag{4.10}$$

$$\cosh(\psi) = \frac{e^\psi + e^{-\psi}}{2} \tag{4.11}$$

and the definition

$$P^+ \equiv \frac{M}{\sqrt{2}} e^\psi. \tag{4.12}$$

(4.10) and (4.11) directly imply

$$\sinh\psi + \cosh\psi = e^\psi, \tag{4.13}$$

hence $\sinh(x)$ and $\cosh(x)$ can be seen as the odd and even part of the exponential function e^x.

Expression (4.9) defines a class of Lorentz frames, the collinear frames, depending on the parameter ψ, or equivalently P^+. The name arises from the fact that in all frames of this class x_1 and x_2 are zero, so all three-momenta are parallel.

[2] Its explicit shape is given in equation (2.39).

[3] In contrast to the notation in [Aivazis et al., 1994a], we reserve the letter ψ for the rapidity of a particle. ζ denotes an arbitrary hyperbolic angle.

[4] From now on, we will suppress the index L for light-cone coordinates. As explained at the end of section 4.1, the two sets of coordinates should be clearly distinguishable by the according momentum components.

One can change between two collinear frames by a boost with the according value of ψ. When using the addition theorems

$$\sinh(\psi_1+\psi_2) = \sinh(\psi_1)\cosh(\psi_2) + \sinh(\psi_2)\cosh(\psi_1) \tag{4.14}$$
$$\cosh(\psi_1+\psi_2) = \cosh(\psi_1)\cosh(\psi_2) + \sinh(\psi_1)\sinh(\psi_2), \tag{4.15}$$

one easily sees that the rapidity behaves additively under two boosts, i.e.

$$\Lambda_{z\text{boost}}(\psi_1)\Lambda_{z-\text{boost}}(\psi_2) = \Lambda_{z\text{-boost}}(\psi_1+\psi_2). \tag{4.16}$$

Changing to another collinear frame therefore simply means adding rapidity to the current one.

It is further possible to express rapidity with the help of momentum coordinates, if we (cf. (4.9)) label P^- as

$$P^- \equiv \frac{M^2}{2P^+}. \tag{4.17}$$

Starting with the definition (4.12) and an arbitrary momentum

$$P^\mu = (E, \boldsymbol{P}), \tag{4.18}$$

we obtain

$$\begin{aligned} e^{2\psi} &= \frac{2}{M^2}(P^+)^2 = \frac{P^+}{P^-} \\ \Leftrightarrow \psi &= \frac{1}{2}\ln\frac{P^+}{P^-} = \frac{1}{2}\ln\frac{E+P^3}{E-P^3}. \end{aligned} \tag{4.19}$$

This motivates an expression in light-cone coordinates that parametrizes P^μ in terms of rapidity and transverse momentum instead of the three-momentum:

$$\begin{aligned} P^\mu &= \left(\frac{E+P^3}{\sqrt{2}}, \frac{E-P^3}{\sqrt{2}}, \boldsymbol{P}_T\right) \\ &= \left(\sqrt{\frac{M^2+\boldsymbol{P}_T^2}{2}}e^{\psi}, \sqrt{\frac{M^2+\boldsymbol{P}_T^2}{2}}e^{-\psi}, \boldsymbol{P}_T\right) \\ &\equiv \left(\frac{E_T}{\sqrt{2}}e^{\psi}, \frac{E_T}{\sqrt{2}}e^{-\psi}, \boldsymbol{P}_T\right), \end{aligned} \tag{4.20}$$

since

$$E^2 = M^2 + \boldsymbol{P}^2 = M^2 + \boldsymbol{P}_T^2 + (P^3)^2 \tag{4.21}$$

and[5] (using (4.19))

$$\begin{aligned} e^{\psi} &= \sqrt{\frac{E+P^3}{E-P^3}} \\ \Leftrightarrow E+P^3 &= (E-P^3)e^{2\psi} = \sqrt{E^2-(P^3)^2}e^{\psi} = \sqrt{M^2+\boldsymbol{P}_T^2}e^{\psi}. \end{aligned} \tag{4.22}$$

[5] The "$-$"-coordinate can be obtained in the same way.

If we use the exponential representation of the hyperbolic cosine (4.11), we can give an alternative formula for the scalar product of two momenta P_1^μ and P_2^μ:

$$P_1 \cdot P_2 = E_{1T} E_{2T} \cosh(\psi_1 - \psi_2) - \boldsymbol{P}_{1T} \cdot \boldsymbol{P}_{2T}. \tag{4.23}$$

For vanishing transverse momenta, we see that $P_1 \cdot P_2 = M^2 \cosh(\psi_1 - \psi_2)$, which has a tremendous similarity to the Euclidian scalar product. The difference lies simply in the hyperbolic cosine, which can be again traced back to the idea that boosts are hyperbolic rotations.

4.3. The Breit frame

In this section we will take a closer look at the collinear frames introduced in the last section. Longer calculations can be found in appendix E.

The relevant physical situation in this work is the scattering of two particles by an exchangend vector boson[6] with momentum q^μ. Having two independent quantities q^μ and P^μ, which the last one labels the momentum of one particle before the scattering[7], we want to keep things as simple as possible. Therefore, we choose a frame where both vectors span the x_0-x_3 plane. The parametrization of P^μ was already introduced in the last section, cf. (4.20). Since we are working in a plane, two coordinates of q^μ are automatically set:

$$\boldsymbol{q}_T = \boldsymbol{0}_T. \tag{4.24}$$

Next we make sure that q^μ is in no case parallel to P^μ by introducing an additional parameter $\eta > 0$. A possible choice is e.g.

$$q^\mu = \left(-\eta P^+, \frac{Q^2}{2\eta P^+}, \boldsymbol{0}_T\right). \tag{4.25}$$

With this setup, the scalar product of q^μ and P^μ does not depend on P^+, in other words is frame independent. Explicitly applying (4.4) gives an implicit equation for η,

$$2P \cdot q = \frac{Q^2}{\eta} - \eta M^2. \tag{4.26}$$

We will come back to this expression later, for now we will take a look at some exemplary collinear frames by choosing explicit values for P^+. In this work, we already discussed the laboratory frame (cf. section 3.1), where

$$\boldsymbol{P} = \boldsymbol{0}. \tag{4.27}$$

For $P^+ = \frac{M}{\sqrt{2}}$, we see that $P^+ = P^-$ and so $P^3 = 0$, thus we exactly reproduced this frame.

Taking the limit $P^+ \to \infty$ leads to the infinite momentum frame, discussed in section 3.2 in the context of the naive parton model. Hence, for the first time, we have come to a point where we produce the naive parton model as a limit of a more general framework. The underlying reason in this and all other cases occuring in this work will be same: We gave particles a non-vanishing mass.

The frame of our choice on the following pages will be yet another one: The Breit frame, or

$$P^+ = \frac{Q}{\sqrt{2}\eta}. \tag{4.28}$$

[6] Up until now we only talked about photons, the ACOT formalism though contains the whole electroweak interaction. More in chapter 5.

[7] Later on, this momentum will be the hadron momentum, therefore we will assign the same letter to it from now on. However, unless it is stated otherwise, all explanations in this chapter are of kinematic nature and thus do not only cover DIS exclusively.

This yields

$$P^\mu = \left(\frac{Q}{\sqrt{2}\eta}, \frac{M^2\eta}{\sqrt{2}Q}, \mathbf{0}_T\right) \tag{4.29}$$

$$q^\mu = \left(-\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0}_T\right) \tag{4.30}$$

and[8]

$$E_P = \frac{1}{2Q}\Delta(-Q^2, M^2, P_X^2) \tag{4.31}$$

$$P^3 = \frac{1}{2Q}\beta_1 \equiv \frac{1}{2Q}(Q^2 - M^2 + P_X^2) \tag{4.32}$$

$$q^0 = 0 \tag{4.33}$$

$$q^3 = -Q, \tag{4.34}$$

where (4.26) was used for the components of P^μ. Obviously, the other two components of both four-momenta are zero. For the sake of completeness we also give the components of the hadron momentum after the scattering, which was labeled by P_X^μ in chapter 3. Momentum conservation implies

$$P_X^\mu = P^\mu + q^\mu = \left(\left(\frac{1}{\eta} - 1\right)\frac{Q}{\sqrt{2}}, \frac{M^2\eta}{\sqrt{2}Q} + \frac{Q}{\sqrt{2}}, \mathbf{0}_T\right), \tag{4.35}$$

or, more convenient, in Minkovski coordinates,

$$E_{P_X} = E_P \tag{4.36}$$

$$P_X^3 = -\frac{1}{2Q}\beta_2 \equiv \frac{1}{2Q}(Q^2 + M^2 - P_X^2). \tag{4.37}$$

Hence, in this frame, the hadron simply "bounces" off the exchanged boson with unchanged energy after the scattering[9].

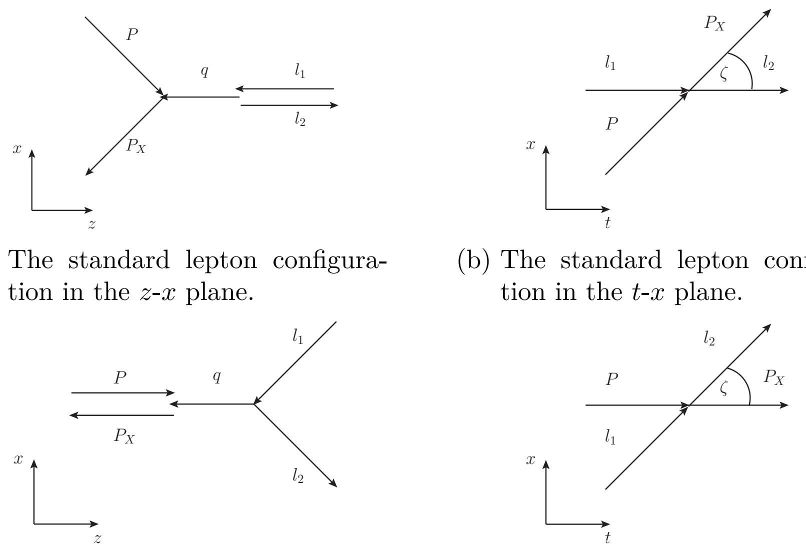
The Breit frame is the natural, i.e. simplest frame with regard to the exchanged gauge boson in a t-channel process. To understand this, we first note that $q^2 < 0$, i.e. q^μ is timelike in every process fulfilling $-Q^2 = t$. To see this, we explicitly calculate q^2, for example in the CM frame, and use that it is a Lorentz scalar, thus frame independent. This allows us to set $q^0 = 0$ in an according frame. If we also set the z-axis along $\boldsymbol{q}$ we have constructed the Breit frame, where only $q^3 \neq 0$.

A similar derivation as the one above can be done from the leptonic perspective. This means that instead of $\boldsymbol{P}$ we can also align $\boldsymbol{l}_1$ along $\boldsymbol{e}_z$. In general, if the initial momentum has only a z-component, so has the final one, since one only adds or substracts $\boldsymbol{q}$. Referring to the typical particles with momenta P^μ and l_1^μ, the Breit frames described above are called standard hadron and lepton configurations.

[8]The Δ-function, cf. (3.15), is used to obtain compact expressions.

[9]This is why this frame is occasionally called the "Brick-wall" frame.

They are illustrated in figure 4.1. It becomes obvious that these configurations are connected by a hyperbolic rotation, i.e. a boost along the x-axis.



(a) The standard lepton configuration in the z-x plane.

(b) The standard lepton configuration in the t-x plane.

(c) The standard hadron configuration in the z-x plane.

(d) The standard hadron configuration in the t-x plane.

Figure 4.1.: The standard hadron and lepton configuration. The (hyperbolic) angle between t-axis and not z-aligned momenta is denoted by ζ.

Since in the standard lepton configuration lepton momenta before and after the scattering (in Minkovski coordinates) are simply[10]

$$l_1^\mu = \frac{Q}{2}(1,0,0,-1) \tag{4.38}$$

$$l_2^\mu = \frac{Q}{2}(1,0,0,+1). \tag{4.39}$$

Boosting l_1^μ and l_2^μ with ζ along $\boldsymbol{e}_x$ gives (cf. (2.39) with exchange of x and z entries)

$$l_1^\mu = \frac{Q}{2}(\cosh\zeta, \sinh\zeta, 0, -1) \tag{4.40}$$

$$l_2^\mu = \frac{Q}{2}(\sinh\zeta, \cosh\zeta, 0, +1). \tag{4.41}$$

In light-cone coordinates, these momenta read

$$l_1^\mu = \frac{Q}{2}\left(\frac{\cosh\zeta - 1}{\sqrt{2}}, \frac{\cosh\zeta + 1}{\sqrt{2}}, \begin{pmatrix}\sinh\zeta \\ 0\end{pmatrix}\right) \tag{4.42}$$

$$l_2^\mu = \frac{Q}{2}\left(\frac{\cosh\zeta + 1}{\sqrt{2}}, \frac{\cosh\zeta - 1}{\sqrt{2}}, \begin{pmatrix}\sinh\zeta \\ 0\end{pmatrix}\right). \tag{4.43}$$

[10] One can obtain these coordinates by using the conditions $\boldsymbol{l}_{1T} = \boldsymbol{l}_{2T} = \boldsymbol{0}$, $q^\mu = l_1^\mu - l_2^\mu$ and $l_{1,2}^2 = m^2 = 0$. Leptons remain massless in the ACOT formalism, cf. chapter 5.

One easily sees that ζ can be obtained by constructing scalar products with the hadron momentum (4.29)[11],

$$\cosh\zeta = \frac{2P\cdot(l_1+l_2)}{\Delta(-Q^2,P^2,P_X^2)} = \frac{\eta^2M^2-Q^2+2\eta(s-M^2)}{\eta^2M^2+Q^2}. \tag{4.44}$$

[11]A more detailed calculation can be found on appendix E.

4.4. Helicity currents, tensors and amplitudes in DIS

The ACOT formalism makes use of the helicity formalism, where the helicity itself was introduced in section 2.3. To have a better understanding of this approach, we take a closer look at the generic DIS process (cf. e.g. figure 3.1) formulated with respect to the helicity formalism. We limit ourselves to LO calculations in this section. Additionally, we assume the exchanged gauge boson is massive, which leads to a third physical helicity eigenvalue $\lambda = 0$ with respective polarization ϵ_0^μ. In the special case of $M_B = 0$, which is the case for QED as a simplification of electroweak interactions, we can simply set $\epsilon_0^\mu = 0$ in all expressions below.

A transformation from the well-known Minkovski coordinates to the helicity basis can be performed by multiplying polarizations with variable helicity. For the leptonic and hadronic currents, this results in[12]

$$\begin{aligned} J_{L}{}^{\rho_2}_{\rho_1 \kappa} &\equiv \epsilon_\lambda^{\star\mu}(l_1, q) \left\langle l_2, \rho_2 \right| J_{L,\mu}(0) \left| l_1, \rho_1 \right\rangle \\ &= \epsilon_\lambda^{\star\mu}(l_1, q) \bar{u}_{\rho_2}(l_2) \Gamma_\mu u_{\rho_1}(l_1) \end{aligned} \tag{4.45}$$

$$J_{H}{}^{\sigma_2}_{\sigma_1 \lambda} \equiv \epsilon_\lambda^{\star\mu}(P, q) \left\langle P_X, \sigma_2 \right| J_{H,\mu}(0) \left| P, \sigma_1 \right\rangle , \tag{4.46}$$

where $\sigma_{1,2} = \pm\frac{1}{2}$ and $\rho_{1,2} = \pm\frac{1}{2}$ denote the helicities of the incoming and outgoing lepton and hadron, while $\kappa = \pm, 0$ and $\lambda = \pm, 0$ are the helicities of the exchanged gauge boson at the leptonic and hadronic vertex. As defined in section 2.3, the $\epsilon_\lambda^\mu(p, q)$ are the polarizations of a gauge boson with momentum q^μ and reference vector p^μ.

For the electroweak theory, we work in unitary gauge[13], which means that the propagator is the same as in the Proca-theory, cf. section 2.4.1.1:

$$\Pi^{\mu\nu} = \frac{-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_B^2}}{Q^2 + M_B^2}. \tag{4.47}$$

As we want to work with helicities as degrees of freedom, we make their appearance in the propagator above obvious by using the completeness relation (2.70). This yields

$$\Pi^{\mu\nu} = -\sum_{\lambda=\pm,0} v_\lambda \epsilon_\lambda^\mu(p, q) \frac{1}{Q^2 + M_B^2} \epsilon_\lambda^{\star\nu}(p, q) + \frac{\epsilon_q^\mu(p, q)\epsilon_q^\nu(p, q)}{M_B^2}. \tag{4.48}$$

Although this is the complete propagator, the second term will not be of any importance in the rest of this work and is therefore omitted for the rest of this work. The reason for that is that the parts of the amplitude belonging to this part of the

[12] In order to carry over these expression to the following chapter 5, we introduce an arbitrary vertex factor Λ^μ instead of a simple γ^μ as well as arbitrary gauge couplings g_B and gauge boson masses M_B included in a general propagator (2.86). Nevertheless, these modifications do not stand in the focus of this section and play no important role in the helicity formalism.

[13] Cf. section 2.6.2 for further explanations.

propagator will always be proportional to the (in our case vanishing) lepton mass. A full calculation can be found in appendix G.2.1.

In the last section, we always defined polarizations with respect to one specific reference momentum p^μ. The completeness relation holds for every set of polarizations, therefore we can choose an arbitrary polarization axis for the expression above, e.g. the hadron momentum $p^\mu = P^\mu$. However, since both leptonic and hadronic currents are involved, we need to change the reference of one polarization vector from P^μ to l_1^μ. To do that, a boost of the polarization into the new direction is necessary. Note that this cannot be acchieved by simply applying a Lorentz boost matrix (2.39), as we did with l_1 and l_2 in section 4.3. Instead, we are in the helicity basis, so what we need is another representation of the group of boosts in two space dimensions[14], i.e. $SO(2,1)$. The group generators are K_3 and $K_\pm \equiv K_1 \pm iK_2$ and therefore the according matrix for a boost along the x-axis "around" the angle ζ in the t-x plane is[15]

$$\begin{aligned} d^1(\zeta) &= \exp(-i\zeta K_2) = \exp\left(-\frac{\zeta}{2}(K_+ - K_-)\right) \\ &= \begin{pmatrix} \frac{1+\cosh\zeta}{2} & -\frac{\sinh\zeta}{\sqrt{2}} & \frac{1-\cosh\zeta}{2} \\ -\frac{\sinh\zeta}{\sqrt{2}} & \cosh\zeta & \frac{\sinh\zeta}{\sqrt{2}} \\ \frac{1-\cosh\zeta}{2} & \frac{\sinh\zeta}{\sqrt{2}} & \frac{1+\cosh\zeta}{2} \end{pmatrix}. \end{aligned} \tag{4.49}$$

Note that the components of this matrix do not refer to spacial or time components but rather to different polarizations. We follow the conventional ordering $\kappa, \lambda = -, 0, +$ from up to down and left to right. The propagator then becomes

$$\Pi^{\mu\nu} = -\sum_{\lambda,\kappa} v_\lambda \epsilon^\mu_\lambda(P,q) \frac{d^1(\zeta)^{\lambda\kappa}}{Q^2 + M_B^2} \epsilon^{\star\nu}_\kappa(l_1,q). \tag{4.50}$$

Starting from the complete amplitude obtained by Feynman rules,

$$\mathcal{M} = g_B^2 J^\star_{H\,\mu} \Pi^{\mu\nu} J_{L\,\nu}, \tag{4.51}$$

we insert the new helicity propagator and currents to obtain

$$\mathcal{M}^{\rho_2,\sigma_2}_{\rho_1,\sigma_1} = -\sum_{\lambda,\kappa} v_\lambda J_{H\,\sigma_1\lambda}^{\,\sigma_2} \frac{g_B^2 d^1(\zeta)^{\lambda\kappa}}{Q^2 + M_B^2} J_{L\,\rho_1\kappa}^{\,\rho_2}. \tag{4.52}$$

We see that choosing helicities as degrees of freedom is no way inferior to the usual Minkovski coordinates. In fact, it is now obvious that there is an equivalent set of Feynman rules for helicity amplitudes. A squared and averaged matrix element

[14] In section 4.3 we saw that the whole process takes place in the x-z plain.

[15] Cf. figure 4.1 for an illustration. We need to change to spherical tensors $K_\pm$, since we deal with angular momentum eigenstates. Note that an explicit calculation of this matrix performed by the author, which can be found in appendix B, does not lead to the result that is given here. Hence, the reproduction of this matrix remains an open problem.

$\overline{|\mathcal{M}|^2}$ would now include a sum over external helicities instead of spins[16]. One could argue that the helicity formalism is the more "physical" approach, since one actually "sees" the effect of scattering in form of the matrix $d^1(\zeta)$ as a part of the amplitude.

[16] Cf. appendix G.2 for an explicit calculation of a squared and averaged amplitude $\overline{|\mathcal{M}|^2}$.

5. The ACOT formalism: General framework

The references for this and the following chapter is the original series of two papers [Aivazis et al., 1994a] and [Aivazis et al., 1994b], in which the ACOT formalism was developed and applied to production of heavy quarks in processes including leptons and hadrons. The general framework containing cross sections, cf. section 5.2, kinematics, cf. section 5.3, and factorization, cf. section 5.4.2, is introduced in [Aivazis et al., 1994a], while NLO calculations and the formulation of a proper renormalization scheme are the topic of [Aivazis et al., 1994b] and hence are postponed to chapter 6.

5.1. A detailed view on mass and energy scales

Similar discussions as the one below are given in section I of [Aivazis et al., 1994b] and section 3.9 of [Collins, 2011].

Chapter 3 gave a general introduction to DIS and the naive parton model. There are several limitations to this approach, one of the most prominent ones being the ignored mass of hadrons and quarks. This makes the naive parton model valid only on scales where $Q \gg m$ for every hadronic or partonic mass scale m. However, the goal of the ACOT formalism is to cover the complete range of Q. In principle, we can identify three regions which need to be treated differently from one another.

To specify them, we consider a scenario which was also the objective of the original ACOT publications [Aivazis et al., 1994a] and [Aivazis et al., 1994b]: Heavy quark production in DIS, meaning that the inclusive final state of DIS must contain a heavy quark. In this context, "heavy" means that its mass m is huge compared to the absolute[1] scale, which is the Landau pole[2] Λ. In other words, $\alpha_s(m) \ll 1$. Usually, charm, bottom and top quarks are considered as heavy in an absolute sense. But this does not mean that such a quark is necessarily heavy in a specific kinematic process, i.e. compared to the respective scale Q. In general, the three scenarios are:

- $Q \ll m$: The heavy quark is also heavy in a kinematic sense. Outside of renormalization terms, it decouples completely from the process, meaning that there are no quark lines inside both the collinear and hard region. The decoupling theorem is explained in detail e.g. in [Appelquist and Carazzone, 1975].

[1] The term "absolute" stands for "only depending on the underlying theory", which is QCD in this case.
[2] Cf. section 2.6.1 for further explanations.

- $Q \sim m$: Q and m both are large. There are heavy quark lines inside the partonic tensor.
- $Q \gg m$: The quark can be considered as light. The naive parton model can be applied and consequently there is a PDF assigned to the quark.

Similar considerations can be made for other processes involving heavy[3] quarks. The derivation in this work is kept completely general. In addition, heavy quarks have nontrivial implications on the renormalization procedure. This discussion is postponed to chapter 6, where we deal with all arising NLO effects.

When allowing wide ranges of values for Q one also needs to deal with electroweak interactions[4]. Ignoring them produces correction terms proportional to a factor of $m_{W,Z}^2/Q^2$, which becomes important when Q^2 is comparable to, if not much larger than $m_{W,Z}^2$. $m_{W,Z}$ are the masses of the exchanged W- or Z-boson, respectively. This new form of interactions requires us to recalculate the leptonic tensor (compared to chapter 3) and also needs to be respected when computing the hadronic one.

5.2. Generalized cross sections

As stated in the introduction above, in the ACOT formalism we also want to include weak interactions. To do that, we need to adjust the related matrix element $\mathcal{M}$. To keep things simple, we only consider LO electroweak interactions, comparable to the naive parton model.

Starting with the matrix element in QED-based DIS (3.1), the vertices and propagator are modified. In other words, in the new matrix element the coupling constants g_B and boson masses M_B are kept variable. As a shorthand notation for vertex and propagator contributions we introduce

$$G_B \equiv \frac{g_B^2}{Q^2 + M_B^2}, \tag{5.1}$$

which reduces to the well-known

$$G_\gamma^2 = \frac{e^4}{Q^4} \tag{5.2}$$

in the case of an exchanged photon. The form of G_B can be easily understood by looking at a propagator of a massive gauge boson, cf. (2.86). The coupling constants of the $W^\pm$- and Z-boson are explained in section 2.6.2, here they are only given again for the sake of clarity:

$$g_W = \frac{g}{2\sqrt{2}} \tag{5.3}$$

$$\text{and } g_Z = \frac{g}{2\cos\theta_W}. \tag{5.4}$$

[3] Here and in the following, when we use the term "heavy" without further specification, we mean it in an absolute sense.

[4] Cf. 2.6.2 for a short introduction.

The masses M_W and M_Z can be found in section 2.6.2.

Following (3.13), the new cross section including weak interaction is now

$$d\sigma = \frac{G_{B_1} G_{B_2}}{F} 4\pi Q^2 L_{\mu\nu} W^{\mu\nu} d\Pi_{\text{LIPS}}. \tag{5.5}$$

where we allow different G_{B_1} and G_{B_2} for the case of γ-Z interference[5]. The tensors in the equation above need to be changed in comparison to chapter 3 and cross section (3.17). In the case of of electroweak interaction, we deal with a non abelian gauge theory, so the Ward-identity (3.7) cannot be applied and we must consider the general hadronic tensor (3.6). Additionally, we need to implement general couplings into the original leptonic tensor (3.3):

$$\begin{aligned} L^{\mu\nu} = \frac{1}{2Q^2} \sum_{s_1,s_2} &\left[\overline{u}_{s_2}(l_2)\gamma^\mu \Big(g_R(1+\gamma^5) + g_L(1-\gamma^5) \Big) u_{s_1}(l_1) \right] \\ &\times \left[\overline{u}_{s_2}(l_2)\gamma^\nu \Big(g_R(1+\gamma^5) + g_L(1-\gamma^5) \Big) u_{s_1}(l_1) \right]^* . \end{aligned} \tag{5.6}$$

The values for the chiral couplings $g_{R,L}$ for the full electroweak theory can be found in table 2.2.

The whole calculation of the full differential cross section in the laboratory frame is given in appendix D.3, at this point we only state the final result:

$$\begin{aligned} \frac{d\sigma}{dE_2 d\cos\theta} = \frac{2E_2^2}{\pi n_l M} G_{B_1} G_{B_2} \Big\{ & g_+^2 \left[2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2} \right] \\ &+ g_-^2 \left[\frac{E_1+E_2}{M} W_3 \sin^2\frac{\theta}{2} \right] \Big\}, \end{aligned} \tag{5.7}$$

where we keep the number of leptonic spin states n_l variable. The reason for this is that in the case of an electroweak coupling, a neutrino can also play the role of the incoming particle. Since neutrinos are, at least in the Standard Model, massless and purely left-handed, there is only one possible spin state. Hence,

$$n_l = \begin{cases} 1 \text{ if the incoming lepton is a neutrino } (\nu_e,\ \nu_\mu,\ \nu_\tau) \\ 2 \text{ if the incoming lepton is a fermion } (e,\ \mu,\ \tau) \end{cases}. \tag{5.8}$$

In addition, we introduced

$$g_\pm^2 \equiv g_R^2 \pm g_L^2 \tag{5.9}$$

and, exactly as in the cross sections of chapter 3, θ is the angle in laboratory frame. As described at the end of appendix D.3, in QED $g_L = g_R = \frac{1}{2}$[6], so, just as expected, (5.7) reduces to the related hadronic cross section (3.17) of the naive parton model.

[5] Cf. chapter 2.6.2 for a more detailed explanation.

[6] This simply means that QED has a chiral symmetry.

We are free to change variables: Instead of the structure functions W_i we can choose an alternative normalization F_i, which was already briefly mentioned in chapter 3 (cf. (3.35) and (3.36)):

$$F_1 \equiv W_1 \tag{5.10}$$
$$F_2 \equiv \frac{\nu}{M} W_2 \tag{5.11}$$
$$F_3 \equiv \frac{\nu}{M} W_3. \tag{5.12}$$

If we also take the derivative with respect to the Bjorken x (3.11) and[7]

$$y \equiv \frac{P \cdot q}{P \cdot l_1} = \left(\frac{\nu}{E_1}\right)_{\text{lab}} \tag{5.13}$$

instead of E_1 and $\cos\theta$ we arrive at the alternative formulation[8]

$$\frac{d\sigma}{dxdy} = \frac{2ME_1}{\pi n_l} G_{B_1} G_{B_2} \left\{ g_+^2 \left[xy^2 F_1 + \left(1 - y - \frac{Mxy}{2E_1}\right) F_2 \right] + g_-^2 \left[xy\left(1 - \frac{y}{2}\right) F_3 \right] \right\}. \tag{5.14}$$

Next, we need to specify all relevant kinematics.

5.3. Kinematics

As introduced in section 4.2 and 4.3, we work with momenta in light-cone coordinates and the Breit frame. Thus, the hadron and gauge boson momenta are

$$P^\mu = \left(\frac{Q}{\sqrt{2}\eta}, \frac{M^2\eta}{\sqrt{2}Q}, \mathbf{0}_T\right) \tag{5.15}$$
$$\text{and } q^\mu = \left(-\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0}_T\right). \tag{5.16}$$

It was already mentioned in section 4.3 that these definitions lead to an implicit definition for η,

$$2P \cdot q = \frac{Q^2}{\eta} - \eta M^2. \tag{5.17}$$

Reformulating this equation gives[9]

$$\frac{1}{\eta^2} - \frac{1}{x\eta} - \frac{M^2}{Q^2} = 0, \tag{5.18}$$

[7] The second equality is only valid in the laboratory frame.

[8] x and y have the advantage of no particular frame dependence unlike the laboratory frame variables E_2 and θ. A detailed derivation can be found in appendix D.3.

[9] Here, we use the definition $x \equiv Q^2/(2P \cdot q)$, cf. (3.11).

which can be solved for $1/\eta$[10]:

$$\frac{1}{\eta} = \frac{1}{2x} + \sqrt{\frac{1}{4x^2} + \frac{M^2}{Q^2}}. \tag{5.19}$$

In the zero target mass limit $M^2/Q^2 \to 0$, we see that $\eta \to x$, which means that η can be used as a generalized version of the scaling variable x for hadron masses different from zero.

Next, we turn to the parton momenta $p^\mu_{1,2}$ before and after the scattering. The reader shall be reminded of the fact that in the standard hadron configuration of the Breit frame, which we will choose in the following, the hadronic and bosonic three-momenta both are collinear with the z-axis. Since we still only allow collinear hadronic and partonic momenta, we can use P^μ and q^μ as a basis of the t-z plane and hence decompose the incoming partonic momentum into

$$p_1^\mu = c_P P^\mu + c_q q^\mu. \tag{5.20}$$

Note that this has not the same simple structure as in the naive parton model anymore.

In addition, we have to find new a definition for ξ, which is compatible with the old one of the naive parton model. The main problem with the old definition were ξ-dependent parton masses, cf. (3.26). The new definition will not have such a clear meaning as a momentum fraction, but will still be used as the convolution variable for the factorization of the hadronic tensor. (5.15) shows that the "$-$"-components of P^μ is suppressed by a factor of M^2/Q^2 (cf. (5.15)). Hence, instead of the momenta itself, we take ξ to be the fraction of their "$+$"-component coordinate,

$$\xi \equiv \frac{p_1^+}{P^+}. \tag{5.21}$$

With this definition, we can explicitly write down the momentum of the incoming parton using the general structure of z-aligned four-momenta in the Breit frame[11]:

$$p_1^\mu = \left(\xi P^+, \frac{m_1^2}{2\xi P^+}, \mathbf{0}_T\right) = \frac{Q}{\sqrt{2}}\left(\frac{\xi}{\eta}, \frac{\eta}{\xi}\frac{m_1^2}{Q^2}, \mathbf{0}_T\right). \tag{5.22}$$

In the expression above, we also introduced an indexed parton mass m_1. It is possible to have a different parton mass m_2 after the scattering due to flavour changing processes in electroweak interaction. As mentioned earlier, the ACOT formalism was originally developed to properly describe the production of heavy quark flavours, meaning that the parton changes into flavors with non-negligible masses within the interaction.

[10] One easily checks that the second solution leads to a negative η and consequently is omitted.

[11] By recalling the definitions of P^μ and p_1^μ, one immediately sees that the old definition of ξ can be reproduced in the limit of vanishing M and m_1.

With the explicit shape of p_1^μ at hand we can now compute c_P and c_q by contractions with P^μ and q^μ[12]:

$$c_P = \frac{\xi^2 Q^2 + \eta^2 m_1^2}{\xi(Q^2+\eta^2 M^2)} \tag{5.23}$$

$$c_q = \frac{\eta(m_1^2 - \xi^2 M^2)}{\xi(Q^2+\eta^2 M^2)}. \tag{5.24}$$

The zero mass limit is now mathematically apparent, too. If we let both m_1 and M go to zero, we see that $c_P \to \xi$ and $c_q \to 0$.

At LO, the only possible subprocess at parton level is $p_1 + B \to p_2$[13], where the outgoing momentum is[14]

$$p_2^\mu = p_1^\mu + q^\mu = \frac{Q}{\sqrt{2}}\left(\frac{\xi}{\eta} - 1, 1 + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}, \mathbf{0}_T\right). \tag{5.25}$$

Therefore, we can express m_2 in terms of m_1 and Q:

$$m_2^2 = p_2^2 = \left(\frac{\xi}{\eta} - 1\right)Q^2 + \left(1 - \frac{\eta}{\xi}\right)m_1^2. \tag{5.26}$$

Using this relation, we can derive

$$\begin{aligned} p_1 \cdot p_2 =& \frac{Q^2}{2}\left[\frac{\xi}{\eta}\left(1 + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}\right) + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}\left(\frac{\xi}{\eta} - 1\right)\right] \\ =& \frac{1}{2}\left[\frac{\xi}{\eta}Q^2 + \left(2 - \frac{\eta}{\xi}\right)m_1^2\right] = \frac{1}{2}\left[Q^2 + m_1^2 + m_2^2\right]. \end{aligned} \tag{5.27}$$

In addition, we can compute all relevant scalar products between lepton and parton momenta using the respective formula for light-cone coordinates (4.5). According to section 4.3, the lepton momenta in the standard hadron configuration are:

$$l_1^\mu = \frac{Q}{2}\left(\frac{\cosh\zeta - 1}{\sqrt{2}}, \frac{\cosh\zeta + 1}{\sqrt{2}}, \begin{pmatrix}\sinh\zeta \\ 0\end{pmatrix}\right) \tag{5.28}$$

$$l_2^\mu = \frac{Q}{2}\left(\frac{\cosh\zeta + 1}{\sqrt{2}}, \frac{\cosh\zeta - 1}{\sqrt{2}}, \begin{pmatrix}\sinh\zeta \\ 0\end{pmatrix}\right), \tag{5.29}$$

where the hyperbolic angle ζ is given by (cf. (4.44))

$$\cosh\zeta = \frac{\eta^2 M^2 - Q^2 + 2\eta(s - M^2)}{\eta^2 M^2 + Q^2}. \tag{5.30}$$

[12] The full calculation can be found in appendix F.
[13] This process is examined in detail in section 5.5.
[14] Here, we use q^μ in light-cone coordinates, cf. equation (4.30).

Consequently, the results are[15]

$$p_1 \cdot l_1 = p_1^+ l_1^- + p_1^- l_1^+ = \frac{1}{2}\frac{\xi}{\eta} Q^2 \left(\frac{\cosh\zeta + 1}{2}\right) + \frac{1}{2}\frac{\eta}{\xi} m_1^2 \left(\frac{\cosh\zeta - 1}{2}\right) \tag{5.31}$$

$$p_1 \cdot l_2 = \frac{1}{2}\frac{\xi}{\eta} Q^2 \left(\frac{\cosh\zeta - 1}{2}\right) + \frac{1}{2}\frac{\eta}{\xi} m_1^2 \left(\frac{\cosh\zeta + 1}{2}\right) \tag{5.32}$$

$$p_2 \cdot l_1 = -\frac{1}{2}\Big((p_2 - l_1)^2 + p_2^2 + l_1^2\Big) = -\frac{1}{2}\Big((p_1 - l_2)^2 + m_2^2\Big)$$

$$= p_1 \cdot l_2 - \frac{m_2^2 - m_1^2}{2} \tag{5.33}$$

$$\text{and } p_2 \cdot l_2 = p_1 \cdot l_1 + \frac{m_2^2 - m_1^2}{2}. \tag{5.34}$$

5.3.1. Kinematic region of ξ

Before further continuing, one has to deal with the effect of massive partons and hadrons on the momentum fraction ξ. This section requires some longer calculations which are outsourced into appendix F.

In the case of massive partons and hadrons, one should expect a shift of the lower border, which is x in the naive parton model. To obtain this new threshold ξ_{th}, we need to look at the on-shell condition of the outgoing parton[16]:

$$0 = p_2^2 - m_2^2 = Q^2 \frac{(\xi - \chi_+)(\xi - \chi_-)}{\eta\xi}, \tag{5.35}$$

with

$$\chi_\pm \equiv \eta \frac{Q^2 - m_1^2 + m_2^2 \pm \Delta(-Q^2, m_1^2, m_2^2)}{2Q^2}. \tag{5.36}$$

From the two roots $\chi_\pm$ only the greater one, $\chi_+ \equiv \chi$, lies between 0 and 1. In fact, we inspected the case of minimal produced mass in the final parton state[17], so we can take χ as the lower border in case of massive partons, $\xi_{\text{th}} = \chi$.

With this calculations, we also found a more suitable shape of the often occurring δ distribution ensuring on-shell particles in phase space integrals:

$$\delta(p_2^2 - m_2^2) = \frac{\delta\left(\frac{\xi}{\chi} - 1\right)}{\Delta(-Q^2, m_1^2, m_2^2)}. \tag{5.37}$$

If we go to the massless limit, $m_1 \to 0$ and $M \to 0$, where only the outgoing parton m_2 is massive, χ simplifies to

$$\chi = x\left(1 + \frac{m_2^2}{Q^2}\right). \tag{5.38}$$

[15] For the scalar product involving p_2^μ we additionally use massless leptons, $l_{1,2}^2 = 0$, and momentum conservation, $p_1 + l_1 = p_2 + l_2$.
[16] Again, we consider the simplest case of LO scattering.
[17] In other words, the center-of-mass energy in the partonic subprocess $\hat{s} = m_2^2$ is minimal.

This is the simplest correction to the naive parton model for productions of heavy quarks. However, the expression for χ above, equation (5.36), was the first one including massive partons in the initial state as well as a non-vanishing target mass. If we also set m_2 to zero, χ reduces to $\eta = x$. Hence, we retained the old $\xi = x$ LO condition of the naive parton model, cf. (3.31).

5.3.2. x, ξ, χ and η

To avoid confusion about the new definitions and variables introduced in this section, let us close it with a little summary on the four quantities given in the header.

We start with the Bjorken-x, which was orginally defined in the context of the naive parton model, cf. (3.11). There is no reason to change this kinematic and Lorentz-invariant quantity, so its definitions stays

$$x \equiv \frac{Q^2}{2P \cdot q}. \tag{5.39}$$

In the naive parton model at LO, x could be directly identified with the fraction of hadronic and partonic four-momentum ξ, cf. (3.31). This definition is not compatible with massive quarks and therefore needs to be adjusted to the ratio of the according light-cone coordinates,

$$\xi \equiv \frac{p_1^+}{P^+}. \tag{5.40}$$

Hence, there is not such a vivid interpretation of ξ as in the naive parton model anymore. However, ξ is still used as the integration variable of the generalized product introduced in the context of factorization, cf. section 3.3.1. It is therefore better to think of ξ as the convolution variable of $\otimes$. As we still want to integrate over all possible partonic momenta, we again define this variable as the fraction of partonic and hadronic coordinates. On of the most direct consequences of the introduction of masses is the change of the lower border in the integration inside $\otimes$, which is simply x in the naive parton model. In the section above, we derived (cf. (5.36)) that it becomes

$$\chi \equiv \eta \frac{Q^2 - m_1^2 + m_2^2 + \Delta(-Q^2, m_1^2, m_2^2)}{2Q^2} \tag{5.41}$$

for all involved masses being non-zero. In this definition, we also encounter the last new variable η, which is probably the most unintuitive one. It was introduced as an arbitrary parameter in section 4.3 when defining q^μ and P^+ in the Breit frame. However, we can solve for η and make it depend on x, cf. (5.19). We obtain

$$\eta = \left[\frac{1}{2x} + \sqrt{\frac{1}{2x} + \frac{M^2}{Q^2}} \right]^{-1}. \tag{5.42}$$

Therefore, η can be seen as a target (i.e. hadron) mass corrected x compared to the naive parton model. As we will see in the next sections, it enters calculations only implicitly inside χ and ζ.

5.4. Helicity in the ACOT formalism

All calculations necessary for the results below can be found in appendix G.

5.4.1. Structure functions and the differential cross section

An introduction to the helicity operator and its eigenvalues and -states can be found in section 2.3. The helicity formalism was introduced in section 4.4. The ACOT formalism uses the bosonic helicities ϵ^μ_λ instead of Minkovski coordinates as degrees of freedom. This gives rise to the new helicity structure functions[18]

$$F_\lambda \equiv \epsilon^{\star\mu}_\lambda(P,q) W_{\mu\nu} \epsilon^\nu_\lambda(P,q), \tag{5.43}$$

which take the place of the so far used F_i. As in section 4.4, we explicitly deal with massive gauge bosons, leading to a third structure function F_0. In the case of QED-interactions which are the only massless electroweak gauge bosons, we set $\epsilon^\mu_0(P,q)$ and therefore F_0 itself to zero.

Polarization vectors, here belonging to a gauge boson with momentum q^μ and spin 1, always need a second reference vector[19] (cf. section 2.3). Since we are characterizing the tensor describing the hadron, the reference is chosen to be P^μ. The ACOT formalism operates in the Breit frame, so P and q have the same shape as in section 4.3. This leads to polarizations $\epsilon^\mu_{\pm,0}(P,q)$ like the ones introduced in section 2.3, cf. (2.66) and (2.67). Explicitly computing (5.43) gives[20]

$$F_\pm = F_1 \mp \frac{1}{2}\sqrt{1+\frac{Q^2}{\nu^2}} F_3 \tag{5.44}$$

$$F_0 = -F_1 + \frac{1}{2x}\left(1+\frac{Q^2}{\nu^2}\right) F_2. \tag{5.45}$$

The factors of $(1+Q^2/\nu^2)$ will be relevant only in the case of target mass corrections, i.e. $M \neq 0$. Otherwise, in the limit $M = Q^2/(2x\nu) \to 0$, we directly see that $Q^2/\nu^2 \to 0$ as well. A simple rearrangement of (5.44) and (5.45) gives

$$F_1 = \frac{1}{2}\Big(F_+ + F_-\Big) \tag{5.46}$$

$$F_2 = \frac{x}{1+\frac{Q^2}{\nu^2}}\Big(2F_0 + F_+ + F_-\Big) \tag{5.47}$$

$$F_3 = \frac{1}{\sqrt{1+\frac{Q^2}{\nu^2}}}\Big(F_- - F_+\Big), \tag{5.48}$$

[18] The right-hand-side contains no sum over the helicity λ. Just as in section 2.3, we will write an explicit $\sum$ to mark situations in which the helicity is summed over.

[19] These references are sometimes called polarization or quantization axis in the literature.

[20] Appendix G.1.1 contains a full calculation.

which allows us to switch between helicity and ordinary structure functions. It is worth noting that this relation is valid up to any order of perturbation theory.

The three helicity structure functions above will be completely sufficient for our purposes. At first sight, this seems contradictory to the fact that the hadronic tensor contains six structure functions in a theory in which chirality is not conserved (cf. e.g. equation (5.52) in the next section.). However, only three structure functions occur when calculating the cross section in the conventional way, cf. (5.14), too. In appendix D.3, the vanishing contributions of the other three structure functions can be traced back to the fact that we work with massless leptons. This does not change of course when going to the helicity basis, but the appearance of the three additional structure functions is a little more subtle: When we introduced the helicity formalism in section 4.4, we already cut away the contribution proportional to $\epsilon_q^\mu(P,q)\epsilon_q^\nu(P,q)$ in the original propagator (4.48). The reason for this was exactly the fact that it only leads to non-vanishing contributions inside the amplitude in the presence of lepton masses[21]. In this case, we would obtain three new structure functions W_{qq}, W_{q0} and W_{0q}, which can be extracted from $W^{\mu\nu}$ via (G.39). The mixing between $\epsilon_q^\mu(P,q)$ and $\epsilon_0^\mu(P,q)$ is due to their same helicity eigenvalue $\lambda = 0$, cf. section 2.3.2.1.

To obtain the cross section expressed in terms of these helicity structure functions, we start with the general formula for a differential cross section (3.13). Explicitly computing the squared and averaged amplitude, which was derived in section 4.4, gives

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{1}{n_l} \sum_{\rho_{1,2}\sigma_{1,2}} \mathcal{M}^{\rho_1,\sigma_1\star}_{\rho_2,\sigma_2} \mathcal{M}^{\rho_2,\sigma_2}_{\rho_1,\sigma_1} \\
=& \frac{32\pi Q^2}{n_l} G_{B_1} G_{B_2} \sum_\lambda \left\{ g_R^2 \left(d^1(\zeta)^{+1\lambda} \right)^2 F_\lambda + g_L^2 \left(d^1(\zeta)^{-1\lambda} \right)^2 F_\lambda \right\} \\
=& \frac{32\pi Q^2}{n_l} G_{B_1} G_{B_2} \times \\
& \left\{ g_R^2 \left[\frac{(1+\cosh\zeta)^2}{4} F_+ + \frac{\sinh^2\zeta}{2} F_0 + \frac{(1-\cosh\zeta)^2}{4} F_- \right] \right. \\
& \left. + g_L^2 \left[\frac{(1-\cosh\zeta)^2}{4} F_+ + \frac{\sinh^2\zeta}{2} F_0 + \frac{(1+\cosh\zeta)^2}{4} F_- \right] \right\}. \qquad (5.49)
\end{aligned}$$

Conservation of helicity and equality between helicity and chirality eigenstates allowed us to bring $\overline{|\mathcal{M}|^2}$ in this compact form. More on this can be found in the explicit derivation in appendix G.2. The final result for the cross section is

[21] Appendix G.2.1 contains an according calculation.

$$\frac{d\sigma}{dxdy} = \frac{yQ^2}{2\pi n_l} G_{B_1} G_{B_2} \left\{ g_+^2 \left[\frac{1}{2} \left(1 + \cosh^2 \zeta \right) \left(F_+ + F_- \right) + \sinh^2 \zeta F_0 \right] + g_-^2 \left[\cosh \zeta \left(F_- - F_+ \right) \right] \right\}, \tag{5.50}$$

where we use the squared couplings $g_\pm = g_R^2 \pm g_L^2$. The hyperbolic angle ζ contains the full kinematic information. It was introduced in section 4.3 and also used when presenting the helicity formalism in section 4.4.

5.4.2. Factorization

Up until now, we have not talked about partons in the helicity formalism. As explained in section 3.3.1, the factorization theorem implies[22]

$$W^{\mu\nu} \equiv \sum_i f_i \otimes \hat{W}_i^{\mu\nu}, \tag{5.51}$$

where the hadronic tensor stays in its most general form, cf. (3.6):

$$W^{\mu\nu} = -g^{\mu\nu} F_1 + \frac{1}{M\nu} P^\mu P^\nu F_2 - \frac{i}{2M\nu} \epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta F_3 + \frac{1}{M^2} q^\mu q^\nu W_4 + \frac{1}{2M^2} (P^\mu q^\nu + q^\mu P^\nu) W_5 + \frac{1}{2M^2} (P^\mu q^\nu - q^\mu P^\nu) W_6. \tag{5.52}$$

We used the alternative structure functions

$$F_1 \equiv W_1 \tag{5.53}$$

$$\text{and } F_{2,3} \equiv \frac{\nu}{M} W_{2,3}, \tag{5.54}$$

which were partially already defined in section 3.2.

The partonic tensor $\hat{W}^{\mu\nu}$ was already a subject of discussion in the introduction to DIS, cf. section 3.2. However, in that section we restricted ourselves to substituting completely calculated expressions. A purely agnostic definition of $\hat{W}^{\mu\nu}$ would be in terms of partonic currents J_i^μ, just as the ones for the leptonic and hadronic tensors, cf. (3.3) and (3.5):

$$\hat{W}_i^{\mu\nu} \equiv \frac{1}{4\pi} (2\pi) \delta(p_2^2 - m_2^2) \frac{1}{n_i} \sum_{s_{1,2}} \langle p_1, s_1 | J_i^\mu | p_2, s_2 \rangle \langle p_2, s_2 | J_i^{\nu\dagger} | p_1, s_1 \rangle, \tag{5.55}$$

[22] In fact, this is not a simple implication. The factorization theorem is proven in certain kinematic configurations, where Q is much greater than all masses involved in the process (Cf. section 3.3.1 for more details). This is sufficient for the naive parton model, but, as explained in section 5.1, not for the ACOT formalism. During the time when the original ACOT papers [Aivazis et al., 1994a] and [Aivazis et al., 1994b] were published, the authors needed to assume that the factorization theorem holds also in regions of lower Q. Later, efforts were made to obtain a general factorization formula that also covers heavy quark masses, cf. [Collins, 1998].

where we introduced the number of spin states n_i of parton type i. We implicitly assume momentum conservation, so $p_2^\mu = p_1^\mu + q^\mu$. What is left is only the on-shell condition, meaning that we fix p_2^2 at m_2^2.

For now, let us assume the most general form of $\hat{W}^{\mu\nu}$ in terms of Lorentz tensors. The parametrization of $\hat{W}^{\mu\nu}$ is closely aligned to the one of $W^{\mu\nu}$ in section 3.1, cf. (3.6). However, we obviously need to change the hadronic momentum P^μ to the partonic one p_1^μ. Using the partonic mass instead of the hadronic one would lead to divergences in regions where the partonic mass is set to zero[23]. Instead, we simply use Q. Hence,

$$\begin{aligned}\hat{W}_i^{\mu\nu} \equiv &- g^{\mu\nu}\hat{W}_1^i + \frac{1}{Q^2}p_1^\mu p_1^\nu \hat{W}_2^i - \frac{i}{2Q^2}\epsilon^{\alpha\beta\mu\nu}p_{1\alpha}q_\beta \hat{W}_3^i + \frac{1}{Q^2}q^\mu q^\nu \hat{W}_4^i \\ &+ \frac{1}{2Q^2}(p_1^\mu q^\nu + q^\mu p_1^\nu)\hat{W}_5^i + \frac{1}{2Q^2}(p_1^\mu q^\nu - q^\mu p_1^\nu)\hat{W}6^i. \end{aligned} \tag{5.56}$$

We naturally define the partonic helicity structure functions to be

$$\hat{W}_\lambda^i \equiv \epsilon_\lambda^{\star\mu}(P,q)\hat{W}_{i\mu\nu}\epsilon_\lambda^\nu(P,q). \tag{5.57}$$

In the same way as for the hadronic structure functions, we can relate classical and helicity partonic structure functions[24]:

$$\hat{W}_\pm^i = \hat{W}_1^i \mp \frac{1}{4}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_i^2}{Q^2}\right)\hat{W}_3^i \tag{5.58}$$

$$\hat{W}_0^i = - \hat{W}_1^i + \left(\frac{\xi^2}{4x^2} + \frac{m_i^2}{Q^2}\right)\hat{W}_2^i. \tag{5.59}$$

m_i is the mass of the respective parton flavor i. Inverting the formulae above gives

$$\hat{W}_1^i = \frac{1}{2}\left(\hat{W}_+^i + \hat{W}_-^i\right) \tag{5.60}$$

$$\hat{W}_2^i = \frac{2x^2}{\xi^2 + 4x^2\frac{m_i^2}{Q^2}}\left(2\hat{W}_0^i + \hat{W}_+^i + \hat{W}_-^i\right) \tag{5.61}$$

$$\hat{W}_3^i = \frac{2\xi\eta}{\xi^2 + \eta^2\frac{m_i^2}{Q^2}}\left(\hat{W}_-^i - \hat{W}_+^i\right). \tag{5.62}$$

When we express these tensors and, even more important, differential cross sections in terms of structure functions W_j^i and $\hat{W}_j^i$, we need to know their behavior under the factorization theorem. Since both hadronic and partonic tensors are inevitably decomposed into different sets of tensors of rank two, there is no reason why one should be able to carry over the simple structure of (5.51) into the factorization of

[23] A detailed discussion on the treatment of parton (or quark) masses can be found in section 6.1.
[24] Cf. Appendix G.1.2 for a derivation.

structure functions. Hence, hadronic and partonic structure functions are entangled[25]:

$$W_j = \sum_i f_i \otimes \sum_k \kappa^j_k \hat{W}^i_k. \tag{5.63}$$

Keep in mind that this relation simplifies again when we go back the naive parton model, i.e. set $p_1^\mu = \xi P^\mu$. In this case, it is trivial that the left- and right-hand-sides are expressed in terms of the same set of tensors. For non-vanishing masses, the situation is more complicated due to a non-trivial decomposition of p_1^μ, cf. equation (5.20).

At first sight, changing to helicity structure functions does not change much. When using the helicity formalism, we need to multiply both sides of (5.51) with polarizations ϵ^μ_λ. The problem is again that we have to take different polarizations for the two sides of the equation, namely $\epsilon^\mu_\lambda(P,q)$ and $\epsilon^\mu_\lambda(p_1,q)$. However, there is one crucial simplification: These two sets of polarizations are actually the same! The reference vector P^μ or p_1^μ is only needed to define the three-dimensional polarization axis. But, since $\boldsymbol{q}$ and $\boldsymbol{P}$ are both aligned with the z-axis, the same is true for $\boldsymbol{p}_1$. Therefore[26],

$$\epsilon^\mu_\lambda(P,q) = \epsilon^\mu_\lambda(p_1,q). \tag{5.64}$$

Consequently, we obtain the straightforward factorization formula

$$F_\lambda = \sum_i f_i \otimes \hat{W}^i_\lambda \tag{5.65}$$

for the helicity structure functions. Applying (5.65) on the cross section (5.50) directly yields

$$\begin{aligned}\frac{d\sigma}{dxdy} =& \frac{yQ^2}{2\pi n_l} G_{B_1} G_{B_2} \times \\ & \sum_i f_i \otimes \Big\{ g_+^2 \Big[\frac{1}{2}\Big(1+\cosh^2\zeta\Big)\Big(\hat{W}^i_+ + \hat{W}^i_-\Big) + \sinh^2\zeta \hat{W}^i_0 \Big] \\ & + g_-^2 \Big[\cosh\zeta\Big(\hat{W}^i_- - \hat{W}^i_+\Big)\Big]\Big\}. \end{aligned} \tag{5.66}$$

[25]Appendix G.3 contains a calculation of the coefficients κ^i_k. It should be noted that the decomposition is only non-diagonal in W_4 and W_5, i.e. in the presence of lepton masses.

[26]There is also a more formal way to show the equality of both polarization sets by simply exploiting their definitions. The reader can again find this calculation in appendix G.3.

5.4.3. The advantages of using the helicity formalism

Let us close this section with a short discussion on the benefits of the helicity formalism.

First, we can express the differential cross section (5.50) in terms of helicity structure functions and the hyperbolic angle ζ, which contains the complete kinematic information.

Second, going from partonic to hadronic tensors in the process of factorization is a lot easier when working with helicity quantities due to the direct connection between both types of structure functions. This can be traced back to the equality of partonic and hadronic sets of polarizations. Consequently, there is no need for lengthy projection operators, as they are used e.g. in [Kretzer and Schienbein, 1998] to extract the conventional structure functions F_i from the hadronic tensor.

Finally, the helicity formalism would reveal its full potential when a preparation of polarized beams is possible in a potential future experiment. In this case, the helicity basis would be the natural (i.e. experimentally induced) choice for the structure functions.

However, the helicity formalism does not decouple completely from the conventional approach. With the help of the identities given in sections 5.4.1 and 5.4.2 it is always possible to switch back and forth between both concepts. At the end of appendix H, we explicitly verify these relations for a certain process at LO by a comparison between helicity structure functions calculated in the next section and ordinary structure functions given in section 2.1 of [Kretzer and Schienbein, 1998].

5.5. $l_1 + p_1 \to l_2 + p_2$ at LO

Let us apply all the acquired knowledge in this chapter on the very general example of $l_1 + p_1 \to l_2 + p_2$, where we follow the convention of denoting the initial partonic momentum with p_1^μ and the final one with p_2^μ. We only exclude the case of antiquarks from our considerations, but, as we will see below, an according generalization is trivial. All longer calculations can be found in appendix H.

As in chapter 3, we could reduce the problem of lepton-hadron scattering with the help of cross section (5.66) to determine proper expressions for the partonic structure functions. A key finding of chapter 3 was the point-like structure of partons. Hence, let us assume that $p_{1,2}$ are momenta of strongly interacting point-like fermions, i.e. quarks. The reader shall be reminded of the fact that we allow $p_1^2 = m_1^2 \neq m_2^2 = p_2^2$, so flavor-changing processes (where the exchanged gauge boson is a W-boson) are possible. To avoid confusion, we will denote the mass of the intial parton with momentum p_1^μ as m_1. The reader shall keep in mind that there is an implicit dependence on the parton flavor i.

Identifying partons with quarks[27] enables us to write the partonic currents in the definition of the partonic tensor (5.56) as

$$J_i^\mu = \overline{u}_{s_2}(p_2)\gamma^\mu\Big(g_{Ri}(1+\gamma^5) + g_{Li}(1-\gamma^5)\Big)u_{s_1}(p_1). \tag{5.67}$$

The subscript i specifies the couplings of the initial parton i to vector bosons, which differ in general from the leptonic ones $g_{R,L}$. Again, all their possible values in the electroweak theory can be found in table 2.2.

Being back on the well-known terrain of fermionic currents and, consequently, scattering between elementary fermions and gauge bosons (described by the Standard Model, cf. section 2.6), we can draw Feynman diagrams for the partonic subprocesses. We follow the convention of cutting out the upper part of a full diagram like $l_1 + p_1 \to l_2 + p_2$. Diagrams reduced in such a way only describe the partonic tensor of the according order in perturbation theory. Hence, when we talk about NLO corrections in the next chapter, we mean corrections to the reduced partonic subprocess. In the case of this section, that process is $B + p_1 \to p_2$, which is shown in figure 5.1.

In this diagram, we also make use the convention of using a cut, which was originally introduced in section 3.3.1, to depict squared amplitudes such as partonic and hadronic tensors. We will make more use of this technique in the following chapter.

[27]There is no LO process with a gluon in the initial or final state due to missing vertices between strong and electroweak vector bosons.

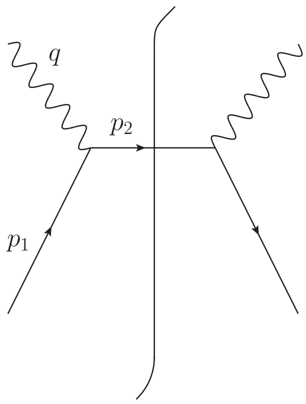


Figure 5.1.: Feynman diagram for the partonic subprocess $B+p_1 \to p_2$ at LO, which directly corresponds to the partonic tensor.

Using the general definition of $\hat{W}_i^{\mu\nu}$ (5.55) and inserting the explicit shape of the partonic current J_i^μ, we obtain

$$\hat{W}_i^{(0)\mu\nu} = 2\delta(p_2^2 - m_2^2)\Big\{g_{+i}^2\Big[-(p_1\cdot p_2)g^{\mu\nu} + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\Big] + g_{-i}^2 i\epsilon^{\mu\nu}_{\alpha\beta}p_1^\alpha p_2^\beta + 2g_{Ri}g_{Li}m_1 m_2 g^{\mu\nu}\Big\} \tag{5.68}$$

at LO. As usual, we use the generalized couplings $g_{\pm i}^2 = g_{Ri}^2 \pm g_{Li}^2$, now depending on the partonic flavor. The partonic tensor above leads to the following partonic helicity structure functions:

$$\hat{W}_\pm^{i(0)} = \delta\Big(\frac{\xi}{\chi} - 1\Big)\frac{1}{\Delta}\Big\{g_{+i}^2\Big(Q^2 + m_1^2 + m_2^2\Big) \pm g_{-i}^2\Delta - 2g_{Li}g_{Ri}m_1 m_2\Big\} \tag{5.69}$$

and

$$\hat{W}_0^{i(0)} = \delta\Big(\frac{\xi}{\chi} - 1\Big)\frac{1}{Q^2\Delta}\Big\{g_{+i}^2\Big(Q^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\Big) + 2g_{Li}g_{Ri}m_1 m_2 Q^2\Big\}, \tag{5.70}$$

where used the abbreviation[28]

$$\Delta \equiv \Delta(-Q^2, m_1^2, m_2^2). \tag{5.71}$$

Additionally, we made use of the identity (5.37) to simplify the on-shell condition. This δ-distribution is the counterpart of the $\delta(1-x)$ in the LO structure functions of

[28] The Δ-function itself is defined in (3.15).

chapter 3. Also necessary when deriving the expressions above are the components of $p_{1,2}$, cf. (5.22) and (5.25), their scalar product (5.27) and the explicit expression for χ, cf. (5.36). In the case of antiquarks, we simply reverse the fermion flow, which introduces an additional minus sign in front of the terms proportional to g_-^2.

Having the explicit $\hat{W}^i_{\pm,0}$ at hand, we can now substitute into (5.66) and obtain a complete differential cross section at leading order. The δ-distribution cancels with the generalized product $\otimes$, which contains an integration over ξ. Hence, similar to the naive parton model, at LO the PDFs $f_i(\xi)$ are evaluated only at one point, namely $\xi = \chi$:

$$W_\lambda^{(0)} = \sum_i f_i(\chi, Q^2)\hat{W}_\lambda^{i(0)}. \tag{5.72}$$

The sum runs over active flavors i, that contribute to the desired process. The end of appendix H also contains the conventional structure functions $\hat{W}_i$ and F_i. Furthermore, it is shown that in the limit of M and $m_{1,2}$ going to zero we reproduce the structure functions of the naive parton model, cf. (3.33) and (3.34).

At first sight, it may seem like there is no dependence on M. In fact, there is no explicit one. However, if one recalls the expressions of η and χ, (5.19) and (5.36), as well as the hyperbolic angle ζ, (5.30), it is evident that there are indeed implicit dependences on the target mass. Hence, the ACOT formalism takes all relevant masses, m_1, m_2 and M, into account and consequently covers all possible regions of phase space. In special cases, e.g. $Q^2 \sim m_1^2$, one can expand in according smallness parameters and consequently obtain approximate, but simpler results[29].

As expected, the only antisymmetric terms under $L \leftrightarrow R$ are the ones that are proportional to the Levi-Civita tensor in (5.68). Additionally, we see that the contributions surviving when sending m_1 and m_2 to zero are the purely left- or right-handed ones of $\hat{W}^i_\pm$, which is completely similar to the case of a massless lepton.

Note that it is of course also possible to contract (5.68) directly with $L^{\mu\nu}$, which is the "classical" way of calculating the squared and averaged matrix element. The result is given in [Aivazis et al., 1994a], equation (C7). As expected, the results are similar to the ones obtained above. However, they are much more lengthy and do not unfold the physical behavior as nicely as the one obtained in the helicity approach.

[29] Sections VI and VII of [Aivazis et al., 1994a] include a more detailed discussion on different historical approaches to approximate mass dependencies of the differential cross section.

6. The ACOT formalism: NLO effects

In this chapter, our objective is to reproduce the main results of [Aivazis et al., 1994b], namely the (non-) subtracted partonic tensor for DIS in the case of gluon initiated processes.

According to section 3.3.3, we can calculate the subtracted hadronic tensor at NLO via

$$\begin{aligned} W^{(1)\mu\nu} =& \sum_i \left(f_i^{(1)} \otimes \hat{W}_i^{(0)\mu\nu} + f_i^{(0)} \otimes \hat{W}_i^{(1)\mu\nu} \right) \\ =& \sum_i \left(f_i \otimes \hat{W}_i^{(0)\mu\nu} + f_i \otimes \hat{\Omega}_i^{(1)\mu\nu} - f_i \otimes \sum_j f_i^{j(1)} \otimes \hat{W}_j^{(0)\mu\nu} \right). \end{aligned} \tag{6.1}$$

Here, we sum over all active flavors i and j, including heavy[1] (anti-)quarks. Contracting with the polarizations of the exchanged gauge boson together with the discussion of section 5.4.2 directly reveals that the exact same formula also connects the hadronic and non-subtracted partonic helicity structure functions.

At the end of section II.E in [Aivazis et al., 1994b], it is argued that quark initiated processes are numerically not comparable to gluon initiated ones at the same order due to much larger values of the gluon PDF. In consequence, [Aivazis et al., 1994b] only deals with gluon initiated NLO partonic tensors, which will also be covered in this section. For the special case of heavy quark production, where we denote the flavor of the heavy quark by q, (6.1) reduces to

$$W^{(1)\mu\nu} = f_q \otimes \hat{W}_q^{(0)\mu\nu} + f_g \otimes \hat{\Omega}_g^{(1)\mu\nu} - f_g \otimes \sum_j f_g^{q(1)} \otimes \hat{W}_q^{(0)\mu\nu} \tag{6.2}$$

under the assumptions explained above. $\hat{W}_q^{(0)\mu\nu}$ is the LO partonic tensor which was already calculated in section 5.5. The sum over j runs over all (anti-) quark flavors for which the heavy quark production is possible (i.e. for which the CKM-matrix entries are non-zero). In the case of a neutral current, the only remaining contribution is the one proportional to the PDF of the heavy quark itself and the corresponding antiquark.

Some years later it was shown by [Kretzer and Schienbein, 1998] that only taking into account the gluon initiated interactions cannot be hold up on a general level. In a full NLO treatment, the quark initiated contributions in equation (6.1), whose ignoring led to equation (6.2), need to be calculated as well. Results presented in the

[1] As explained in section 5.1, we use the term "heavy" in an absolute sense. This means that the existence of heavy quark PDFs is allowed when Q exceeds the respective quark mass.

conventional approach, i.e. for the structure functions F_i, can be found in [Kretzer and Schienbein, 1998]. Using the formulae (5.44) and (5.45), they can be easily converted to helicity structure functions F_λ. The same is true for partonic helicity structure functions, where the according indentities are given in (5.58) and (5.59).

As it was already discussed earlier, the ACOT papers, [Aivazis et al., 1994a] and [Aivazis et al., 1994b], focused on the special case of heavy quark production. However, to obtain as general results as possible, all quark masses in the partonic NLO processes (for example depicted in figure 6.1) are kept non-zero and arbitrary. After obtaining the non-subtracted partonic tensor under these prerequisites, we then deduce the subtracted partonic tensor for the special case of heavy quark production. But before turning to the explicit calculations, let us first set the ground by some general consideration on the renormalization scheme of choice.

6.1. The CWZ renormalization scheme

The Collins-Wilczek-Zee (CWZ) renormalization scheme has its origins in the 1970's, cf. [Collins et al., 1978]. Conceptual introductions can be found e.g. in section 2 of [Collins and Tung, 1986], section III. of [Collins, 1998] or section 3.9 et seq. in [Collins, 2011]. A general introduction to renormalization was already given in section 2.5.

In general, non-leading order calculations introduce logarithms with arguments containing fractions of all energy scales of the process[2]. In a general case involving hadrons, these scales are the virtuality Q, all masses of heavy quarks participating in the process and the renormalization scale μ (assuming $\overline{\text{MS}}$ renormalization[3]) as well as the factorization scale μ_f. The basic problem, which was already described in section 5.1, is that Q covers a wide range of values, including regions where certain quarks are negligible or not. Transferred to the logarithms from above, this leads to large non-leading terms spoiling the power series in α_s. The task of a renormalization scheme which deals with processes involving heavy quarks is to take care of this problem, preferably in an elegant way. Below, we present the CWZ renormalization scheme and its advantages as one of the suitable candidates for such a scheme. After that, we describe its role in the ACOT formalism.

Let us start with the harmless regions of Q: As already stated in section 5.1, if we have $Q \ll m$, where m is a heavy quark mass, the decoupling theorem (cf. [Appelquist and Carazzone, 1975]) applies and the heavy quark does not participate in the process. Hence, m does not appear in any logarithms. If $Q \gg m$, we can set m to zero and no logarithms containing m occur. Consequently, we need to regulate mass divergencies inside the PDFs and the partonic tensor via subtractions, cf. section 3.3.3.

What is left are the intermediate regions. In general, we are free to choose μ so that large logarithms are avoided. For example, in the two situations above we set μ to be of order Q. If $Q \sim m$, the natural choice for μ is again of order Q. Problems arise when there is more than only one quark. Consider the situation of two widely spread quark masses m_1 and $m_2 \gg m_1$, as it is the case in QCD e.g. for the bottom and top quark (cf. table 2.1). When Q is much larger or smaller than both m_1 and m_2, the considerations above apply again. But if $Q \sim m_1$, it is not possible to find a suitable choice for μ. Some large logarithms, either of the form $\ln\left(m_2^2/\mu^2\right)$ (for $\mu \sim Q$) or $\ln\left(Q^2/\mu^2\right)$ (for $\mu \sim m_2$), will remain.

There is a variety of approaches tackling this problem. Among these, the CWZ renormalization schemes is one of most famous ones. However, most of them share the same basic idea: Instead of dealing with one large and complex renormalization scheme, a number of subschemes is introduced to take care of different energy regions. This makes it possible to to have one choice for the renormalization scale, i.e. μ can

[2] For this chapter, there is no need to assume a particular process like DIS.

[3] As it is done throughout the literature, the term "$\overline{\text{MS}}$ renormalization" is used to describe dimensional regularization in conjunction with $\overline{\text{MS}}$ subtraction.

be of order Q everywhere. In the CWZ scheme, every subscheme is characterized by a different number of active and passive quark flavors. The active flavors are made up by all quarks that can be treated as light (or, as in our case, even as massless) at the respective energy scale, while the passive flavors are the remaining heavy quarks. Graphs containing only active flavors are renormalized via $\overline{\text{MS}}$. For the remaining ones, zero-momentum subtractions are used instead. This has the advantage of no further mass divergencies in any graphs as well as gauge invariance in all subschemes. Additionally, the RGEs of a CWZ subscheme coincide completely with RGEs of a system made of only active quarks renormalized with $\overline{\text{MS}}$ due mass independent RGE parameters and the decoupling theorem.

Pairs of subschemes (e.g. going from three to four active flavors) are connected by threshold conditions at their coinciding borders. These equations connect the (in general differing) coefficients of both subschemes, like α_s or RGE parameters. As they are not relevant for this work, we only refer to the works given at the beginning of this section. There is some freedom of choice for the thresholds itself. A possible configuration is to set all thresholds equal to the respective heavy quark masses.

When this formalism is extended to factorization and PDFs, one sometimes speaks of the "ACOT scheme". The direct implications of the use of subschemes as they are introduced above are the following: First, we set the factorization scale to $\mu_f \sim Q$. Then, for a certain value of Q, we determine the active flavors. A PDF is assigned to each of them, which evolves according to the usual DGLAP equation (cf. section 3.3.2) due to $\overline{\text{MS}}$ renormalization[4]. In consequence, there are threshold conditions for PDFs as well. Opposed to active flavors, inactive ones only appear through flavor changing processes inside the hard scattering. Although these consideration will not enter explicitly inside the following calculations, they form the very basis of this whole chapter.

[4]Questions may arise concerning the number of active flavors taken into account for an evolution at energies around a threshold. Just like the exact locations of a threshold, the choice for that is to some extend arbitrary and an open point of discussion. The authors of the ACOT scheme argue in section VI of [Aivazis et al., 1994b] for an evolution in both competing schemes and a comparison afterwards to identify the threshold region. According to them, both schemes are equally valid in this interval.

6.2. The unsubtracted partonic tensor

Figure 6.1 shows all contributions to the partonic tensor. Both charged and neutral currents are allowed.

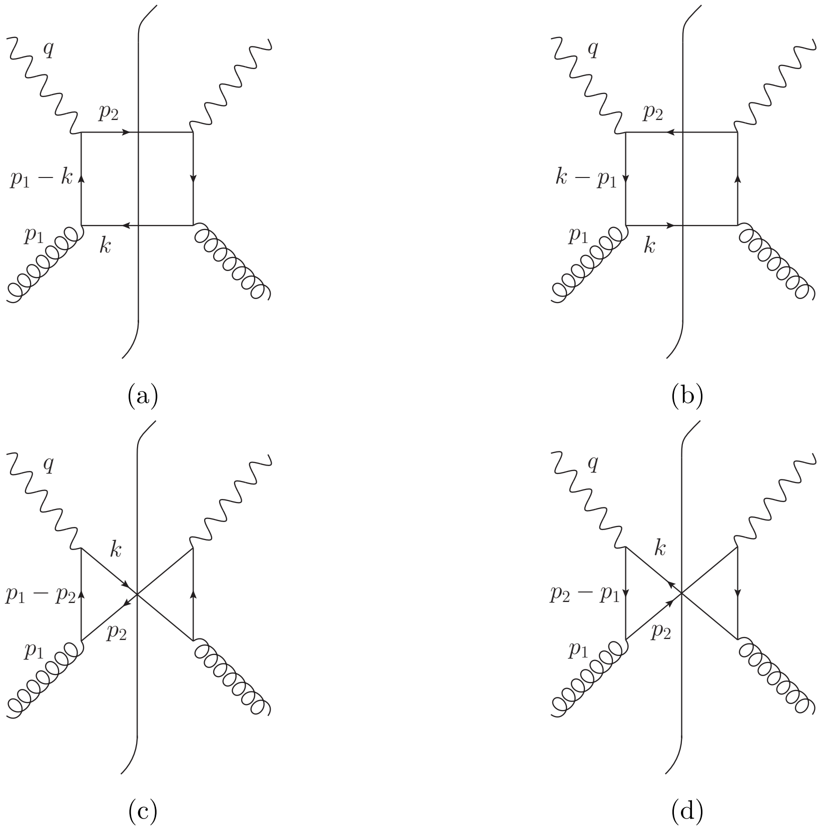


Figure 6.1.: All gluon initiated NLO diagrams. Diagrams (a) and (b) correspond to squared amplitudes, while (c) and (d) are interferences.

As suggested in section 3.3.3, we start by calculating $\hat{\Omega}_g^{(1)\mu\nu}$, or rather the non-subtracted helicity structure functions $\hat{\Omega}_\lambda^{g(1)}$. For reasons of clarity, we split up the helicity structure functions into the parts which depend on different combinations of chiral couplings:

$$\hat{\Omega}_\lambda^{g(1)} \equiv \frac{\alpha_s}{2\pi} \sum_\kappa C_\kappa \hat{\Omega}_{\lambda\kappa}^{g(1)}, \tag{6.3}$$

where the possible combinations (similar to the LO contribution, cf. section 5.5) are

$$C_{s,a} \equiv g_\pm^2 \tag{6.4}$$

$$\text{and } C_x \equiv 2 g_R g_L. \tag{6.5}$$

These couplings shall not be confused with the chiral couplings introduced in chapter 5. They do neither depend in the initial parton, a gluon in this case, nor the lepton in the upper half of the diagram 3.3, but, however, on the quarks in the final state. Consequently, they take the same role as the couplings $g_{Ri,Li}$ in the LO partonic tensor $\hat{W}_{\lambda}^{i(0)}$, calculated in section 5.5.

As explained in the section above, we choose a completely general approach by setting all quark masses non-zero. Hence, due to all final states having non-vanishing masses, no divergences occur in the phase-space integration. This means that there is no need for the use of a regulator[5] and the integration can be done in four space-time dimensions. We choose the partonic center-of-mass (CM) frame to evaluate the integrals. Appendix I contains the complete calculation, starting from the computation of Feynman diagrams in figure 6.1. The final results, in terms of Lorentz invariant variables, are

$$\begin{aligned}\hat{\Omega}^{g(1)}_{\pm s} =& \left[\left(\frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2}\right)^2 - \frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_t \\ &+\left[\left(\frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2}\right)^2 - \frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_u \\ &-\frac{\Delta_g}{\hat{s}}\left(\frac{\hat{s}-Q^2}{\hat{s}+Q^2}\right)^2 \end{aligned} \tag{6.6}$$

$$\begin{aligned}\hat{\Omega}^{g(1)}_{\pm a} =& \pm\left\{\left[\left(\frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2}\right)^2 + 2\frac{m_k^2(m_2^2-m_k^2)}{\left(\hat{s}+Q^2\right)^2} - \frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_t\right. \\ &-\left[\left(\frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2}\right)^2 + 2\frac{m_2^2(m_k^2-m_2^2)}{\left(\hat{s}+Q^2\right)^2} - \frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_u \\ &\left.+\frac{2\Delta_g(m_2^2-m_k^2)}{\left(\hat{s}+Q^2\right)^2}\right\} \end{aligned} \tag{6.7}$$

$$\hat{\Omega}^{g(1)}_{\pm x} = \frac{2m_k m_2}{\left(\hat{s}+Q^2\right)^2}\left\{\left(\hat{s}-m_2^2-m_k^2\right)\left(L_t+L_u\right) - 2\Delta_g\right\} \tag{6.8}$$

and

[5] Cf. sections 2.5.1 and 2.5.2 for further explanations on divergences and regulators.

$$\hat{\Omega}_{0s}^{g(1)} = \frac{1}{Q^2\left(\hat{s}+Q^2\right)^2}\Bigg\{\Bigg[\frac{1}{2}\left(m_2^2+m_k^2\right)\left(\hat{s}^2+4Q^2\hat{s}-3Q^4\right)$$
$$+2m_2^2m_k^2\left(\hat{s}-3Q^2\right)+\left(m_2^4+m_k^4\right)\left(Q^2-\hat{s}\right)$$
$$+m_2^6-m_2^4m_k^2-m_2^2m_k^4+m_k^6\Bigg]\left(L_t+L_u\right)$$
$$+2Q^2\left(\hat{s}-Q^2\right)\left(m_k^2-m_2^2\right)\left(L_t-L_u\right)$$
$$+2\Delta_g\left[(m_2^2-m_k^2)^2-Q^2(m_2^2+m_k^2)+2Q^4\right]\Bigg\} \tag{6.9}$$

$$\hat{\Omega}_{0a}^{g(1)} = 0 \tag{6.10}$$

$$\hat{\Omega}_{0x}^{g(1)} = \frac{m_k m_2}{Q^2\left(\hat{s}+Q^2\right)^2}\Bigg\{\Bigg[-\hat{s}^2+2Q^2\left(m_2^2+m_k^2-2\hat{s}\right)-Q^4\Bigg]\left(L_t+L_u\right)+4Q^2\Delta_g\Bigg\}, \tag{6.11}$$

where we used[6]

$$\Delta_g \equiv \Delta(\hat{s}, m_2^2, m_k^2). \tag{6.12}$$

Two different logarithms,

$$L_t = \ln\frac{\left(\hat{s}-m_2^2+m_k^2+\Delta_g\right)^2}{4\hat{s}m_k^2} \tag{6.13}$$

$$\text{and } L_u = \ln\frac{\left(\hat{s}+m_2^2-m_k^2+\Delta_g\right)^2}{4\hat{s}m_2^2}, \tag{6.14}$$

arise from the t- and u-channel diagrams[7] in figure 6.1. They also contain the full divergences when setting one or both masses to zero. In this sense, m_2 and m_k can be thought of regulators when dealing with processes where one or both outgoing quarks are actually taken to be massless. Consequently, we can describe the divergent behavior of the partonic tensor and, even more important, obtain the subtracted partonic tensor $W_\lambda^{g(1)}$ by studying $L_{t,u}$ under the desired conditions. This is exactly the objective of the next section.

[6] The Δ-function itself is defined in (3.15).

[7] Physically, they correspond to virtual quark (t-channel) or antiquark (u-channel) scattering off the electroweak boson, as it can be seen in figure 6.1. This will become important in the context of subtractions, especially in section 6.3.2.

6.3. The subtraction process at the example of heavy quark production

In principle, with equation (3.51) we already have a procedure at hand[8] to determine the subtracted partonic structure functions $\hat{W}_\lambda^{g(1)}$. However, let us perform a consistency check for the case of heavy quark production to see whether the unsubtracted structure functions of the last section actually lead to the same subtraction terms as equation (3.51). Looking at the NLO structure functions above, we see that the mass divergencies are condensed into the logarithms $L_{t,u}$.

6.3.1. Charged currents

For flavor changing processes, mass divergencies and therefore subtraction terms occur when the heavy quark is only heavy on an absolute scale and/or the second quark is light. Let us first consider the second case, as the first one will only be a trivial generalization.

For $m_k \to 0$, the general expressions for $L_{t,u}$, (6.13) and (6.14), simplify to

$$L_t = \ln \frac{\left(\hat{s} - m_2^2 + \hat{s} - m_2^2\right)^2}{4\hat{s}m_k^2} = \ln \frac{\left(\hat{s} - m_2^2\right)^2}{\hat{s}m_k^2} \tag{6.15}$$

$$\text{and } L_u = \ln \frac{\left(\hat{s} + m_2^2 + \hat{s} - m_2^2\right)^2}{4\hat{s}m_2^2} = \ln \frac{\hat{s}}{m_2^2}, \tag{6.16}$$

when we only retain a factor of m_k^2 in L_t, where it would lead to a divergent behavior otherwise. Here, we can use it as a regulator for the collinear divergence according to section 2.5.1. Next, we introduce the factorization scale μ_f via

$$L_t = \ln \frac{\left(\hat{s} - m_2^2\right)^2}{m_k^2} = \ln \frac{\mu_f^2}{m_k^2} + \ln \frac{\left(\hat{s} - m_2^2\right)^2}{\mu_f^2}. \tag{6.17}$$

One can interpret this equation as an explicit manifestation of the separation of long- and short-scale physics with the factorization scale as their threshold. As it was already discussed in section 3.3.3, the unsubtracted partonic tensor contains both the long- and short-scale behavior of the process. To obtain the subtracted tensor, we need to subtract the long-scale part (the mass divergence, to put it in other words), which is contained in the first logarithm. In the following, we give all divergent parts of the structure functions (6.6) to (6.11)[9]:

$$\hat{\Omega}_{\pm s}^{g(1)\text{div}} = P_{g\to q}^{(1)}(z_m) \ln \frac{\mu_f^2}{m_k^2} \tag{6.18}$$

[8] Although we dealt with massless partons in the whole chapter 3, the CWZ scheme ensures that active quarks (with negligible masses) are treated exactly as in the massless theory.

[9] For the first two terms, it is helpful to exploit the symmetry $P_{g\to q}^{(1)}(1 - z_m) = P_{g\to q}^{(1)}(z_m)$. The rest of the calculation is straightforward.

$$\hat{\Omega}^{g(1)\text{div}}_{\pm a} = \pm P^{(1)}_{g\to q}(z_m)\ln\frac{\mu_f^2}{m_k^2} \tag{6.19}$$

$$\hat{\Omega}^{g(1)\text{div}}_{0s} = \frac{m_2^2}{Q^2}P^{(1)}_{g\to q}(z_m)\ln\frac{\mu_f^2}{m_k^2} \tag{6.20}$$

$$\hat{\Omega}^{g(1)\text{div}}_{0a} = \hat{\Omega}^{g(1)\text{div}}_{0x} = \hat{\Omega}^{g(1)\text{div}}_{\pm x} = 0, \tag{6.21}$$

including the leading order gluon to quark splitting function[10]

$$P^{(1)}_{g\to q}(z_m) = z_m^2 - z_m + \frac{1}{2} \tag{6.22}$$

and the mass corrected[11] partonic version of the Bjorken-x,

$$z_m \equiv z\left(1+\frac{m_2^2}{Q^2}\right) = \frac{m_2^2+Q^2}{\hat{s}+Q^2}. \tag{6.23}$$

z itself is defined in the context of the naive parton model, equation (3.28). Calculating it e.g. in the partonic CMS frame (cf. appendix I.2.1), we obtain

$$z = \frac{Q^2}{2p_1\cdot q} = \frac{Q^2}{\hat{s}+Q^2}. \tag{6.24}$$

Following equation (6.3), the full mass divergent helicity structure functions can be written as

$$\hat{\Omega}^{g(1)\text{div}}_{\pm} = \frac{\alpha_s}{2\pi}\left(g_+^2 \pm g_-^2\right)P^{(1)}_{g\to q}(z_m)\ln\frac{\mu_f^2}{m_k^2} \tag{6.25}$$

$$\text{and } \hat{\Omega}^{g(1)\text{div}}_{0} = \frac{\alpha_s}{2\pi}g_+^2\frac{m_2^2}{Q^2}P^{(1)}_{g\to q}(z_m)\ln\frac{\mu_f^2}{m_k^2}. \tag{6.26}$$

We are now in the position to explicitly validate the general formula (6.2), which reduces to

$$\hat{\Omega}^{g(1)\text{sub}}_{\lambda} = f_q^{g(1)}\otimes\hat{W}^{q(0)}_{\lambda} \tag{6.27}$$

in the case of helicity structure functions and heavy quark production with one heavy quark with mass m_2[12]. To do so, we first copy the NLO PDF (3.50):

$$f_q^{g(1)} = \frac{\alpha_s}{2\pi}P^{(1)}_{g\to q}(z_m)\ln\frac{\mu_f^2}{m_k^2}. \tag{6.28}$$

[10] For an introduction on splitting functions, see section 3.3.2.

[11] A mass correction can be applied identically to the Bjorken-x at hadron level. There, we arrived at a similar formula for χ, equation (5.38), in the case of only the outgoing quark being massive.

[12] We need the subtraction term containing the quark (and not the antiquark), since the the t-channel diverges. Diagrammatically, this can be nicely seen e.g. in figure 6.1.

Next, we need the LO quark initiated partonic tensors (5.69) and (5.70) in the case of an initial quark flavor $i = q$ with mass $m_1 = m_k \to 0$ and couplings $g_{Ri,Li} \equiv g_{R,L}$:

$$\hat{W}_{\pm}^{q(0)}\Big|_{m_k=0} = \delta\left(\frac{\xi}{\chi} - 1\right)\left(g_+^2 \pm g_-^2\right) \tag{6.29}$$

$$\hat{W}_{0}^{q(0)}\Big|_{m_k=0} = \delta\left(\frac{\xi}{\chi} - 1\right) g_+^2 \frac{m_2^2}{Q^2}. \tag{6.30}$$

A convolution with (6.28) directly leads to the fact that

$$\hat{\Omega}_{\lambda}^{g(1)\mathrm{sub}} = \hat{\Omega}_{\lambda}^{g(1)\mathrm{div}} \tag{6.31}$$

is indeed fulfilled in the limit $m_k \to 0$. In a massless theory, everything would be said at this point. For us, the last remaining question is: How can the divergent term on the right-hand-side above be extended into arbitrary energy regions to give a generalized subtraction term and therefore the subtracted tensor?

As it is explained at the end of section 6.1, we set $\mu_f \sim Q$. Due to the choice of the CWZ renormalization scheme and $\mu \sim Q$, m_k vanishes in our approach if $Q \gg m_k$. All this put together implies that a subtraction is only necessary in this energy region. To achieve this, an additional Heavyside function $\theta(\mu_f - m_k)$ is multiplied to the divergent term. Since $\hat{W}_{\pm}^{q(0)}$ has a finite mass limit[13], there is no danger in keeping the $m_k = 0$ limit in the equation above. However, if Q is only marginally greater than m_k, this leads to huge and avoidable deviations. Therefore, we allow m_k to be non-zero in the LO partonic tensor and obtain

$$\hat{\Omega}_{\lambda}^{g(1)\mathrm{sub}} = \frac{\alpha_s}{2\pi} \hat{W}_{\lambda}^{q(0)}(Q^2, m_k, m_2) P_{g\to q}^{(1)}(z_m) \ln \frac{\mu_f^2}{m_k^2} \theta(\mu_f - m_k), \tag{6.32}$$

where we explicitly denoted the dependencies of $\hat{W}_{\lambda}^{q(0)}$ for the sake of clarity. It should be also noted that the same choice regarding the value of m_k needs to be made in the first term of the perturbative expansion of the hadronic tensor, cf. equation (6.2). Accordingly, the subtracted NLO partonic helicity structure functions are

$$\hat{W}_{\lambda}^{g(1)} = \hat{\Omega}_{\lambda}^{g(1)} - \hat{\Omega}_{\lambda}^{g(1)\mathrm{sub}}, \tag{6.33}$$

where the first terms are given in section 6.2 and the second terms can be found right above.

When further increasing μ_f, we might exceed the $\mu_f = m_2$ threshold. In that case, similar subtractions have to made with $m_2 \to 0$. When calculating the subtraction terms via equation (6.2), the antiquark LO structure functions need to be used instead of their quark counterparts due to reversed fermion flow in the u-channel compared to the t-channel.

[13] We showed this in section 3.3.3 on general terms and explicitly saw it in the calculation above.

6.3.2. Neutral currents

In the case of a neutral exchanged current, no flavor changing is possible. Therefore, we set $m_2 = m_k \equiv m$. This greatly simplifies the general unsubtracted structure functions (6.6) to (6.11):

$$\hat{\Omega}^{g(1)}_{\pm s} = \frac{1}{\left(\hat{s}+Q^2\right)^2}\left\{\left(\hat{s}^2+Q^4\right)L - \frac{\Delta_g}{\hat{s}}\left(\hat{s}-Q^2\right)^2\right\} \tag{6.34}$$

$$\hat{\Omega}^{g(1)}_{\pm x} = \frac{4m^2}{\left(\hat{s}+Q^2\right)^2}\left\{\left(\hat{s}-2m^2\right)L - \Delta_g\right\} \tag{6.35}$$

$$\hat{\Omega}^{g(1)}_{0s} = \frac{1}{Q^2\left(\hat{s}+Q^2\right)^2}\left\{2m^2\left[\hat{s}^2+4Q^2\left(\hat{s}-m^2\right)-3Q^4\right]L + 4Q^2\Delta_g\left(Q^2-m^2\right)\right\} \tag{6.36}$$

$$\hat{\Omega}^{g(1)}_{0x} = \frac{2m^2}{Q^2\left(\hat{s}+Q^2\right)^2}\left\{\left[-\hat{s}^2+4Q^2\left(m^2-\hat{s}\right)-Q^4\right]L + 2Q^2\Delta_g\right\} \tag{6.37}$$

$$\hat{\Omega}^{g(1)}_{\lambda a} = 0, \tag{6.38}$$

where the Δ-function reduces to

$$\Delta_g = \Delta(\hat{s}, m^2, m^2) = \sqrt{\hat{s}\left(\hat{s}-4m^2\right)}. \tag{6.39}$$

Additionally, the two logarithms L_t and L_u are equal:

$$L_t = L_u \equiv L = \ln\frac{\hat{s}+\sqrt{\hat{s}\left(\hat{s}-4m^2\right)}-2m^2}{2m^2} = 2\ln\frac{\sqrt{\hat{s}}+\sqrt{\hat{s}-4m^2}}{2m}. \tag{6.40}$$

The divergencies and subtraction terms are calculated in exactly the same way as in section 6.3.1. In the limit $m \to 0$, the logarithm above reduces to

$$L = \ln\frac{\hat{s}}{m^2} = \ln\frac{\mu_f^2}{m^2} + \ln\frac{\hat{s}}{\mu_f^2}, \tag{6.41}$$

again containing the whole mass divergence. In the equation above, we have already introduced the factorization scale μ_f in the same way as before. The divergent parts of the structure functions are

$$\hat{\Omega}^{g(1)\text{div}}_{\pm s} = 2P^{(1)}_{g\to q}(z)\ln\frac{\mu_f^2}{m^2} \tag{6.42}$$

$$\text{and } \hat{\Omega}^{g(1)\text{div}}_{0s} = \hat{\Omega}^{g(1)\text{div}}_{\lambda a} = \hat{\Omega}^{g(1)\text{div}}_{\lambda x} = 0, \tag{6.43}$$

where $P^{(1)}_{g\to q}(z)$ is defined in equation (6.22) and z in (6.24). Substituting the latter expression into the first one, we obtain

$$P^{(1)}_{g\to q}(z) = \frac{\hat{s}^2 + Q^4}{2\left(\hat{s} + Q^2\right)^2}, \tag{6.44}$$

which justifies the replacement done in equation (6.42). Consequently, the full diverging parts of the structure functions are

$$\hat{\Omega}^{g(1)\mathrm{div}}_{\pm} = \frac{\alpha_s}{\pi} g_+^2 P^{(1)}_{g\to q}(z) \ln\frac{\mu_f^2}{m^2} \tag{6.45}$$

$$\text{and } \hat{\Omega}^{g(1)\mathrm{div}}_{0} = 0. \tag{6.46}$$

Let us again cross check this result with the general formula (6.2). Although numerically equal, we deal with two divergences, namely in the t- and u-channel, which need to be subtracted in an independent way. Looking at figure 6.1, we can deduce that the subtractions are proportional to the quark (t-channel) and antiquark (u-channel) LO partonic tensor (cf. section 5.5) with equal and vanishing masses. Therefore, the explicit expressions (5.69) and (5.70) reduce to

$$\hat{W}^{q(0)}_{\pm}\Big|_{m=0} = \delta\left(\frac{\xi}{\chi} - 1\right)\left(g_+^2 \pm g_-^2\right), \tag{6.47}$$

$$\hat{W}^{\overline{q}(0)}_{\pm}\Big|_{m=0} = \delta\left(\frac{\xi}{\chi} - 1\right)\left(g_+^2 \mp g_-^2\right) \tag{6.48}$$

$$\text{and } \hat{W}^{q(0)}_{0}\Big|_{m=0} = \hat{W}^{\overline{q}(0)}_{0}\Big|_{m=0} = 0, \tag{6.49}$$

where $\overline{q}$ indicates the contributions belonging to the antiquark. The PDFs are again[14]

$$f^{g(1)}_{q} = f^{g(1)}_{\overline{q}} = \frac{\alpha_s}{2\pi} P^{(1)}_{g\to q}(z) \ln\frac{\mu_f^2}{m^2}. \tag{6.50}$$

Consequently, the subtraction terms are

$$\hat{\Omega}^{g(1)\mathrm{sub}}_{\pm} = f^{q(1)}_{g} \otimes \left(\hat{W}^{q(0)}_{\pm} + \hat{W}^{\overline{q}(0)}_{\pm}\right) = \frac{\alpha_s}{\pi} g_+^2 P^{(1)}_{g\to q}(z) \ln\frac{\mu_f^2}{m^2} \tag{6.51}$$

$$\text{and } \hat{\Omega}^{g(1)\mathrm{sub}}_{0} = f^{q(1)}_{g} \otimes \left(\hat{W}^{q(0)}_{0} + \hat{W}^{\overline{q}(0)}_{0}\right) = 0 \tag{6.52}$$

in the $\overline{\mathrm{MS}}$ limit, which agrees to the divergent parts above. As in the charged current case, we can generalize this result to

$$\hat{\Omega}^{g(1)\mathrm{sub}}_{\lambda} = \frac{\alpha_s}{2\pi}\left(\hat{W}^{q(0)}_{\lambda}(Q^2, m) + \hat{W}^{\overline{q}(0)}_{\lambda}(Q^2, m)\right) P^{(1)}_{g\to q}(z) \ln\frac{\mu_f^2}{m^2}\theta(\mu_f - m), \tag{6.53}$$

[14] Cf. equation (6.28), with the only change that z is not mass corrected, i.e. z_m is replaced by z.

which is valid on the whole energy scale. In addition, it is worth noting that the subtracted structure functions reduce to[15]

$$\hat{W}_{\pm}^{g(1)} = \frac{\alpha_s}{2\pi} g_+^2 \left(2P_{g\to q}^{(1)}(z) \ln \frac{\hat{s}}{\mu_f^2} - (1-2z)^2 \right) \tag{6.54}$$

$$\text{and } \hat{W}_0^{g(1)} = 2\frac{\alpha_s}{\pi} g_+^2 z(1-z) \tag{6.55}$$

in the limit of $m \to 0$. Using the conversion formulae (5.60) to (5.62), we obtain

$$\hat{W}_1^i = \frac{\alpha_s}{2\pi} g_+^2 \left(2P_{g\to q}^{(1)}(z) \ln \frac{\hat{s}}{\mu_f^2} - (1-2z)^2 \right) \tag{6.56}$$

$$\hat{W}_2^i = \frac{\alpha_s}{2\pi} g_+^2 (2z)^2 \left(2P_{g\to q}^{(1)}(z) \ln \frac{\hat{s}}{\mu_f^2} + 8z(1-z) - 1 \right) \tag{6.57}$$

$$\hat{W}_3^i = 0 \tag{6.58}$$

for the ordinary structure functions. Here we are in the situation of the naive, massless parton model (renormalized via $\overline{\text{MS}}$). The CWZ renormalization scheme induces that both results agree. However, a comparison with results obtained in the ordinary formalism is not straightforwardly possible due to the use of different conventions and renormalization schemes in the literature, cf. e.g. [Altarelli et al., 1979].

6.3.3. Are subtracted structure functions finite?

The whole procedure of applying subtractions to partonic and hadronic tensors was established to cancel mass divergences, which alters the tensor in such a way that it, as desired, only represents the hard region of the respective process. However, this does not strictly imply that there are no divergences left. A closer look at the remaining logarithms in the subtracted structure functions is necessary to obtain further insights.

As explained in section 6.1, we typically set $\mu_f \sim Q$. It is therefore instructive to isolate terms proportional to $\ln Q^2/\mu_f^2$ in the logarithms inside the subtracted terms, namely

$$\ln \frac{\hat{s}+Q^2}{\mu_f^2} \quad \text{and} \quad \ln \frac{\hat{s}}{\mu_f^2}$$

for charged and neutral currents, respectively. In order to do so, the identity

$$\hat{s} = (1-z)\left(\hat{s}+Q^2\right) = \frac{1-z}{z} Q^2 \tag{6.59}$$

[15] Here, we again use the explicit expression for z, equation (6.24).

is helpful. We obtain

$$\ln \frac{\hat{s} + Q^2}{\mu_f^2} = \ln \frac{Q^2}{\mu_f^2} - \ln z \tag{6.60}$$

$$\text{and} \;\; \ln \frac{\hat{s}}{\mu_f^2} = \ln \frac{Q^2}{\mu_f^2} + \ln \frac{1-z}{z}. \tag{6.61}$$

It has become evident that there are indeed residual divergences in the case $z \to 1$ and $z \to 0$. They correspond to additional collinear and soft divergences in the massless case.

7. Conclusion

We close this thesis with a short recapitulation of what has been done and an outlook of what would be the next steps along the road.

The ACOT formalism introduces two very instructive techniques: The helicity formalism and, its actual purpose, the treatment of heavy masses. The first one offers the ability to use an often easier way of calculation. The results, namely the helicity structure functions, are at the same time more closely linked to the actual physical process. Moreover, helicity tensors and structure functions represent the natural choice in experiments that include polarized lepton beams. In this work, we gave a derivation at LO and NLO using this formalism, showing all its foundations and characteristics. Additionally, we outlined the relations and conversions between the helicity and the classical approach at several points, so that this work does not loose its connection to the majority of publications in the field.

Furthermore, we described a very general treatment of parton masses for partonic tensors and structure functions. This, together with the additional consideration of target mass corrections as well as massive gauge bosons sets the ground for a complete analysis of masses coming from all possible origins inside hadronic processes, e.g. DIS. Nevertheless, due to the work that was done in the introductory chapters, the conventions used in this work are also compatible with the rather simple case of massless DIS involving QED interactions. We gave a comprehensive calculation of gluon initiated processes at NLO, on which basis some equations from the original ACOT publication could have been adjusted (cf. appendix J.2) and most of the derivations were carried out in more detail. However, what remains to be done is a NLO calculation of quark initiated processes in the helicity formalism for the sake of completeness. The respective quantities were already computed in [Kretzer and Schienbein, 1998] using the conventional way. In fact, via the conversion formulae for hadronic and partonic quantities (where the latter ones were derived exclusively in this work) it is already possible at this point to obtain the according helicity structure function with a quick and simple calculation.

After establishing all these theoretical concepts, numerical studies should be the next point on the list. At the end of [Aivazis et al., 1994a], it is shown that the ignorance of masses leads to a possible deviation of over 25% at an LO cross section for heavy quark production compared to the massive case. [Aivazis et al., 1994b] already includes some graphs of (ordinary) structure functions also in the last sections, showing their behavior on the Q scale in the case of heavy quark production. More extensive studies, that cover not only heavy quark production, together with comparisons to actual experimental data would be in order to validate the ACOT scheme and work out its effects on helicity structure functions in detail.

A. Conventions and Identities

A.1. General Conventions

We set, as usual in theoretical particle physics, $\hbar = c = 1$ throughout the whole work. This implies that all relevant physical quantities are given in units of energy, e.g. $[M] = \mathrm{GeV}$.

Three-vectors are denoted by a bold symbol, e.g. $\boldsymbol{p}$. In contrast, four-vectors are always printed in normal font and, if one wants to refer to their components, indexed with a greek letter, e.g. p^μ, where $\mu = 0, ..., 3$. An exception is made only for the letters λ and κ, which stand for helicities $+$, $-$ and 0. In contrast, components of a three-vector or spacial components of four-vector are indexed with an arabic letter, e.g. p^k, where $k = 1, 2, 3$.

Contractions of tensors in the Lorentz basis (and only in the Lorentz basis) will be denoted in the Einstein notation. This means that there are implicit sums over every index which appears twice in one term, e.g.

$$p^\mu p_\mu \equiv \sum_\mu p^\mu p_\mu. \tag{A.1}$$

Every other sum is stated explicitly. We use the Minkovski metric

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{A.2}$$

For the Levi-Civita tensor, which we will use exclusively with a rank of four, we use the convention

$$\epsilon^{0123} = 1. \tag{A.3}$$

Together with its anti-cyclic properties, this is sufficient to compute every component of the contravariant counterpart.

By far the most important identity for the well-known δ-distribution, which is implicitly defined by

$$\int dx f(x) \delta(x - x_0) = f(x_0), \tag{A.4}$$

is

$$\delta\Big(g(x)\Big) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{A.5}$$

where the x_i are roots of $g(x)$.

We use the general definition for a Lorentz invariant phase space (LIPS) integral with N particles in the final state:

$$\int d\Pi_{\mathrm{LIPS}} \equiv (2\pi)^4 \int \prod_{i=1}^{N} \frac{d^4 p_i}{(2\pi)^3} \frac{1}{2E_i} \delta\Big(\sum_{i=1}^{N} p_i\Big) \tag{A.6}$$

A.2. Dirac matrices

Dirac matrices γ^μ are defined by the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{A.7}$$

When using (A.7) and the cyclic property of the trace, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, one obtains the well-known trace identities

$$\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big] = 4g^{\mu\nu} \tag{A.8}$$

$$\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big] = 4\Big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\Big), \tag{A.9}$$

$$\begin{aligned}\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta\Big] = 4\Big\{ & g^{\mu\nu}\Big(g^{\rho\sigma}g^{\alpha\beta} - g^{\rho\alpha}g^{\sigma\beta} + g^{\rho\beta}g^{\sigma\beta}\Big) \\ & - g^{\mu\rho}\Big(g^{\nu\rho}g^{\alpha\beta} - g^{\nu\alpha}g^{\sigma\beta} + g^{\nu\beta}g^{\sigma\alpha}\Big) \\ & + g^{\mu\sigma}\Big(g^{\nu\rho}g^{\alpha\beta} - g^{\nu\alpha}g^{\rho\beta} + g^{\nu\beta}g^{\rho\alpha}\Big) \\ & - g^{\mu\alpha}\Big(g^{\nu\rho}g^{\sigma\beta} - g^{\nu\sigma}g^{\rho\beta} + g^{\nu\beta}g^{\rho\sigma}\Big) \\ & + g^{\mu\beta}\Big(g^{\nu\rho}g^{\sigma\alpha} - g^{\nu\sigma}g^{\rho\alpha} + g^{\nu\alpha}g^{\rho\sigma}\Big)\Big\}\end{aligned} \tag{A.10}$$

$$\mathrm{Tr}\Big[\text{odd number of Dirac matrices}\Big] = 0. \tag{A.11}$$

Traces including

$$\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.12}$$

which fulfills

$$(\gamma^5)^2 = 1, \tag{A.13}$$

can be computed when we additionally exploit

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{A.14}$$

Then,

$$\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\nu\Big] = 0 \tag{A.15}$$

$$\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big] = -4i\epsilon^{\mu\nu\rho\sigma} \tag{A.16}$$

$$\begin{aligned}\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta\Big] = -4i\Big(& g^{\mu\nu}\epsilon^{\rho\sigma\alpha\beta} - g^{\mu\rho}\epsilon^{\nu\sigma\alpha\beta} + g^{\nu\rho}\epsilon^{\mu\sigma\alpha\beta} \\ & + g^{\alpha\beta}\epsilon^{\mu\nu\rho\sigma} - g^{\sigma\beta}\epsilon^{\mu\nu\rho\alpha} + g^{\sigma\alpha}\epsilon^{\mu\nu\rho\beta}\Big)\end{aligned} \tag{A.17}$$

$$\mathrm{Tr}\Big[\gamma^5 \cdot \text{odd number of Dirac matrices}\Big] = 0. \tag{A.18}$$

Also useful are the contraction identities

$$\gamma^\mu \gamma_\mu = 4 \tag{A.19}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{A.20}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\mu\rho} \tag{A.21}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu, \tag{A.22}$$

which can be deduced from the Clifford algebra as well. If not stated otherwise, we use the Weyl representation for Dirac matrices:

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \overline{\sigma}^k & 0 \end{pmatrix}, \tag{A.23}$$

which implies

$$\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{A.24}$$

The well-known Pauli matrices are defined as

$$\sigma^0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.25}$$

and

$$\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3) = (\sigma^0, \boldsymbol{\sigma}), \quad \overline{\sigma}^\mu \equiv (\sigma^0, -\boldsymbol{\sigma}). \tag{A.26}$$

Every time we add a 1 to a matrix, for example in the projection operators[1] $P_{R,L} = 1 \pm \gamma^5$, the reader shall think of it as an unity matrix with according dimension. For projection operators in Dirac traces, we often use the commutation relations

$$\gamma^\mu(1+\gamma^5) = (1-\gamma^5)\gamma^\mu \tag{A.27}$$
$$\gamma^\mu(1-\gamma^5) = (1+\gamma^5)\gamma^\mu \tag{A.28}$$

which follow from the anti-commutator (A.14). The products between projection operators are

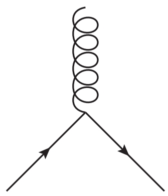
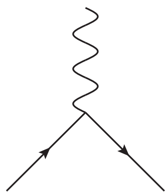
$$(1+\gamma^5)(1-\gamma^5) = (1-\gamma^5)(1+\gamma^5) = 0 \tag{A.29}$$
$$(1+\gamma^5)^2 = 2(1+\gamma^5) \tag{A.30}$$
$$(1-\gamma^5)^2 = 2(1-\gamma^5). \tag{A.31}$$

[1]Note that the conventional factor of 1/2 is absorbed into $g_{R,L}$ in this work.

A.3. Feynman rules

The Feynman rules used throughout this work are given in table A.1. For incoming (outgoing) states, fermions are described by the spinors[2] $u(p)$ ($\overline{u}(p)$), Anti-fermions by the spinors[3] $v(p)$ ($\overline{v}(p)$) and bosons by the polarization states[4] ϵ^μ ($\epsilon^{\star\mu}$).

Table A.1.: All relevant Feynman rules for this work. More informations can be found in the according sections of chapter 2.

Quark propagator		$i\delta_{ij}\frac{\not{p}+m}{p^2-m^2}$
Gluon propagator (covariant gauge)		$i\delta^{ab}\frac{-g^{\mu\nu}+(1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2}$
Photon propagator (covariant gauge)		$i\frac{-g^{\mu\nu}+(1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2}$
W- and Z-propagator (unitary gauge)		$i\frac{-g^{\mu\nu}+\frac{p^\mu p^\nu}{M^2_{W,Z}}}{p^2-M^2_{W,Z}}$
Gluon-Quark vertex		$ig_s T^a_{ij}\gamma^\mu$
Electroweak Boson-Fermion vertex		$ig_B\Big(g_R(1+\gamma^5)+g_L(1-\gamma^5)\Big)\gamma^\mu \equiv ig_B\Gamma^\mu$

[2]Cf. section 2.2 for more details.
[3]Cf. section 2.2 for more details.
[4]Cf. section 2.3.2.1 for more details.

B. Boosts in Minkovski space and in the spherical basis

In this work we encounter boost matrices in two different representations: On one hand, we have the Lorentz boost matrix (2.39), here in four-dimensional space-time. On the other hand, in the context of the helicity formalism, there is also the boost matrix (4.49) in the basis of the three polarizations $\epsilon_{\lambda=\pm,0}$, cf. (2.66) and (2.67).

B.1. Boosts in Minkovski space

Let us begin with the calculation of the boost in Minkovski coordinates,

$$\Lambda_{z\text{-boost}}(\zeta) = \exp(-i\zeta K_3). \tag{B.1}$$

We see that we only need the explicit shape of K_3 in the representation of four-dimensional space-time, which is[1]

$$K_3 = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{B.2}$$

Since for $n \in \mathbb{N}$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (n \neq 0) \tag{B.3}$$

$$\text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{B.4}$$

we can expand the exponential in (B.1), which gives

$$\Lambda_{z\text{-boost}}(\zeta) = \mathbb{1} + \sum_{n\in\mathbb{N}} \frac{(-\zeta)^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{(-\zeta)^{2n}}{(2n)!} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.5}$$

[1] Cf. e.g. [Schwartz, 2014], p. 160

We can also split up $\exp(\psi)$ into a sum of $\sinh(\psi)$ and $\cosh(\psi)$, cf. (4.13). Because $\sinh(\psi)$ is by definition odd (cf. (4.10)) and $\cosh(\psi)$ even (cf. (4.11)), we can identify the according even and odd parts inside the series expansion of e^{ψ} with the hyperbolic functions. Exactly these series expansions can also be found in the expression above, which yields

$$\Lambda_{z\text{-boost}}(\zeta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \sinh\zeta \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \cosh\zeta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\zeta & 0 & 0 & -\sinh\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix}. \tag{B.6}$$

B.2. Boosts in the spherical basis for $j = 1$

Before doing the same calculation for $d^1(\zeta)$ we need to make clear with what generators we work in this case. In order to do so we will first take a closer look at a highly related case, the group $SO(3)$, or, more precisely, a rotation around the x-axis $d^1_J(\theta)$[2] instead of a boost. Note that the polarizations ϵ^{μ}_{λ} are nothing else than the spin-1 eigenstates[3] $|j, j_3\rangle = |1, \lambda\rangle$ of J_3 and J^2 (cf. section 2.1.3 for further explanation). Hence, we can find the according generator matrices of $J_\pm$ and J_3 (which we also label with $J_\pm$ and J_3, respectively) by making use of equations (2.23) and (2.26), e.g.[4]

$$J_{3--} = \langle 1, -|\, J_3 \,|1, -\rangle = -1 \,\langle 1, -|1, -\rangle = -1. \tag{B.7}$$

This lets us compute $d^1(\theta)$ by exploiting similar series expansions as for $\Lambda_{z\text{-boost}}$:

$$d^1_J(\theta) = \exp(-i\theta J_2) = \exp\left(-\frac{\theta}{2}(J_+ - J_-)\right)$$

$$= \ldots = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}. \tag{B.8}$$

There is also a connection between this matrix and an arbitrary rotation in three dimensions. The latter can be described with the help of the three Euler angles α, β and γ, representing the rotations around the three axis. The resulting matrix in an arbitrary representation j can be split up into three rotations,

$$D^j(\alpha, \beta, \gamma) = R^j_3(\alpha) R^j_2(\beta) R^j_3(\gamma) = \exp(-i\alpha J_3) d^j_J(\beta) \exp(-i\gamma J_3). \tag{B.9}$$

[2] As in the case of $d^1(\zeta)$, the index 1 stands for $j = 1$.
[3] Here, we can identify the different spin states with the according helicity states.
[4] The full generator matrices can be found in [Tung, 1985], p.106. Keep in mind that we label the helicity in a descending order, so e.g. J_{3--} is the third diagonal element of J_3.

One can then obtain any desired matrix element via

$$D^j(\alpha,\beta,\gamma)_{mn} = \langle j,m|\exp(-i\alpha J_3)d_J^j(\beta)\exp(-i\gamma J_3)|j,n\rangle$$
$$= \exp(-i\alpha m)d_J^j(\beta)_{mn}\exp(-i\gamma n). \quad \text{(B.10)}$$

All this is explained in more detail in [Tung, 1985], chapter 7.

Let us now turn to $d^1(\zeta)$ again. In this case, we consider boosts, therefore we need to look at the (vector-like) generators K_i instead of J_i. However, the computation of $d^1(\zeta)$ closely follows the one of $d_J^1(\theta)$. Again, we are interested in the (spherical) generators $K_\pm$. Hence, we are in a situation where all the information covered in section 2.1.2.2 becomes useful. We define according to (2.32) and (2.33)

$$K_\pm \equiv K_1 \pm iK_2 \quad \text{(B.11)}$$

and

$$K_0 \equiv K_3. \quad \text{(B.12)}$$

Hence,

$$K_2 = -\frac{i}{2}(K_+ - K_-). \quad \text{(B.13)}$$

Since we deal with irreducible tensors we can apply the Wigner-Eckart theorem to derive the desired matrix elements:

$$\langle j',j_3'|K_0|j,j_3\rangle = K_{j'j}\langle j'j_3'(1,j)0j_3\rangle \quad \text{(B.14)}$$
$$\langle j',j_3'|K_\pm|j,j_3\rangle = \pm\sqrt{2}K_{j'j}\langle j'j_3'(1,j)\pm j_3\rangle. \quad \text{(B.15)}$$

One can find a detailed explanation of the reduced matrix elements $K_{j'j}$ in appendix VII of [Tung, 1985]. In this calculation, we deal with gauge bosons of the standard model, so we have

$$j = j' = 1. \quad \text{(B.16)}$$

For our sakes, the only relevant $K_{j'j}$ is consequently

$$K_{11} = \sqrt{2}i \quad \text{(B.17)}$$

As mentioned in section 2.1.2.2, the $\langle j'j_3'(1,j)qj_3\rangle$ are Clebsch-Gordan coefficients. One can look up the j and j' dependent expressions in appendix V of [Tung, 1985]. By using the selection rule (2.35), we see that all but two matrix elements of $K_\pm$ are zero. The non-vanishing components are

$$\langle 1,1|K_+|1,0\rangle \equiv K_{+0+} = -\sqrt{2}K_{11}\langle 11(1,1)+0\rangle = -\sqrt{2}i \quad \text{(B.18)}$$
$$\langle 1,0|K_+|1,-1\rangle \equiv K_{+-0} = -\sqrt{2}K_{11}\langle 10(1,1)+[-1]\rangle = -\sqrt{2}i \quad \text{(B.19)}$$
$$\langle 1,0|K_-|1,1\rangle \equiv K_{-+0} = \sqrt{2}K_{11}\langle 10(1,1)-1\rangle = -\sqrt{2}i \quad \text{(B.20)}$$
$$\langle 1,-1|K_-|1,0\rangle \equiv K_{-0-} = \sqrt{2}K_{11}\langle 1[-1](1,1)-0\rangle = -\sqrt{2}i. \quad \text{(B.21)}$$

This yields the following shape of the generator matrices in spin-1 basis:

$$K_+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \tag{B.22}$$

$$K_- = \sqrt{2}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.23}$$

Therefore, we have

$$K_2 = -\frac{i}{2}(K_+ - K_-) = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{B.24}$$

However, this does not lead to the desired final result as given in [Aivazis et al., 1994a]. For this to happen, the generator has to be

$$K_2 = -\frac{i}{2}(K_+ - K_-) = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{B.25}$$

instead. Then, we would have

$$\begin{aligned} d^1(\zeta) = \exp(-i\zeta K_2) &= \exp\Big(-\frac{\zeta}{2}(K_+ - K_-)\Big) \\ &= \mathbb{1} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n!}\Big(\frac{\zeta}{\sqrt{2}}\Big)^n \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n \\ &\equiv \mathbb{1} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n!}\Big(\frac{\zeta}{\sqrt{2}}\Big)^n k^n. \end{aligned} \tag{B.26}$$

Since

$$k^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{B.27}$$

$$\text{and} \quad k^3 = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 2k, \tag{B.28}$$

we can deduce that for $n \in \mathbb{N}$

$$k^{2n} = 2^{n-1}k^2 \quad (n \neq 0) \tag{B.29}$$

$$\text{and } k^{2n+1} = 2^n k. \tag{B.30}$$

With this information we are now able to compute the final boost matrix:

$$\begin{aligned}
d^1(\zeta) &= \mathbb{1} + \sum_{n\in\mathbb{N}} \frac{1}{(2n+1)!}\left(\frac{\zeta}{\sqrt{2}}\right)^{2n+1} 2^n k + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{(2n)!}\left(\frac{\zeta}{\sqrt{2}}\right)^{2n} 2^{n-1} k^2 \\
&= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{N}} \frac{1}{(2n+1)!}\left(\frac{\zeta}{\sqrt{2}}\right)^{2n+1} \left(\sqrt{2}\right)^{2n+1} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&\quad + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{(2n)!}\left(\frac{\zeta}{\sqrt{2}}\right)^{2n} \left(\sqrt{2}\right)^{2n} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+\cosh\zeta}{2} & -\frac{\sinh\zeta}{\sqrt{2}} & \frac{1-\cosh\zeta}{2} \\ -\frac{\sinh\zeta}{\sqrt{2}} & \cosh\zeta & \frac{\sinh\zeta}{\sqrt{2}} \\ \frac{1-\cosh\zeta}{2} & \frac{\sinh\zeta}{\sqrt{2}} & \frac{1+\cosh\zeta}{2} \end{pmatrix}. \qquad \text{(B.31)}
\end{aligned}$$

This leaves the question why the derivation at the start of this chapter does not lead to the same boost matrix as in the underlying literature. Concerning this topic, [Aivazis et al., 1994a] only refers to works of the same authors, namely [Olness and Tung, 1987] and [Tung, 1985], where $d^1(\zeta)$ is only stated, but not explicitly calculated. The author was not able to find any contradictions in his calculation of K_2 in the spin-1 basis, so this problem remains open. No matter if the matrix above is correct or not, the shape of K_2 given in equation (B.24) yields alternating signs in its even and odd powers, respectively. Therefore, $d^1(\zeta)$ would contain no hyperbolic cosines and sines, which cannot be true.

C. Helicity eigenvalues and eigenstates

We will calculate eigenvalues and eigenstates of the helicity operator

$$h = \frac{\boldsymbol{p} \cdot \boldsymbol{S}}{|\boldsymbol{p}|}, \tag{C.1}$$

which was introduced in section 2.3.1, for particles in the spin-1/2 and spin-1 representation of the Lorentz group, cf. section 2.1.3.

C.1. Spin-1

For $s = 1$, we need the four-dimensional representation of $\boldsymbol{S}$. This is nothing less than the rotational part of the Lorentz group generators J_i, which were introduced in section 2.1.3. Since (in the hadronic configuration) all hadronic and bosonic momenta are aligned to the z-axis, we will restrict ourselves to momenta fulfilling[1]

$$\frac{\boldsymbol{q}}{|\boldsymbol{q}|} = \boldsymbol{e}_z \tag{C.2}$$

in this calculation. Then the helicity operator simply reduces to

$$h = J_3 \tag{C.3}$$

and we are left with calculating eigenvalues and eigenstates of the spin alligned to the z-axis.

The explicit form of J_3 in four dimensions is

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.4}$$

The eigenvalues are roots of

$$\det\Big(\lambda \mathbb{1} - J_3\Big) = \lambda^2(\lambda^2 - 1), \tag{C.5}$$

which implies that

$$\lambda = 0, \pm 1 \tag{C.6}$$

[1]To follow the convention of the rest of this work, we will denote the bosonic (spin-1) momentum with q^μ rather than p^μ.

are possible helicities. Eigenvectors[2] ϵ^μ_λ need to fulfill the relation

$$\left(S^3\right)^{\mu\nu}\epsilon_{\lambda\nu} = \lambda\epsilon^\mu_\lambda. \tag{C.7}$$

For $\lambda = 1$, this leads to

$$\epsilon^0_1 = \epsilon^3_1 = 0 \tag{C.8}$$
$$\epsilon^1_1 = -ic \tag{C.9}$$
$$\epsilon^2_1 = c, \tag{C.10}$$

with an arbitrary normalization c. The most common choice is the Condon-Shortley

$$c = -\frac{i}{\sqrt{2}}, \tag{C.11}$$

leading to

$$\epsilon^\mu_1 = \frac{1}{\sqrt{2}}(0, -1, -i, 0). \tag{C.12}$$

The same procedure and choice of normalization for $\lambda = -1$ yields

$$\epsilon^\mu_{-1} = \frac{1}{\sqrt{2}}(0, 1, -i, 0). \tag{C.13}$$

In this work, these two vectors will be called $\epsilon^\mu_\pm$. They are independent of the reference vector, which was introduced in section 2.3.2.

For $\lambda = 0$ (which has a degeneracy of 2 and thus two linear independent eigenvectors), it is obvious that every choice is suitable which fulfills

$$\epsilon^1_0 = \epsilon^2_0 = 0. \tag{C.14}$$

In the ACOT formalism, the two eigenvectors of $\lambda = 0$ are[3]

$$\epsilon^\mu_0 = \frac{Q^2 P^\mu + (P\cdot q)q^\mu}{Q\sqrt{(P\cdot q)^2 + Q^2M^2}} \tag{C.15}$$
$$\text{and } \epsilon^\mu_q = \frac{q^\mu}{Q}. \tag{C.16}$$

The subscript of the second vector is there only to distinguish it from the first one and to indicate that it is aligned to q^μ.

These helicity eigenvectors are commonly used as polarizations, i.e. they form the basis in which the photon field A^μ is quantized. As explained in section 2.4.1.1, out of the two vectors above only ϵ^μ_0 is a physical polarization, since it fulfills $p_\mu\epsilon^\mu_0 = 0$.

[2] We will omit the obvious dependencies on momenta throughout this section.
[3] Again, the reader shall be reminded of the fact that in the hadronic configuration P^μ and q^μ have vanishing transverse momenta (cf. section 4.3), so (C.14) is fulfilled.

To show that the four vectors above indeed form an at least nearly orthonormal basis, we first verify orthogonality. Since S^3 is unitary, eigenvectors to different eigenvalues are automatically orthogonal. In addition, one easily derives that

$$\epsilon_{0\mu}\epsilon_q^\mu = \frac{Q^2(P\cdot q)-(P\cdot q)Q^2}{Q^2\sqrt{(P\cdot q)^2+Q^2M^2}} = 0. \tag{C.17}$$

However, not all vectors are normalized to 1:

$$\epsilon_0^2 = \frac{Q^4M^2 - Q^2(P\cdot q)^2 + 2Q^2(P\cdot q)^2}{Q^2\Big((P\cdot q)^2+Q^2M^2\Big)} = 1, \tag{C.18}$$

$$\text{but } \epsilon_\pm^2 = \frac{1}{2}(1+1) = -1 \tag{C.19}$$

$$\text{and } \epsilon_q^2 = \frac{-Q^2}{Q^2} = -1. \tag{C.20}$$

In the completeness relation

$$\sum_{\lambda=\pm,0,q} v_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu = g^{\mu\nu}, \tag{C.21}$$

this is taken into account via a factor

$$v_\lambda = \begin{cases} 1 & \text{for } \lambda = 0 \\ -1 & \text{for } \lambda = \pm, q \end{cases}. \tag{C.22}$$

C.2. Spin-$\frac{1}{2}$

Next, let us verify that for $s = \frac{1}{2}$ the helicity eigenvalues λ are always ± 1 by diagonalizing

$$h = \frac{\boldsymbol{p} \cdot \boldsymbol{\sigma}}{|\boldsymbol{p}|} = \frac{1}{|\boldsymbol{p}|} \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}, \tag{C.23}$$

where we inserted the Pauli matrices $\boldsymbol{\sigma}$ for the general spin operator $\boldsymbol{S}$ due to $s = \frac{1}{2}$. Accordingly, the eigenvalues are

$$\lambda_{\pm} = \pm \frac{1}{2|\boldsymbol{p}|} \sqrt{4(p_1^2 + p_2^2 + p_3^2)} = \pm 1. \tag{C.24}$$

The corresponding eigenstates fulfill

$$\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \xi(p) = \pm |\boldsymbol{p}|(p) \tag{C.25}$$

and, since

$$\overline{u}_\lambda u_{\lambda'} = 2m_l \delta_{\lambda\lambda'} \tag{C.26}$$

must hold, also[4]

$$\xi^\dagger \xi = 1 \Rightarrow |\xi_1^2| + |\xi_2^2| = 1. \tag{C.27}$$

For $\xi_{\frac{1}{2}}(p)$, the first condition gives

$$\xi_{+\frac{1}{2}}(\boldsymbol{p})_2 = \frac{p_1 - ip_2}{|\boldsymbol{p}| + p_3} \xi_{+\frac{1}{2}}(p)_1. \tag{C.28}$$

In combination with the proper normalization, we obtain

$$\xi_{+\frac{1}{2}}(\boldsymbol{p})_1 = \frac{|\boldsymbol{p}| + p_3}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}| + p_3)}}, \tag{C.29}$$

and thus

$$\xi_{+\frac{1}{2}}(\boldsymbol{p}) = \frac{1}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}| + p_3)}} \begin{pmatrix} |\boldsymbol{p}| + p_3 \\ p_1 + ip_2 \end{pmatrix}. \tag{C.30}$$

The same procedure applied on $\xi_{-\frac{1}{2}}(p)$ yields

$$\xi_{-\frac{1}{2}}(\boldsymbol{p}) = \frac{1}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}| + p_3)}} \begin{pmatrix} -p_1 + ip_2 \\ |\boldsymbol{p}| + p_3. \end{pmatrix}. \tag{C.31}$$

[4] The orthogonality condition is automatically fulfilled by eigenvectors belonging to two different eigenvalues of a hermitian matrix.

However, the eigenstates actually do not depend on $|\boldsymbol{p}|$, but only on the momentum direction, since this is also the case for the helicity operator itself. To make this obvious, we change to spherical coordinates, where

$$p^\mu = (E, |\boldsymbol{p}|\sin\theta\cos\phi, |\boldsymbol{p}|\sin\theta\sin\phi, |\boldsymbol{p}|\cos\theta). \tag{C.32}$$

In this coordinates, the only dimensional variable $|\boldsymbol{p}|$ must obviously vanish and we are left with the always non-ambigous results

$$\xi_{+\frac{1}{2}}(\boldsymbol{p}) = \frac{1}{\sqrt{2(1+\cos\theta)}}\begin{pmatrix} 1+\cos\theta \\ (\cos\phi + i\sin\phi)\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{C.33}$$

and

$$\begin{aligned}\xi_{-\frac{1}{2}}(\boldsymbol{p}) =& \frac{1}{\sqrt{2(1+\cos\theta)}}\begin{pmatrix} (-\cos\phi + i\sin\phi)\sin\theta \\ 1+\cos\theta \end{pmatrix} \\ =& \begin{pmatrix} -\Big(\cos(-\phi) + i\sin(-\phi)\Big)\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}\end{aligned} \tag{C.34}$$

after using the identities $\sqrt{1+\cos\theta} = \sqrt{2}\cos\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. Hence, we reproduced the results (2.73) and (2.74).

D. Calculation of all differential cross sections containing $L^{\mu\nu}$ and $W^{\mu\nu}$

Starting with the general expression (3.3), we need to bring the leptonic tensor $L^{\mu\nu}$ into a more convenient shape before proceeding. This calculation is for example written down in [Halzen and Martin, 1984], p. 122 et seq.

If we use the fact that complex and hermitian conjugation are the same for complex numbers[1],

$$\begin{aligned}\left[\overline{u}_{s_2}(l_2)\gamma^\mu u_{s_1}(l_1)\right]^* &= u^\dagger_{s_1}(l_1\gamma^{\mu\dagger}\gamma^{0\dagger}u_{s_2}(l_2)\\ &= u^\dagger_{s_1}(l_1\gamma^0\gamma^\mu\gamma^0\gamma^0 u_{s_2}(l_2) = \overline{u}_{s_1}(l_1)\gamma^\mu u_{s_2}(l_2),\end{aligned} \tag{D.1}$$

explicitely write down all vector and matrix components, i.e.

$$\overline{u}(l_2)\gamma^\mu u(l_1) = \overline{u}(l_2)_\alpha \gamma^\mu_{\alpha\beta} u(l_1)_\beta \tag{D.2}$$

and use the completeness relation (2.55), we see that $L^{\mu\nu}$ can be written as a trace of a product of Dirac matrices:

$$\begin{aligned}L^{\mu\nu} &= \frac{1}{Q^2}\frac{1}{2s_1+1}\sum_{s_2}\overline{u}^{s_2}_\alpha(l_2)\gamma^\mu_{\alpha\beta}\sum_{s_1}u^{s_1}_\beta(l_1)\overline{u}^{s_1}_\delta(l_1)\gamma^\nu_{\delta\sigma}u^{s_2}_\sigma(l_2)\\ &= \frac{1}{2Q^2}(\not{l}_2+m)_{\sigma\alpha}\gamma^\mu_{\alpha\beta}(\not{l}_1+m)_{\beta\delta}\gamma^\nu_{\delta\sigma}\\ &= \frac{1}{Q^2}\frac{1}{2s_1+1}\operatorname{Tr}\Big((\not{l}_2+m)\gamma^\mu(\not{l}_1+m)\gamma^\nu\Big)\\ &= \frac{1}{2Q^2}\Big(l_{1\alpha}l_{2\beta}\operatorname{Tr}\Big(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big)+m^2\operatorname{Tr}\Big(\gamma^\mu\gamma^\nu\Big)\Big).\end{aligned} \tag{D.3}$$

Using the trace identities for Dirac matrices given in Appendix A.2 yields

$$L^{\mu\nu}_m = \frac{2}{Q^2}\Big(l_1^\mu l_2^\nu + l_2^\mu l_1^\nu - (l_1\cdot l_2 - m^2)g^{\mu\nu}\Big). \tag{D.4}$$

Throughout the whole work lepton masses are ne, so the final result for our purposes is

$$L^{\mu\nu}_0 = \frac{2}{Q^2}\Big(l_1^\mu l_2^\nu + l_2^\mu l_1^\nu - l_1\cdot l_2 g^{\mu\nu}\Big), \tag{D.5}$$

which is the leptonic tensor $L^{\mu\nu}$ in chapter 3.

[1] We take advantage of $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $\gamma^{0\dagger} = \gamma^0$.

D.1. $\overline{|\mathcal{M}|}^2 \propto L_{\mu\nu}L^{\mu\nu}$

With these two shapes of the leptonic tensor, the first matrix element $\mathcal{M}$, contributing to the pure leptonic cross section (3.16), can now be calculated. Keep in mind that the "lower" lepton turns into a massive hadron in section 3.1, so we need one massless and one massive leptonic tensor:

$$\begin{aligned}
\overline{|\mathcal{M}|}^2 &= \frac{e^4}{Q^4} Q^4 L_{0,\mu\nu} L_M^{\mu\nu} \\
&= \frac{4e^4}{Q^4}\Big((l_1 \cdot p_1)(l_2 \cdot p_2) + (l_1 \cdot p_2)(l_2 \cdot p_1) - (l_1 \cdot l_2)(p_1 \cdot p_2) + M^2 l_1 \cdot l_2 \\
&\qquad + (l_1 \cdot p_1)(l_2 \cdot p_2) + (l_1 \cdot p_2)(l_2 \cdot p_1) - (l_1 \cdot l_2)(p_1 \cdot p_2) + M^2 l_1 \cdot l_2 \\
&\qquad - (l_1 \cdot l_2)(p_1 \cdot p_2) - (l_1 \cdot l_2)(p_1 \cdot p_2) + 4(l_1 \cdot l_2)(p_1 \cdot p_2) - 4M^2 l_1 \cdot l_2 \Big) \\
&= \frac{8e^4}{Q^4}\Big((l_1 \cdot p_1)(l_2 \cdot p_2) + (l_1 \cdot p_2)(l_2 \cdot p_1) - M^2 l_1 \cdot l_2 \Big),
\end{aligned} \tag{D.6}$$

where the momentum of the second lepton was assigned with $p_{1,2}$.

To evaluate the scalar products we go to the laboratory frame for a fixed target experiment, where[2]

$$p_1^\mu = (M, \mathbf{0}). \tag{D.7}$$

Additionally, we are free to set $\boldsymbol{l}_1$ along the z-axis, thus

$$l_1^\mu = (E_1, 0, 0, E_1). \tag{D.8}$$

In general, we can express the final leptonic momentum as

$$l_2^\mu = (E_2, \boldsymbol{l}_2). \tag{D.9}$$

For reasons of three-momentum conservation, $\boldsymbol{l}_1 = \boldsymbol{p}_2 + \boldsymbol{l}_2$ must hold and the angle between $\boldsymbol{l}_1$ and $\boldsymbol{p}_2$ must be the same as the one between $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, in the following denoted by $\theta_{\text{lab}} \equiv \theta$. The scattering in the laboratory frame is shown in figure D.1. For $\overline{|\mathcal{M}|}^2 \propto L_{\mu\nu}L^{\mu\nu}$ we, as described in section 3.1, replace the hadron in the diagram with a lepton of mass M, i.e. we replace $P \leftrightarrow p_1$ and $P_X \leftrightarrow p_2$.

Using the laboratory frame described above we can calculate all scalar products in the squared amplitude (D.6)[3]:

$$l_1 \cdot l_2 = E_1 E_2 (1 - \cos\theta) \tag{D.10}$$
$$l_1 \cdot p_1 = M E_1 \tag{D.11}$$
$$l_2 \cdot p_1 = M E_2 \tag{D.12}$$
$$l_1 \cdot p_2 = l_1 \cdot (l_1 + p_1 - l_2) = M E_1 - l_1 \cdot l_2 \tag{D.13}$$
$$l_2 \cdot p_2 = l_2 \cdot (l_1 + p_1 - l_2) = M E_2 + l_1 \cdot l_2. \tag{D.14}$$

[2] The subscript "lab" will be omitted throughout the whole calculation.

[3] As always for massless particles, $l_{1,2}^2 = 0$ and $E_{1,2} = |\boldsymbol{l}_{1,2}|$ holds. We also made use of momentum conservation, i.e. $l_1 + p_1 = l_2 + p_2$.

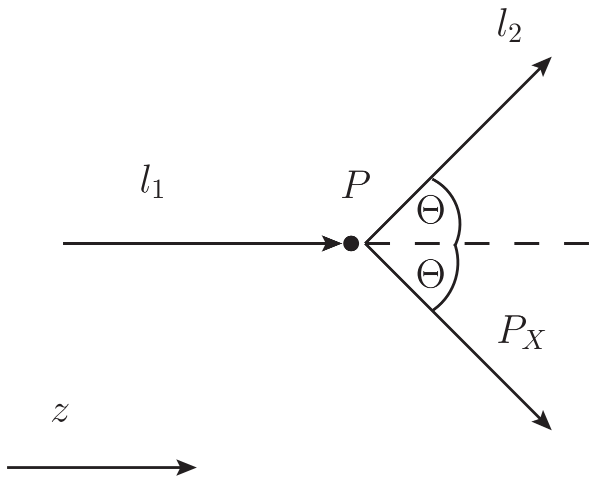


Figure D.1.: lepton-hadron scattering in the laboratory frame for a fixed target experiment.

The Lorentz invariant Mandelstam variables for the DIS process
$l_1 + P \to l_2 + P_X$ are

$$s = (l_1 + P)^2 = 2l_1 \cdot P + M^2 = 2ME_1 + M^2 \tag{D.15}$$

$$-Q^2 = t = (l_1 - l_2)^2 = -2l_1 \cdot l_2 = -2E_1E_2(1 - \cos\theta) \tag{D.16}$$

$$u = (P - l_2)^2 = -2l_2 \cdot P + M^2 = -2ME_2 + M^2. \tag{D.17}$$

Replacing $P \leftrightarrow p_1$ and $P_X \leftrightarrow p_2$ is again possible. We are now able to give the fully calculated squared amplitude by using the identity

$$1 - \cos\theta = 1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta}{2} \tag{D.18}$$

and replacing $l_1 \cdot l_2$ by $Q^2/2$, cf. (D.16):

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{8e^4}{Q^4}\Big(ME_1\Big(ME_2 + \frac{1}{2}Q^2\Big) + ME_2\Big(ME_1 - \frac{1}{2}Q^2\Big) - \frac{1}{2}M^2Q^2\Big) \\
&= \frac{16e^4}{Q^4}M^2E_1E_2\Big(1 - \frac{1}{4}\frac{Q^2}{E_1E_2} + \frac{1}{4}\frac{Q^2}{M}\frac{E_2 - E_1}{E_1E_2}\Big) \\
&= \frac{4e^4}{\sin^4\frac{\theta}{2}}M^2E_1E_2\Big(\frac{Q^2}{2M^2}\frac{M(E_1 - E_2)}{2E_1E_2} + 1 - \sin^2\frac{\theta}{2}\Big) \\
&= \frac{4e^4}{E_1E_2\sin^4\frac{\theta}{2}}M^2\Big(\frac{Q^2}{2M^2}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\Big).
\end{aligned} \tag{D.19}$$

In the last step we made also use of (3.20) and $x = 1$ (which holds only in this specific case), cf. e.g. (3.21) and (3.22). This yields

$$M\nu = M(E_1 - E_2) = \frac{1}{2}Q^2\Big(= 2E_1E_2\sin^2\frac{\theta}{2}\Big). \tag{D.20}$$

In order to obtain the full differential cross section (3.13) we also need to take a closer look at the flux factor F and the Lorentz invariant phase space $d\Pi_{\text{LIPS}}$. Starting with F, which was defined in (3.14), we make use of the fact that $m = 0$ and (D.15), so

$$F = 2\Delta(s, m^2 = 0, M^2) = 2\sqrt{s^2 + M^4 - 2sM^2} = 2(s - M^2) = 4ME_1. \tag{D.21}$$

Note that F is Lorentz invariant, since it only contains Lorentz scalars. For $d\Pi_{\text{LIPS}}$, we start with the general expression for a two particle phase space,

$$d\Pi_{\text{LIPS}} = (2\pi)^4 \frac{d^3\boldsymbol{l}_2}{(2\pi)^3 2E_2} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2p_{2,0}} \delta^{(4)}(l_1 + p_1 - l_2 - p_2). \tag{D.22}$$

One can show that $d^3\boldsymbol{p}/(2p_0)$ is covariant by performing the integration over p_0 in the following expression:

$$\begin{aligned}\int d^4p\,\delta(p^2 - M^2) &= \int d^4p\,\delta(p_0^2 - \boldsymbol{p}^2 - M^2) \\ &= \int d^4p \frac{1}{2\sqrt{\boldsymbol{p}^2 + M^2}} \left[\delta\Big(p_0 - \sqrt{\boldsymbol{p}^2 + M^2}\Big) + \delta\Big(p_0 + \sqrt{\boldsymbol{p}^2 + M^2}\Big)\right] \\ &= \int \frac{d^3p}{2p_0},\end{aligned} \tag{D.23}$$

where in the last step we could ignore the second δ distribution since $p_0 > 0$. Using this relation, we can integrate over d^4p_2 by using $\delta^{(4)}(l_1 + p_1 - l_2 - p_2)$. This gives

$$\begin{aligned}d\Pi_{\text{LIPS}} &= \frac{1}{8\pi^2}\frac{1}{E_2} d^3\boldsymbol{l}_2 \delta\Big((l_2 - l_1 - p_1)^2 - M^2\Big) \\ &= \frac{1}{8\pi^2}\frac{1}{E_2} E_2^2 dE_2 d\cos\theta d\varphi \delta(M^2 - 2l_2 \cdot l_1 - 2l_2 \cdot p_1 + 2l_1 \cdot p_1 - M^2) \\ &= \frac{1}{8\pi^2} E_2 dE_2 d\cos\theta d\varphi \delta(Q^2 - 2ME_2 + 2ME_1) \\ &= \frac{1}{8\pi}\frac{E_2}{M} dE_2 d\cos\theta \delta\Big(E_1 - E_2 - \frac{Q^2}{2M}\Big).\end{aligned} \tag{D.24}$$

The trivial integration over φ leads to a factor of 2π. Consequently, the complete differential cross section as given in eq. (3.16) is

$$\begin{aligned}d\sigma &= \frac{\overline{|\mathcal{M}|^2}}{F} d\Pi_{\text{LIPS}} \\ \Leftrightarrow \frac{d\sigma}{dE_2 d\cos\theta} &= \frac{2\pi\alpha^2}{E_1^2 \sin^4\frac{\theta}{2}} \left(\frac{Q^2}{2M^2}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) \delta\Big(E_1 - E_2 - \frac{Q^2}{2M}\Big),\end{aligned} \tag{D.25}$$

where α is the well-known fine structure constant of QED,

$$\alpha = \frac{e^2}{4\pi}. \tag{D.26}$$

Note that one could easily perform the E_2-integration by exploiting the δ distribution. This is not done because the additional derivative becomes useful in the comparison with the parton cross section (3.32).

D.2. $\overline{|\mathcal{M}|^2} \propto L_{\mu\nu}W^{\mu\nu}$ for QED interactions

Next, we will turn to expressions involving the hadronic tensor $W^{\mu\nu}$, starting with the hadronic tensor (3.10), where we already applied the Ward identity (2.115). Furthermore, we consider only pure QED interaction, so the leptonic tensor[4] $L_{\mu\nu}$ stays the same. Hence, the aim is to reproduce the cross section (3.17). The normalized product of leptonic and hadronic tensor is

$$\begin{aligned}
\frac{Q^2}{2}L_{\mu\nu}W^{\mu\nu} = &\Big(l_{1\mu}l_{2\nu} + l_{2\mu}l_{1\nu} - l_1\cdot l_2 g_{\mu\nu}\Big)\\
&\times\left(-\left(g^{\mu\nu} + \frac{1}{Q^2}q^\mu q^\nu\right)W_1 + \frac{1}{M^2}\left(P^\mu + \frac{P\cdot q}{Q^2}q^\mu\right)\left(P^\nu + \frac{P\cdot q}{Q^2}q^\nu\right)W_2\right)\\
= &\,2\Bigg[-(l_1\cdot l_2)W_1 - \frac{1}{Q^2}(l_1\cdot q)(l_2\cdot q)W_1\\
&+\frac{1}{M^2}\left(l_1\cdot P + \frac{P\cdot q}{Q^2}l_1\cdot q\right)\left(l_2\cdot P + \frac{P\cdot q}{Q^2}l_2\cdot q\right)W_2\Bigg]\\
&+4l_1\cdot l_2 W_1 + \frac{1}{Q^2}(l_1\cdot l_2)q^2 W_1\\
&-\frac{1}{M^2}\left(P^\mu + \frac{P\cdot q}{Q^2}q^\mu\right)\left(P_\mu + \frac{P\cdot q}{Q^2}q_\mu\right)(l_1\cdot l_2)W_2\\
= &\,2\Bigg[\frac{1}{2}(l_1\cdot l_2)W_1 - \frac{1}{Q^2}(l_1\cdot q)(l_2\cdot q)W_1\\
&+\frac{1}{M^2}\bigg((l_1\cdot P)(l_2\cdot P) + \frac{P\cdot q}{Q^2}(l_1\cdot P)(l_2\cdot q)\\
&+\frac{P\cdot q}{Q^2}(l_1\cdot q)(l_2\cdot P) + \frac{(P\cdot q)^2}{Q^4}(l_1\cdot q)(l_2\cdot q)\bigg)W_2\Bigg]\\
&-\frac{1}{M^2}\left[M^2 + \frac{(P\cdot q)^2}{Q^2}\right](l_1\cdot l_2)W_2.
\end{aligned}\tag{D.27}$$

We have to handle three new scalar products[5] involving q^μ:

$$l_1\cdot q = l_1\cdot(l_1 - l_2) = -l_1\cdot l_2 = -\frac{1}{2}Q^2 \tag{D.28}$$

$$l_2\cdot q = \frac{1}{2}Q^2 \tag{D.29}$$

$$P\cdot q = P\cdot l_1 - P\cdot l_2 = M(E_1 - E_2) = M\nu. \tag{D.30}$$

[4] From here on, only the massless lepton takes part in the process, so we set $L_0^{\mu\nu} \equiv L^{\mu\nu}$ as in chapter 3.

[5] All scalar products calculated in the section above are still valid if we replace $p_1 \leftrightarrow P$ and $p_2 \leftrightarrow P_X$. Note that the equality $x = 1 \Leftrightarrow M\nu = \frac{1}{2}Q^2$ does not hold in the case of lepton-hadron scattering.

This implies

$$\begin{aligned}
\frac{Q^2}{2}L_{\mu\nu}W^{\mu\nu} &= 2\left[\frac{1}{2}Q^2W_1 + \left(E_1E_2 + \frac{1}{2}E_1\nu - \frac{1}{2}E_2\nu - \frac{1}{4}\nu^2\right)W_2\right] \\
&\quad - \left[1 + \frac{\nu^2}{Q^2}\right]\frac{1}{2}Q^2W_2 \\
&= Q^2W_1 + 2E_1E_2\left[1 + \frac{1}{2E_1E_2}\nu^2 - \frac{1}{4E_1E_2}\nu^2\right. \\
&\qquad\qquad\qquad \left. - \frac{1}{4}\frac{Q^2}{E_1E_2} - \frac{1}{4E_1E_2}\nu^2\right]W_2 \\
&= Q^2W_1 + 2E_1E_2\left[1 - \frac{1}{4}\frac{Q^2}{E_1E_2}\right]W_2 \\
&= 4E_1E_2\left(W_1\sin^2\frac{\theta}{2} + \frac{1}{2}W_2\cos^2\frac{\theta}{2}\right).
\end{aligned} \tag{D.31}$$

Hence,

$$\frac{4\pi e^4}{Q^2}L_{\mu\nu}W^{\mu\nu} = \frac{8\pi}{E_1E_2\sin^4\frac{\theta}{2}}\left(W_1\sin^2\frac{\theta}{2} + \frac{1}{2}W_2\cos^2\frac{\theta}{2}\right). \tag{D.32}$$

There is no change to F, but the phase space needs to be adjusted. $W^{\mu\nu}$ includes an integration over all final momenta P_X, i.e. it contains the hadronic part of the phase space, so there is only the one-particle leptonic phase space left:

$$d\Pi_{\text{LIPS}} = \frac{d^3l_2}{(2\pi)^3 2E_2} = \frac{1}{8\pi^2}E_2 dE_2 d\cos\theta. \tag{D.33}$$

This leads to

$$\begin{aligned}
d\sigma &= \frac{1}{F}\frac{4\pi e^4}{Q^2}L_{\mu\nu}W^{\mu\nu}d\Pi_{\text{LIPS}} \\
\Leftrightarrow \frac{d\sigma}{dE_2 d\cos\theta} &= \frac{4\pi\alpha^2}{E_1^2\sin^4\frac{\theta}{2}}\frac{1}{M}\left(W_1\sin^2\frac{\theta}{2} + \frac{1}{2}W_2\cos^2\frac{\theta}{2}\right).
\end{aligned} \tag{D.34}$$

D.3. $\overline{|\mathcal{M}|^2} \propto L_{\mu\nu}W^{\mu\nu}$ for general chiral couplings

Let us turn to the more generalized electroweak coupling to reproduce the cross section (5.7) relevant for the ACOT formalism. Electroweak coupling obeys a general chiral structure, cf. section 2.6.2, which implies that the leptonic tensor needs to be adjusted. Hence, we start with a slightly modified leptonic tensor, now including generic chiral couplings[6] $g_{R,L}$,

$$L^{\mu\nu} = \frac{1}{n_l Q^2} \sum_{s_1,s_2} \left[\overline{u}_{s_2}(l_2)\gamma^\mu\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)u_{s_1}(l_1)\right] \times \left[\overline{u}_{s_2}(l_2)\gamma^\nu\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)u_{s_1}(l_1)\right]^*. \tag{D.35}$$

In addition, we keep the number of spin states n_l variable. The procedure is very similar to the one at the start of this appendix. First we take a closer look at the second bracket above:

$$\begin{aligned}
\Big[\ldots\Big]^* &= u^\dagger_{s_1}(l_1)\Big(g_R(1+\gamma^5)^\dagger + g_L(1-\gamma^5)^\dagger\Big)\gamma^{\nu\dagger}\gamma^{0\dagger}u_{s_2}(l_2) \\
&= u^\dagger_{s_1}(l_1)\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)\gamma^0\gamma^\nu u_{s_2}(l_2) \\
&= u^\dagger_{s_1}(l_1)\gamma^0\Big(g_R(1-\gamma^5) + g_L(1+\gamma^5)\Big)\gamma^\nu u_{s_2}(l_2) \\
&= \overline{u}_{s_1}(l_1)\gamma^\nu\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)u_{s_2}(l_2),
\end{aligned} \tag{D.36}$$

where we used $\gamma^{5\dagger} = \gamma^5$, $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$ and $\{\gamma^\mu, \gamma^5\} = 0$. Further following the calculation of the QED leptonic tensor yields

$$\begin{aligned}
L^{\mu\nu} &= \frac{1}{n_l Q^2}\sum_{s_2}\overline{u}_{s_2}(l_2)_\alpha\gamma^\mu_{\alpha\beta}\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)_{\beta\delta} \\
&\quad \times \sum_{s_1} u_{s_1}(l_1)_\delta\overline{u}_{s_1}(l_1)_\epsilon\gamma^\nu_{\epsilon\zeta}\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)_{\zeta\eta}u_{s_2}(l_2)_\eta \\
&= \frac{1}{n_l Q^2}\not{k}_{2\eta\alpha}\gamma^\mu_{\alpha\beta}\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)_{\beta\delta} \\
&\quad \times \not{k}_{1\delta\epsilon}\gamma^\nu_{\epsilon\zeta}\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)_{\zeta\eta} \\
&= \frac{1}{n_l Q^2} l_{1\alpha} l_{2\beta}\Big\{g_R^2 \operatorname{Tr}\Big[\gamma^\alpha\gamma^\mu(1+\gamma^5)\gamma^\beta\gamma^\nu(1+\gamma^5)\Big] \\
&\qquad + g_L^2 \operatorname{Tr}\Big[\gamma^\alpha\gamma^\mu(1-\gamma^5)\gamma^\beta\gamma^\nu(1-\gamma^5)\Big]\Big\}
\end{aligned}$$

[6]Note that all these couplings do not include the coupling constants g_B of the gauge boson B which prefix the cross section, cf. e.g. (5.7).

$$= \frac{2}{n_l Q^2} l_{1\alpha} l_{2\beta} \Big\{ g_+^2 \operatorname{Tr}\Big(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big) + g_-^2 \operatorname{Tr}\Big(\gamma^5\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big)\Big\}$$

$$= \frac{8}{n_l Q^2}\Big\{ g_+^2 \Big(l_1^\mu l_2^\nu + l_2^\mu l_1^\nu - l_1\cdot l_2 g^{\mu\nu}\Big) + i g_-^2 l_{1\alpha} l_{2\beta}\epsilon^{\alpha\beta\mu\nu}\Big\}, \tag{D.37}$$

where we defined

$$g_\pm^2 \equiv g_R^2 \pm g_L^2 \tag{D.38}$$

and used the identities for (traces of) Dirac matrices given in appendix A.2. As expected, (D.37) reduces to $L_0^{\mu\nu}$ (cf. (D.5)), since in the case of QED interaction[7]

$$g_R = g_L = \frac{1}{2}, \tag{D.39}$$

thus

$$g_+^2 = \frac{1}{2} \text{ and } g_-^2 = 0. \tag{D.40}$$

We are now able to calculate the product of the new leptonic and the hadronic tensor. Note that we are not allowed to use the Ward identity (2.115) because we are dealing with a non-abelian gauge theory, so we need to take the unreduced hadronic tensor (3.6) into consideration:

$$\begin{aligned}
\frac{n_l Q^2}{8} L_{\mu\nu} W^{\mu\nu} = &\Big\{ g_+^2 \Big(l_{1\mu} l_{2\nu} + l_{2\mu} l_{1\nu} - l_1\cdot l_2 g_{\mu\nu}\Big) + i g_-^2 l_{1\alpha} l_{2\beta}\epsilon^{\alpha\beta\mu\nu}\Big\} \\
&\times \Big\{ -g^{\mu\nu} W_1 + \frac{1}{M^2} P^\mu P^\nu W_2 - \frac{i}{2M^2}\epsilon^{\gamma\delta\mu\nu} P_\gamma q_\delta W_3 + \frac{1}{M^2} q^\mu q^\nu W_4 \\
&\quad + \frac{1}{2M^2}(P^\mu q^\nu + q^\mu P^\nu) W_5 + \frac{1}{2M^2}(P^\mu q^\nu - q^\mu P^\nu) W_6 \Big\}
\end{aligned} \tag{D.41}$$

$$\begin{aligned}
= &\, g_+^2 \Big\{ 2(l_1\cdot l_2) W_1 + \Big[\frac{2}{M^2}(l_1\cdot P)(l_2\cdot P) - l_1\cdot l_2\Big] W_2 \\
&+ \Big[\frac{2}{M^2}(l_1\cdot q)(l_2\cdot q) + \frac{Q^2}{M^2}(l_1\cdot l_2)\Big] W_4 \\
&+ \Big[(P\cdot l_1)(q\cdot l_2) + (P\cdot l_2)(q\cdot l_1) - \frac{1}{M^2}(l_1\cdot l_2)(P\cdot q)\Big] W_5 \Big\} \\
&+ g_-^2 \Big\{ \frac{1}{2M^2} l_1^\alpha l_2^\beta P_\gamma q_\delta \epsilon_{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\mu\nu} W_3 \\
&\quad - \frac{i}{2M^2} l_1^\alpha l_2^\beta (P_\mu q_\nu - q_\mu P_\nu)\epsilon^{\alpha\beta\mu\nu} W_6 \Big\},
\end{aligned} \tag{D.42}$$

where we used the fact that every product of one symmetric and one antisymmetric tensor vanishes. Before proceeding, we first need to look at the Levi-Civita tensors

[7] This can be easily deduced by looking at table 2.2 and replacing the factors of Q_i by a 1 for the fractional leptonic charge.

in (D.42). The product of two of these tensors via a contraction of two indices can be simplified:

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\mu\nu} = 2! \cdot g^{\gamma\delta}_{\alpha\beta} = 2(g^\gamma_\alpha g^\delta_\beta - g^\gamma_\beta g^\delta_\alpha). \tag{D.43}$$

Substituting the scalar products calculated in the two sections above and further simplifying gives

$$\begin{aligned}
\frac{n_l Q^2}{8} L_{\mu\nu}W^{\mu\nu} = g_+^2 \Big\{ & Q^2 W_1 + \left[\frac{2}{M^2}M^2 E_1 E_2 - \frac{1}{2}Q^2\right] W_2 \\
& + \left[\frac{2}{M^2}\left(-\frac{1}{4}Q^2\right) + \frac{Q^2}{M^2}\left(\frac{1}{2}Q^2\right)\right] W_4 \\
& + \left[\frac{Q^2}{2M^2}(ME_1 - ME_2) - \frac{1}{M^2}\left(\frac{1}{2}Q^2 M\nu\right)\right] W_5 \Big\} \\
+ g_-^2 \Big\{ & \frac{1}{M^2}\Big((l_1 \cdot P)(l_2 \cdot q) - (l_2 \cdot P)(l_1 \cdot q)\Big) W_3 \\
& - \frac{i}{2M^2} l_{1\alpha} l_{2\beta} \Big(P_\mu (l_{2\nu} - l_{1\nu}) - (l_{2\mu} - l_{1\mu}) P_\nu\Big) \epsilon^{\alpha\beta\mu\nu} W_6 \Big\} \\
= 2E_1E_2 g_+^2 & \left\{ 2W_1 \sin^2\frac{\theta}{2} + \left[1 - \frac{Q^2}{4E_1E_2}\right] W_2 \right\} + g_-^2 \left\{ \frac{Q^2}{2M}(E_1 + E_2) W_3 \right\} \\
= 2E_1E_2 & \left\{ g_+^2 \left[2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2}\right] + g_-^2 \left[\frac{E_1 + E_2}{M} W_3 \sin^2\frac{\theta}{2}\right] \right\}.
\end{aligned} \tag{D.44}$$

The contribution proportional to W_6 vanishes since we can express q^μ in terms of l_1^μ and l_2^μ. Hence, products between symmetric and antisymmetric tensors occur again. The fact that there are no contributions of W_4 and W_5 is not universal but follows from the assumption of massless leptons[8].

The flux factor (D.21) and the phase space (D.33) stay the same compared to the previous calculation. Instead of a factor of e^4/Q^4, we multiply two generalized propagators $G_{B_{1,2}}$ (cf. (5.1)) to take the electroweak couplings into account:

$$\begin{aligned}
d\sigma &= \frac{G_{B_1}G_{B_2}}{F} 4\pi Q^2 L_{\mu\nu}W^{\mu\nu} d\Pi_{\text{LIPS}} \\
\Leftrightarrow \frac{d\sigma}{dE_2 d\cos\theta} &= \frac{2E_2^2}{\pi n_l M} G_{B_1}G_{B_2} \left\{ g_+^2 \left[2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2}\right] \right. \\
& \qquad \left. + g_-^2 \left[\frac{E_1 + E_2}{M} W_3 \sin^2\frac{\theta}{2}\right] \right\}.
\end{aligned} \tag{D.45}$$

As it should, this cross section reduces to the one of the previous chapter (D.34) if one sets

$$G_{B_1}G_{B_2} = G_\gamma^2 = \frac{e^4}{Q^4} = \frac{4\pi\alpha^2}{E_1^2E_2^2\sin^4\frac{\theta}{2}}, \quad g_+^2 = \frac{1}{2} \text{ and } g_-^2 = 0 \tag{D.46}$$

[8] One can easily check this by calculating the scalar products in both prefactors again without demanding $l_1^2 = l_2^2 = 0$.

for pure QED-interaction. This expression can be reformulated using the alternative structure functions

$$F_1 \equiv W_1, \quad F_2 \equiv \frac{\nu}{M} W_2, \quad F_3 \equiv \frac{\nu}{M} W_3 \tag{D.47}$$

and the derivative with respect to the Bjorken x,

$$x \equiv \frac{Q^2}{2(P \cdot q)} = \frac{Q^2}{2M\nu} = \frac{2E_1E_2}{M\nu} \sin^2\frac{\theta}{2} = \frac{E_1E_2}{M\nu}(1-\cos\theta), \tag{D.48}$$

as well as

$$y \equiv \frac{P \cdot q}{P \cdot l_1} = \frac{M\nu}{ME_1} = \frac{\nu}{E_1}. \tag{D.49}$$

We first need to compute both differentials in terms of x and y:

$$dE_2 = d\Big(E_1(1-y)\Big) = -E_1 dy \tag{D.50}$$

$$d\cos\theta = d\left(1 - \frac{M\nu}{E_1E_2}x\right) = -\frac{My}{E_1(1-y)}dx. \tag{D.51}$$

If we substitute

$$E_2 = E_1(1-y) \tag{D.52}$$

and

$$\sin^2\frac{\theta}{2} = \frac{My}{2E_1(1-y)}x, \tag{D.53}$$

we arrive at

$$\begin{aligned}
\frac{d\sigma}{dxdy} &= \frac{My}{1-y}\frac{d\sigma}{dE_2 d\cos\theta} \\
&= \frac{My}{1-y}\frac{E_1^2(1-y)^2}{\pi n_l M} G_{B_1}G_{B_2}\Bigg\{g_+^2\left[2F_1\frac{2My}{2E_1(1-y)}x + \frac{M}{E_1y}F_2\left(1 - \frac{My}{2E_1(1-y)}x\right)\right] \\
&\qquad + g_-^2\left[\frac{E_1(2-y)}{M}\frac{M}{E_1y}F_3\frac{My}{2E_1(1-y)}x\right]\Bigg\} \\
&= \frac{2ME_1}{\pi n_l}G_{B_1}G_{B_2}\left\{g_+^2\left[xy^2F_1 + \left(1 - y - \frac{Mxy}{2E_1}\right)F_2\right] + g_-^2\left[xy\left(1-\frac{y}{2}\right)F_3\right]\right\}.
\end{aligned} \tag{D.54}$$

E. Side calculations: The Breit frame

In the following we give the side calculations for section 4.3. Let us start with the Minkovski coordinates of the hadron momenta P and P_X. We begin with the with the light-cone coordinates (4.29)

$$E_P = \frac{1}{\sqrt{2}}(P^+ + P^-) = \frac{1}{2Q}\left(\frac{Q^2}{\eta} + \eta M^2\right). \tag{E.1}$$

On the other hand, we have (using the definition of the Δ-function (3.15) and the implicit definition of η, (4.26))

$$\begin{aligned} \Delta(-Q^2, P^2, P_X^2) &= \Delta(-Q^2, M^2, (P+q)^2) = \Delta(-Q^2, M^2, M^2 + 2P\cdot q - Q^2) \\ &= \Big[Q^4 + M^4 + (M^2 + 2P\cdot q - Q^2)^2 + 2M^2Q^2 \\ &\quad + 2Q^2(M^2 + 2P\cdot q - Q^2) - 2M^2(M^2 + 2P\cdot q - Q^2)\Big]^{\frac{1}{2}} \\ &= \sqrt{4(P\cdot q)^2 + 4M^2Q^2} = \sqrt{\frac{Q^4}{\eta^2} - 2M^2Q^2 + \eta^2M^4 + 4M^2Q^2} \\ &= \frac{Q^2}{\eta} + \eta M^2, \end{aligned} \tag{E.2}$$

which makes the identity

$$E_P = \frac{1}{2Q}\Delta(-Q^2, P^2, P_X^2) \tag{E.3}$$

obvious. Continuing with P^3, the expression in terms of light-cone coordinates gives

$$P^3 = \frac{1}{\sqrt{2}}(P^+ - P^-) = \frac{1}{2Q}\left(\frac{Q^2}{\eta} - \eta M^2\right) = \frac{1}{2Q}2P\cdot q. \tag{E.4}$$

By simplifying the variable β_1 we obtain

$$\beta_1 = Q^2 - P^2 + P_X^2 = Q^2 - M^2 + M^2 + 2P\cdot q - Q^2 = 2P\cdot q, \tag{E.5}$$

validating its use as the third spacial component of P^μ. The equality of E_P and E_{P_X} can be seen directly by the fact that in the Breit frame q^0 vanishes. For P_X^3, we use that momentum conservation, $P_X^\mu = P^\mu + q^\mu$, so

$$P_X^3 = P^3 + q^3 = \frac{P\cdot q}{Q} - Q = \frac{1}{2Q}(2P\cdot q - 2Q^2). \tag{E.6}$$

Again it is simple to connect this expression to β_2:

$$-\beta_2 = -Q^2 - P^2 + P_X^2 = -Q^2 - M^2 + M^2 + 2P\cdot q - Q^2 = 2P\cdot q - 2Q^2. \quad \text{(E.7)}$$

The last equality that needs some lines of calculation is the explicit expression for the hyperbolic angle, (4.44). As explained at the end of section 4.3, ζ can be obtained in the hadronic configuration of the Breit frame via a scalar product between P^μ and $l_1^\mu + l_2^\mu$. We can reexpress the sum of leptonic momenta as

$$l_1^\mu + l_2^\mu = 2l_1^\mu - q^\mu. \quad \text{(E.8)}$$

The (Lorentz invariant) quantity $P\cdot l_1$ was implicitly calculated in appendix D.1. We can rewrite (D.15) to

$$2P\cdot l_1 = s - M^2. \quad \text{(E.9)}$$

When we also use (4.26) for the scalar product between P^μ and q^μ as well as the expression (E.2) from above, we obtain

$$\cosh\zeta = \frac{2P\cdot(l_1+l_2)}{\Delta(-Q^2,P^2,P_X^2)} = \frac{\eta^2M^2 - Q^2 + 2\eta(s-M^2)}{\eta^2M^2+Q^2}. \quad \text{(E.10)}$$

F. Side calculations: Kinematics in the ACOT formalism

In the following we give the side calculations for section 5.3. We start with the calculation of c_P and c_q. Contracting (5.20) with P^μ and q^μ gives

$$\begin{aligned}
&p_1 \cdot P = M^2 c_P + c_q P \cdot q \\
\Leftrightarrow &\xi P^+ \frac{M^2}{2P^+} + P^+ \frac{m_1^2}{2\xi P^+} = M^2 c_P + \frac{1}{2}\left(\frac{Q^2}{\eta} - \eta M^2\right) c_q \\
\Leftrightarrow &c_P = \frac{1}{2}\left(\xi + \frac{1}{\xi}\frac{m_1^2}{M^2}\right) - \frac{1}{2}\left(\frac{Q^2}{\eta M^2} - \eta\right) c_q
\end{aligned} \tag{F.1}$$

and

$$\begin{aligned}
&p_1 \cdot q = c_P P \cdot q - Q^2 c_q \\
&\Leftrightarrow -\frac{Q}{\sqrt{2}}\frac{m_1^2}{2\xi P^+} + \xi P^+ \frac{Q}{\sqrt{2}} = \frac{1}{2} c_P \left(\frac{Q^2}{\eta} - \eta M^2\right) - Q^2 c_q \\
&\Leftrightarrow \frac{1}{2}\left(\frac{\xi}{\eta} Q^2 - \frac{\eta}{\xi} m_1^2\right) = \frac{1}{2} c_P \left(\frac{Q^2}{\eta} - \eta M^2\right) - Q^2 c_q .
\end{aligned} \tag{F.2}$$

Substituting (F.1) into (F.2) yields

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\xi}{\eta} Q^2 - \frac{\eta}{\xi} m_1^2\right) = \frac{1}{4}\left[\left(\xi + \frac{1}{\xi}\frac{m_1^2}{M^2}\right) - \left(\frac{Q^2}{\eta M^2} - \eta\right) c_q\right]\left(\frac{Q^2}{\eta} - \eta M^2\right) - Q^2 c_q \\
\Leftrightarrow &\frac{1}{2}\left(\frac{\xi}{\eta} Q^2 - \frac{\eta}{\xi} m_1^2\right) = \frac{1}{4}\left[\left(\frac{\xi}{\eta} Q^2 + \frac{1}{\eta\xi}\frac{m_1^2}{M^2} Q^2 - \xi\eta M^2 - \frac{\eta}{\xi} m_1^2\right)\right. \\
&\qquad\qquad\qquad\qquad \left. - \left(\frac{Q^4}{\eta^2 M^2} - 2Q^2 + \eta^2 M^2\right) c_q\right] - Q^2 c_q \\
\Leftrightarrow &\frac{1}{4}\left[\frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_1^2}{Q^2} - \frac{1}{\eta\xi}\frac{m_1^2}{M^2} + \eta\xi\frac{M^2}{Q^2}\right] = -\left(\frac{1}{4}\frac{Q^2}{\eta^2 M^2} + \frac{1}{4}\frac{\eta^2 M^2}{Q^2} + \frac{1}{2}\right) c_q \\
&\qquad\qquad = -\frac{(Q^2 + \eta^2 M^2)^2}{4Q^2\eta^2 M^2} c_q
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow c_q &= -\frac{Q^2\eta^2 M^2}{(Q^2+\eta^2M^2)^2}\left[\frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_1^2}{Q^2} - \frac{1}{\eta\xi}\frac{m_1^2}{M^2} + \eta\xi\frac{M^2}{Q^2}\right] \\
&= \frac{\eta}{\xi(Q^2+\eta^2M^2)^2}\left[\eta^2 m_1^2 M^2 + m_1^2 Q^2 - \xi^2\eta^2 M^4 - \xi^2 Q^2 M^2\right] \\
&= \frac{\eta(m_1^2-\xi^2M^2)}{\xi(Q^2+\eta^2M^2)}. && \text{(F.3)}
\end{aligned}
$$

Substituting back into (F.1) leads to

$$
\begin{aligned}
c_P &= \frac{1}{2}\left(\xi + \frac{1}{\xi}\frac{m_1^2}{M^2}\right) - \frac{1}{2}\frac{\frac{Q^2m_1^2}{M^2} - \xi^2Q^2 - \eta^2m_1^2 + \eta^2\xi^2M^2}{\xi(Q^2+\eta^2M^2)} \\
&= \frac{1}{2}\frac{1}{\xi(Q^2+\eta^2M^2)}\left[\xi^2Q^2 + \frac{Q^2m_1^2}{M^2} + \eta^2\xi^2M^2 + \eta^2m_1^2 - \frac{Q^2m_1^2}{M^2}\right. \\
&\qquad\qquad \left. + \xi^2Q^2 + \eta^2m_1^2 - \eta^2\xi^2M^2\right] \\
&= \frac{\xi^2Q^2+\eta^2m_1^2}{\xi(Q^2+\eta^2M^2)}. && \text{(F.4)}
\end{aligned}
$$

Now we turn to section 5.3.1. We use the definition of p_2^μ, cf. (5.25), the scalar product in ligth-cone coordinates (4.5) and the Δ-function (3.15) to reformulate the on-shell condition:

$$
\begin{aligned}
0 &= p_2^2 - m_2^2 = Q^2\left(\frac{\xi}{\eta} - 1\right)\cdot\left(1 + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}\right) - m_2^2 \\
&= Q^2\frac{\xi^2 + \eta\xi\frac{m_1^2}{Q^2} - \eta^2\frac{m_1^2}{Q^2} - \eta\xi - \eta\xi\frac{m_2^2}{Q^2}}{\eta\xi} = Q^2\frac{\xi^2 + \eta\left(\frac{m_1^2-m_2^2}{Q^2} - 1\right)\xi - \eta^2\frac{m_1^2}{Q^2}}{\eta\xi}. && \text{(F.5)}
\end{aligned}
$$

Solving the numerator for ξ yields

$$
\begin{aligned}
\xi = \chi_\pm &\equiv \frac{\eta}{2}\left(1 - \frac{m_1^2-m_2^2}{Q^2}\right) \pm \eta\sqrt{\frac{1}{4}\left(1 - \frac{m_1^2-m_2^2}{Q^2}\right)^2 + \frac{m_1^2}{Q^2}} \\
&= \eta\frac{Q^2 - m_1^2 + m_2^2}{2Q^2} \pm \frac{\eta}{Q^2}\sqrt{\frac{1}{4}Q^4 - \frac{1}{2}Q^2(m_1^2-m_2^2) + \frac{1}{4}(m_1^2-m_2^2)^2 + Q^2m_1^2} \\
&= \eta\frac{Q^2 - m_1^2 + m_2^2 + m_2^2 \pm \sqrt{Q^4 + m_1^4 + m_2^4 + 2Q^2m_1^2 + 2Q^2m_2^2 - 2m_1^2m_2^2}}{2Q^2} \\
&= \eta\frac{Q^2 - m_1^2 + m_2^2 \pm \Delta(-Q^2, m_1^2, m_2^2)}{2Q^2}, && \text{(F.6)}
\end{aligned}
$$

which implies

$$
p_2^2 - m_2^2 = Q^2\frac{(\xi-\chi_+)(\xi-\chi_-)}{\eta\xi}. \qquad \text{(F.7)}
$$

As in the main section, instead of χ_+ we will use χ in the following lines. Let us now show that $\chi_- < 0$ and hence can be ignored as a possible value for ξ for the rest of this calculation and section 5.3[1]:

$$\begin{aligned}\chi_- &= \frac{\eta}{2Q^2}\Big(Q^2 - m_1^2 + m_2^2 - \sqrt{Q^4 + m_1^4 + m_2^4 + 2Q^2 m_1^2 + 2Q^2 m_2^2 - 2m_1^2 m_2^2}\Big) \\ &\leq \frac{\eta}{2Q^2}\Big(Q^2 - m_1^2 + m_2^2 - \sqrt{Q^4 + m_2^4 + 2Q^2 m_2^2}\Big) = -\frac{\eta}{2Q^2} m_1^2 < 0. \end{aligned} \tag{F.8}$$

In the first step we used

$$m_1^4 + 2Q^2 m_1^2 - 2m_1^2 m_2^2 = m_1^2(m_1^2 + 2Q^2 - m_2^2) > 0, \tag{F.9}$$

which holds since[2]

$$p_1 \cdot q = \xi P \cdot q = -\frac{\xi Q^2}{2x} < 0 \tag{F.10}$$

and thus

$$\begin{aligned} m_2^2 = p_2^2 = (p_1 + q)^2 = m_1^2 + 2p_1 \cdot q - Q^2 < m_1^2 - Q^2 < m_1^2 + 2Q^2 \\ \Leftrightarrow m_1^2 + 2Q^2 - m_2^2 > 0. \end{aligned} \tag{F.11}$$

Using (F.7), the associated δ distribution is

$$\begin{aligned}\delta(p_2^2 - m_2^2) &= \frac{\eta}{Q^2}\delta\Big(\Big(1 - \frac{\chi}{\xi}\Big)\Big(\xi - \chi_-\Big)\Big) \\ &= \frac{\eta}{Q^2}\delta\Big(\xi + \frac{\chi\chi_-}{\xi} - (\chi + \chi_-)\Big) \equiv \frac{\eta}{Q^2}\delta\Big(g(\xi)\Big). \end{aligned} \tag{F.12}$$

The derivative of the argument is

$$g'(\xi) = 1 - \frac{\chi\chi_-}{\xi^2} \tag{F.13}$$

and the roots of $g(\xi)$ are still χ and χ_-. We are now able to simplify the δ-distribution by using the identity (A.5). Since we have shown before that χ_- does not lie in the domain of ξ, we can ignore its contribution proportional to $\delta\Big(g(\xi)\Big)$. Then the identity (A.5) leads to[3]

$$\begin{aligned}\delta(p_2^2 - m_2^2) &= \frac{\eta}{Q^2}\frac{\delta(\xi - \chi)}{1 - \frac{\chi_-}{\chi}} = \frac{\eta\chi}{Q^2}\frac{\delta(\xi - \chi)}{\chi - \chi_-} \\ &= \chi\frac{\delta(\xi - \chi)}{\Delta(-Q^2, m_1^2, m_2^2)} = \frac{\delta\Big(\frac{\xi}{\chi} - 1\Big)}{\Delta(-Q^2, m_1^2, m_2^2)}, \end{aligned} \tag{F.14}$$

where we used the explicit expressions for χ and χ_-, cf. (F.6), in the second last step.

[1]Here we assume $Q^2 > 0$ and $\eta > 0$, which is both valid by definition.
[2]Here we use the definition of $x \in [0, 1)$, cf. (3.11).
[3]Due to $\chi_- < \chi \Rightarrow \frac{\chi_-}{\chi} < 1$ we can make the simplification $\big|f'(\chi)\big| = f'(\chi)$.

G. Side calculations: Helicity formalism in the ACOT formalism

In the following we give the side calculations for section 5.4.1.

G.1. Conversion formulae for structure functions

G.1.1. Hadronic structure functions

Expressions where the helicity structure functions F_λ are given in terms of the standard structure functions F_i can be obtained by exploiting their definition (5.43). To do so, let us first give the eplicit definitions of q^μ, P^μ, $\epsilon^\mu_\pm(P,q)$ and $\epsilon^\mu_0(P,q)$, as derived in chapter 4, again[1]:

$$P^\mu = \frac{1}{2Q}(\Delta(-Q^2, P^2, P_X^2), 0, 0, \beta_1) \tag{G.1}$$

$$q^\mu = -Q(0,0,0,1) \tag{G.2}$$

$$\epsilon^\mu_\pm(P,q) = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0) \tag{G.3}$$

$$\epsilon^\mu_0(P,q) = \frac{Q^2 P^\mu + (P\cdot q)q^\mu}{Q\sqrt{(P\cdot q)^2 + Q^2M^2}}. \tag{G.4}$$

We will also need the complex conjugate of the polarizations:

$$\epsilon^{\star\mu}_\pm(P,q) = \frac{1}{\sqrt{2}}(0, \pm 1, i, 0) \tag{G.5}$$

$$\epsilon^{\star\mu}_0(P,q) = \epsilon^\mu_0(P,q). \tag{G.6}$$

Let us start with $F_\pm$. To calculate the full expression,

$$F_\pm = \epsilon^{\star\mu}_\pm(P,q) W_{\mu\nu} \epsilon^\nu_\pm(P,q), \tag{G.7}$$

we first look at the contractions with the single constituents of

$$\begin{aligned} W^{\mu\nu} \equiv & -g^{\mu\nu}F_1 + \frac{1}{M\nu}P^\mu P^\nu F_2 - \frac{i}{2M\nu}\epsilon^{\alpha\beta\mu\nu}P_\alpha q_\beta F_3 + \frac{1}{M^2}q^\mu q^\nu W_4 \\ & + \frac{1}{2M^2}(P^\mu q^\nu + q^\mu P^\nu)W_5 + \frac{1}{2M^2}(P^\mu q^\nu - q^\mu P^\nu)W_6. \end{aligned} \tag{G.8}$$

[1] In the following, we will need the Minkovski-coordinates of the mentioned momenta and polarizations.

For the one proportional to F_1 we get

$$\epsilon_{\pm}^{\star\mu}(P,q) g_{\mu\nu} \epsilon_{\pm}^{\nu}(P,q) = \frac{1}{2}\Big(-(\pm 1)^2 + i^2\Big) = -1, \tag{G.9}$$

which is simply the normalization of both polarizations. Note that the scalar products $P_\mu \epsilon_{\pm}^{\mu}(P,q)$ and $q_\mu \epsilon_{\pm}^{\mu}(P,q)$ both do not contribute, which can be directly deduced from the coordinate-wise expressions above[2]. These vanishing scalar products directly imply vanishing contributions of F_2, W_4, W_5 and W_6. What remains is the prefactor[3] of F_3,

$$\epsilon_{\pm\mu}^{\star}(P,q) \epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta \epsilon_{\pm\nu}(P,q). \tag{G.10}$$

To compute this Lorentz invariant quantity, we work in the hadronic configuration of the Breit frame, cf. 4.3, where the components are given above. From the only non-vanishing coordinate of q^μ we can deduce that $\beta = 3$ needs to be fulfilled for a non-zero contribution. Since the Levi-Civita tensor forbids two indices with the same value, this immediately implies $\alpha = 0$, since $P^1 = P^2 = 0$. This leaves the two combinations $(1,2)$ and $(2,1)$ for (μ,ν). Hence, the expression above simplifies to

$$\begin{aligned}
\epsilon_{\pm\mu}^{\star}(P,q) \epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta \epsilon_{\pm\nu}(P,q) =& \epsilon^{0312} P^0 (-q^3)(-\epsilon_{\pm}^{\star 1}(P,q))(-\epsilon_{\pm}^{2}(P,q)) \\
&+ \epsilon^{0321} P^0 (-q^3)(-\epsilon_{\pm}^{2\star}(P,q))(-\epsilon_{\pm}^{1}(P,q)) \\
=& \frac{\Delta(-Q^2, P^2, P_x^2)}{2Q} \cdot Q \cdot (\pm 1) \cdot (-i) \\
&- \frac{\Delta(-Q^2, P^2, P_X^2)}{2Q} \cdot Q \cdot i \cdot (\pm 1) \\
=& \mp \frac{1}{2} i \Delta(-Q^2, P^2, P_X^2).
\end{aligned} \tag{G.11}$$

Including the prefactors inside $W_{\mu\nu}$ and using the alternative expressions[4]

$$\Delta(-Q^2, P^2, P_X^2) = 2P \cdot q = \frac{Q^2}{\eta} + \eta M^2, \tag{G.12}$$

$$\frac{1}{\eta} = \frac{1}{2x} + \sqrt{\frac{1}{4x^2} + \frac{M^2}{Q^2}} \tag{G.13}$$

$$\text{and} \quad x = \frac{Q^2}{2M\nu}, \tag{G.14}$$

[2] This fact should not surprise, since it is part of the definiton of transverse polarizations.
[3] We choose the polarization to be covariant so that we can use the conventions that were determined in appendix A for the contravariant components of the Levi-Civita tensor.
[4] The first one was derived in appendix E.

we arrive at

$$
\begin{aligned}
-\epsilon^{\star}_{\pm\mu}(P,q)\frac{i}{2M^2}\epsilon^{\alpha\beta\mu\nu}P_\alpha q_\beta \epsilon_{\pm\nu}(P,q) &= \mp\frac{1}{4M\nu}\left[Q^2\left(\frac{1}{2x}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}\right)+\frac{M^2}{\frac{1}{2x}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}}\right] \\
&= \mp\frac{1}{4}\left[1+\sqrt{1+\frac{Q^2}{\nu^2}}+\frac{M^2}{Q^2}\frac{2x}{\frac{1}{2x}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}}\right] \\
&= \mp\frac{1}{4}\left[1+\sqrt{\frac{Q^2}{\nu^2}}+\frac{\frac{1}{2x}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}+2x\frac{M^2}{Q^2}}{\frac{1}{2x}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}}\right] \\
&= \mp\frac{1}{4}\left[\sqrt{1+\frac{Q^2}{\nu^2}}+\frac{1+\sqrt{1+\frac{Q^2}{\nu^2}}+\frac{Q^2}{\nu^2}}{1+\sqrt{1+\frac{Q^2}{\nu^2}}}\right] \\
&= \mp\frac{1}{4}\left[\sqrt{1+\frac{Q^2}{\nu^2}}+\sqrt{1+\frac{Q^2}{\nu^2}}\frac{1+\sqrt{1+\frac{Q^2}{\nu^2}}+\frac{Q^2}{\nu^2}}{\sqrt{1+\frac{Q^2}{\nu^2}}+1+\frac{Q^2}{\nu^2}}\right] \\
&= \mp\frac{1}{2}\sqrt{1+\frac{Q^2}{\nu^2}}. \qquad \text{(G.15)}
\end{aligned}
$$

Apparently, the final result is

$$
F_\pm = F_1 \mp \frac{1}{2}\sqrt{1+\frac{Q^2}{\nu^2}}F_3. \qquad \text{(G.16)}
$$

Continuing with the same procedure for F_0, we obtain[5]

$$
\epsilon_0^{\star\mu}(P,q)g_{\mu\nu}\epsilon_0^{\nu}(P,q) = \frac{Q^4M^2 - Q^2(P\cdot q)^2 + 2Q^2(P\cdot q)^2}{Q^2\Big((P\cdot q)^2 + Q^2M^2\Big)} = 1 \qquad \text{(G.17)}
$$

and

$$
\begin{aligned}
&\epsilon_0^{\star\mu}(P,q)P_\mu P_\nu \epsilon_0^{\nu}(P,q) = \frac{\Big(Q^2M^2+(P\cdot q)^2\Big)^2}{Q^2\Big((P\cdot q)^2+Q^2M^2\Big)} \\
&= M^2 + \frac{1}{4Q^2}\left(\frac{Q^2}{\eta}-\eta M^2\right)^2 = \frac{M^2}{2}+\frac{\eta^2}{4}\left(\frac{Q^2}{\eta^4}+\frac{M^4}{Q^2}\right) \\
&= \frac{M^2}{2}+\frac{\eta^2}{4}\left(\frac{M^4}{Q^2}+2\frac{M^2}{x}\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}+2\frac{M^2}{x^2}+\frac{Q^2}{x^3}\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}+\frac{Q^2}{2x^4}+\frac{M^4}{Q^2}\right) \\
&= \frac{\eta^2 M}{2x}\left(\frac{Mx}{\eta^2}+\frac{M^3x}{Q^2}+M\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2\nu}{Q^2}+2\frac{M\nu^2}{Q^2}\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2\nu^3}{Q^4}\right)
\end{aligned}
$$

[5] Again, this is only the normalization of $\epsilon_0^{\star\mu}(P,q)$.

$$
\begin{aligned}
&= \frac{\eta^2 M\nu}{2x}\left(\frac{M}{\nu}\left(\frac{1}{2x}+\frac{M^2}{Q^2}+\sqrt{\frac{1}{4x^2}+\frac{M^2}{Q^2}}\right)+\frac{M^2}{2\nu^2}+\frac{M}{\nu}\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2}{Q^2}\right.\\
&\quad\left.+2\frac{M\nu}{Q^2}\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2\nu^2}{Q^4}\right)\\
&= \frac{M\nu}{2x}\frac{1}{\frac{M^2\nu^2}{Q^4}\left(2+\frac{Q^2}{\nu^2}+2\sqrt{1+\frac{Q^2}{\nu^2}}\right)}\left[\frac{M}{\nu}\left(\frac{M\nu}{Q^2}+\frac{M}{2\nu}+\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}\right)+\frac{M^2}{2\nu^2}\right.\\
&\quad\left.+\frac{M}{\nu}\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2}{Q^2}+2\frac{M\nu}{Q^2}\sqrt{\frac{M^2\nu^2}{Q^4}+\frac{M^2}{Q^2}}+2\frac{M^2\nu^2}{Q^4}\right]\\
&= \frac{M\nu}{2x}\left(1+\frac{Q^2}{\nu^2}\right)\frac{\frac{Q^2}{\nu^2}+\frac{Q^4}{2\nu^4}+\frac{Q^2}{\nu^2}\sqrt{1+\frac{Q^2}{\nu^2}}+\frac{Q^4}{2\nu^4}+\frac{Q^2}{\nu^2}\sqrt{1+\frac{Q^2}{\nu^2}}+2\frac{Q^2}{\nu^2}+2\sqrt{1+\frac{Q^2}{\nu^2}}+2}{2+2\frac{Q^2}{\nu^2}+\frac{Q^2}{\nu^2}+\frac{Q^4}{\nu^4}+2\sqrt{1+\frac{Q^2}{\nu^2}}+2\frac{Q^2}{\nu^2}\sqrt{1+\frac{Q^2}{\nu^2}}}\\
&= \frac{M\nu}{2x}\left(1+\frac{Q^2}{\nu^2}\right). \qquad \text{(G.18)}
\end{aligned}
$$

In addition, we immediately see that

$$\epsilon_0^{\star\mu}(P,q)\epsilon^{\alpha\beta\mu\nu}P_\alpha q_\beta \epsilon_0^\mu(P,q)=0 \qquad \text{(G.19)}$$

since $\epsilon_0^\mu(P,q)$ is a linear combination of P^μ and q^μ[6]. The same holds for the contributions proportional to W_4, W_5 and W_6:

$$\epsilon_0^{\star\mu}(P,q)q_\mu q_\nu\epsilon_0^\nu(P,q)=\left(\frac{Q^2(P\cdot q)-Q^2(P\cdot q)}{Q^2((P\cdot q)^2+Q^2M^2)}\right)^2=0, \qquad \text{(G.20)}$$

$$
\begin{aligned}
\epsilon_0^{\star\mu}(P,q)\Big(P_\mu q_\nu+q_\mu q_\nu\Big)\epsilon_0^\nu(P,q) &= 2\epsilon_0^\mu(P,q)P_\mu P_\nu\epsilon_0^\nu(P,q)\\
&= 2\frac{\Big(Q^2M^2+(P\cdot q)^2\Big)\cdot\Big(Q^2(P\cdot q)-Q^2(P\cdot q)\Big)}{Q^2\big((P\cdot q)^2+Q^2M^2\big)}\\
&= 0 \qquad \text{(G.21)}
\end{aligned}
$$

$$\text{and } \epsilon_0^{\star\mu}(P,q)\Big(P_\mu q_\nu-q_\mu P_\nu\Big)\epsilon_0^\nu(P,q)=\epsilon_0^\mu(P,q)\epsilon_0^\nu(P,q)\Big(P_\mu q_\nu+q_\mu P_\nu\Big)=0, \qquad \text{(G.22)}$$

where we used identity (G.6). This fact directly leads to a product of one symmetric and one antisymmetric tensor in the last equation, which is obviously zero.

All this summed up gives

$$F_0=-F_1+\frac{1}{2x}\left(1+\frac{Q^2}{\nu^2}\right)F_2. \qquad \text{(G.23)}$$

[6]The same argument was used in appendix D.3.

G.1.2. Partonic structure functions

The exact same procedure as in the hadronic case can be applied to partonic structure functions. In this case, we use the Lorentz tensor decomposition of the partonic tensor,

$$\begin{aligned}\hat{W}_i^{\mu\nu} \equiv & - g^{\mu\nu}\hat{W}_1^i + \frac{1}{Q^2}p_1^\mu p_1^\nu \hat{W}_2^i - \frac{i}{2Q^2}\epsilon^{\alpha\beta\mu\nu}p_{1\alpha}q_\beta \hat{W}_3^i + \frac{1}{Q^2}q^\mu q^\nu \hat{W}_4^i \\ & + \frac{1}{2Q^2}(p_1^\mu q^\nu + q^\mu p_1^\nu)\hat{W}_5^i + \frac{1}{2Q^2}(p_1^\mu q^\nu - q^\mu p_1^\nu)\hat{W}6^i.\end{aligned} \tag{G.24}$$

In the hadronic configuration of the Breit frame the partonic momentum is[7]

$$p_1^\mu = \frac{Q}{2}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_i^2}{Q^2}, 0, 0, \frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_i^2}{Q^2}\right). \tag{G.25}$$

We set $p_1^2 = m_i^2$ to make the dependence on the partonic flavor explicit. It will be also helpful to use the partonic Bjorken-x, cf. equation (3.28):

$$z = \frac{Q^2}{2p_1 \cdot q} = \frac{x}{\xi}. \tag{G.26}$$

We choose the partonic momentum as the reference vector for the polarizations so that the scalar product between q^μ and p_1^μ is the only one that appears in the calculation:

$$\epsilon_\pm^\mu(p_1, q) = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0) \tag{G.27}$$

$$\epsilon_0^\mu(p_1, q) = \frac{Q^2 p_1^\mu + (p_1 \cdot q)q^\mu}{Q\sqrt{(p_1 \cdot q)^2 + Q^2 m_i^2}}. \tag{G.28}$$

Let us start with

$$\hat{W}_\pm^i = \epsilon_{\pm\mu}^\star(p_1, q)\hat{W}_i^{\mu\nu}\epsilon_{\pm\nu}(p_1, q). \tag{G.29}$$

Replacing hadronic with partonic quantities does not change the fact that many contractions vanish, as explained in the previous section. In addition, we still have

$$\epsilon_\pm^{\star\mu}(p_1, q)g_{\mu\nu}\epsilon_\pm^\mu(p_1, q) = -1. \tag{G.30}$$

What is left is the prefactor of $\hat{W}_3^i$. The calculation is completely similar to the hadronic case, except for a partonic instead of a hadronic momentum component. Therefore, the final result is

$$\hat{W}_\pm^i = \hat{W}_1^i \mp \frac{1}{4}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_i^2}{Q^2}\right)\hat{W}_3^i. \tag{G.31}$$

[7] The light-cone coordinates of p_1^μ are given in equation (5.22). To obtain the according Minkovski coordinates, we use the relation (4.3).

For

$$\hat{W}_0^i = \epsilon_{0\mu}^{\star}(p_1,q)\hat{W}_i^{\mu\nu}\epsilon_{0\nu}(p_1,q), \tag{G.32}$$

the procedure is identical. The normalization is

$$\epsilon_0^{\mu}(p_1,q)g_{\mu\nu}\epsilon_0^{\mu}(p_1,q) = 1 \tag{G.33}$$

and the only non-vanishing contraction gives

$$\begin{aligned}\epsilon_0^{\mu}(p_1,q)p_{1\mu}p_{1\nu}\epsilon_0^{\nu}(p_1,q) =& \left(\frac{Q^2m_i^2 + (p_1\cdot q)^2}{Q\sqrt{(p_1\cdot q)^2 + Q^2m_i^2}}\right)^2 = \frac{1}{Q^2}\Big((p_1\cdot q)^2 + m_i^2Q^2\Big)\\ =& \frac{Q^2}{(2z)^2} + m_i^2.\end{aligned} \tag{G.34}$$

Hence,

$$\hat{W}_0^i = -\hat{W}_1^i + \left(\frac{\xi^2}{4x^2} + \frac{m_i^2}{Q^2}\right)\hat{W}_2^i. \tag{G.35}$$

G.2. Helicity currents, amplitudes and the differential cross section

The general formula for differential cross sections is (cf. (3.13))

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{F} d\Pi_{\text{LIPS}}. \tag{G.36}$$

At first, we will derive the squared and averaged amplitude. According to section 2.3, an LO amplitude in DIS, formulated in the helicity formalism, is

$$\mathcal{M}^{\rho_2,\sigma_2}_{\rho_1,\sigma_1} = \sum_{\kappa,\lambda} J_{H}{}^{\sigma_2}_{\sigma_1\kappa} \frac{g_B^2 d^1(\zeta)^{\kappa\lambda}}{Q^2 + M_B^2} J_{L}{}^{\rho_2}_{\rho_1\lambda}. \tag{G.37}$$

Squaring and averaging yields[8]

$$\begin{aligned}\overline{|\mathcal{M}|^2} =& \frac{1}{n_l} \sum_{\rho_{1,2},\sigma_{1,2}} \mathcal{M}^{\rho_1,\sigma_1\star}_{\rho_2,\sigma_2} \mathcal{M}^{\rho_2,\sigma_2}_{\rho_1,\sigma_1} \\ =& \frac{1}{n_l} G_{B_1} G_{B_2} \sum_{\rho_{1,2},\lambda,\lambda',\kappa,\kappa'} v_\lambda v_{\lambda'} \left(J_{L}{}^{\rho_2}_{\rho_1\lambda'}\right)^\star J_{L}{}^{\rho_1}_{\rho_2\lambda} d^1(\zeta)^{\lambda\kappa} d^1(\zeta)^{\lambda'\kappa'} 4\pi W_{\lambda'\lambda},\end{aligned} \tag{G.38}$$

since $d^1(\zeta)$ is orthogonal. In addition, we used the notation $G_{B_{1,2}}$ for the bosonic couplings which were introduced in section 5. The hadronic helicity tensor is defined as

$$\begin{aligned}W_{\lambda'\lambda} \equiv& \frac{1}{4\pi} \sum_{\sigma_{1,2}} \left(J_{H}{}^{\sigma_2}_{\sigma_1\lambda'}\right)^\star J_{H}{}^{\sigma_1}_{\sigma_2\lambda} \\ =& \frac{1}{4\pi} \sum_{\sigma_{1,2}} \epsilon^{\star\mu}_\lambda(P,q) \langle P,\sigma_1| J_{H,\mu}(0)^\dagger |P_X,\sigma_2\rangle \langle P_X,\sigma_2| J_{H,\nu}(0) |P,\sigma_1\rangle \epsilon^\nu_{\lambda'}(P,q) \\ =& \epsilon^{\star\mu}_\lambda(P,q) W_{\mu\nu} \epsilon^\nu_{\lambda'}(P,q),\end{aligned} \tag{G.39}$$

where the factor of 4π arises due to the definition of $W_{\mu\nu}$, cf. equation (3.5). Note that

$$W_{\lambda\lambda} = F_\lambda. \tag{G.40}$$

Let us start with a closer look at

$$J_{L}{}^{\rho_2}_{\rho_1\lambda} = \overline{u}_{\rho_2}(l_2)\epsilon^{\star\mu}_\lambda(l_1,q)\Gamma_\mu u_{\rho_1}(l_1), \tag{G.41}$$

where the generalized coupling is

$$\Gamma^\mu = \gamma^\mu [g_R(1+\gamma^5) + g_L(1-\gamma^5)]. \tag{G.42}$$

The reader shall be reminded of the fact that helicity eigenstates can be identified with chirality eigenstates in the limit of vanishing lepton masses[9], which is the

[8] Keep in mind that the sums over external helicities are implicit through the Einstein notation.
[9] Cf. section 2.3 for further information.

case in the ACOT formalism. The generalized coupling above can be decomposed into chirality projection operators proportional to their according couplings $g_{R,L}$. This leads to interactions only between purely left- and right- handed particles. For the gauge boson, a coupling to longitudinal degrees of freedom is proportional to a particle's masses and therefore vanishes in the presence of massless leptons. Moreover, mixed polarizations lead to zeros, what can be clearly seen if one e.g. multiplies (G.56) with (G.58) in the calculation below. Hence, the only two relevant configurations of the leptonic helicity current above are $J_L{}_{-\frac{1}{2}-}^{-\frac{1}{2}}$ and $J_L{}_{+\frac{1}{2}+}^{+\frac{1}{2}}$.

We start our explicit calculation for the case of all helicities being left-handed. $J_L{}_{\rho_1\lambda}^{\rho_2}$ is a Lorentz scalar, so it must be frame independent. Therefore, we choose a frame in which the lepton momenta $l_{1,2}^{\mu}$ are as simple as possible: The standard lepton configuration (cf. section 4.3), where

$$l_1^{\mu} = \frac{Q}{2}(1,0,0,-1) \tag{G.43}$$

$$\text{and } l_2^{\mu} = \frac{Q}{2}(1,0,0,1). \tag{G.44}$$

We already calculated the helicity spinors for an arbitrary momentum in appendix C.2:

$$\xi_{+\frac{1}{2}}(\boldsymbol{p}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{G.45}$$

$$\xi_{-\frac{1}{2}}(\boldsymbol{p}) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{G.46}$$

For $l_{1,2}$, θ takes the values 0 and π, while ϕ can be set to 0 in both cases. Hence, we obtain

$$\xi^1_{+\frac{1}{2}} \equiv \xi_{+\frac{1}{2}}(l_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{G.47}$$

$$\xi^1_{-\frac{1}{2}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{G.48}$$

and

$$\xi^2_{+\frac{1}{2}} \equiv \xi_{+\frac{1}{2}}(l_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{G.49}$$

$$\xi^2_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{G.50}$$

According to section 2.3.2, the full helicity spinors are[10]

$$u_{-\frac{1}{2}}(l_1) = \begin{pmatrix} \sqrt{l_1 \cdot \sigma}\xi^1_{-\frac{1}{2}} \\ \sqrt{l_1 \cdot \overline{\sigma}}\xi^1_{-\frac{1}{2}} \end{pmatrix} = \sqrt{\frac{Q}{2}} \begin{pmatrix} \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \sqrt{Q} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix} = \sqrt{Q} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{G.51}$$

and (following the exact same steps)

$$u_{-\frac{1}{2}}(l_2) = \begin{pmatrix} \sqrt{l_2 \cdot \sigma}\xi^2_{-\frac{1}{2}} \\ \sqrt{l_2 \cdot \overline{\sigma}}\xi^2_{-\frac{1}{2}} \end{pmatrix} = \sqrt{\frac{Q}{2}} \begin{pmatrix} \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \sqrt{Q} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \sqrt{Q} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{G.52}$$

which gives

$$\overline{u}_{-\frac{1}{2}}(l_2) = u^\dagger_{-\frac{1}{2}}\gamma^0 = \sqrt{Q}\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \sqrt{Q}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}. \tag{G.53}$$

We immediately see that

$$(1+\gamma^5)u_{-\frac{1}{2}}(l_1) = 2\text{diag}(0,0,1,1)u_{-\frac{1}{2}}(l_1) = 0 \tag{G.54}$$

and

$$(1-\gamma^5)u_{-\frac{1}{2}}(l_1) = 2\text{diag}(1,1,0,0)u_{-\frac{1}{2}}(l_1) = 2u_{-\frac{1}{2}}(l_1), \tag{G.55}$$

meaning that $u_{-\frac{1}{2}}(l_1)$ is indeed a purely left-handed spinor. Using the Condon-Shortley convention introduced in section 2.3, we obtain

$$\epsilon^{\star\mu}_{-}\gamma_\mu = \frac{1}{\sqrt{2}}(0,-1,i,0)\cdot\gamma = \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2) = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{G.56}$$

[10] Taking the square root of both matrices can be simply achieved by taking the square root of their components because they are already diagonalized.

All this multiplied together yields

$$J_{L_{-\frac{1}{2}-}^{-\frac{1}{2}}} = \sqrt{Q}\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\sqrt{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix} g_L 2\sqrt{Q}\begin{pmatrix}-1\\ 0\\ 0\\ 0\end{pmatrix} = g_L\sqrt{8}Q. \quad \text{(G.57)}$$

For $J_{L_{+\frac{1}{2}+}^{+\frac{1}{2}}}$, we have

$$u_{+\frac{1}{2}}(l_1) = \sqrt{Q}\begin{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\end{pmatrix}\end{pmatrix} = \sqrt{Q}\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix} \quad \text{(G.58)}$$

$$\Rightarrow (1+\gamma^5)u_{+\frac{1}{2}}(l_1) = 2u_{+\frac{1}{2}}(l_1) \text{ and } (1-\gamma^5)u_{+\frac{1}{2}}(l_1) = 0, \quad \text{(G.59)}$$

$$u_{+\frac{1}{2}}(l_2) = \sqrt{Q}\begin{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix}\end{pmatrix} = \sqrt{Q}\begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix}$$

$$\Rightarrow \overline{u}_{+\frac{1}{2}}(l_2) = \sqrt{Q}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} \quad \text{(G.60)}$$

$$\text{and } \epsilon_+^{\star\mu}\gamma_\mu = \frac{1}{\sqrt{2}}(0,1,i,0)\cdot\gamma = -\frac{1}{\sqrt{2}}(\gamma^1 + i\gamma^2) = \sqrt{2}\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \quad \text{(G.61)}$$

Hence,

$$J_{L_{+\frac{1}{2}+}^{+\frac{1}{2}}} = \sqrt{Q}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\sqrt{2}\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} g2\sqrt{Q}\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix} = -g\sqrt{8}Q. \quad \text{(G.62)}$$

Of course this kind of explicit calculations are not possible for the hadronic part. However, since we already know the leptonic currents in the pure left- and right-handed case, helicity conservation also implies that the only contributions on the hadronic side are of type[11] $W_{\lambda\lambda} = F_\lambda$. These helicity structure functions were already covered in great detail in appendix G.1.1.

This directly implies that the two factors in (G.38), which keep track of the different normalizations of $\epsilon^\mu_\lambda(P,q)$, cancel:

$$v_\lambda v_{\lambda'} \to v_\lambda^2 = 1, \quad \text{(G.63)}$$

[11] Due to their identical helicity eigenvalue, mixings between $\epsilon^\mu_0(P,q)$ and $\epsilon^\mu_q(P,q)$ are not forbidden. We cover these cases in the following section.

since $|v_\lambda|= 1$ for all helicities λ.

With all these informations, we can simplify the squared amplitude:

$$\begin{aligned}\overline{|\mathcal{M}|^2} =&\frac{32\pi Q^2}{n_l} G_{B_1} G_{B_2} \sum_\lambda \Big\{ g_R^2 \Big(d^1(\zeta)^{+\lambda}\Big)^2 F_\lambda + g_L^2 \Big(d^1(\zeta)^{-\lambda}\Big)^2 F_\lambda \Big\} \\ =&\frac{32\pi Q^2}{n_l} G_{B_1} G_{B_2} \Big\{ g_R^2 \Big[F_+ \frac{(1+\cosh\zeta)^2}{4} + F_0 \frac{\sinh^2\zeta}{2} + F_- \frac{(1-\cosh\zeta)^2}{4} \Big] \\ &+ g_L^2 \Big[F_+ \frac{(1-\cosh\zeta)^2}{4} + F_0 \frac{\sinh^2\zeta}{2} + F_- \frac{(1+\cosh\zeta)^2}{4} \Big] \Big\},\end{aligned} \tag{G.64}$$

where we used the explicit matrix elements of $d^1(\zeta)$, cf. (4.49). The expression in the curly brackets can be further simplified by inserting the alternative couplings $g_\pm = g_R^2 \pm g_L^2$:

$$\begin{aligned}\Big\{ \dots \Big\} =&\Big\{ \frac{1}{4} g_+^2 (1+\cosh^2\zeta) F_+ - \frac{1}{2} g_-^2 \cosh\zeta F_+ + \frac{1}{2} g_+^2 \sinh^2\zeta F_0 \\ &+ \frac{1}{4} g_+^2 (1+\cosh^2\zeta) F_- + \frac{1}{2} g_-^2 \cosh\zeta F_- \Big\} \\ =&\frac{1}{2} \Big\{ g_+^2 \Big[\frac{1}{2} (1+\cosh^2\zeta (F_+ + F_-)) + \sinh^2\zeta F_0 \Big] \\ &+ g_-^2 \Big[\cosh\zeta (F_- - F_+) \Big] \Big\}.\end{aligned} \tag{G.65}$$

Together with the Lorentz invariant Flux factor

$$F = 4ME_1 \tag{G.66}$$

and phase space

$$d\Pi_{\text{LIPS}} = \frac{ME_1 y}{8\pi^2} dx dy, \tag{G.67}$$

both the same as in appendix D.3[12] we arrive at

$$\begin{aligned}\frac{d\sigma}{dxdy} = \frac{\overline{|\mathcal{M}^2|}}{F} d\Pi_{\text{LIPS}} =&\frac{yQ^2}{2\pi n_l} G_{B_1} G_{B_2} \Big\{ g_+^2 \Big[\frac{1}{2} \Big(1+\cosh^2\zeta\Big) \Big(F_+ + F_-\Big) + \sinh^2\zeta F_0 \Big] \\ &+ g_-^2 \Big[\cosh\zeta \Big(F_- - F_+ \Big) \Big] \Big\}.\end{aligned} \tag{G.68}$$

G.2.1. The massive lepton case

We claimed in section 5.4.1 that when considering massive leptons we obtain new contributions proportional to W_{qq}, W_{0q} and W_{q0} in the squared amplitude due to

[12] There, the same differential cross section in terms of the usual structure functions was derived.

an omitted contribution of

$$\frac{\epsilon_q^\mu(P,q)\epsilon_q^\nu(P,q)}{M^2} = \frac{q^\mu q^\nu}{M^2 Q^2} \tag{G.69}$$

in the full propagator (2.86) of a massive gauge boson. In this section, we will justify this claim with a short calculation. Since every structure function above contains at least one contribution from $\epsilon_q^\mu(P,q)$[13], at least one $J_{L\,\rho_1 q}^{\rho_2}$ occurs as well. In the case of massive leptons, we are no longer in a situation where helicities correspond to pure left- or right-handed couplings.

As an example, let us explicitly compute $J_{L\,-\frac{1}{2}q}^{+\frac{1}{2}}$. In the massive case, also the leptonic momenta $l_{1,2}$ change. In the leptonic configuration of the Breit frame we still deal with vanishing transverse momenta. Together with momentum conservation, $l_1^\mu = l_2^\mu + q^\mu$, and on-shell conditions $l_{1,2}^2 = m_l^2$ we obtain

$$l_1^\mu = \left(\sqrt{m_l^2 + \frac{Q^2}{4}}, 0, 0, -\frac{Q}{2}\right) \tag{G.70}$$

$$\text{and } l_2^\mu = \left(\sqrt{m_l^2 + \frac{Q^2}{4}}, 0, 0, +\frac{Q}{2}\right). \tag{G.71}$$

Therefore,

$$\begin{aligned} u_{-\frac{1}{2}}(l_1) &= \begin{pmatrix} \sqrt{l_1\cdot\sigma}\,\xi^1_{-\frac{1}{2}} \\ \sqrt{l_1\cdot\bar\sigma}\,\xi^1_{-\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\begin{pmatrix} \sqrt{m_l^2+\frac{Q^2}{4}}+\frac{Q}{2} & 0 \\ 0 & \sqrt{m_l^2+\frac{Q^2}{4}}-\frac{Q}{2} \end{pmatrix}}\cdot\begin{pmatrix}-1\\0\end{pmatrix} \\ \sqrt{\begin{pmatrix} \sqrt{m_l^2+\frac{Q^2}{4}}-\frac{Q}{2} & 0 \\ 0 & \sqrt{m_l^2+\frac{Q^2}{4}}+\frac{Q}{2} \end{pmatrix}}\cdot\begin{pmatrix}-1\\0\end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{\sqrt{m_l^2+\frac{Q^2}{4}}+\frac{Q}{2}} \\ 0 \\ -\sqrt{\sqrt{m_l^2+\frac{Q^2}{4}}-\frac{Q}{2}} \\ 0 \end{pmatrix} \end{aligned} \tag{G.72}$$

[13] As explained in section 2.3.2.1, the reason for the mixing between $\epsilon_q^\mu(P,q)$ and $\epsilon_0^\mu(P,q)$ occurs because both polarizations correspond to the helicity $\lambda = 0$.

$$\text{and } u_{+\frac{1}{2}}(l_2) = \begin{pmatrix} \sqrt{l_2 \cdot \sigma}\xi^2_{+\frac{1}{2}} \\ \sqrt{l_2 \cdot \overline{\sigma}}\xi^2_{+\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\begin{pmatrix} \sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2} & 0 \\ 0 & \sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2} \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{\begin{pmatrix} \sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2} & 0 \\ 0 & \sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2} \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2}} \\ 0 \\ \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2}} \\ 0 \end{pmatrix} \tag{G.73}$$

$$\Rightarrow \overline{u}_{+\frac{1}{2}}(l_2) = \begin{pmatrix} \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2}} & 0 & \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2}} & 0 \end{pmatrix}. \tag{G.74}$$

Hence, we arrive at

$$\begin{aligned} J_{L^{+\frac{1}{2}}_{-\frac{1}{2}q}} &= \overline{u}_{+\frac{1}{2}}(l_2)\epsilon_q^{\star\mu}(P,q)\Gamma_\mu\Big)u_{-\frac{1}{2}}(l_1) \\ &= \begin{pmatrix} \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2}} & 0 & \sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2}} & 0 \end{pmatrix} \\ &\quad \times (-1) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} 2 \begin{pmatrix} -g_L\sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2}} \\ 0 \\ -g_R\sqrt{\sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2}} \\ 0 \end{pmatrix} \\ &= 2\sqrt{\left(\sqrt{m_l^2 + \frac{Q^2}{4}} + \frac{Q}{2}\right)\left(\sqrt{m_l^2 + \frac{Q^2}{4}} - \frac{Q}{2}\right)}(g_R - g_L) \\ &= 2m_l(g_R - g_L). \end{aligned} \tag{G.75}$$

Since $J_{L^{\sigma_2}_{\sigma_1 q}}$ is a Lorentz scalar, this result is valid in every Lorentz frame. The other values of $J_{L^{\rho_2}_{\rho_1 q}}$ can be calculated in a similar way and reveal the same behavior regarding m_l. Note that also in the case of chirality conservation (e.g. in QED), where $g_R = g_L$, the current above vanishes.

In summary, we saw that the non-zero contributions of $J_{L^{\sigma_2}_{\sigma_1 q}}$ are proportional to the lepton mass and hence vanish in the limit of massless leptons. This is the reason why we can safely ignore the second term in the propagator (2.86).

G.3. Factorization of structure functions

We start by computing the coefficients κ^i_k of identity (5.63). First, we see that in the general tensor compositions of $W^{\mu\nu}$ and $\hat{W}^{\mu\nu}$, (5.52) and (5.56), only the metric tensor $g^{\mu\nu}$ appears on both sides. Hence, F_1 and $\hat{W}_1$ are simply connected with a factor of 1:

$$F_1 = \sum_i f_i \otimes \hat{W}_1. \tag{G.76}$$

For the remaining structure functions, we boil equation (5.51) down to a scalar equation by contracting with $P_\mu P_\nu$. Obviously, only the symmetric contributions survive and we obtain

$$\begin{aligned}\frac{M^3}{\nu}F_2 + \frac{(P\cdot q)^2}{M^2}W_4 + (P\cdot q)W_5 = \sum_i f_i \otimes \Bigg\{ &\frac{(p_1\cdot P)^2}{Q^2}\hat{W}_2 \\ &+ \frac{(P\cdot q)^2}{Q^2}\hat{W}_4 + \frac{(p_1\cdot P)(P\cdot q)}{Q^2}\hat{W}_5\Bigg\}.\end{aligned} \tag{G.77}$$

We see that each structure function on the left-hand-side corresponds to one unique order of $(P\cdot q)$. The trick is now to express every other scalar product in terms of $(P\cdot q)$ and, after that, also sort the right-hand-side in orders of it. The decomposition (cf. (5.20))

$$p_1^\mu = c_P P^\mu + c_q q^\mu \tag{G.78}$$

yields

$$p_1\cdot P = c_P M^2 + c_q (P\cdot q) \tag{G.79}$$

$$\text{and } p_1\cdot q = c_P (P\cdot q) - c_q Q^2. \tag{G.80}$$

Hence,

$$\begin{aligned}\frac{M^3}{\nu}F_2 + (P\cdot q)W_5 + \frac{(P\cdot q)^2}{M^2}W_4 = \sum_i f_i \otimes \Bigg\{ &c_P^2\frac{M^4}{Q^2}\hat{W}_2 \\ &+ \left[2c_Pc_q\frac{M^2}{Q^2}\hat{W}_2 + c_P\frac{M^2}{Q^2}\hat{W}_5\right](P\cdot q) \\ &+ \left[c_q^2\frac{1}{Q^2}\hat{W}_2 + \frac{1}{Q^2}\hat{W}_4 + c_q\frac{1}{Q^2}\hat{W}_5\right](P\cdot q)^2\Bigg\}.\end{aligned} \tag{G.81}$$

Evaluating order by order yields[14]

$$F_2 = \sum_i f_i \otimes \frac{c_P^2}{2x} \hat{W}_2 \tag{G.82}$$

$$W_4 = \sum_i f_i \otimes \left\{ \frac{c_q^2 M^2}{Q^2} \hat{W}_2 + \frac{M^2}{Q^2} \hat{W}_4 + \frac{c_q M^2}{Q^2} \hat{W}_5 \right\} \tag{G.83}$$

$$W_5 = \sum_i f_i \otimes \left\{ \frac{2 c_P c_q M^2}{Q^2} \hat{W}_2 + \frac{c_P M^2}{Q^2} \hat{W}_5 \right\}, \tag{G.84}$$

where one can read of all coefficients for the symmetric tensor contributions. As explained in section 5.4.2, these equations become diagonal when going back to the naive parton model, i.e. setting $c_P = \xi$ and $c_q = 0$ again.

For the antisymmetric tensor, we contract with $(P_\mu q_\nu - q_\mu P_\nu)$. Since $\epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta P_\mu q_\nu$ and $\epsilon^{\alpha\beta\mu\nu} P_\alpha q_\beta q_\mu P_\nu$ are products of symmetric and antisymmetric tensors itself, there are no contributions of W_3 and $\hat{W}_3$ (and obviously also no contributions from symmetric parts of the decomposition). Therefore, W_6 is diagonal and we obtain

$$\begin{aligned} \frac{1}{M^2}\Big(-M^2 Q^2 - (P\cdot q)^2\Big) W_6 &= \sum_i f_i \otimes \left(-(p_1 \cdot P) - c_P \frac{(p_1\cdot q)(P\cdot q)}{Q^2} \right) \hat{W}_6 \\ &= \sum_i f_i \otimes \left(-c_P M^2 - c_P \frac{(P\cdot q)^2}{Q^2} \right) \hat{W}_6 \\ \Leftrightarrow \qquad W_6 &= \sum_i f_i \otimes \frac{c_P M^2}{Q^2} \hat{W}_6. \end{aligned} \tag{G.85}$$

Finally, we contract with $\epsilon_{\mu\nu\gamma\delta} P^\gamma q^\delta$ and use the identity (D.43), which yields

$$\begin{aligned} \frac{1}{M\nu}\Big(g^\alpha_\delta g^\beta_\delta - g^\beta_\gamma g^\alpha_\delta\Big) P_\alpha q_\beta P^\gamma q^\delta F_3 &= \sum_i f_i \otimes \frac{1}{Q^2}\Big(g^\alpha_\delta g^\beta_\delta - g^\gamma_\beta g^\alpha_\delta\Big) p_{1\alpha} q_\beta p_1^\gamma q^\delta \hat{W}_3 \\ \Leftrightarrow \qquad \frac{1}{M\nu}\Big(M^2 Q^2 + (P\cdot q)^2\Big) F_3 &= \sum_i f_i \otimes \left(c_P M^2 + c_P \frac{(P\cdot q)^2}{Q^2} \right) \hat{W}_3 \\ \Leftrightarrow \qquad \Big(M^2 Q^2 + (P\cdot q)^2\Big) F_3 &= \sum_i f_i \otimes \left(c_P \frac{M^2 Q^2}{2x} + c_P \frac{(P\cdot q)^2}{2x} \right) \hat{W}_3 \\ \Leftrightarrow \qquad F_3 &= \sum_i f_i \otimes \frac{c_P}{2x} \hat{W}_3. \end{aligned} \tag{G.86}$$

In the case of helicity structure functions, the only necessary equality that needs to be shown is the one between both polarization sets $\epsilon^\mu_\lambda(P,q)$ and $\epsilon^\mu_\lambda(p_1,q)$. When we recall their definitions, which are given in section 2.3.1, we see that the reference vector, P^μ or p_1^μ in our case, only enters in the definition of the longitudinal polarization,

$$\epsilon^\mu_0(P,q) = \frac{Q^2 P^\mu + (P\cdot q) q^\mu}{Q\sqrt{(P\cdot q)^2 + Q^2 P^2}}. \tag{G.87}$$

[14] For the first equation, we used the connection between x and ν, cf. (3.23).

Thus, all we need to do is insert p_1^μ into the formula above and simplify. Besides of (G.79), all we need is the identity

$$p_1^2 = c_P^2 P^2 + 2c_P c_q (P \cdot q) - c_q^2 Q^2, \tag{G.88}$$

where we used the decomposition of p_1^μ, cf. (G.78). Then, some algebra yields the desired result:

$$\begin{aligned}
\epsilon_0^\mu(p_1, q) =& \frac{Q^2 p_1^\mu + (p_1 \cdot q) q^\mu}{Q\sqrt{(p_1 \cdot q)^2 + Q^2 p_1^2}} \\
=& \frac{Q^2\Big(c_P P^\mu + c_q q^\mu\Big) + \Big(c_P (P \cdot q) - c_q Q^2\Big) q^\mu}{Q\sqrt{c_q^2 Q^4 - 2c_P c_q Q^2 (P \cdot q) + c_P^2 (P \cdot q)^2 + Q^2\Big(c_P^2 P^2 + 2c_P c_q (P \cdot q) - c_q^2 Q^2\Big)}} \\
=& \frac{Q^2 P^\mu + (P \cdot q) q^\mu}{Q\sqrt{(P \cdot q)^2 + Q^2 P^2}} = \epsilon_0^\mu(P, q).
\end{aligned} \tag{G.89}$$

H. LO calculation in the ACOT formalism

In the following we give the side calculations for section 5.5. Obtaining the partonic tensor with general fermionic currents (5.68) is closely related to the leptonic tensor calculated in appendix D.3. The only difference is the non-vanishing mass of $p_{1,2}$. Therefore, following the exact same steps as in Appendix D.3, the spin averaged sum in (5.55) becomes

$$\begin{aligned}
&\mathrm{Tr}\Big[(\not p_1 + m_1)\gamma^\mu\Big((g_{Ri}(1+\gamma^5) + g_{Li}(1-\gamma^5)\Big)(\not p_2 + m_2)\gamma^\nu\Big(g_{Ri}(1+\gamma^5) + g_{Li}(1-\gamma^5)\Big)\Big] \\
=&p_{1\alpha}p_{2\beta}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu(...)\gamma^\beta\gamma^\nu(...)\Big] + m_1 p_{2\alpha}\,\mathrm{Tr}\Big[\gamma^\mu(...)\gamma^\beta\gamma^\nu(...)\Big] \\
&+ m_2 p_{1\alpha}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu(...)\gamma^\nu(...)\Big] + m_1 m_2\,\mathrm{Tr}\Big[\gamma^\mu(...)\gamma^\nu(...)\Big],
\end{aligned} \tag{H.1}$$

where the masses enter the calculation via the completeness relation (2.55). We use (...) for the full electroweak coupling, which is written out in the first line. We immediately see that the second and third term vanish, since a trace over an odd number of Dirac matrices (irrespective of whether or not there is an additional γ^5 matrix in it) vanishes. For the remaining two traces we use the identities given in appendix D.3 for the projection operators $(1 \pm \gamma^5)$ and obtain

$$\begin{aligned}
... =&2p_{1\alpha}p_{2\beta}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\Big(g_{Ri}^2(1+\gamma^5) + g_{Li}^2(1-\gamma^5)\Big)\gamma^\beta\gamma^\nu\Big] \\
&+ 2m_1 m_2 g_{Ri} g_{Li}\Big\{\mathrm{Tr}\Big[\gamma^\mu(1+\gamma^5+1-\gamma^5)\gamma^\nu\Big]\Big\} \\
=&2p_{1\alpha}p_{2\beta}\Big\{\Big(g_{Ri}^2 + g_{Li}^2\Big)\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big] + \Big(g_{Ri}^2 - g_{Li}^2\Big)\,\mathrm{Tr}\Big[\gamma^5\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big]\Big\} \\
&+ 4m_1 m_2 g_{Ri} g_{Li}\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big] \\
=&8\Big\{\Big(g_{Ri}^2 + g_{Li}^2\Big)\Big(-g^{\mu\nu}(p_1\cdot p_2) + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\Big) + \Big(g_{Ri}^2 - g_{Li}^2\Big)i\epsilon^{\alpha\beta\mu\nu}p_{1\alpha}p_{2\beta} \\
&+ 2g_{Ri}g_{Li}m_1 m_2 g^{\mu\nu}\Big\}.
\end{aligned} \tag{H.2}$$

Here, we could simply copy the steps done in appendix D.3 for the first line, which corresponds to the completely massless case. In the last line, we used the trace identities given in appendix A.2. The complete tensor then is

$$\begin{aligned}\hat{W}_i^{(0)\mu\nu} =& \frac{1}{4\pi}(2\pi)\delta(p_2^2 - m_2^2)\frac{1}{n_i}\sum_{s_{1,2}} \langle p_1, s_1| J_i^\mu |p_2, s_2\rangle \langle p_2, s_2| J_i^{\star\nu} |p_1, s_1\rangle \\ =& 2\delta(p_2^2 - m_2^2)\Big\{ g_{Ri}^2\Big[-(p_1\cdot p_2)g^{\mu\nu} + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu + i\epsilon^{\mu\nu}_{\alpha\beta}p_1^\alpha p_2^\beta\Big] \\ & + g_{Li}^2\Big[-(p_1\cdot p_2)g^{\mu\nu} + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - i\epsilon^{\mu\nu}_{\alpha\beta}p_1^\alpha p_2^\beta\Big] \\ & + 2g_{Ri}g_{Li}m_1m_2g^{\mu\nu}\Big\}. \end{aligned} \tag{H.3}$$

where we used that the number of spin states of a fermionic parton is $n_i = 2$ and sorted all tensors in terms of left- and right-handed couplings.

To obtain the according helicity structure functions $\hat{W}_\lambda$ (cf. section 5.4.1), we contract with the polarizations $\epsilon_\lambda^\mu(p_1, q)$. According to section 2.3.2, these polarizations have the following form:

$$\epsilon_\pm^\mu(p_1, q) = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0) \tag{H.4}$$

$$\epsilon_0^\mu(p_1, q) = \frac{Q^2 p_1^\mu + (p_1\cdot q)q^\mu}{Q\sqrt{(p_1\cdot q)^2 + Q^2p_1^2}}. \tag{H.5}$$

In this section, we will work in Minkovski coordinates. Thus, applying the transformation from light-cone back to Minkovski basis given in section 4.1 on the partonic momenta (5.22) and (5.25), we obtain[1]

$$\begin{aligned} p_1^\mu =& \frac{Q}{2}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}, 0, 0, \frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_1^2}{Q^2}\right) \\ \text{and } p_2^\mu =& \frac{Q}{2}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}, 0, 0, \frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_1^2}{Q^2} - 2\right). \end{aligned} \tag{H.6}$$

Their scalar product was already calculated in light-cone-coordinates, cf. (5.27). It reads

$$p_1\cdot p_2 = \frac{1}{2}(Q^2 + m_1^2 + m_2^2). \tag{H.7}$$

Also helpful will be a reformulated expression of m_2, cf. (5.26):

$$\frac{\eta}{\xi}m_1^2 = \left(\frac{\xi}{\eta} - 1\right)Q^2 + m_1^2 - m_2^2. \tag{H.8}$$

Finally, note that the δ-distribution in $\hat{W}_i^{\mu\nu}$ sets ξ to

$$\chi = \eta\frac{Q^2 - m_1^2 + m_2^2 + \Delta}{2Q^2}, \tag{H.9}$$

[1] Note that the given p_2^μ was derived from momentum conservation, $p_2^\mu = p_1^\mu + q^\mu$. We are allowed to use momentum conservation because it was implicitly assumed in the partonic tensor, cf. (5.55).

cf. (5.36), where we defined[2]

$$\Delta \equiv \Delta(-Q^2, m_1^2, m_2^2). \tag{H.10}$$

Let us start with $\hat{W}_{\pm}^{i(0)}$. We easily see that

$$\epsilon_{\pm}^{\star\mu}(p_1,q) g_{\mu\nu} \epsilon_{\pm}^{\nu}(p_1,q) = -\frac{1}{2}\Big((\pm 1)^2 + i(-i)\Big) = -1, \tag{H.11}$$

which is simply the normalization of $\epsilon_{\pm}^{\mu}(p_1,q)$. Additionally, by making use of the explicit components, we obtain

$$\epsilon_{\pm}^{\star\mu}(p_1,q) p_{1\mu} p_{1\nu} \epsilon_{\pm}^{\nu}(p_1,q) = \epsilon_{\pm}^{\star\mu}(p_1,q) p_{2\mu} p_{1\nu} \epsilon_{\pm}^{\nu}(p_1,q) = 0. \tag{H.12}$$

Again using the explicit components together with the identities (H.8) and (H.9), we are able to compute

$$\begin{aligned}
&\epsilon_{\pm\mu}^{\star}(p_1,q)\epsilon^{\alpha\beta\mu\nu} p_{1\alpha} p_{2\beta} \epsilon_{\pm\nu}(p_1,q) \\
=&\frac{Q}{2}\left(\frac{\xi}{\eta} + \frac{\eta}{\eta}\frac{m_1^2}{Q^2}\right)\left(-\frac{Q}{2}\left(\frac{\xi}{\eta} - \frac{\eta}{\xi}\frac{m_1^2}{Q^2} - 2\right)\right)\left[\epsilon^{0312}\cdot\frac{1}{2}\cdot(\pm 1)\cdot(-i) + \epsilon^{0321}\cdot\frac{1}{2}\cdot i\cdot\cdot(\pm 1)\right] \\
&+\left(-\frac{Q}{2}\left(\frac{\xi}{\eta} - \frac{\eta}{\eta}\frac{m_1^2}{Q^2}\right)\right)\frac{Q}{2}\left(\frac{\xi}{\eta} + \frac{\eta}{\xi}\frac{m_1^2}{Q^2}\right)\left[\epsilon^{3012}\cdot\frac{1}{2}\cdot(\pm 1)\cdot(-i) + \epsilon^{3021}\cdot\frac{1}{2}\cdot i\cdot(\pm 1)\right] \\
=&\mp i\frac{Q^2}{4}\left\{-\left[\frac{\xi^2}{\eta^2} - \frac{\eta^2}{\xi^2}\frac{m_1^4}{Q^4} - 2\frac{\xi}{\eta} - 2\frac{\eta}{\xi}\frac{m_1^2}{Q^2}\right] + \left[\frac{\xi^2}{\eta^2} - \frac{\eta^2}{\eta^2}\frac{m_1^4}{Q^4}\right]\right\} \\
=&\mp\frac{i}{2}\left\{\frac{\eta}{\xi}m_1^2 + \frac{\xi}{\eta}Q^2\right\} = \pm\frac{i}{2}\left\{-Q^2 + m_1^2 - m_2^2 + 2\frac{\xi}{\eta}Q^2\right\} \\
=&\mp\frac{i}{2}\left\{-Q^2 + m_1^2 - m_2^2 + Q^2 - m_1^2 + m_2^2 + \Delta\right\} = \pm\frac{i}{2}\Delta
\end{aligned} \tag{H.13}$$

for the last tensor. Summing up all this gives

$$\hat{W}_{\pm}^{i(0)} = \delta\left(\frac{\xi}{\chi} - 1\right)\frac{1}{\Delta}\left\{g_{+i}^2\left(Q^2 + m_1^2 + m_2^2\right) \pm g_{-i}^2\Delta - 2g_{Li}g_{Ri}m_1m_2\right\}, \tag{H.14}$$

where we also used identity (H.7) and the alternative expression (5.37) for the on-shell condition derived in appendix F.

Turning to $\hat{W}_0^i$, the normalization of $\epsilon_0^{\mu}(p_1,q)$ is[3]

$$\epsilon_{0\mu}(p_1,q) g^{\mu\nu} \epsilon_{0\nu}(p_1,q) = 1, \tag{H.15}$$

cf. (G.17)[4]. Since $\epsilon_0^{\mu}(p_1,q)$ is a linear combination of p_1^{μ} and q^{μ}, contracting with $\epsilon^{\alpha\beta\mu\nu}p_{1\alpha}p_{2\beta}$ is zero.

[2]Cf. (3.15) for the full definition of $\Delta(a,b,c)$.
[3]Note that $\epsilon_0^{\mu}(p_1,q)$ is real and therefore does not need to be complex conjugated.
[4]The reader shall be reminded of the fact that the polarizations with p_1^{μ} and P^{μ} as a reference are equal, as it was explained in section 5.4.2.

For the last two tensors, we need to evaluate two more scalar products in the hadronic configuration with the help of (H.8):

$$p_1 \cdot q = \frac{1}{2}\Big(\frac{\eta}{\xi}m_1^2 - 2Q^2 + \frac{\xi}{\eta}Q^2\Big) = \frac{1}{2}\Big(m_2^2 - m_1^2 - Q^2\Big) \tag{H.16}$$

$$\text{and } p_2 \cdot q = \frac{1}{2}\Big(\frac{\xi}{\eta}Q^2 - \frac{\eta}{\xi}m_1^2\Big) = \Big(m_2^2 - m_1^2 + Q^2\Big), \tag{H.17}$$

since

$$q^\mu = Q(0,0,0,-1). \tag{H.18}$$

Hence,

$$\begin{aligned}\epsilon_0^\mu(p_1,q)p_{1\mu}p_{2\nu}\epsilon_0^\nu(p_1,q) &= \frac{Q^2p_1^2 + (p_1\cdot q)^2}{Q\sqrt{(p_1\cdot q)^2 + Q^2p_1^2}} \cdot \frac{Q^2(p_1\cdot p_2) + (p_1\cdot q)(p_2\cdot q)}{Q\sqrt{(p_1\cdot q)^2 + Q^2p_1^2}} \\ &= \frac{1}{Q^2}\Big(-\frac{1}{2}\Big(Q^2 + m_1^2 + m_2^2\Big) \\ &\quad + \frac{1}{4}\Big(m_2^2 - m_1^2 - Q^2\Big)\cdot\Big(m_2^2 - m_1^2 + Q^2\Big)\Big) \\ &= \frac{Q^2\Big(m_1^2 + m_2^2\Big) + \Big(m_1^2 - m_2^2\Big)^2}{2Q^2}.\end{aligned} \tag{H.19}$$

And because

$$\epsilon_0^\mu(p_1,q)p_{1\mu}p_{2\nu}\epsilon_0^\nu(p_1,q) = \epsilon_0^\mu(p_1,q)p_{2\mu}p_{1\nu}\epsilon_0^\nu(p_1,q), \tag{H.20}$$

the final result is

$$\hat{W}_0^{i(0)} = \delta\Big(\frac{\xi}{\chi} - 1\Big)\frac{1}{Q^2\Delta}\Big\{g_{+i}^2\Big(Q^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\Big) + 2g_{Li}g_{Ri}m_1m_2Q^2\Big\}. \tag{H.21}$$

Let us close this calculation with a check of the conversion formulae between helicity and ordinary structure functions given in sections 5.4.1 and 5.4.2. We will compare our results with the structure functions $F_{1,2,3}$ given in section 2.1, equations (2) of [Kretzer and Schienbein, 1998]. As explained in section 5.5, we obtain the hadronic helicity structure functions $F_\lambda^{(0)}$ by replacing the δ-distribution in $\hat{W}_\lambda^{i(0)}$ with a factor of $f_i(\chi, Q^2)$. Applying the conversion formulae gives

$$\begin{aligned}F_1^{(0)} &= \frac{1}{2}\Big(F_+ + F_-\Big) \\ &= \frac{1}{\Delta}\sum_i\Big\{g_{+i}^2\Big(Q^2 + m_1^2 + m_2^2\Big) - 2g_{Li}g_{Ri}m_1m_2\Big\}f_i(\chi, Q^2)\end{aligned} \tag{H.22}$$

$$
\begin{aligned}
F_2^{(0)} =& \frac{x}{1+\frac{Q^2}{\nu^2}}\Big(2F_0 + F_+ + F_-\Big)\\
=& \frac{2x}{1+\frac{Q^2}{\nu^2}} \frac{1}{Q^2\Delta} \sum_i g_{+i}^2 \Big\{ Q^2\Big(m_1^2+m_2^2\Big) + \big(m_1^2-m_2^2\big)^2 \\
&\qquad\qquad + Q^2\Big(Q^2+m_1^2+m_2^2\Big)\Big\} f_i(\chi,Q^2)\\
=& \frac{1}{1+\frac{Q^2}{\nu^2}} \frac{2x\Delta}{Q^2} \sum_i g_{+i}^2 f_i(\chi,Q^2) \qquad \text{(H.23)}\\
F_3^{(0)} =& \frac{1}{\sqrt{1+\frac{Q^2}{\nu^2}}}\Big(F_- - F_+\Big) = -\frac{2}{\sqrt{1+\frac{Q^2}{\nu^2}}} \sum_i g_{-i}^2 f_i(\chi,Q^2). \qquad \text{(H.24)}
\end{aligned}
$$

These results directly correspond to the ones given in [Kretzer and Schienbein, 1998][5], if one, according to their assumptions, ignores target mass corrections. In that case, Q^2/ν^2 vanishes, as described below equation (5.45).

If we additionally let both masses $m_{1,2}$ go to zero, which yields $\Delta = Q^2$ and $\chi = x$, and set $g_{Ri} = g_{Li} = \frac{1}{2}Q_i$ as well as $F_0^{(0)} = 0$ (due to the massless gauge boson) for QED interactions, we end up with

$$
F_1^{(0)} = \frac{1}{2}\sum_i Q_i^2 f_i(x,Q^2),\;\; F_2^{(0)} = x\sum_i Q_i^2 f_i(x,Q^2) \text{ and } F_3^{(0)} = 0. \qquad \text{(H.25)}
$$

As expected, we reproduced the structure functions of the naive parton model, cf. equations (3.33) and (3.34)[6].

Finally, we also verify the conversion formulae on the parton level. According to section 5.4.2 and appendix G.3, the relations between partonic and hadronic ordinary structure functions are

$$
F_1^{(0)} = \sum_i f_i \otimes \hat{W}_1^{i(0)} \qquad \text{(H.26)}
$$

$$
F_2^{(0)} = \sum_i f_i \otimes \frac{\xi^2}{2x}\hat{W}_2^{i(0)} \qquad \text{(H.27)}
$$

$$
\text{and } F_3^{(0)} = \sum_i f_i \otimes \frac{\xi}{2x}\hat{W}_3^{i(0)}, \qquad \text{(H.28)}
$$

For the sake of simplicity, we set all masses to zero again, which leads to $c_P = \xi$, $\Delta = Q^2$ and $\eta = x$. The conversion formula (5.60) yields

$$
\hat{W}_1^i = \frac{1}{2}\Big(\hat{W}_+^i + \hat{W}_-^i\Big)\delta\Big(\frac{\xi}{x}-1\Big)\frac{1}{\Delta}\Bigg(g_{+i}^2\Big(Q^2+m_1^2+m_2^2\Big) - 2g_{Li}g_{Ri}m_1m_2Q^2\Bigg) \qquad \text{(H.29)}
$$

[5]There, vector and axial vector couplings are used. However, the connections between their and our conventions are simple: $S_+ = 2g_{+i}^2$, $R_+ = -g_{-i}^2$ and $S_- = 4g_{Ri}g_{Li}$. Furthermore, they neglect the sum over all quark flavors.

[6]We implicitly set F_3 to zero in that section due to chirality conservation in QED.

when inserting the explicit expressions that were derived in this chapter. Substituting this into (H.26) directly reproduces (H.22). The same procedure for the second structure functions, now using formula (5.61), gives

$$\hat{W}_2^i = \frac{2x^2}{\xi^2}\left(2\hat{W}_0^i + \hat{W}_+^i + \hat{W}_-^i\right) = \delta\left(\frac{\xi}{x} - 1\right) g_{+i}^2 \frac{4x^2}{\xi^2}, \tag{H.30}$$

again agreeing with (H.23). For the last structure function, equation (5.62) yields

$$\hat{W}_3^i = \frac{2x}{\xi}\left(\hat{W}_-^i - \hat{W}_+^i\right) = -\delta\left(\frac{\xi}{x} - 1\right) g_{-i}^2 \frac{4x}{\xi}. \tag{H.31}$$

Inserting this into (H.28) correctly reproduces (H.24).

I. Calculating the gluon initiated partonic tensor and structure functions at NLO

I.1. The partonic tensor

Applying Feynman rules on the two left sides of the cuts in figure 6.1, we obtain the amplitudes

$$\mathcal{M}_t^\mu = - g_s T_a^{ij} \epsilon_m^\alpha(p_1) \overline{u}(p_2) \Gamma^\mu \frac{\not{p}_1 - \not{k} + m_k}{(p_1-k)^2 - m_k^2} \gamma_\alpha v(k) \tag{I.1}$$

$$\text{and } \mathcal{M}_u^\mu = g_s T_a^{ij} \epsilon_m^\beta(p_1) \overline{u}(p_2) \gamma_\beta \frac{\not{p}_2 - \not{p}_1 + m_2}{(p_2-p_1)^2 - m_2^2} \Gamma^\mu v(k), \tag{I.2}$$

where we first ignore the polarization of the exchanged vector boson B. Note that the general coupling Γ^μ (more precisely the right- and left-handed couplings g_R and g_L) implicitly depends on the quark flavor(s) of k and p_2. We do not give an explicit reference vector for the gluon polarization $\epsilon_m^\mu(p_1)$ as it will not be important in this context. The additional minus sign in $\mathcal{M}_u^\mu$ follows from the exchange of identical fermion lines in the final state.

Following the definition (5.55), we can compute the unsubtracted partonic tensor via

$$\begin{aligned}\hat{\Omega}_g^{(1)\mu\nu} =& \frac{1}{4\pi}\frac{1}{n_g}\frac{T_R}{N_C C_F} \sum_{\text{final states}} \int d\Pi_{\text{LIPS}} \Big(\mathcal{M}_t + \mathcal{M}_u\Big)^\mu \Big(\mathcal{M}_t^\star + \mathcal{M}_u^\star\Big)^\nu \\ =& \frac{1}{4\pi}\frac{1}{n_g}\frac{T_R}{N_C C_F} \sum_{\text{final states}} \int d\Pi_{\text{LIPS}} \Big(\mathcal{M}_t^\mu \mathcal{M}_t^{\star\nu} + \mathcal{M}_u^\mu \mathcal{M}_u^{\star\nu} + \mathcal{M}_t^\mu \mathcal{M}_u^{\star\nu} + \mathcal{M}_u^\mu \mathcal{M}_t^{\star\nu}\Big),\end{aligned} \tag{I.3}$$

where the four terms in the phase space integral correspond to the four graphs in figure 6.1. Additional to $n_g = 2$ spin states we average over $N_C C_F / T_R = 8$ color states of the gluon.

As denoted in their subscripts, the amplitudes correspond to the (partonic) t- and u-channel process. According to the general partonic process $q + p_1 \to p_2 + k$, we can define Mandelstam variables

$$\hat{s} = (q+p_1)^2 = (p_2+k)^2 \tag{I.4}$$

$$\hat{t} = (q-p_2)^2 = (p_1-k)^2 \tag{I.5}$$

$$\hat{u} = (q-k)^2 = (p_1-p_2)^2. \tag{I.6}$$

We start with the calculation of the first term in (I.3):

$$\begin{aligned}\frac{1}{n_g}\frac{T_R}{N_C C_F}&\sum_{\text{final states}} \mathcal{M}_t^{\mu}\mathcal{M}_t^{\star\nu}\\ &= -\frac{1}{2}T_R\alpha_s\frac{1}{\Big((p_1-k)^2-m_k^2\Big)^2}\\ &\quad\times \mathrm{Tr}\Big[(\not{p}_2+m_2)\Gamma^{\mu}(\not{p}_1-\not{k}+m_k)\gamma^{\alpha}(\not{k}-m_k)\gamma_{\alpha}(\not{p}_1-\not{k}+m_k)\Gamma^{\nu}\Big],\end{aligned} \tag{I.7}$$

where we followed the same steps that were done in appendix D when we calculated squared matrix elements. We used that the massless gluon has $n_g = 2$ spin states. Therefore, the sum over final states contains a sum over $m = \pm 1$ and we can use the completeness relation[1]

$$\sum_{m=\pm 1} \epsilon_m^{\star\alpha}(p_1)\epsilon_m^{\beta}(p_1,q) = -g^{\alpha\beta} + \sum_{m=0,p_1} v_m \epsilon_m^{\star\alpha}(p_1)\epsilon_m^{\beta}(p_1). \tag{I.8}$$

Due to $\epsilon^{\mu}_{0,p_1}(p_1,q) \propto p_1$ (cf. 2.3.2.1) and the Ward identity[2] (2.115), it is sufficient to substitute

$$\sum_{m=\pm 1} \epsilon_m^{\star\alpha}(p_1)\epsilon_m^{\beta}(p_1) = -g^{\alpha\beta} \tag{I.9}$$

for the whole amplitude. For the (anti-) spinors, we made use of the completeness relations[3]

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m \tag{I.10}$$

$$\text{and} \sum_s v_s(p)\overline{v}_s(p) = \not{p} - m. \tag{I.11}$$

At last, the color factor is

$$T_a^{ij}T_a^{ji} = \mathrm{Tr}\Big[T_a T_a\Big] = N_C C_F. \tag{I.12}$$

In general, we simplify traces like the one in (I.7) by following these steps:

1. Multiply out the argument of the trace and factor out every slashed momentum, e.g. $\not{k} = k_\mu\gamma^\mu$. Then use the linearity of the trace operation to obtain a sum of traces over Dirac matrices only. Traces over an odd number of Dirac matrices vanish and therefore can be ignored right from the beginning.

[1] Cf. section 2.3.2.1 for further information.

[2] Strictly speaking, this Ward only holds in the case of abelian gauge theories, i.e. not for QCD. In this particular case, QCD-enhanced Ward identities will reduce to (2.115) since the conversion of all gluon lines to photon lines (a process called abelization) would lead to a proper QED diagram. In other words, the use of an abelian Ward identity is allowed because of no non-abelian (i.e. gluon-gluon) interactions in the regarded process.

[3] Cf. section 2.2 for further information.

2. Use the commutation relations given in appendix A.2 to be able to multiply both generalized couplings $\Gamma^\mu = \gamma^\mu\Big(g_R(1+\gamma^5) + g_L(1-\gamma^5)\Big)$. Two scenarios are possible:
   - If there is an even number of Dirac matrices in between, we obtain a factor of[4] $2(g_+^2 + g_-^2\gamma^5)$, which in combination with the other Dirac matrices can be multiplied out and thus written as a sum of two traces. Then commute the γ^5 matrix to the front of the trace argument, picking up an extra minus sign in front of the g_-^2 if there is an odd number of Dirac matrices in front of the γ^5.
   - For an odd number of Dirac matrices between both couplings, there is a factor of $4g_Rg_L$ after multiplying them.
3. Use the contraction identities, which are again given in appendix A.2, to simplify $\gamma^\alpha...\gamma_\alpha$, where the dots can stand for up to three Dirac matrices.

In the end, this gives

$$\begin{aligned}
&\mathrm{Tr}\Big[(\not{p}_2 + m_2)\Gamma^\mu(\not{p}_1 - \not{k} + m_k)\gamma^\alpha(\not{k} - m_k)\gamma_\alpha(\not{p}_1 - \not{k} + m_k)\Gamma^\nu\Big] \\
&= -4p_{2\beta}(p_1-k)_\delta k_\rho (p_1-k)_\sigma \Big\{ g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\rho\gamma^\sigma\gamma^\nu\Big] + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\rho\gamma^\sigma\gamma^\nu\Big]\Big\} \\
&\quad - 8m_k^2 p_{2\beta}(p_1-k)_\sigma \Big\{ g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\Big] + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\Big]\Big\} \\
&\quad - 4m_k^2 p_{2\beta}k_\rho \Big\{ g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\nu\Big] + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\nu\Big]\Big\} \\
&\quad - 8m_k^2 p_{2\beta}(p_1-k)_\delta \Big\{ g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\nu\Big] + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\nu\Big]\Big\} \\
&\quad - 8g_Rg_Lm_km_2k_\rho(p_1-k)_\sigma\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu\Big] \\
&\quad - 16g_Rg_Lm_km_2(p_1-k)_\delta(p_1-k)_\sigma\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\nu\Big] \\
&\quad - 16g_Rg_Lm_k^3m_2\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big]. \qquad \text{(I.13)}
\end{aligned}$$

Applying the trace identities for Dirac matrices and contracting everything yields

$$\begin{aligned}
&\mathrm{Tr}\Big[(\not{p}_2 + m_2)\Gamma^\mu(\not{p}_1 - \not{k} + m_k)\gamma^\alpha(\not{k} - m_k)\gamma_\alpha(\not{p}_1 - \not{k} + m_k)\Gamma^\nu\Big] \\
&= 32\Big\{ -g_+^2\Big[\Big(m_k^2(p_2\cdot k - p_1\cdot p_2) - (p_1\cdot p_2)(p_1\cdot k)\Big)g^{\mu\nu} \qquad \text{(I.14)} \\
&\qquad + \Big(m_k^2 + p_1\cdot k\Big)\Big(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\Big) - m_k^2\Big(k^\mu p_2^\nu + k^\nu p_2^\mu\Big)\Big]
\end{aligned}$$

[4]We use the generalized couplings $g_\pm \equiv g_R^2 \pm g_L^2$, as defined in equation (5.9).

$$+ ig_-^2 \left[\left(p_1 \cdot k + m_k^2 \right) p_{1\rho} p_{2\sigma} + m_k^2 p_{2\rho} k_\sigma \right] \epsilon^{\rho\sigma\mu\nu}$$
$$\left. + 2 g_R g_L m_k m_2 \left(p_1 \cdot k - m_k^2 \right) g^{\mu\nu} \right\}. \tag{I.15}$$

Hence, the final result is

$$\begin{aligned}
&\frac{1}{n_g} \frac{T_R}{N_C C_F} \sum_{\text{final states}} \mathcal{M}_t^\mu \mathcal{M}_t^{\star\nu} \\
&= 8 g_s^2 \frac{1}{\left((p_1 - k)^2 - m_k^2 \right)^2} \Bigg\{ g_+^2 \Bigg[\left(m_k^2 (p_2 \cdot k - p_1 \cdot p_2) - (p_1 \cdot p_2)(p_1 \cdot k) \right) g^{\mu\nu} \\
&\qquad + \left(m_k^2 + p_1 \cdot k \right) \left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu \right) - m_k^2 \left(k^\mu p_2^\nu + k^\nu p_2^\mu \right) \Bigg] \\
&\qquad - ig_-^2 \left[\left(p_1 \cdot k + m_k^2 \right) p_{1\rho} p_{2\sigma} + m_k^2 p_{2\rho} k_\sigma \right] \epsilon^{\rho\sigma\mu\nu} \\
&\qquad - 2 g_R g_L m_k m_2 \left(p_1 \cdot k - m_k^2 \right) g^{\mu\nu} \Bigg\}.
\end{aligned} \tag{I.16}$$

The second term in (I.3) can be obtained from the first one by simply exchanging $p_2 \leftrightarrow k$ and $m_2 \leftrightarrow m_k$ and reverse the fermion flow, which leads to an additional minus sign in the antisymmetric part[5]:

$$\begin{aligned}
&\frac{1}{n_g} \frac{T_R}{N_C C_F} \sum_{\text{final states}} \mathcal{M}_u^\mu \mathcal{M}_u^{\star\nu} \\
&= 8 g_s^2 \frac{1}{\left((p_2 - p_1)^2 - m_2^2 \right)^2} \Bigg\{ g_+^2 \Bigg[\left(m_2^2 (p_2 \cdot k - p_1 \cdot k) - (p_1 \cdot k)(p_1 \cdot p_2) \right) g^{\mu\nu} \\
&\qquad + \left(m_2^2 + p_1 \cdot p_2 \right) \left(p_1^\mu k^\nu + p_1^\nu k^\mu \right) - m_2^2 \left(k^\mu p_2^\nu + k^\nu p_2^\mu \right) \Bigg] \\
&\qquad + ig_-^2 \left[\left(p_1 \cdot p_2 + m_2^2 \right) p_{1\rho} k_\sigma - m_2^2 p_{2\rho} k_\sigma \right] \epsilon^{\rho\sigma\mu\nu} \\
&\qquad - 2 g_R g_L m_k m_2 \left(p_1 \cdot p_2 - m_2^2 \right) g^{\mu\nu} \Bigg\}.
\end{aligned} \tag{I.17}$$

[5] Of course one can also calculate the second term by hand. The computation is quite similar to the squared t-channel above, which is why we do not explicitly write it down here. To obtain the final result in the given form, it is helpful to use $\mathrm{Tr}\left[\gamma^5 \gamma^\beta \gamma^\delta \gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma\right] = -\mathrm{Tr}\left[\gamma^5 \gamma^\sigma \gamma^\beta \gamma^\delta \gamma^\mu \gamma^\rho \gamma^\nu\right]$ (using the same assignment of indices as above) after splitting up the initial trace into Dirac traces.

The same procedure for the interference terms in (I.3) gives

$$
\begin{aligned}
&\frac{1}{n_g}\frac{T_R}{N_C C_F}\sum_{\text{final states}} \mathcal{M}_t^{\mu}\mathcal{M}_u^{\star\nu} \\
&= -\frac{1}{4}g_s^2\frac{1}{\Big((p_1-k)^2-m_k^2\Big)\Big((p_2-p_1)^2-m_2^2\Big)} \\
&\qquad\times\Bigg\{-4p_{2\beta}(p_1-k)_\delta k_\rho (p_2-p_1)_\sigma\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\nu\gamma^\rho\Big] \\
&\qquad\qquad + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\nu\gamma^\rho\Big]\Big) \\
&\qquad + 8g_R g_L m_k m_2 p_{2\beta}(p_1-k)_\delta\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\delta\gamma^\nu\Big] \\
&\qquad + 16 g_R g_L m_k m_2 k^\nu p_{2\beta}\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\Big] \\
&\qquad - 8m_k^2(p_2-p_1)^\nu p_{2\beta}\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\mu\Big]\Big) \\
&\qquad + 8m_2^2 k^\nu (p_1-k)_\delta\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\delta\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\delta\Big]\Big) \\
&\qquad - 16 g_R g_L m_k m_2 (p_2-p_1)^\nu (p_1-k)_\delta\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\delta\Big] \\
&\qquad - 8 g_R g_L m_k m_2 k_\rho (p_2-p_1)_\sigma\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho\Big] \\
&\qquad + 4m_k^2 m_2^2\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\nu\Big]\Big)\Bigg\} \\
&= -4g_s^2\frac{1}{\Big((p_1-k)^2-m_k^2\Big)\Big((p_2-p_1)^2-m_2^2\Big)} \\
&\qquad\times\Bigg\{g_+^2\Big[\Big(2(p_2\cdot k)^2-2(p_1\cdot p_2)(p_2\cdot k)-2(p_1\cdot k)(p_2\cdot k) \\
&\qquad\qquad + m_k^2 p_1\cdot p_2 + m_2^2 p_1\cdot k\Big)g^{\mu\nu} \\
&\qquad\quad - 2(p_1\cdot p_2)k^\mu k^\nu - 2(p_1\cdot k)p_2^\mu p_2^\nu \\
&\qquad\quad + \Big(2p_2\cdot k + m_2^2\Big)p_1^\mu k^\nu - m_2^2 p_1^\nu k^\mu - m_k^2 p_1^\mu p_2^\nu + \Big(2p_2\cdot k + m_k^2\Big)p_1^\nu p_2^\mu \\
&\qquad\quad - 2(p_2\cdot k)p_2^\mu k^\nu + 2\Big(p_1\cdot p_2 + p_1\cdot k - p_2\cdot k\Big)p_2^\nu k^\mu\Big] \\
&\qquad + ig_-^2\Big[\Big((p_1\cdot k - p_2\cdot k)p_{1\rho}p_{2\sigma} - (p_1\cdot p_2 - p_2\cdot k)p_{1\rho}k_\sigma \\
&\qquad\qquad + (p_1\cdot p_2 + p_1\cdot k - 2p_2\cdot k)p_{2\rho}k_\sigma\Big)\epsilon^{\rho\sigma\mu\nu} \\
&\qquad\qquad + \Big(-p_1^\mu - p_2^\mu + k^\mu\Big)p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\nu} + \Big(p_1^\nu - p_2^\nu + k^\nu\Big)p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\mu}\Big] \\
&\qquad + 2g_R g_L m_k m_2\Big[\Big(2p_2\cdot k - p_1\cdot k - p_1\cdot p_2\Big)g^{\mu\nu} \\
&\qquad\qquad + 2p_1^\mu p_1^\nu + p_2^\mu p_1^\nu - p_2^\nu p_1^\mu + p_1^\mu k^\nu - p_1^\nu k^\mu\Big]\Bigg\}
\end{aligned}
\tag{I.18}
$$

and

$$
\begin{aligned}
&\frac{1}{n_g}\frac{T_R}{N_C C_F}\sum_{\text{final states}}\mathcal{M}_u^{\mu}\mathcal{M}_t^{\star\nu} \\
&= -\frac{1}{4}g_s^2\frac{1}{\Big((p_1-k)^2-m_k^2\Big)\Big((p_2-p_1)^2-m_2^2\Big)} \\
&\quad\times\Bigg\{-4p_{2\beta}(p_2-p_1)_\delta k_\rho(p_1-k)_\sigma\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\rho\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\nu\Big] \\
&\qquad\qquad + g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\rho\gamma^\mu\gamma^\delta\gamma^\sigma\gamma^\nu\Big]\Big) \\
&\qquad - 8m_k^2p_{2\beta}(p_2-p_1)^\mu\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\nu\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\beta\gamma^\nu\Big]\Big) \\
&\qquad + 16g_Rg_Lm_km_2k^\mu p_{2\beta}\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\Big] \\
&\qquad + 8g_Rg_Lm_km_2p_{2\beta}(p_1-k)_\sigma\,\mathrm{Tr}\Big[\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\Big] \\
&\qquad - 8g_Rg_Lm_km_2(p_2-p_1)_\delta k_\rho\,\mathrm{Tr}\Big[\gamma^\rho\gamma^\mu\gamma^\delta\gamma^\nu\Big] \\
&\qquad - 16g_Rg_Lm_km_2(p_2-p_1)^\mu(p_1-k)_\sigma\,\mathrm{Tr}\Big[\gamma^\sigma\gamma^\nu\Big] \\
&\qquad + 8m_2^2k^\mu(p_1-k)_\sigma\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\sigma\gamma^\nu\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\sigma\gamma^\nu\Big]\Big) \\
&\qquad - 4m_k^2m_2^2\Big(g_+^2\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\Big]+g_-^2\,\mathrm{Tr}\Big[\gamma^5\gamma^\mu\gamma^\nu\Big]\Big)\Bigg\} \\
&= -4g_s^2\frac{1}{\Big((p_1-k)^2-m_k^2\Big)\Big((p_2-p_1)^2-m_2^2\Big)} \\
&\quad\times\Bigg\{g_+^2\Big[\Big(2(p_2\cdot k)^2-2(p_1\cdot p_2)(p_2\cdot k)-2(p_1\cdot k)(p_2\cdot k) \\
&\qquad\qquad + m_k^2p_1\cdot p_2+m_2^2p_1\cdot k\Big)g^{\mu\nu} \\
&\qquad\quad - 2(p_1\cdot p_2)k^\mu k^\nu-2(p_1\cdot k)p_2^\mu p_2^\nu \\
&\qquad\quad - m_2^2p_1^\mu k^\nu+\Big(2p_2\cdot k+m_2^2\Big)p_1^\nu k^\mu+\Big(2p_2\cdot k+m_k^2\Big)p_1^\mu p_2^\nu-m_k^2p_1^\nu p_2^\mu \\
&\qquad\quad + 2\Big(p_1\cdot p_2+p_1\cdot k-p_2\cdot k\Big)p_2^\mu k^\nu-2(p_2\cdot k)p_2^\nu k^\mu\Big] \\
&\qquad + ig_-^2\Big[\Big(-(p_2\cdot k)p_{1\rho}p_{2\sigma}+(p_2\cdot k)p_{1\rho}k_\sigma \\
&\qquad\qquad + (p_1\cdot p_2+p_1\cdot k-2p_2\cdot k)p_{2\rho}k_\sigma\Big)\epsilon^{\rho\sigma\mu\nu} \\
&\qquad\quad + \Big(p_2^\mu-k^\mu\Big)p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\nu}+\Big(p_2^\nu-k^\nu\Big)p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\mu}\Big] \\
&\qquad + 2g_Rg_Lm_km_2\Big[\Big(2p_2\cdot k-p_1\cdot k-p_1\cdot p_2\Big)g^{\mu\nu} \\
&\qquad\qquad + 2p_1^\mu p_1^\nu-p_2^\mu p_1^\nu+p_2^\nu p_1^\mu-p_1^\mu k^\nu+p_1^\nu k^\mu\Big]\Bigg\}.
\end{aligned}
\tag{I.19}
$$

Using the notation

$$\hat{\Omega}_g^{(1)\mu\nu} = \alpha_s \sum_{\kappa=s,a,x} C_\kappa \int d\Pi_{\text{LIPS}} \hat{\Omega}_{g\kappa}^{(1)\mu\nu}, \tag{I.20}$$

with

$$C_{s,a} \equiv g_\pm^2 \tag{I.21}$$
$$C_x \equiv 2 g_R g_L, \tag{I.22}$$

we finally obtain

$$\begin{aligned}
\hat{\Omega}_{gs}^{(1)\mu\nu} =& 2\Bigg\{ \frac{1}{(p_1\cdot k)^2}\Big[\Big(m_k^2(p_2\cdot k - p_1\cdot p_2) - (p_1\cdot p_2)(p_1\cdot k)\Big)g^{\mu\nu} \\
&+ \Big(m_k^2 + p_1\cdot k\Big)\Big(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\Big) - m_k^2\Big(k^\mu p_2^\nu + k^\nu p_2^\mu\Big)\Big] \\
&+ \frac{1}{(p_1\cdot p_2)^2}\Big[\Big(m_2^2(p_2\cdot k - p_1\cdot k) - (p_1\cdot p_2)(p_1\cdot k)\Big)g^{\mu\nu} \\
&+ \Big(m_2^2 + p_1\cdot p_2\Big)\Big(p_1^\mu k^\nu + p_1^\nu k^\mu\Big) - m_2^2\Big(k^\mu p_2^\nu + k^\nu p_2^\mu\Big)\Big] \\
&- \frac{1}{(p_1\cdot p_2)(p_1\cdot k)}\Big[\Big(2(p_2\cdot k)^2 - 2(p_1\cdot p_2)(p_2\cdot k) - 2(p_1\cdot k)(p_2\cdot k) \\
&+ m_k^2 p_1\cdot p_2 + m_2^2 p_1\cdot k\Big)g^{\mu\nu} \\
&- 2(p_1\cdot p_2)k^\mu k^\nu - 2(p_1\cdot k)p_2^\mu p_2^\nu \\
&+ (p_2\cdot k)\Big(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\Big) + (p_2\cdot k)\Big(p_1^\mu k^\nu + p_1^\nu k^\mu\Big) \\
&+ \Big(p_1\cdot p_2 + p_1\cdot k - 2p_2\cdot k\Big)\Big(p_2^\mu k^\nu + p_2^\nu k^\mu\Big)\Big]\Bigg\}
\end{aligned} \tag{I.23}$$

$$\begin{aligned}
\hat{\Omega}_{ga}^{(1)\mu\nu} =& i\Bigg\{-2\frac{\Big(p_1\cdot k + m_k^2\Big)p_{1\rho}p_{2\sigma} + m_k^2 p_{2\rho}k_\sigma}{(p_1\cdot k)^2}\epsilon^{\rho\sigma\mu\nu} \\
&+ 2\frac{\Big(p_1\cdot p_2 + m_2^2\Big)p_{1\rho}k_\sigma - m_2^2 p_{2\rho}k_\sigma}{(p_1\cdot p_2)^2}\epsilon^{\rho\sigma\mu\nu} \\
&- \frac{1}{(p_1\cdot p_2)(p_1\cdot k)}\Big[\Big((p_1\cdot k - 2p_2\cdot k)p_{1\rho}p_{2\sigma} - (p_1\cdot p_2 - 2p_2\cdot k)p_{1\rho}k_\sigma \\
&+ 2(p_1\cdot p_2 + p_1\cdot k - 2p_2\cdot k)p_{2\rho}k_\sigma\Big)\epsilon^{\rho\sigma\mu\nu} \\
&- p_1^\mu p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\nu} + p_1^\nu p_{1\rho}p_{2\sigma}k_\beta\epsilon^{\rho\sigma\beta\mu}\Big]\Bigg\}
\end{aligned} \tag{I.24}$$

$$\hat{\Omega}_{gx}^{(1)\mu\nu} = 2m_k m_2 \Bigg\{ -\frac{p_1 \cdot k - m_k^2}{(p_1 \cdot k)^2} g^{\mu\nu} - \frac{p_1 \cdot p_2 - m_2^2}{(p_1 \cdot p_2)^2} g^{\mu\nu} - \frac{\Big(2p_2 \cdot k - p_1 \cdot k - p_1 \cdot p_2\Big) g^{\mu\nu} + 2p_1^\mu p_1^\nu}{(p_1 \cdot k)(p_1 \cdot p_2)} \Bigg\} \tag{I.25}$$

after adding up all four contributions from above.

I.2. Helicity structure functions

I.2.1. Kinematics: The CM frame

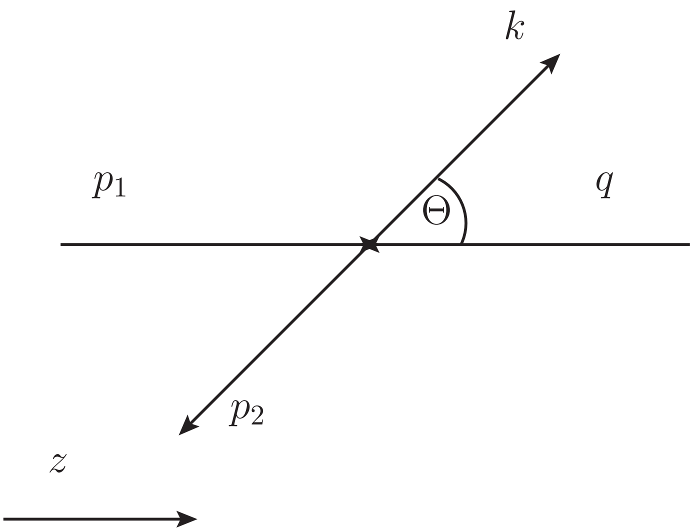


Figure I.1.: Kinematics in the center-of-mass (CM) frame of the partonic subprocess.

We will perform the phase space integration in the center-of-mass (CMS) frame of the partonic subprocess $q + p_1 \to p_2 + k$, thus

$$\boldsymbol{q}_{\mathrm{CM}} = -\boldsymbol{p}_{1\mathrm{CM}} \text{ and } \boldsymbol{k}_{\mathrm{CM}} = -\boldsymbol{p}_{2\mathrm{CM}}. \tag{I.26}$$

Furthermore, we choose $\boldsymbol{q}$ and $\boldsymbol{p}_1$ to be aligned to the z-axis. The scattering shall take place in the x-z-plane. Hence,

$$q^\mu_{\mathrm{CM}} = (E_q, 0, 0, q), \tag{I.27}$$
$$p^\mu_{1\mathrm{CM}} = (q, 0, 0, -q) \tag{I.28}$$
$$p^\mu_{2\mathrm{CM}} = (E_2, -p\sin\theta, 0, -p\cos\theta) \tag{I.29}$$
$$k^\mu_{\mathrm{CM}} = (E_k, p\sin\theta, 0, p\cos\theta), \tag{I.30}$$

where θ denotes the scattering angle and E_q, q, E_k, E_2 and p are all positive and real. We will suppress the "CM" subscript from now on.

In the calculation below, scalar products will occur between all these momenta.

They are:

$$q \cdot p_1 = qE_q + q^2 = \sqrt{\hat{s}}q \tag{I.31}$$
$$q \cdot p_2 = E_q E_2 + qp\cos\theta \tag{I.32}$$
$$q \cdot k = E_q E_k - qp\cos\theta \tag{I.33}$$
$$p_1 \cdot p_2 = qE_2 - qp\cos\theta \tag{I.34}$$
$$p_1 \cdot k = qE_k + qp\cos\theta \tag{I.35}$$
$$p_2 \cdot k = E_2 E_k + p^2. \tag{I.36}$$

To obtain frame-independent expressions, we can relate the entries of the CM-momenta above with the Lorentz scalars $\hat{s}$, Q^2, m_k^2 and m_2^2. For the incoming momenta, reformulating

$$\hat{s} = (p_1^\mu + q^\mu)(p_{1\mu} + q_\mu) = 2(qE_q + q^2) - Q^2 \tag{I.37}$$
$$\text{and } Q^2 = -q^\mu q_\mu = q^2 - E_q^2 \tag{I.38}$$

yields

$$\sqrt{Q^2 + E_q^2} = \frac{\hat{s} - Q^2}{2E_q} - E_q$$
$$\Rightarrow Q^2 + E_q^2 = \frac{\left(\hat{s} - Q^2\right)^2}{4E_q^2} - \hat{s} + Q^2 + E_q^2$$
$$\Rightarrow E_q = \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}} \tag{I.39}$$
$$\text{and } q = \sqrt{Q^2 + E_q^2} = \frac{\hat{s} + Q^2}{2\sqrt{s}}. \tag{I.40}$$

The on-shell conditions for the outgoing momenta are

$$E_{k,2} = \sqrt{p^2 + m_{k,2}^2}. \tag{I.41}$$

Therefore,

$$\hat{s} = (p_2^\mu + k^\mu)(p_{2\mu} + k_\mu) = m_k^2 + m_2^2 + 2(E_1 E_2 + p^2)$$
$$= m_k^2 + m_2^2 + 2\left(\sqrt{p^4 + p^2(m_k^2 + m_2^2) + m_k^2 m_2^2} + p^2\right)$$
$$\Rightarrow 4(m_k^2 + m_2^2 + p^4 + p^2(m_k^2 + m_2^2)) = \hat{s}^2 + m_k^4 + m_2^4 + 4p^4 - 2m_k^2\hat{s}^2 - 2m_2^2\hat{s}$$
$$- 4\hat{s}p^2 + 2m_k^2 m_2^2 + 4m_k^2 p^2 + 4m_2^2 p^2$$
$$\Leftrightarrow p = \frac{\Delta_g}{2\sqrt{\hat{s}}}, \tag{I.42}$$

where we defined[6]

$$\Delta_g \equiv \Delta(\hat{s}, m_2^2, m_k^2). \tag{I.43}$$

[6]The Δ-function itself is defined in (3.15).

Substituting this expression back into the on-shell conditions of E_k and E_2 yields

$$E_k = \frac{\hat{s} - m_2^2 + m_k^2}{2\sqrt{\hat{s}}} \tag{I.44}$$

$$\text{and } E_2 = \frac{\hat{s} + m_2^2 - m_k^2}{2\sqrt{\hat{s}}}. \tag{I.45}$$

As they should, these energies, as well as the ones of the incoming particles, add up to the complete center-of-mass energy:

$$E_k + E_2 = E_q + q = \sqrt{\hat{s}}. \tag{I.46}$$

I.2.2. The phase space integral

Next, we turn to the phase space integration itself. We can simplify the integration measure by exploiting the δ-distribution before dealing the structure functions. Starting from the general definition (A.6), we obtain

$$\begin{aligned}\int d\Pi_{\text{LIPS}} =& \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\int \frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}(2\pi)^4\delta^{(4)}(q+p_1-p_2-k)\\ =&\frac{1}{16\pi^2 E_2 E_k}\int p^2 dp d\cos\theta d\phi \delta\Big(E_q + q - \sqrt{p^2+m_k^2} - \sqrt{p^2+m_2^2}\Big)\\ =&\frac{1}{8\pi}\frac{p}{\sqrt{\hat{s}}}\int_{-1}^{+1} d\cos\theta,\end{aligned} \tag{I.47}$$

when we switch to spherical coordinates for p_2. The integration over ϕ can be executed already at this point because there are no according dependencies inside the partonic tensor. We canceled three δ-distributions in the second line for implicit three-momentum conservation. The momenta in the CM frame, which were introduced in Appendix I.2.1, already fulfill this condition. Using the identity (A.5), we reformulated the remaining δ-distribution to

$$\begin{aligned}\delta\Big(E_q + q - \sqrt{p^2+m_k^2} - \sqrt{p^2+m_2^2}\Big) \equiv&\delta\big(g(p)\big) = \frac{1}{|g'(p)|}\delta\Big(p - \frac{\Delta_g}{2\sqrt{\hat{s}}}\Big)\\ =&\frac{E_2 E_k}{p\sqrt{\hat{s}}}\delta\Big(p - \frac{\Delta_g}{2\sqrt{\hat{s}}}\Big),\end{aligned} \tag{I.48}$$

so that it cancels the p integration. Since we already used momentum conservation to find an explicit value for p in (I.42), no calculation is needed to find the positive root[7] of $g(p)$. For the rest of the calculation, p is identified as a constant and not as an integration variable anymore.

[7] The negative root does not lay in the integration domain $0 < p < \infty$, so that the second term in the sum over the roots of $f(p)$ can be safely ignored.

I.2.3. Calculating $\hat{\Omega}^{g(1)}_{\pm}$

We are now ready to perform the actual phase space integration. Since q is still aligned with the z-axis[8], we are in the position to choose the same set of polarizations as for the LO calculation[9].

Let us start with the transverse contributions

$$\hat{\Omega}^{g(1)}_{\pm} \equiv \frac{\alpha_s}{2\pi} \sum_{\kappa} C_\kappa \hat{\Omega}^{g(1)}_{\pm\kappa}, \tag{I.49}$$

where

$$\hat{\Omega}^{g(1)}_{\pm\kappa} \equiv \frac{1}{4} \frac{p}{\sqrt{\hat{s}}} \int_{-1}^{+1} d\cos\theta \epsilon^\star_{\pm\mu} \hat{\Omega}^{(1)\mu\nu}_{g\kappa} \epsilon_{\pm\nu}. \tag{I.50}$$

The transverse polarizations are

$$\epsilon^\mu_\pm(p_1, q) \equiv \epsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0), \tag{I.51}$$

which are normalized to

$$\epsilon^{\star\mu}_\pm \epsilon_{\pm\mu} = \epsilon^\star_{\pm\mu} g^{\mu\nu} \epsilon_{\pm\nu} = -1. \tag{I.52}$$

We start by calculating all scalar products between polarizations and momenta in the CM frame:

$$\epsilon_{\pm\mu} p_1^\mu = \epsilon^\star_{\pm\mu} p_1^\mu = 0 \tag{I.53}$$

$$\epsilon_{\pm\mu} p_2^\mu = \epsilon^\star_{\pm\mu} p_2^\mu = \pm \frac{1}{\sqrt{2}} p \sin\theta \tag{I.54}$$

$$\epsilon_{\pm\mu} k^\mu = \epsilon^\star_{\pm\mu} k^\mu = \mp \frac{1}{\sqrt{2}} p \sin\theta. \tag{I.55}$$

Furthermore, again in the CM frame, contracting the polarizations, momenta and Levi-Civita-tensors gives

$$\begin{aligned}
\epsilon^\star_{\pm\mu} p_{1\rho} p_{2\sigma} \epsilon^{\rho\sigma\mu\nu} \epsilon_{\pm\nu} =& \frac{i}{2} \Big(\epsilon^{0312} qp\cos\theta(\mp 1) + \epsilon^{3012} qE_2(\mp 1) \\
&+ \epsilon^{0321} qp\cos\theta(\mp 1)(-1) + \epsilon^{3021} qE_2(\mp 1)(-1) \Big) \\
=& \pm iq(E_2 - p\cos\theta)
\end{aligned} \tag{I.56}$$

$$\begin{aligned}
\epsilon^\star_{\pm\mu} p_{1\rho} k_{\sigma} \epsilon^{\rho\sigma\mu\nu} \epsilon_{\pm\nu} =& \frac{i}{2} \Big(\epsilon^{0312} q(-p\cos\theta)(\mp 1) + \epsilon^{3012} qE_k(\mp 1) \\
&+ \epsilon^{0321} q(-p\cos\theta)(\mp 1)(-1) + \epsilon^{3021} qE_k(\mp 1)(-1) \Big) \\
=& \pm iq(E_k + p\cos\theta)
\end{aligned} \tag{I.57}$$

[8] This is obviously a statement which is true for all orders in pertubation theory.

[9] Cf. sections 2.3.2.1 and 2.3 for further informations on polarizations and their role in the helicity formalism.

$$\begin{aligned}
\epsilon^{\star}_{\pm\mu} p_{2\rho} k_{\sigma} \epsilon^{\rho\sigma\mu\nu} \epsilon_{\pm\nu} =& \frac{i}{2}\Big(\epsilon^{0312} E_2(-p\cos\theta)(\mp 1) + \epsilon^{3012} p\cos\theta E_k(\mp 1) \\
&+ \epsilon^{0321} E_2(-p\cos\theta)(\mp 1)(-1) + \epsilon^{3021} p\cos\theta E_k(\mp 1)(-1)\Big) \\
=& \pm ip\sqrt{\hat{s}}\cos\theta. \qquad \text{(I.58)}
\end{aligned}$$

Using these relations, we obtain

$$\begin{aligned}
\epsilon^{\star}_{\pm\mu}\hat{\Omega}^{(1)\mu\nu}_{gs}\epsilon_{\pm\nu} =& \frac{2}{q^2}\Bigg\{\frac{1}{\Big(E_k + p\cos\theta\Big)^2}\Big[-m_k^2\Big(E_2E_k + p^2 - qE_2 + qp\cos\theta\Big) \\
&+ \Big(qE_2 - qp\cos\theta\Big)\Big(qE_k + qp\cos\theta\Big) + m_k^2 p^2\sin^2\theta\Big] \\
&+ \frac{1}{\Big(E_2 - p\cos\theta\Big)^2}\Big[-m_2^2\Big(E_2E_k + p^2 - qE_k - qp\cos\theta\Big) \\
&+ \Big(qE_k + qp\cos\theta\Big)\Big(qE_2 - qp\cos\theta\Big) + m_2^2 p^2\sin^2\theta\Big] \\
&- \frac{1}{\Big(E_k + p\cos\theta\Big)\Big(E_2 - p\cos\theta\Big)} \\
&\times\Big[-2\Big(E_2E_k + p^2\Big)^2 + 2\Big(E_2E_k + p^2\Big)qE_2 \\
&+ 2\Big(E_2E_k + p^2\Big)qE_k - m_k^2\Big(qE_2 - qp\cos\theta\Big) \\
&- m_2^2\Big(qE_k + qp\cos\theta\Big) - qp^2E_2\sin^2\theta - qp^2E_k\sin^2\theta \\
&+ \Big(-qE_2 - qE_k + 2E_2E_k + 2p^2\Big)p^2\sin^2\theta\Big]\Bigg\} \qquad \text{(I.59)}
\end{aligned}$$

$$\begin{aligned}
\epsilon^{\star}_{\pm\mu}\hat{\Omega}^{(1)\mu\nu}_{ga}\epsilon_{\pm\nu} =& \frac{2}{q^2(E_k + p\cos\theta)^2}\Big\{\pm q\Big(qE_k + qp\cos\theta + m_k^2\Big)(E_2 - p\cos\theta) \pm m_k^2 p\sqrt{\hat{s}}\cos\theta\Big\} \\
&- \frac{2}{q^2(E_2 - p\cos\theta)^2}\Big\{\pm q\Big(qE_2 - qp\cos\theta + m_2^2\Big)(E_k + p\cos\theta) \mp m_2^2 p\sqrt{\hat{s}}\cos\theta\Big\} \\
&+ \frac{1}{q^2(E_k + p\cos\theta)(E_2 - p\cos\theta)} \\
&\times\Big\{\pm q\Big(qE_k + qp\cos\theta - 2E_2E_k - 2p^2\Big)(E_2 - p\cos\theta) \\
&\mp q\Big(qE_2 - qp\cos\theta - 2E_2E_k - 2p^2\Big)(E_k + p\cos\theta) \\
&\pm 2p\sqrt{\hat{s}}\Big(\sqrt{\hat{s}}q - 2E_2E_k - 2p^2\Big)\cos\theta\Big\} \qquad \text{(I.60)}
\end{aligned}$$

$$\epsilon^{\star}_{\pm\mu}\hat{\Omega}^{(1)\mu\nu}_{gx}\epsilon_{\pm\nu} = \frac{2m_k m_2}{q^2}\Bigg\{\frac{qE_k + qp\cos\theta - m_k^2}{\Big(E_k + p\cos\theta\Big)^2} + \frac{qE_2 - qp\cos\theta - m_2^2}{\Big(E_2 - p\cos\theta\Big)^2} + \frac{2(E_2E_k + p^2) - \sqrt{\hat{s}}q}{\Big(E_k + p\cos\theta\Big)\Big(E_2 - p\cos\theta\Big)}\Bigg\}. \tag{I.61}$$

We can identify several generic integrals, which can be solved by substituting the denominator:

$$I_{t0} \equiv \int_{-1}^{+1} d\cos\theta \frac{1}{\Big(E_k + p\cos\theta\Big)^2} = \frac{1}{p}\int_{E_k-p}^{E_k+p} du \frac{1}{u^2} = \frac{2}{m_k^2} \tag{I.62}$$

$$I_{t1} \equiv \int_{-1}^{+1} d\cos\theta \frac{\cos\theta}{\Big(E_k + p\cos\theta\Big)^2} = \frac{1}{p^2}\int_{E_k-p}^{E_k+p} du \frac{u - E_k}{u^2} = \frac{1}{p}\left(\frac{1}{p}L_t - 2\frac{E_k}{m_k^2}\right) \tag{I.63}$$

$$I_{t2} \equiv \int_{-1}^{+1} d\cos\theta \frac{\cos^2\theta}{\Big(E_k + p\cos\theta\Big)^2} = \frac{1}{p^3}\int_{E_k-p}^{E_k+p} du \frac{u^2 - 2E_k u + E_k^2}{u^2} = \frac{2}{p^2}\left(-\frac{E_k}{p}L_t + 1 + \frac{E_k^2}{m_k^2}\right) \tag{I.64}$$

$$I_{u0} \equiv \int_{-1}^{+1} d\cos\theta \frac{1}{\Big(E_2 - p\cos\theta\Big)^2} = \frac{1}{p}\int_{E_2-p}^{E_2+p} du \frac{1}{u^2} = \frac{2}{m_2^2} \tag{I.65}$$

$$I_{u1} \equiv \int_{-1}^{+1} d\cos\theta \frac{\cos\theta}{\Big(E_2 - p\cos\theta\Big)^2} = -\frac{1}{p^2}\int_{E_2-p}^{E_2+p} du \frac{u - E_2}{u^2} = -\frac{1}{p}\left(\frac{1}{p}L_u - 2\frac{E_2}{m_2^2}\right) \tag{I.66}$$

$$I_{u2} \equiv \int_{-1}^{+1} d\cos\theta \frac{\cos^2\theta}{\Big(E_2 - p\cos\theta\Big)^2} = \frac{1}{p^3}\int_{E_2-p}^{E_2+p} du \frac{u^2 - 2E_2 u + E_2^2}{u^2} = \frac{2}{p^2}\left(-\frac{E_2}{p}L_u + 1 + \frac{E_2^2}{m_2^2}\right). \tag{I.67}$$

We defined the two logarithms

$$L_t \equiv \ln\frac{E_k + p}{E_k - p} \tag{I.68}$$

$$\text{and } L_u \equiv \ln\frac{E_2 + p}{E_2 - p}, \tag{I.69}$$

arising from the t- and u-channel of the underlying process (cf. figure 6.1). Expressing them in terms of Lorentz scalars yields

$$L_t = \ln \frac{(E_k + p)^2}{E_k^2 - p^2} = \ln \frac{\left(\hat{s} - m_2^2 + m_k^2 + \Delta_g\right)^2}{4\hat{s}m_k^2} \tag{I.70}$$

and, equivalently,

$$L_u = \ln \frac{\left(\hat{s} + m_2^2 - m_k^2 + \Delta_g\right)^2}{4\hat{s}m_2^2}. \tag{I.71}$$

To integrate the interference terms, we use partial fraction decompositions:

$$\begin{aligned}
I_{i0} \equiv& \int_{-1}^{+1} d\cos\theta \frac{1}{\left(E_k + p\cos\theta\right)\left(E_2 - p\cos\theta\right)} \\
=& \frac{1}{\sqrt{\hat{s}}} \int_{-1}^{+1} d\cos\theta \left(\frac{1}{E_k + p\cos\theta} + \frac{1}{E_2 - p\cos\theta}\right) \\
=& \frac{1}{\sqrt{\hat{s}}p}\left(L_t + L_u\right)
\end{aligned} \tag{I.72}$$

$$\begin{aligned}
I_{i1} \equiv& \int_{-1}^{+1} d\cos\theta \frac{\cos\theta}{\left(E_k + p\cos\theta\right)\left(E_2 - p\cos\theta\right)} \\
=& \frac{1}{\sqrt{\hat{s}}p} \int_{-1}^{+1} d\cos\theta \left(-\frac{E_k}{E_k + p\cos\theta} + \frac{E_2}{E_2 - p\cos\theta}\right) \\
=& \frac{1}{\sqrt{\hat{s}}p^2}\left(-E_k L_t + E_2 L_u\right) \\
I_{i2} \equiv& \int_{-1}^{+1} d\cos\theta \frac{\cos^2\theta}{\left(E_k + p\cos\theta\right)\left(E_2 - p\cos\theta\right)} \\
=& \frac{1}{\sqrt{\hat{s}}p^2} \int_{-1}^{+1} d\cos\theta \left(\frac{E_k^2}{E_k + p\cos\theta} + \frac{E_2^2}{E_2 - p\cos\theta} - \sqrt{s}\right) \\
=& \frac{1}{\sqrt{\hat{s}}p^3}\left(E_k^2 L_t + E_2^2 L_u - 2\sqrt{\hat{s}}p\right).
\end{aligned} \tag{I.73}$$

Inserting these integrals into the contracted tensors (I.59), (I.60) and (I.61) gives the final results. For the symmetric contribution, this yields

$$
\begin{aligned}
\hat{\Omega}^{g(1)}_{\pm s} =& \frac{p}{2\sqrt{\hat{s}}q^2}\Bigg\{\Big[(q^2-m_k^2)E_2E_k + m_k^2 qE_2\Big]I_{t0} \\
&+\Big[-m_k^2 q + q^2(E_2-E_k)\Big]pI_{t1} + \Big[-q^2-m_k^2\Big]p^2 I_{t2} \\
&+\Big[(q^2-m_2^2)E_2E_k + m_2^2 qE_k\Big]I_{u0} \\
&+\Big[m_2^2 q + q^2(E_2-E_k)\Big]pI_{u1} + \Big[-q^2-m_2^2\Big]p^2 I_{u2} \\
&+\Big[2(E_2E_k+p^2-\sqrt{\hat{s}}q)E_2E_k + m_k^2 qE_2 + m_2^2 qE_k\Big]I_{i0} \\
&+\Big[m_2^2-m_k^2\Big]qpI_{i1} + \Big[-2\sqrt{\hat{s}}q + 2E_2E_k + 2p^2\Big]p^2 I_{i2}\Bigg\} \\
=& \frac{1}{2\sqrt{\hat{s}}q^2}\Bigg\{\Big(\sqrt{\hat{s}}q^2 + 2m_k^2E_k + 2E_k^2E_2 + 2p^2E_k - 2\sqrt{\hat{s}}qE_k\Big)L_t \\
&+ \text{ similar terms proportional to } L_u \\
&+ 2p\Big(4\sqrt{\hat{s}}q - q^2 - 2(E_2E_k+p^2) - \hat{s} - m_k^2 - m_2^2\Big)\Bigg\} \\
=& \left[\left(\frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2}\right)^2 - \frac{\hat{s}-m_2^2+m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_t \\
&+\left[\left(\frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2}\right)^2 - \frac{\hat{s}+m_2^2-m_k^2}{\hat{s}+Q^2} + \frac{1}{2}\right]L_u \\
&- \frac{\Delta_g}{\hat{s}}\left(\frac{\hat{s}-Q^2}{\hat{s}+Q^2}\right)^2 .
\end{aligned}
\tag{I.74}
$$

Here and in the following, the "similar terms proportional to L_u" are the contributions proportional to L_t with exchanged $m_k \leftrightarrow m_2$ and $E_k \leftrightarrow E_2$. When inserting the explicit CM expressions in the last step, we could do the following simplification:

$$
\begin{aligned}
E_2E_k + p^2 &= \frac{1}{4\hat{s}}\Big(\hat{s}^2 - m_k^4 - m_2^4 + 2m_k^2m_2^2 + \hat{s}^2 + m_k^4 + m_2^4 - 2m_k^2\hat{s}^2 - 2m_2^2\hat{s} - 2m_k^2m_2^2\Big) \\
&= \frac{1}{2}\Big(\hat{s} - m_2^2 - m_k^2\Big)
\end{aligned}
\tag{I.75}
$$

This relation will also be useful in the other calculations.

The antisymmetric contribution gives

$$\begin{aligned}
\hat{\Omega}^{g(1)}_{\pm a} &= \pm \frac{p}{2\sqrt{\hat{s}}q^2}\Bigg\{\Big[qE_k + m_k^2\Big]qE_2 I_{t0} + \Big[q^2(E_2 - E_k) + m_k^2(\sqrt{\hat{s}} - q)\Big]pI_{t1} - q^2p^2 I_{t2} \\
&\qquad - \Big[qE_2 + m_2^2\Big]qE_k I_{u0} - \Big[q^2(E_2 - E_k) + m_2^2(q - \sqrt{\hat{s}})\Big]pI_{u1} + q^2p^2 I_{u2} \\
&\qquad + \Big[(E_k - E_2)(E_2E_k + p^2)\Big]qI_{i0} + \Big[2(E_2E_k + p^2)(q - \sqrt{\hat{s}}) + \hat{s}q\Big]pI_{i1}\Bigg\} \\
&= \pm \frac{1}{2\sqrt{\hat{s}}q^2}\Bigg\{\Bigg[\sqrt{\hat{s}}q^2 + m_k^2(\sqrt{\hat{s}} - q) + \frac{q}{2s}(m_k^2 - m_2^2)(\hat{s} - m_2^2 - m_k^2) \\
&\qquad + (\hat{s} - m_2^2 - m_k^2)(\sqrt{\hat{s}} - q)\frac{E_k}{\sqrt{\hat{s}}} - \sqrt{\hat{s}}qE_k\Bigg]L_t \\
&\qquad - \text{similar terms proportional to } L_u + \sqrt{\hat{s}}p(E_2 - E_k)\Bigg\} \\
&= \pm\Bigg\{\Bigg[\frac{s^2 + m_2^4 - m_k^4 + 2sm_k^2 - 2sm_2^2}{\Big(\hat{s} + Q^2\Big)^2} - \frac{\hat{s} - m_2^2 + m_k^2}{\hat{s} + Q^2} + \frac{1}{2}\Bigg]L_t \\
&\qquad - \text{similar terms proportional to } L_u + \frac{p}{\sqrt{\hat{s}}q^2}(m_2^2 - m_k^2)\Bigg\} \\
&= \pm\Bigg\{\Bigg[\left(\frac{\hat{s} - m_2^2 + m_k^2}{\hat{s} + Q^2}\right)^2 + 2\frac{m_k^2(m_2^2 - m_k^2)}{\Big(\hat{s} + Q^2\Big)^2} - \frac{\hat{s} - m_2^2 + m_k^2}{\hat{s} + Q^2} + \frac{1}{2}\Bigg]L_t \\
&\qquad - \Bigg[\left(\frac{\hat{s} + m_2^2 - m_k^2}{\hat{s} + Q^2}\right)^2 + 2\frac{m_2^2(m_k^2 - m_2^2)}{\Big(\hat{s} + Q^2\Big)^2} - \frac{\hat{s} + m_2^2 - m_k^2}{\hat{s} + Q^2} + \frac{1}{2}\Bigg]L_u + \frac{2\Delta_g(m_2^2 - m_k^2)}{\Big(\hat{s} + Q^2\Big)^2}\Bigg\}.
\end{aligned} \tag{I.76}$$

Coming to the last contribution, we have

$$\begin{aligned}
\hat{\Omega}^{g(1)}_{\pm x} &= \frac{m_k m_2 p}{2\sqrt{\hat{s}}q^2}\Bigg\{\Big[qE_k - m_k^2\Big]I_{t0} + qpI_{t1} + \Big[qE_2 - m_2^2\Big]I_{u0} - qpI_{u1} \\
&\qquad + \Big[2(E_2E_k + p^2) - \sqrt{\hat{s}}q\Big]I_{i0}\Bigg\} \\
&= \frac{m_k m_2}{q^2}\Bigg\{\frac{E_2E_k + p^2}{\hat{s}}\Big(L_t + L_u\Big) - \frac{p}{\sqrt{\hat{s}}}\left(\frac{m_k^2}{m_k^2} + \frac{m_2^2}{m_2^2}\right)\Bigg\} \\
&= \frac{2m_k m_2}{\Big(\hat{s} + Q^2\Big)^2}\Bigg\{\Big(\hat{s} - m_2^2 - m_k^2\Big)\Big(L_t + L_u\Big) - 2\Delta_g\Bigg\}.
\end{aligned} \tag{I.77}$$

I.2.4. Calculating $\hat{\Omega}_0^{g(1)}$

The last remaining polariziation is the longitudinal one (cf. (2.67)),

$$\epsilon_0^\mu(p_1, q) \equiv \epsilon_0^\mu = \frac{Q^2 p_1^\mu + (q \cdot p_1) q^\mu}{Q(q \cdot p_1)}, \tag{I.78}$$

which is real, $\epsilon_0^{\star\mu} = \epsilon_0^\mu$, and normalized to

$$\epsilon_0^\mu \epsilon_{0\mu} = \epsilon_{0\mu} g^{\mu\nu} \epsilon_{0\nu} = 1. \tag{I.79}$$

Similar to the computation in the section above, the objective is to find the contributions to the $\lambda = 0$ structure function:

$$\hat{\Omega}_{0\kappa}^{g(1)} \equiv \frac{1}{4} \frac{p}{\sqrt{\hat{s}}} \int_{-1}^{+1} d\cos\theta \epsilon_{0\mu}^\star \hat{\Omega}_{g\kappa}^{(1)\mu\nu} \epsilon_{0\nu}. \tag{I.80}$$

In the CM frame, the relevant scalar products are

$$\epsilon_0^\mu p_{1\mu} = \frac{\sqrt{\hat{s}} q}{Q}, \tag{I.81}$$

$$\epsilon_0^\mu p_{2\mu} = \left(\frac{Q}{\sqrt{\hat{s}}} + \frac{E_q}{Q} \right) E_2 + \left(\frac{q}{Q} - \frac{Q}{\sqrt{\hat{s}}} \right) p \cos\theta, \tag{I.82}$$

$$\text{and } \epsilon_0^\mu k_\mu = \left(\frac{Q}{\sqrt{\hat{s}}} + \frac{E_q}{Q} \right) E_k + \left(\frac{Q}{\sqrt{s}} - \frac{q}{Q} \right) p \cos\theta. \tag{I.83}$$

It is also helpful to evaluate some further expressions, namely[10]

$$\epsilon_{0\mu} p_2^\mu p_2^\nu \epsilon_{0\nu} = \frac{1}{Q^2} \left(q^2 E_2^2 + 2q E_q E_2 p \cos\theta + E_q^2 p^2 \cos^2\theta \right), \tag{I.84}$$

$$\epsilon_{0\mu} k^\mu k^\nu \epsilon_{0\nu} = \frac{1}{Q^2} \left(q^2 E_k^2 - 2q E_q E_k p \cos\theta + E_q^2 p^2 \cos^2\theta \right) \tag{I.85}$$

$$\text{and } \epsilon_{0\mu} p_2^\mu k^\nu \epsilon_{0\nu} = \frac{1}{Q^2} \left(q^2 E_2 E_k + q E_q (E_k - E_2) p \cos\theta - E_q^2 p^2 \cos^2\theta \right), \tag{I.86}$$

before turning to the fully contracted symmetric tensor (I.23):

$$\begin{aligned} \epsilon_{0\mu} \hat{\Omega}_{gs}^{(1)\mu\nu} \epsilon_{0\nu} = \frac{2}{q^2} \Bigg\{ & \frac{1}{(E_k + p\cos\theta)^2} \\ & \times \Big[m_k^2 \Big(E_2 E_k + p^2 - q E_2 + q p \cos\theta \Big) - \Big(q E_2 - p q \cos\theta \Big) \Big(q E_k + q p \cos\theta \Big) \\ & + 2 \frac{\sqrt{\hat{s}} q}{Q} \Big(m_k^2 + q E_k + q p \cos\theta \Big) \epsilon_{0\mu} p_2^\mu - 2 m_k^2 \epsilon_{0\mu} p_2^\mu k^\nu \epsilon_{0\nu} \Big] \end{aligned}$$

[10]Here, the simplest approach is to substitute the Lorentz-invariant expressions for E_q (I.39) as well as q (I.40) and then convert back at the end.

$$
\begin{aligned}
&+\frac{1}{(E_2-p\cos\theta)^2}\\
&\quad\times\Big[m_2^2\Big(E_2E_k+p^2-qE_k-qp\cos\theta\Big)-\Big(qE_2-pq\cos\theta\Big)\Big(qE_k+qp\cos\theta\Big)\\
&\qquad+2\frac{\sqrt{\hat{s}}q}{Q}\Big(m_2^2+qE_2-qp\cos\theta\Big)\epsilon_{0\mu}k^\mu-2m_2^2\epsilon_{0\mu}p_2^\mu k^\nu\epsilon_{0\nu}\Big]\\
&+\frac{1}{(E_2-p\cos\theta)(E_k+p\cos\theta)}\\
&\quad\times\Big[-2\Big(E_2E_k+p^2\Big)^2+2\sqrt{\hat{s}}q\Big(E_2E_k+p^2\Big)-m_k^2\Big(qE_2-qp\cos\theta\Big)\\
&\qquad-m_2^2\Big(qE_k+qp\cos\theta\Big)+2\Big(qE_2-qp\cos\theta\Big)\epsilon_{0\mu}k^\mu k^\nu\epsilon_{0\nu}\\
&\qquad+2\Big(qE_k+qp\cos\theta\Big)\epsilon_{0\mu}p_2^\mu p_2^\nu\epsilon_{0\nu}-2\frac{\sqrt{\hat{s}}q}{Q}\Big(E_2E_k+p^2\Big)\epsilon_{0\mu}p_2^\mu\\
&\qquad-2\frac{\sqrt{\hat{s}}q}{Q}\Big(E_2E_k+p^2\Big)\epsilon_{0\mu}k^\mu-2\Big(\sqrt{\hat{s}}q-2\Big(E_2E_k+p^2\Big)\Big)\epsilon_{0\mu}p_2^\mu k^\nu\epsilon_{0\nu}\Big]\Big\}. \qquad \text{(I.87)}
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\hat{\Omega}_{0s}^{g(1)}=\frac{p}{2\sqrt{\hat{s}}q^2}\Bigg\{&\Big[m_k^2\Big(E_2E_k+p^2\Big)+m_k^2qE_2+q^2E_2E_k+2m_k^2\frac{\sqrt{\hat{s}}q}{Q^2}E_qE_2\\
&\quad+2\frac{\sqrt{\hat{s}}q^2}{Q^2}E_qE_2E_k-2m_k^2\frac{q^2}{Q^2}E_2E_k\Big]I_{t0}\\
&+\Big[-m_k^2q-\sqrt{\hat{s}}q^2+2m_k^2\frac{\sqrt{\hat{s}}q^2}{Q^2}+2\frac{\sqrt{\hat{s}}q^3}{Q^2}E_k+2q^2E_2\\
&\quad+2\frac{\sqrt{\hat{s}}q^2}{Q^2}E_qE_2+2m_k^2\frac{q}{Q^2}E_q(E_2-E_k)\Big]pI_{t1}\\
&+\Big[-q^2+2\frac{\sqrt{\hat{s}}q^3}{Q^2}+2m_k^2\frac{E_q^2}{Q^2}\Big]p^2I_{t2}\\
&+\Big[m_2^2\Big(E_2E_k+p^2\Big)+m_2^2qE_k+q^2E_2E_k+2m_2^2\frac{\sqrt{\hat{s}}q}{Q^2}E_qE_k\\
&\quad+2\frac{\sqrt{\hat{s}}q^2}{Q^2}E_qE_2E_k-2m_2^2\frac{q^2}{Q^2}E_2E_k\Big]I_{u0}\\
&+\Big[m_2^2q+\sqrt{\hat{s}}q^2-2m_2^2\frac{\sqrt{\hat{s}}q^2}{Q^2}-2\frac{\sqrt{\hat{s}}q^3}{Q^2}E_2-2q^2E_k\\
&\quad-2\frac{\sqrt{\hat{s}}q^2}{Q^2}E_qE_k+2m_2^2\frac{q}{Q^2}E_q(E_2-E_k)\Big]pI_{u1}\\
&+\Big[-q^2+2\frac{\sqrt{\hat{s}}q^3}{Q^2}+2m_2^2\frac{E_q^2}{Q^2}\Big]p^2I_{u2}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-2\Big(E_2E_k+p^2\Big)^2-m_k^2qE_2-m_2^2qE_k \\
&\quad -2\frac{\hat{s}q}{Q^2}E_q\Big(E_2E_k+p^2\Big)+4\frac{q^2}{Q^2}\Big(E_2E_k+p^2\Big)E_2E_k\Big]I_{i0} \\
&+\Big[m_k^2q-m_2^2q+2\frac{q^3}{Q^2}\Big(E_2^2-E_k^2\Big)+2\frac{\sqrt{\hat{s}}q^2}{Q^2}E_q(E_2-E_k) \\
&\quad +4\frac{q}{Q^2}E_q\Big(E_2E_k+p^2\Big)(E_k-E_2)\Big]pI_{i1} \\
&+\Big[4\frac{\hat{s}q}{Q^2}E_q-4\frac{E_q^2}{Q^2}\Big(E_2E_k+p^2\Big)\Big]p^2I_{i2}\Big\} \\
=\frac{p}{\sqrt{\hat{s}}q^2}&\Big\{\frac{1}{p}\Big[-\frac{1}{2}m_k^2q+\frac{1}{2}\sqrt{\hat{s}}q^2+m_k^2\frac{\sqrt{\hat{s}}q^2}{Q^2}+\frac{\sqrt{\hat{s}}q^2}{Q^2}E_qE_2+m_k^2\frac{q}{Q^2}E_q(E_2-E_k) \\
&\quad -\frac{\sqrt{\hat{s}}q^3}{Q^2}E_k-2m_k^2\frac{1}{Q^2}E_q^2E_k-\frac{1}{\sqrt{\hat{s}}}\Big(E_2E_k+p^2\Big)^2-\frac{1}{2}m_k^2\frac{q}{\sqrt{\hat{s}}}E_2-\frac{1}{2}m_2^2\frac{q}{\sqrt{\hat{s}}}E_k \\
&\quad -\frac{\sqrt{\hat{s}}q}{Q^2}E_q\Big(E_2E_k+p^2\Big)+2\frac{q^2}{\sqrt{\hat{s}}Q^2}\Big(E_2E_k+p^2\Big)E_2E_k+\frac{1}{2}\Big(m_2^2-m_k^2\Big)\frac{q}{\sqrt{\hat{s}}}E_k \\
&\quad -\frac{q^3}{\sqrt{\hat{s}}Q^2}\Big(E_2^2-E_k^2\Big)E_k-\frac{q^2}{Q^2}E_q(E_2-E_k)E_k \\
&\quad -2\frac{q}{\sqrt{\hat{s}}Q^2}E_q\Big(E_2E_k+p^2\Big)(E_k-E_2)E_k \\
&\quad +2\frac{\sqrt{\hat{s}}q}{Q^2}E_qE_k^2-2\frac{1}{\sqrt{\hat{s}}Q^2}E_q^2\Big(E_2E_k+p^2\Big)E_k^2\Big]L_t \\
&+\text{ similar terms proportional to } L_u \\
&+2\Big(E_2E_k+p^2\Big)+2\sqrt{\hat{s}}q+2\frac{\hat{s}q}{Q^2}E_q-4\frac{q^2}{Q^2}E_2E_k+\sqrt{\hat{s}}q^2\Big(\frac{E_k}{m_k^2}+\frac{E_2}{m_2^2}\Big) \\
&-2\frac{\hat{s}q^2}{Q^2}-\frac{m_2^2+m_k^2}{m_2^2m_k^2}q^2E_2E_k+2\frac{q}{Q^2}E_q(E_2-E_k)^2-2q^2+4\frac{\sqrt{\hat{s}}q^3}{Q^2} \\
&+2\Big(m_k^2+m_2^2\Big)\frac{1}{Q^2}E_q^2-q^2\Big(\frac{E_k^2}{m_k^2}+\frac{E_2^2}{m_2^2}\Big)+2\frac{1}{Q^2}E_q^2\Big(E_2^2+E_k^2\Big) \\
&-4\frac{\hat{s}}{Q^2}qE_q+4\frac{1}{Q^2}E_q^2\Big(E_2E_k+p^2\Big)\Big\}. \qquad \text{(I.88)}
\end{aligned}
$$

After substituting the CM expressions, we arrive at

$$\begin{aligned}\hat{\Omega}^{g(1)}_{0s} = \frac{1}{Q^2\Big(\hat{s}+Q^2\Big)^2}\Bigg\{&\Bigg[\frac{1}{2}\Big(m_2^2+m_k^2\Big)\Big(\hat{s}^2+4Q^2\hat{s}-3Q^4\Big)\\ &+2m_2^2m_k^2\Big(\hat{s}-3Q^2\Big)+\Big(m_2^4+m_k^4\Big)\Big(Q^2-\hat{s}\Big)\\ &+m_2^6-m_2^4m_k^2-m_2^2m_k^4+m_k^6\Bigg]\Big(L_t+L_u\Big)\\ &+2Q^2\Big(\hat{s}-Q^2\Big)\Big(m_k^2-m_2^2\Big)\Big(L_t-L_u\Big)\\ &+2\Delta_g\Big[(m_2^2-m_k^2)^2-Q^2(m_2^2+m_k^2)+2Q^4\Big]\Bigg\}.\end{aligned} \tag{I.89}$$

For the antisymmetric contribution, we note that $\epsilon_{0\mu}\epsilon_{0\nu}$ is a symmetric tensor. In addition, it is easy to see that $\hat{\Omega}^{(1)\mu\nu}_{ga}$ (I.24) is antisymmetric. This immediately yields

$$\hat{\Omega}^{g(1)}_{0a} = 0. \tag{I.90}$$

For the mixed contribution, we first contract (I.25):

$$\begin{aligned}\epsilon_{0\mu}\hat{\Omega}^{(1)\mu\nu}_{gx}\epsilon_{0\nu} = \frac{2m_km_2}{q^2}\Bigg\{&\frac{m_k^2-qE_k-qp\cos\theta}{\Big(E_k+p\cos\theta\Big)^2}+\frac{m_2^2-qE_2+qp\cos\theta}{\Big(E_2-p\cos\theta\Big)^2}\\ &+\frac{\sqrt{\hat{s}}q-2\Big(E_2E_k+p^2+\frac{\hat{s}q^2}{Q^2}\Big)}{\Big(E_k+p\cos\theta\Big)\Big(E_2-p\cos\theta\Big)}\Bigg\}.\end{aligned} \tag{I.91}$$

Hence, the full expression is

$$\begin{aligned}\hat{\Omega}^{g(1)}_{0x} = \frac{m_km_2p}{2\sqrt{\hat{s}}q^2}\Bigg\{&\Big[m_k^2-qE_k\Big]I_{t0}-qpI_{t1}+\Big[m_2^2-qE_2\Big]I_{u0}-qpI_{u1}\\ &+\Big[\sqrt{\hat{s}}q-2\Big(E_2E_k+p^2+\frac{\hat{s}q^2}{Q^2}\Big)\Big]I_{i0}\Bigg\}.\end{aligned} \tag{I.92}$$

Despite an overall factor of -1 and one additional term, the expression above is equal to $\hat{\Omega}^{g(1)}_{\pm x}$ (I.77). Therefore, we can directly conclude that

$$\hat{\Omega}^{g(1)}_{0x} = \frac{m_km_2}{Q^2\Big(\hat{s}+Q^2\Big)^2}\Bigg\{\Bigg[-\hat{s}^2+2Q^2\Big(m_2^2+m_k^2-2\hat{s}\Big)-Q^4\Bigg]\Big(L_t+L_u\Big)+4Q^2\Delta_g\Bigg\}. \tag{I.93}$$

J. Ambiguities in ...

J.1. ... [Aivazis et al., 1994a]

- Whenever a sum over polarizations occurs (the first time in equation (6) on page 3086), there is a factor missing which accounts for the normalization of $\epsilon^{\mu}_{\lambda}(P, q)$. In this work, this factor is denoted by v_{λ} and is introduced in section 2.3.2.1.

- page 3096, equation (B9) and below: According to the definition (B7), the coupling g_L (g_R) is missing inside $j^{LL}_{L}(Q)$ ($j^{RR}_{R}(Q)$). In this work, the leptonic helicity currents are derived in appendix G.2.

- page 3096, equation (B10): "...$d^1(\psi)^{-1}\ {}_m d^1(-\psi)^n\ {}_{-1}$..." should be "...$d^1(\psi)^{-1}\ {}_m d^1(\psi)^n\ {}_{-1}$...". Additionally, the second contribution proportional to $d^1(\psi)^{+1}\ {}_m d^1(\psi)^n\ {}_{+1}$ is missing. In this work, this equation is derived in appendix G.2.

- page 3097, equation (B12): Substituting this equation into (A10) on page 3093 gives a differential cross section which differs by a factor of 1/2 from the one given in (14) on page 3088. In appendix G.2 of this work, the latter equation could be reproduced with a slightly different approach. This leads to the assumption that equation (B12) is wrong by a factor of 1/2. However, since this part is kept relatively short in the original paper, the non-matching prefactors could also be due to a not explicitly stated (re-)definition.

- There are missing partonic indices a (in this work called i) of the partonic tensor $\hat{W}^{\mu\nu}_{i}$ at some places, e.g. in appendix C on page 3098 following. There, the dependencies on a occur in the partonic chirality couplings g_{R_a,L_a} inside the structure functions.

J.2. ... [Aivazis et al., 1994b]

- page 3107, equation (9): "... $- f_N^g \otimes f_g^{Q(0)} \otimes \omega_{BQ}^{\lambda(0)}$..." should be "... $- f_N^g \otimes f_g^{Q(1)} \otimes \omega_{BQ}^{\lambda(0)}$...". In this work, subtraction terms at NLO are introduced in section 3.3.3. This specific equation can be found in (6.2).

- page 3107, equations (8) and (9): Terms involving the convolution $f_g^{Q(0)} \otimes \omega_{BQ}^{\lambda(0)}$ are only valid in the case of a neutral exchanged current. And even in that case, an additional term covering the antiquark subtractions should be added for the sake of clarity. For flavour changing processes, contributions proportional to different flavours than Q should also be taken into account. This is covered in detail in the context of equation (6.2).

- page 3109, equation (14): "... $= \frac{\alpha_s(\mu)}{\pi}$..." should be "... $= \frac{\alpha_s(\mu)}{2\pi}$...". Otherwise, the subtraction term (29) would not be equal to the one in (9), using the PDF given in equation (7). In this work, this equation can be found in (6.3), while the subtraction term derivation is done in section 6.3.1.

- page 3112, equation (31): The variable s should be replaced by $\hat{s}$.

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Plagiatserklärung

Hiermit versichere ich, dass die vorliegende Arbeit über „Deep Inelastic Scattering with Massive Quarks at Next-to-leading Order in QCD“ selbstständig verfasst worden ist, dass keine anderen Quellen und Hilfsmittel als die angegebenen benutzt worden sind und dass die Stellen der Arbeit, die anderen Werken - auch elektronischen Medien - dem Wortlaut oder Sinn nach entnommen wurden, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht worden sind.

Münster, 02.07.2020 ______________________________

Ich erkläre mich mit einem Abgleich der Arbeit mit anderen Texten zwecks Auffindung von Übereinstimmungen sowie mit einer zu diesem Zweck vorzunehmenden Speicherung der Arbeit in eine Datenbank einverstanden.

Münster, 02.07.2020 ______________________________