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Bachelor's thesis

Partial wave analysis of dark matter annihilation's.

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Abstract

In this thesis we will derive a partial wave formalism for the invariant amplitude of interactions with two initial particles, both fermions. The expansion for the total angular momentum can be done by expanding the amplitude for Wigner D-functions. To expand the amplitude for small orbital angular momenta one can use a Taylor expansion for small momenta (or relative velocities) and identify the terms with the s-wave, p-wave etc. In the end we will apply this formalism to the MSSM and give two examples for Neutralino annihilation's into fermion pairs.

1 Introduction

Over the last decades the search for dark matter has been an important topic in particle physics. Due to astronomical observations we assume that about 84 % of the matter in the universe is dark matter. There are three ways to look for dark matter particles. They may be produced in collider experiments. Secondly it may be possible to detect them by nuclear recoil experiments. For the purpose of this thesis we are interested in the third way, indirect detection, where one tries to detect dark matter by products of self annihilation and decay processes[1]. In the context of indirect detection the cross section σv is often expanded for partial waves.

The goal of this thesis is to establish a partial wave formalism for the invariant amplitude of interactions with two initial particles, both fermions. Since we are working with particles with spin, we need to consider the coupling of spin and orbital angular momentum. We will expand the amplitudes for both total and orbital angular momentum. Firstly we construct two particle states with total and orbital angular momentum and use these to expand the amplitude for total angular momenta. Secondly we derive a formalism to expand the amplitude for orbital angular momenta. This is done by separating the Dirac equation into a radial and an angular part and then solving the equations for each partial wave. This gives us the momentum dependence of the initial wave. This dependence can be used to expand the amplitude. In the end we apply the formalism to the MSSM and expand the invariant amplitude of Neutralino annihilation via Higgs and Z^0 bosons into fermions pairs.

2 Notation and convention

We will use natural units ($c = \hbar = 1$). If not stated otherwise p denotes the four vector and $\vec{p}$ the three vector. The normal vectors in the direction of a vector $\vec{v}$ will be denoted with a hat ($\hat{v} := \frac{\vec{v}}{|\vec{v}|}$). We denote the invariant δ-function by

$$\tilde{\delta}(\vec{p}' - \vec{p}) = (2\pi)^3 2E\delta^{(3)}(\vec{p}' - \vec{p}) \tag{2.1}$$

and the invariant phase-space element by

$$\tilde{dp} = \frac{1}{(2\pi)^3 2E} d^3p. \tag{2.2}$$

We will use the standard representation of the Dirac-matrices and therefore

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.3}$$

3 Angular momentum and spin

3.1 Angular momentum

"A vector operator $\vec{J}$ is an angular momentum if its components are observables satisfying the commutation relations:"[2]

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{3.1}$$

The Casimir operator $\vec{J}^2 := J^2$ always commutes with every component

$$\left[J_i, J^2\right] = 0. \tag{3.2}$$

Therefore we can diagonalize J^2 and one component of $\vec{J}$ simultaneously. It is customary to use J_z. The eigenvectors are $|j,m\rangle$ and the eigenvalue equations are

$$J^2 |j,m\rangle = j(j+1) |j,m\rangle, \tag{3.3}$$

$$J_z |j,m\rangle = m |j,m\rangle. \tag{3.4}$$

j can take any positive integer or half integer value and $m \in \{-j, ..., j-1, j\}$. Two important angular momentum operators are the orbital angular momentum $\vec{L} := \vec{Q} \times \vec{P}$ and the spin operator $S_i = \frac{1}{2}\sigma_i$, where $\vec{Q}$ and $\vec{P}$ are respectively the position and momentum operators and σ_i are the Pauli matrices.

In a system with two angular momenta j_1 and j_2 we define the total angular momentum operator $\vec{J}$ as the sum of the two angular momentum operators $\vec{J^{(1)}}$ and $\vec{J^{(2)}}$

$$\vec{J} = \vec{J^{(1)}} + \vec{J^{(2)}}. \tag{3.5}$$

It is easy to see that J^2 does not commute with $\vec{J^{(1)}}_z$ and $\vec{J^{(2)}}_z$. The eigenstates of J^2 and J_z are connected to the eigenstates of $(J^{(1)})^2$, $J_z^{(1)}$, $(J^{(2)})^2$ and $J_z^{(2)}$ by Clebsch Gordan coefficients as follows

$$|j, m\rangle := |j, m, j_1, j_2\rangle = \sum_{m_1, m_2} (j, m|j_1, m_1, j_2, m_2) |j_1, m_2, j_2, m_2\rangle \,, \tag{3.6}$$

where $|j_1, m_1, j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$ are the eigenstates of the two separated angular momenta and $|j, m\rangle$ are the eigenstates of the total angular momentum. A few properties of the Clebsch Gordan coefficients are listed in appendix B.3.

3.2 Rotation and angular momentum

Let us look at the Rotation of a vector $\vec{v}$ in three-dimensional space around an axis $\hat{n}$ by an angle φ

$$\vec{v} \to \vec{v}' = R(\hat{n}, \varphi)\vec{v}. \tag{3.7}$$

This Rotation of the vector can equivalently be expressed by the Euler Angles α, β, γ

$$R(\hat{n}, \varphi)\vec{v} = R(\hat{e_z}, \alpha)R(\hat{e_x}, \beta)R(\hat{e_z}, \gamma)\vec{v} := R(\alpha, \beta, \gamma)\vec{v}. \tag{3.8}$$

We will denote the unitary representation of $R(\alpha, \beta, \gamma)$ with $\mathcal{R}(\alpha, \beta, \gamma)$. The generators of rotation are the components of the angular momentum operator $\vec{J}$. Therefore we can write

$$\mathcal{R}(\alpha, \beta, \gamma) = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z}. \tag{3.9}$$

Rotations of angular momentum eigenstates motivate the definition of the Wigner D-function:

$$D^{(j)}_{m',m}(\alpha, \beta, \gamma) := \langle j', m'| \mathcal{R}(\alpha, \beta, \gamma) |j, m\rangle = e^{-i(\alpha m + \gamma m')} d^{(j)}_{m',m}(\beta)\delta_{j',j}. \tag{3.10}$$

With the Wigner D-function the rotation of a wave function with spin can be written as

$$\mathcal{R}(\alpha, \beta, \gamma) |\vec{r}, s, m_s\rangle = \sum_{m_s'} D^{(s)}_{m_s', m_s} |R(\alpha, \beta, \gamma)\vec{r}, s, m_s'\rangle \,. \tag{3.11}$$

A few useful relations for the Wigner D-function are listed in appendix B.1.

3.3 Spin and helicity

As already mentioned spin is a form of angular momentum. In contrast to orbital angular momentum, it is not related to rotations and in fact has no classical analogon. Every particle has an definite (total) spin s. This is often used to classify particles: Particles with $s = \frac{1}{2}, \frac{3}{2}, ...$ are called "fermions" and particles where the spin is an integer are called "bosons".
In eq. (3.11) we see that the third component of the spin is not invariant under rotations. This is one of the main motivations to introduce helicity. The helicity operator Λ is defined as

$$\Lambda = \hat{p} \cdot \vec{\sigma}, \tag{3.12}$$

with the eigenvalue equation

$$\Lambda \left|\vec{p}, \lambda\right\rangle = \lambda \left|\vec{p}, \lambda\right\rangle . \tag{3.13}$$

λ is called the helicity. A appealing property of λ is that it remains unchanged under rotations. When a Lorentz-transformation in the direction of $\vec{p}$ is applied the helicity only changes its sign, when the direction of $\vec{p}$ is reversed and $|\lambda|$ does not change in both cases[3]. Therefore it is convenient to work with helicity instead of the third spin component for many application in particle physics.

4 Expansion for total angular momentum

4.1 Construction of states

4.1.1 Helicity states

We will begin with the one particle states $|\vec{p}, \lambda\rangle$ where $\vec{p}$ denotes the momentum and λ the helicity with the following normalization

$$\langle\vec{p}\,', \lambda'|\vec{p}, \lambda\rangle = \tilde{\delta}(\vec{p}\,' - \vec{p})\delta_{\lambda'\lambda}. \tag{4.1}$$

The two particle states for particle a, b are given by

$$\left|\vec{p}_a, \lambda_a, \vec{p}_b, \lambda_b\right\rangle = \left|\vec{p}_a, \lambda_a\right\rangle \otimes \left|\vec{p}_b, \lambda_b\right\rangle . \tag{4.2}$$

We adapt the normalization of the single particle state

$$\langle\vec{p}\,'_a, \lambda'_a, \vec{p}\,'_b, \lambda'_b|\vec{p}_a, \lambda_a, \vec{p}_b, \lambda_b\rangle = \tilde{\delta}(\vec{p}\,'_a - \vec{p}_a)\tilde{\delta}(\vec{p}\,'_b - \vec{p}_b)\delta_{\lambda'_a\lambda_a}\delta_{\lambda'_b\lambda_b}. \tag{4.3}$$

Often calculations are easier in the center of momentum frame (c.m.). Therefore we write our two particle states as follows

$$|\vec{p}_a, \lambda_a, \vec{p}_b, \lambda_b\rangle = \frac{1}{4\pi}\sqrt{\frac{M_{cm}}{|\vec{p}_1|}}\,|\Omega, \lambda_1, \lambda_2\rangle \otimes |P\rangle \tag{4.4}$$

where P is the c.m. momentum $P^\mu = p_1^\mu + p_2^\mu$, $M_{cm}^2 = P^2$, $\vec{p}_1 = -\vec{p}_2$ and λ_1 are the momenta and helicities of particle a in the c.m. frame, particle b respectively. Ω denotes the spherical angles of $\vec{p}_1$ in the c.m. frame: $\vec{p}_1 = (|\vec{p}_1|, \Omega) = (|\vec{p}_1|, \theta, \phi)$. The normalization factor has been chosen, so that

$$\langle P| \otimes \langle \Omega', \lambda_1', \lambda_2'|\Omega, \lambda_1, \lambda_2\rangle \otimes |P\rangle = (2\pi)^4\delta^{(4)}(P' - P)\delta^{(2)}(\Omega' - \Omega). \tag{4.5}$$

The helicity is rotational invariant[3]. Therefore we can write

$$\begin{aligned}
|\Omega, \lambda_1, \lambda_2\rangle = |\phi, \theta, \lambda_1, \lambda_2\rangle &= \mathcal{R}(\phi, \theta, 0)\,|\phi' = 0, \theta' = 0, \lambda_1, \lambda_2\rangle && (4.6)\\
&= \sum_{j,m} |j, m\rangle \langle j, m|\,\mathcal{R}(\phi, \theta, 0)\,|\phi' = 0, \theta' = 0\rangle && (4.7)\\
&= \sum_{j,m,j',m'} |j, m\rangle \langle j, m|\,\mathcal{R}(\phi, \theta, 0)\,|j', m'\rangle \langle j', m'|\phi' = 0, \theta' = 0\rangle && (4.8)\\
&= \sum_{j,m} N_j D^{(j)}_{m\lambda}(\Omega)\,|j, m, \lambda_1, \lambda_2\rangle && (4.9)
\end{aligned}$$

where $\lambda = \lambda_1 - \lambda_2$ and $N_{j'}\delta_{m',\lambda} := \langle j', m'|\phi' = 0, \theta' = 0\rangle$. We have used eq. (3.10) and the completeness relations for $|j, m, \lambda_1, \lambda_2\rangle$. We have suppressed the indices λ_1, λ_2. The δ-function arises because for $\phi = 0, \theta = 0$ the helicity is the spin projected to the $\pm$z-axis and therefore the third component of the total spin of the state $|\phi = 0, \theta = 0, \lambda_1, \lambda_2\rangle$ equals λ. We demand the following normalization

$$\langle j', m', \lambda_1', \lambda_2'|j, m, \lambda_1, \lambda_2\rangle = \delta_{j',j}\delta_{m',m}\delta_{\lambda_1',\lambda_1}\delta_{\lambda_2',\lambda_2}. \tag{4.10}$$

4.1.2 Canonical states

For our canonical states we begin with the angular states in eq. (4.6). For $\phi = 0$ and $\theta = 0$ the particles move in $\pm z$-direction and we already know the z-components of the spins of each particle. Considering the rotation of a wave function with spin

$$\mathcal{R}(\Omega)\,|s, m\rangle = \sum_{m'} D^{(s)}_{m'm}(\Omega)\,|s, m'\rangle \tag{4.11}$$

we can write

$$|\Omega,\lambda_1,\lambda_2\rangle = \mathcal{R}(\phi,\theta,0)\,|\phi'=0,\theta'=0,\lambda_1,\lambda_2\rangle$$
$$= \sum_{m_1,m_2} D^{(s_1)}_{m_1,\lambda_1}(\Omega) D^{(s_2)}_{m_2,-\lambda_2}(\Omega)\,|\Omega,m_1,m_2\rangle \tag{4.12}$$

and respectively

$$|\Omega,m_1,m_2\rangle = \sum_{\lambda_1,\lambda_2} D^{(s_1)}_{m_1,\lambda_1}(\Omega)^* D^{(s_2)}_{m_2,-\lambda_2}(\Omega)^*\,|\Omega,\lambda_1,\lambda_2\rangle\,. \tag{4.13}$$

The according spins s_1, s_2 of the particles couple to a total spin s

$$|\Omega,s,m_s\rangle = \sum_{m_1,m_2} (s_1,m_1,s_2,m_2|s,m_s)\,|\Omega,m_1,m_2\rangle\,. \tag{4.14}$$

Using the definition of the spherical harmonics

$$Y_{l,m_l}(\Omega) = \langle\theta,\phi|l,m_l\rangle \tag{4.15}$$

we obtain the following

$$|\Omega,s,m_s\rangle = \sum_{l,m_l} Y_{l,m_l}(\Omega)^*\,|l,m_l,s,m_s\rangle \tag{4.16}$$
$$\Leftrightarrow |l,m_l,s,m_s\rangle = \int d\Omega Y_{l,m_l}(\Omega)\,|\Omega,s,m_s\rangle\,. \tag{4.17}$$

Coupling the orbital angular momentum and the spin we obtain the angular momentum states

$$|j,m,l,s\rangle = \sum_{m_l,m_s} (l,m_l,s,m_s|j,m)\,|l,m_l,s,m_s\rangle\,. \tag{4.18}$$

4.1.3 Connection between helicity and canonical states

To find the connection between helicity and canonical states we first substitute eq. (4.9) into eq. (4.13)

$$|\Omega,m_1,m_2\rangle = \sum_{\lambda_1,\lambda_2}\sum_{j,m} N_j D^{(s_1)}_{m_1,\lambda_1}(\Omega)^* D^{(s_2)}_{m_2,-\lambda_2}(\Omega)^* D^{(j)}_{m\lambda}(\Omega)\,|j,m,\lambda_1,\lambda_2\rangle\,. \tag{4.19}$$

Using eqs. (B.3) to (B.5) and the orthogonality relation of the Clebsch Gordon-coefficients we can write

$$
\begin{aligned}
& D^{(s_1)}_{m_1,\lambda_1} D^{(s_2)}_{m_2,-\lambda_2} D^{*(j)}_{m\lambda} && (4.20)\\
&= \sum_{s,m_s,\lambda'} (s_1,m_1,s_2,m_2|s,m_s)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda') D^{(s)}_{m_s,\lambda'} D^{*(j)}_{m\lambda} && (4.21)\\
&= \sum_{s,m_s} \sum_{l,m_l} (s_1,m_1,s_2,m_2|s,m_s)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda) \times \\
&\quad \times \frac{2l+1}{2j+1} (s,m_s,l,m_l|j,m)(s,\lambda,l,0|j,\lambda) \sqrt{\frac{4\pi}{2l+1}} Y_{l,m_l}. && (4.22)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
|\Omega, m_1, m_2\rangle = \sum_{\substack{\lambda_1,\lambda_2\\ j,m}} \sum_{\substack{l,m_l\\ s,m_s}} \sqrt{\frac{2l+1}{2j+1}} (s,m_s,l,m_l|j,m)(s,\lambda,l,0|j,\lambda) \times \\
\times (s_1,m_1,s_2,m_2|s,m_s)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda) Y_{l,m_l}(\Omega)^* \left|j,m,\lambda_1,\lambda_2\right\rangle . \quad (4.23)
\end{aligned}
$$

Substituting this equation into the results of section 4.1.2 we gain the following equation

$$
\begin{aligned}
|j,m,l,s\rangle = & \sum_{\substack{m_l,m_s\\ m_1,m_2}} \int d\Omega Y_{l,m_l}(\Omega)(l,m_l,s,m_s|j,m)(s_1,m_1,s_2,m_2|s,m_s) \times \\
& \times \sum_{\substack{\lambda_1,\lambda_2\\ j',m'}} \sum_{\substack{l',m'_l\\ s',m'_s}} \sqrt{\frac{2l'+1}{2j'+1}} (s',m'_s,l',m'_l|j',m')(s,\lambda,l',0|j',\lambda) \times \\
& \times (s_1,m_1,s_2,m_2|s',m'_s)(s_1,\lambda_1,s_2,-\lambda_2|s',\lambda) \times \\
& \times Y_{l',m'_l}(\Omega)^* \left|j',m',\lambda_1,\lambda_2\right\rangle && (4.24)\\
= & \sum_{\substack{m_l,m_s\\ m_1,m_2}} \sum_{\substack{\lambda_1,\lambda_2\\ j',m'}} \sum_{s',m'_s} \sqrt{\frac{2l+1}{2j'+1}} (l,m_l,s,m_s|j,m)(s_1,m_1,s_2,m_2|s,m_s) \times \\
& \times (s,m'_s,l,m_l|j',m')(s,\lambda,l,0|j',\lambda)(s_1,m_1,s_2,m_2|s',m'_s) \times \\
& \times (s_1,\lambda_1,s_2,-\lambda_2|s',\lambda) \left|j',m',\lambda_1,\lambda_2\right\rangle && (4.25)\\
= & \sum_{m_l,m_s} \sum_{\substack{\lambda_1,\lambda_2\\ j',m'}} \sqrt{\frac{2l+1}{2j'+1}} (l,m_l,s,m_s|j,m)(s,m_s,l,m_l|j',m') \times \\
& \times (s,\lambda,l,0|j',\lambda)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda) \left|j',m',\lambda_1,\lambda_2\right\rangle && (4.26)\\
= & \sum_{\lambda_1,\lambda_2} \sqrt{\frac{2l+1}{2j+1}} (s,\lambda,l,0|j,\lambda)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda) \left|j,m,\lambda_1,\lambda_2\right\rangle && (4.27)
\end{aligned}
$$

or respectively

$$|j,m,\lambda_1,\lambda_2\rangle = \sum_{l,s}\sqrt{\frac{2l+1}{2j+1}}(s,\lambda,l,0|j,\lambda)(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda)\,|j,m,l,s\rangle\,. \tag{4.28}$$

4.2 Expansion of the invariant amplitude

Our invariant amplitude $\mathcal{M}_{fi}$ is defined by[4]:

$$(2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)\mathcal{M}_{fi} = \langle\vec{p}_c,\lambda_c,\vec{p}_d,\lambda_d|\,T\,|\vec{p}_a,\lambda_a,\vec{p}_b,\lambda_b\rangle \tag{4.29}$$

where $S = \mathbb{1} + iT$. S denotes the invariant S-matrix. Using eq. (4.4) we obtain

$$\mathcal{M}_{fi} = (4\pi)^2\frac{M_{cm}}{\sqrt{|\vec{p}_i||\vec{p}_f|}}\,\langle\Omega,\lambda_3,\lambda_4|\,T(M_{cm})\,|0,0,\lambda_1,\lambda_2\rangle \tag{4.30}$$

where $\vec{p}_{i,f}$ are the momenta of the particles a,c in the c.m. frame. Using eq. (4.9) we can rewrite eq. (4.30) as follows

$$\begin{aligned}\mathcal{M}_{fi} = (4\pi)^2\frac{M_{cm}}{\sqrt{|\vec{p}_i||\vec{p}_f|}}\sum_{j,m}\sum_{j',m'}N_jN_{j'}D^{(j)}_{m\lambda_f}(\Omega)^*D^{(j')}_{m'\lambda_i}(0,0,0)\times\\ \times\langle j,m,\lambda_3,\lambda_4|\,T(M_{cm})\,|j',m',\lambda_1,\lambda_2\rangle\end{aligned} \tag{4.31}$$

where $\lambda_i = \lambda_1 - \lambda_2$ and $\lambda_f = \lambda_3 - \lambda_4$. Since angular momentum is conserved we derive $j' = j$ and $m' = m$. With eqs. (A.6) and (B.2) we obtain

$$\begin{aligned}\mathcal{M}_{fi} =&(4\pi)^2\frac{M_{cm}}{\sqrt{|\vec{p}_i||\vec{p}_f|}}\times\\ &\times\sum_j(2j+1)D^{(j)}_{\lambda_i,\lambda_f}(\Omega)^*\,\langle j,\lambda_i,\lambda_3,\lambda_4|\,T(M_{cm})\,|j,\lambda_i,\lambda_1,\lambda_2\rangle\,.\end{aligned} \tag{4.32}$$

Using eq. (4.28) for the initial state we can derive

$$\begin{aligned}\mathcal{M}_{fi} =&(4\pi)^2\frac{M_{cm}}{\sqrt{|\vec{p}_i||\vec{p}_f|}}\sum_j\sum_{l,s}\sqrt{(2l+1)(2j+1)}(s,\lambda_i,l,0|j,\lambda_i)\times\\ &\times(s_1,\lambda_1,s_2,-\lambda_2|s,\lambda_i)D^{(j)}_{\lambda_i,\lambda_f}(\Omega)^*\,\langle j,\lambda_i,\lambda_3,\lambda_4|\,T(M_{cm})\,|j,\lambda_i,l,s\rangle\end{aligned} \tag{4.33}$$

$$:= \sum_{j,l,s}A(^{2s+1}l_j)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}D^{(j)}_{\lambda_i,\lambda_f}(\Omega)^*. \tag{4.34}$$

A similar derivation for this expansion can be found in [4].

4.3 The cross section

The unpolarized cross section is given by

$$\sigma = \frac{1}{4} \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_c,\lambda_d}} \frac{1}{F} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4} |\mathcal{M}_{fi}|^2 (2\pi)\delta(k_1^2 - m_f^2)(2\pi)\delta(k_2^2 - m_f^2) \times \\ \times (2\pi)^4 \delta(p_1 + p_2 - k_1 - k_2) \tag{4.35}$$

$$= \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_c,\lambda_d}} \frac{|\vec{p_f}|}{4FE_f} \int \frac{d\Omega}{32\pi^2} |\mathcal{M}_{fi}|^2 \tag{4.36}$$

where F is the flux. We see that we use the squared amplitude. So in principle the $A(^{2s+1}l_j)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ could interfere with each other. Carrying out the integral and using eq. (B.1) we obtain

$$\sigma = \frac{|\vec{p_f}|}{32\pi F E_f} \sum_j \frac{1}{2j+1} \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_c,\lambda_d}} \left| \sum_{l,s} A(^{2s+1}l_j)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \right|^2 . \tag{4.37}$$

For an unknown velocity distribution it is customary to use σv instead of the normal cross section. v is the relative velocity. With a Flux of $F = sv$ [5] we obtain

$$\sigma v = \frac{|\vec{p_f}|}{32\pi E_f s} \sum_j \frac{1}{2j+1} \sum_{\substack{\lambda_a,\lambda_b\\ \lambda_c,\lambda_d}} \left| \sum_{l,s} A(^{2s+1}l_j)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \right|^2 . \tag{4.38}$$

5 Expansion for orbital angular momentum

Several sources of literature state[6, 7], that the annihilation cross section for particles with orbital angular momentum l and relative velocity v is expressed in first non vanishing order as $\sigma v \propto v^{2l}$. For particles described by the Schrödinger equation this can easily be seen, by expanding the initial plane wave into partial waves

$$e^{i\vec{k}\vec{r}} = \sum_{l,m_l} 4\pi i^l j_l(kr) Y_{l,m_l}(\hat{r}) Y_{l,m_l}(\hat{k})^* . \tag{5.1}$$

and using the power series expansion of the spherical Bessel functions $j_l(kr)$ given in eq. (C.4). For non relativistic particles $k \propto v$, so the cross section can be expanded as stated above. However none of the sources mentioned above, gives an explanation, why this should be true for particles with spin. The goal of this section is to examine how the plane wave solutions of the free Dirac equation can be expanded into partial waves.

5.1 Solving the Dirac equation

5.1.1 Separating angular and radial part

We will start with the Dirac equation for a free particle

$$(i\partial\!\!\!/ - \mu)\psi = 0, \tag{5.2}$$

where μ denotes the mass of the particle. The usual way to obtain the solutions is to take the ansatz

$$\psi = u(\vec{k}, \lambda)e^{-ikx}, \tag{5.3}$$

where x is the position four vector. The solutions for $u(\vec{k}, \lambda)$ are shown in section 6. To obtain the partial wave amplitudes we use a differing ansatz:

$$\psi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} e^{-iEt} \tag{5.4}$$

where $\phi^{(i)}$ are two spinors. This ansatz leads us to two coupled differential equations

$$i\vec{\sigma}\vec{\nabla}\phi^{(2)} = (E - \mu)\phi^{(1)}, \tag{5.5}$$

$$i\vec{\sigma}\vec{\nabla}\phi^{(1)} = (E + \mu)\phi^{(2)}. \tag{5.6}$$

Now we will expand the $\phi^{(i)}$ further, to separate the radial and the angular dependence:

$$\phi^{(i)} = \sum_{m_s} \phi^{(i)}_{m_s} |s, m_s\rangle = \sum_{m_s} \langle \vec{r}|\phi^{(i)}_{m_s}\rangle |s, m_s\rangle \tag{5.7}$$

$$= \sum_{j,m,l} \langle \Omega_r|j, m, l, s\rangle \sum_{m_s} \langle r| \langle j, m, l, s|\phi^{(i)}_{m_s}\rangle |s, m_s\rangle . \tag{5.8}$$

s is fixed to $\frac{1}{2}$ and therefore we do not sum over s. r denotes the length of the position three vector ($r := \frac{\vec{x}}{|\vec{x}|}$) and Ω_r the direction of this vector. We will use the notation

$$\xi_{j,m,l}(\Omega_r) := \langle \Omega_r|j, m, l, s\rangle , \tag{5.9}$$

$$f^{(i)}_{j,m,l}(r) := \sum_{m_s} \langle r| \langle j, m, l, s|\phi^{(i)}_{m_s}\rangle |s, m_s\rangle . \tag{5.10}$$

Note that $\xi_{j,m,l}(\Omega)$ is a two spinor and $f^{(i)}_{j,m,l}(r)$ is a one dimensional function. We will use $\hat{r} := \hat{x}$. The angular and radial part of $\vec{\sigma} \cdot \vec{\nabla}$ can be separated as shown in appendix D. The result is

$$\vec{\sigma} \cdot \vec{\nabla} = \vec{\sigma} \cdot \hat{r} \left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma} \cdot \vec{L} \right) . \tag{5.11}$$

Now we want find out how the spinors $\xi_{j,m,l}(\Omega_r)$ from eq. (5.9) behave, when the operators $\vec{\sigma}\cdot\hat{r}$ and $\vec{\sigma}\cdot\vec{L}$ are applied. We begin with the second operator. With $\vec{J}=\vec{L}+\vec{S}$ and $\vec{S}=\frac{\vec{\sigma}}{2}$ we see

$$\vec{\sigma}\cdot\vec{L} = J^2 - L^2 - S^2. \tag{5.12}$$

Considering $\xi_{j,m,l}(\Omega_r)$ is the representation of $|j,m,l,s\rangle$ in the position space, we see immediately

$$(\vec{\sigma}\cdot\vec{L})\xi_{j,m,l}(\Omega_r) = \left[j(j+1) - l(l+1) - \frac{3}{4}\right]\xi_{j,m,l}(\Omega_r) \tag{5.13}$$

$$:= -(\varkappa+1)\xi_{j,m,l}(\Omega_r), \tag{5.14}$$

$\varkappa \in \mathbb{Z}/\{0\}$. It is possible to distinctively find the corresponding j and l from $\varkappa$ using[8]

$$j = |\varkappa| - \frac{1}{2}, \tag{5.15}$$

$$l = \begin{cases} \varkappa & \text{if} \quad \varkappa > 0 \\ -\varkappa - 1 & \text{if} \quad \varkappa < 0 \end{cases}. \tag{5.16}$$

So from now on we will use the notation

$$\xi_{j,m,l}(\Omega_r) := \xi_{\varkappa,m}(\Omega_r) \tag{5.17}$$

$$f^{(i)}_{j,m,l}(r) := f^{(i)}_{\varkappa,m}(r) \tag{5.18}$$

In addition we must evaluate $\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r)$. This can be done either by using the WignerEckart theorem and Racah coefficients[8] or by brute force (see appendix D). The result is

$$\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) = -\xi_{-\varkappa,m}(\Omega_r). \tag{5.19}$$

With this findings we can combine eqs. (5.5) and (5.6) with eq. (5.8) and obtain

$$\sum_{\substack{j_1,j_2\\ m_1 m_2\\ l_1,l_2}} i\left(\frac{\partial}{\partial r} + \frac{\varkappa_2+1}{r}\right) f^{(2)}_{\varkappa_2,m_2}(r)\xi_{-\varkappa_2,m_2}(\Omega_r) + (E-\mu) f^{(1)}_{\varkappa_1,m_1}(r)\xi_{\varkappa_1,m_1}(\Omega_r) = 0, \tag{5.20}$$

$$\sum_{\substack{j_1,j_2\\ m_1 m_2\\ l_1,l_2}} i\left(\frac{\partial}{\partial r} + \frac{\varkappa_1+1}{r}\right) f^{(1)}_{\varkappa_1,m_1}(r)\xi_{-\varkappa_1,m_1}(\Omega_r) + (E+\mu) f^{(2)}_{\varkappa_2,m_2}(r)\xi_{\varkappa_2,m_2}(\Omega_r) = 0. \tag{5.21}$$

$\xi_{\varkappa,m}(\Omega_r)$ are linearly independent for any combination of $\varkappa$ and m. This implies

$$\varkappa_1 = -\varkappa_2 =: \varkappa, \tag{5.22}$$

$$m_1 = m_1 =: m \tag{5.23}$$

and that each summand has to vanish. So we have two coupled differential equations

$$i\left(\frac{\partial}{\partial r} + \frac{1-\varkappa}{r}\right) f^{(2)}_{-\varkappa,m}(r) + (E-\mu) f^{(1)}_{\varkappa,m}(r) = 0, \tag{5.24}$$

$$i\left(\frac{\partial}{\partial r} + \frac{1+\varkappa}{r}\right) f^{(1)}_{\varkappa,m}(r) + (E+\mu) f^{(2)}_{-\varkappa,m}(r) = 0. \tag{5.25}$$

Up to this point a similar derivation can be found in [8].

5.1.2 Radial dependence of partial waves

Using the definitions

$$f^{(1)}_{\varkappa,m}(r) := \frac{i}{r}\sqrt{E+\mu} F_{\varkappa,m}(r), \tag{5.26}$$

$$f^{(2)}_{-\varkappa,m}(r) := \frac{1}{r}\sqrt{E-\mu} G_{\varkappa,m}(r) \tag{5.27}$$

and the relativistic energy momentum relation

$$E^2 = \vec{k}^2 + \mu^2, \tag{5.28}$$

we can rewrite those equations. We obtain

$$\left(\frac{\partial}{\partial r} - \frac{\varkappa}{r}\right) G_{\varkappa,m}(r) + |\vec{k}| F_{\varkappa,m}(r) = 0, \tag{5.29}$$

$$\left(\frac{\partial}{\partial r} + \frac{\varkappa}{r}\right) F_{\varkappa,m}(r) - |\vec{k}| G_{\varkappa,m}(r) = 0. \tag{5.30}$$

Uncoupling this equations leaves us with

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\varkappa(\varkappa+1)}{r^2} + \vec{k}^2\right) F_{\varkappa,m}(r) = 0, \tag{5.31}$$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\varkappa(\varkappa+1)}{r^2} + \vec{k}^2\right) G_{\varkappa,m}(r) = 0. \tag{5.32}$$

Note that eq. (5.16) implies

$$\varkappa(\varkappa+1) = l(l+1). \tag{5.33}$$

This shows that the radial dependence for any total angular momentum does only depend on the orbital angular momentum. The normalizable solution to eqs. (5.31) and (5.32) are $rj_l(|\vec{k}|r)$ where $j_l(|\vec{k}|r)$ are spherical Bessel functions (see appendix C). Now we can write down the partial wave expanded solution of the Dirac equation for a free particle:

$$\psi = \sum_{j,m,l} \begin{pmatrix} ia^{(1)}_{\varkappa,m}\sqrt{E+\mu}\xi_{\varkappa,m}(\Omega_r) \\ a^{(2)}_{\varkappa,m}\sqrt{E-\mu}\xi_{-\varkappa,m}(\Omega_r) \end{pmatrix} j_l(|\vec{k}|r)e^{-iEt}. \tag{5.34}$$

The spinor $u(\vec{k},\lambda)$ is of the form[9]

$$u(\vec{k},\lambda) = \begin{pmatrix} \sqrt{E+\mu}\chi_\lambda(\hat{k}) \\ \sqrt{E-\mu}\vec{\sigma}\hat{k}\chi_\lambda(\hat{k}) \end{pmatrix} \tag{5.35}$$

where $\chi_\lambda(\hat{k})$ are two spinors and independent from $|\vec{k}|$. The exponential function in eq. (5.3) can be expanded as follows[9]

$$e^{i\vec{k}\vec{x}}e^{-iEt} = \sum_{n,m_n} 4\pi i^n j_n(|\vec{k}|r)Y_{n,m_n}(\Omega_r)Y_{n,m_n}(\hat{k})^* e^{-iEt}. \tag{5.36}$$

With eq. (5.3) it follows

$$\psi = \sum_l j_l(|\vec{k}|r)e^{-iEt} \begin{pmatrix} \sqrt{E+\mu}\sum_{j,m} ia^{(1)}_{\varkappa,m}\xi_{\varkappa,m}(\Omega_r) \\ \sqrt{E-\mu}\sum_{j,m} a^{(2)}_{\varkappa,m}\xi_{-\varkappa,m}(\Omega_r) \end{pmatrix} \tag{5.37}$$

$$= \sum_n j_n(|\vec{k}|r)e^{-iEt} \begin{pmatrix} \sqrt{E+\mu}\sum_{m_n} 4\pi i^n Y_{n,m_n}(\Omega_r)Y_{n,m_n}(\hat{k})^*\chi_\lambda(\hat{k}) \\ \sqrt{E-\mu}\sum_{m_n} 4\pi i^n Y_{n,m_n}(\Omega_r)Y_{n,m_n}(\hat{k})^*\vec{\sigma}\hat{k}\chi_\lambda(\hat{k}) \end{pmatrix}. \tag{5.38}$$

Since the spherical Bessel functions are linearly independent it follows $n = l$ and the coefficients must be equal.

5.2 Momentum dependence of two particle wave and invariant amplitude

In the invariant amplitude $\mathcal{M}_{fi}$ we work with two particle states. These are the tensor product of the one particle states. For simplicity reasons we work in the cm frame with momentum $\pm\vec{k}$ for the two particles. In the position space representation the two particle state is

$$\langle\vec{r_1},\vec{r_2}|\left(|\vec{k},\lambda_a\rangle\otimes|-\vec{k},\lambda_b\rangle\right) = u(\vec{k},\lambda_a)\otimes u(-\vec{k},\lambda_b)\sum_{\substack{l_1,m_{l_1}\\ l_2,m_{l_2}}}(4\pi)^2 i^{l_1+l_2}\times$$
$$\times\, j_{l_1}(|\vec{k}|r_1)Y_{l_1,m_{l_1}}(\hat{r_1})Y_{l_1,m_{l_1}}(\hat{k})^*\times$$
$$\times\, j_{l_2}(|\vec{k}|r_2)Y_{l_2,m_{l_2}}(-\hat{r}_2)Y_{l_2,m_{l_2}}(\hat{k})^*. \tag{5.39}$$

With eq. (C.6) and the comment below we can write this as

$$\langle\vec{r_1},\vec{r_2}|\vec{k},\lambda_a,-\vec{k},\lambda_b\rangle = u(\vec{k},\lambda_a)\otimes u(-\vec{k},\lambda_b)\times$$
$$\times\sum_{l,m_l}4\pi i^l j_l(|\vec{k}||\vec{r_1}-\vec{r_2}|)Y_{l,m_l}\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|}\right)Y_{l,m_l}(\hat{k})^*, \tag{5.40}$$

where l is the angular momentum you obtain, when coupling l_1 and l_2. So l is in fact the orbital angular momentum of the two particle state. So this equation is what we were looking for. The spherical Bessel functions have a power series expansion (see. eq. (C.4)) which shows that for each l the according spherical Bessel function only contains terms such as $c_{l,n}|\vec{k}|^{l+2n}$ with $n\in\mathbb{N}$ and $c_{l,n}$ are coefficients. Now let us take a closer look at $u(\vec{k},\lambda)$. In the standard representation they can be written as[9]

$$u(\vec{k},\lambda) = \sqrt{E+\mu}\begin{pmatrix}\chi_\lambda(\hat{k})\\ 2\lambda\frac{|\vec{k}|}{E+\mu}\chi_\lambda(\hat{k})\end{pmatrix}. \tag{5.41}$$

In the non relativistic limit ($E\approx\mu+\frac{\vec{k}^2}{2\mu}$) the lower component is suppressed by $\frac{|\vec{k}|}{E+\mu}\leq\frac{|\vec{k}|}{2\mu}$. Therefore we will neglect this component. Note that this expansion is only valid, if $\frac{|\vec{k}|}{2\mu}\ll 1$. Thus $u(\vec{k},\lambda_a)\otimes u(-\vec{k},\lambda_b)$ reduces to

$$(E+\mu)\,\chi_{\lambda_a}(\hat{k})\otimes\chi_{\lambda_b}(-\hat{k})\approx\left(2\mu+\frac{\vec{k}^2}{2\mu}\right)\chi_{\lambda_a}(\hat{k})\otimes\chi_{\lambda_b}(-\hat{k}). \tag{5.42}$$

Note that the partial waves are orthogonal since $\chi_\lambda^\dagger(\hat{k})\chi_\lambda(\hat{k}) = 1$ and the spherical harmonics $Y_{l,m_l}(\hat{k})$ are orthogonal. Inserting this approximation into eq. (5.40) and remembering that we expanded $|\vec{p}_a,\lambda_a,\vec{p}_b,\lambda_b\rangle$ we see that the invariant amplitude has a partial wave expansion such as

$$\mathcal{M}_{fi}\approx a+b|\vec{k}|+c|\vec{k}|^2+d|\vec{k}|^3+\mathcal{O}\left(|\vec{k}|^4\right), \tag{5.43}$$

where a is the contribution of the s-wave, $b|\vec{k}|$ arises from the p-wave, $c|\vec{k}|^2$ is a mixture of the s-wave and the d-wave etc. This implies that the expansion of the cross section, mentioned in the beginning of this chapter, is in fact true for non relativistic interaction of spin $\frac{1}{2}$ particles. So for lighter (faster moving) dark matter particles this expansion does not work.

6 Explicit plane wave functions with spin $\frac{1}{2}$

We will later determine the invariant amplitude with Feynman diagrams. The amplitude will contain the plane wave solutions for particles with spin $\frac{1}{2}$. We will now express this plane-wave solutions in terms of Wigner D-functions. The states for spin $\frac{1}{2}$ particles are given by the Dirac equation. For a free particle this equation is normally solved by the ansatz:

$$\psi = u(\vec{k}, \lambda)e^{-ipx} \tag{6.1}$$

for particles and

$$\psi = v(\vec{k}, \lambda)e^{-ipx} \tag{6.2}$$

for antiparticles where x is the position four vector. The explicit form for the four-component helicity spinors are[9]

$$u(p, \lambda) = \begin{pmatrix} \sqrt{E+\mu}\chi_\lambda(\Omega) \\ 2\lambda\sqrt{E-\mu}\chi_\lambda(\Omega) \end{pmatrix} \tag{6.3}$$

for particles and

$$v(p, \lambda) = \begin{pmatrix} \sqrt{E-\mu}\chi_{-\lambda}(\Omega) \\ -2\lambda\sqrt{E+\mu}\chi_{-\lambda}(\Omega) \end{pmatrix} \tag{6.4}$$

for anti-particles, with

$$\chi_{\frac{1}{2}}(\Omega) = \begin{pmatrix} D^{(\frac{1}{2})}_{\frac{1}{2},\frac{1}{2}}(\Omega) \\ D^{(\frac{1}{2})}_{-\frac{1}{2},\frac{1}{2}}(\Omega) \end{pmatrix} \tag{6.5}$$

and

$$\chi_{-\frac{1}{2}}(\Omega) = \begin{pmatrix} D^{(\frac{1}{2})}_{\frac{1}{2},-\frac{1}{2}}(\Omega) \\ D^{(\frac{1}{2})}_{-\frac{1}{2},-\frac{1}{2}}(\Omega) \end{pmatrix}. \tag{6.6}$$

μ denotes the mass of the particle and E the Energy of the particle. For $\lambda = \pm\frac{1}{2}$ we can write

$$\chi_\lambda(\Omega) = \begin{pmatrix} D^{\left(\frac{1}{2}\right)}_{\frac{1}{2},\lambda}(\Omega) \\ D^{\left(\frac{1}{2}\right)}_{-\frac{1}{2},\lambda}(\Omega) \end{pmatrix}. \tag{6.7}$$

7 Supersymmetric Dark Matter

7.1 The Minimal Supersymmetric Standard Model

Supersymmetry is a principle that introduces a relation between fermions and bosons. There are a number of supersymmetric theories. In this thesis we will focus on the "Minimal Supersymmetric Standard Model (MSSM)." To realize the relation between fermions and bosons, we introduce the operator Q that transforms fermionic states into bosonic states and vice versa

$$Q\,|\text{Boson}\rangle = |\text{Fermion}\rangle\,, \qquad Q\,|\text{Fermion}\rangle = |\text{Boson}\rangle\,. \tag{7.1}$$

The irreducible representations of the supersymmetry algebra are called supermultiplets. Each supermultiplet contains both boson and fermion states knows as superpartners of each other. The "supersymmetry generators" Q,$Q^\dagger$ commute with the generators of gauge transformations. Thus the each particle must have the same electric charge, weak isospin, etc as its superpartner. For supersymmetry to work, there must be two Higgs-supermultiplets with weak hypercharge $Y = \pm\frac{1}{2}$. This leads to four Higgs bosons: two electrical neutral (H_u^0, H_d^0) and two electrical charged (H_u^+, H_d^-). The simplest possibility for a supermultiplet is: The superpartner of each fermion is a scalar particle (spin 0) and the superpartner of each vector boson and each Higgs bosons are fermions with spin $\frac{1}{2}$. Table 1 shows the particle content of the Minimal Supersymmetric Standard Model.[10]

7.2 R-Parity

If we want the MSSM to predict a dark matter particle, that does not decay into Standard Model particles, we need to introduce a symmetry that prevents this decay. Therefore we introduce the R-parity of a particle

$$P_R = (-1)^{3(B-L)+2s} \tag{7.2}$$

where s is the spin of the particle. All the Standard Model particle as well as the Higgs bosons have an even R-parity ($P_R = +1$) and the superpartners

SM-Particle	Symbol	Spin	Superpartner	Symbol	Spin
Quark	q	$\frac{1}{2}$	Squark	$\tilde{q}$	0
Lepton	l	$\frac{1}{2}$	Slepton	$\tilde{l}$	0
W boson	$W^{\pm}$ W^0	1	Wino	$\tilde{W}^{\pm}$ $\tilde{W}^0$	$\frac{1}{2}$
B boson	B^0	1	Bino	$\tilde{B}^0$	$\frac{1}{2}$
Gluon	g	1	Gluino	$\tilde{g}$	$\frac{1}{2}$
Higgs	H_u H_d	0	Higgsino	$\tilde{H}_u$ $\tilde{H}_d$	$\frac{1}{2}$

Table 1: Particle content of the MSSM.

of these particles have an odd R-parity ($P_R = -1$). Particles with an odd R-parity are also called supersymmetric particles. Motivated by the hope that the MSSM will provide a good dark matter candidate, we define the MSSM to conserve R-parity. This has the consequence that the lightest supersymmetric particles (LSP) is stable and all other supersymmetric paricles must eventually decay into the LSP. Therefore this LSP is a attractive candidate for non-baryonic dark matter.[10]

7.3 Neutralinos as LSP

There are four supersymmetric fermions that have neither electrical nor color charge: the Bino $\tilde{B}^0$, the neutral Wino $\tilde{W}^0$ and the two neutral Higgsinos $\tilde{H}_u^0$, $\tilde{H}_d^0$. These gauge eigenstates mix are not the mass eigenstates. The mass eigenstates are called Neutralinos $\tilde{\chi}_i^0$. There are four Neutralinos so $i = 1, 2, 3, 4$. By convention these are labeled such that the masses $m_{\tilde{\chi}_i^0}$ have an ascending order: $m_{\tilde{\chi}_1^0} < m_{\tilde{\chi}_2^0} < m_{\tilde{\chi}_3^0} < m_{\tilde{\chi}_4^0}$. Since the LPS has not been observed, it is usually assumed that the lightest Neutralino $\tilde{\chi}_1^0$ is the lightest supersymmetric particle. Neutralinos are weakly interacting massive particles (WIMPs). They are their own antiparticles.[10]

particle masses		vertex factors	
$m_{\tilde{\chi}_1^0}$	698.2 GeV	$\tilde{\chi}_1^0 + \tilde{\chi}_1^0 \to h^0$	$i \cdot 0.06063(P_R + P_L)$
m_h	124.3 GeV	$h^0 \to b + b$	$i \cdot 0.01667(P_R + P_L)$
m_Z	91.2 GeV	$\tilde{\chi}_1^0 + \tilde{\chi}_1^0 \to Z^0$	$i \cdot 0.00292(P_R - P_L)$
m_f	4.2 GeV	$Z^0 \to b + b$	$i(-0.05348P_R + 0.30701P_L)$

Table 2: Particle masses and vertex factors of the MSSM[11].

8 Example calculations for neutralino annihilation

8.1 Conventions and procedure

In the following sections we will compute the invariant amplitude for annihilation processes of the lightest Neutralino $\tilde{\chi}_1^0$. In the final state we will have an Standard model fermion f and the corresponding antiparticle $\bar{f}$. The calculation will be done in the cm system. Therefore we denote the momenta of the first Neutralino with $\vec{p_i} = |\vec{p_i}|\hat{e}_z$, the helicity with λ_a and for the second Neutralino $-\vec{p_i}$ and λ_b. For the fermion (antifermion) we use $\vec{p_f}(-\vec{p_f})$ and $\lambda_c(\lambda_d)$. The mass of the Neutralinos is $m_{\tilde{\chi}}$ and the mass of the fermion m_f. The Energy of the incoming particles is $E_i = \sqrt{|\vec{p}|_i^2 + m_{\tilde{\chi}}^2}$ and analogue E_f for the outgoing particle. In addition we use the Mandelstamm variable $s = 4E_i^2 = 4E_f^2$. c_i, c_{ij} are coupling coefficients that can be determined by the vertex factors of the MSSM. The cm momentum is $P = (2E_i; \vec{0})$. The projection operators are $P_{R/L} = \frac{1}{2}(1 \pm \gamma^5)$. m_h denotes the mass of the Higgs boson and m_Z the mass of the Z^0. Finally we define $\lambda_i = \lambda_a - \lambda_b$ and $\lambda_f = \lambda_c - \lambda_d$. The four spinors $u(\vec{p}, \lambda)$ and $v(\vec{p}, \lambda)$ have already been introduced in section 6. We will calculate the invariant amplitude using this spinors and then use eq. (B.5) to couple the Wigner D-functions. We will use the results in eq. (4.34) to determine the distributing total angular momenta and section 5.2 for the orbital angular momentum. We will suppress the helicity indices for the partial wave amplitudes, therefore $A(^{2s+1}l_j)_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = A(^{2s+1}l_j)$. In both of the cases we will look at, one can easily see at the interference term cancel each other out, when averaging over the helicities of the initial particles.

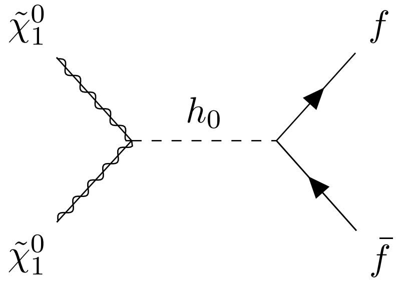

Figure 1: Feynman diagram for Neutralino annihilation into SM-fermions with Higgs boson in s-channel.

8.2 Neutralino annihilation into SM-fermion via Higgs boson

Figure 1 shows the Feynman diagram for Neutralino annihilation into SM-fermions with Higgs boson in s-channel. The invariant amplitude is

$$\mathcal{M}_{fi} = \frac{c_2}{s - m_h^2}\bar{u}(\vec{p}_f, \lambda_c)v(-\vec{p}_f, \lambda_d)\bar{v}((\vec{p}_i, \lambda_a)(c_{11}P_R + c_{12}P_L)u(-\vec{p}_i, \lambda_b) \tag{8.1}$$

$$= -2c_2\delta_{\lambda_i,0}\delta_{\lambda_f,0}|\vec{p}_f| \left(\frac{(c_{11} + c_{12})|\vec{p}_i| + (c_{11} - c_{12})2\lambda_a E_i}{s - m_h^2} \right). \tag{8.2}$$

The amplitude has no angular dependence. Therefore only the total angular momentum $j = 0$ contributes. Now we can express s, E_i and $|\vec{p}_f|$ in terms of $|\vec{p}_i|$ by using the energy momentum relation. Taylor expanding this expression for $|\vec{p}_i| \ll m_{\tilde{\chi}}$ gives us the orbital angular dependence and we can determine the amplitudes $A(^{2s+1}l_j)$ from eq. (4.34):

$$A(^1S_0): \quad -2c_2\delta_{\lambda_i,0}\delta_{\lambda_f,0}2\lambda_a \frac{(c_{11} - c_{12})m_{\tilde{\chi}}\sqrt{m_{\tilde{\chi}}^2 - m_f^2}}{4m_{\tilde{\chi}}^2 - m_h^2} \times \\ \times \left[1 + \left(\frac{1}{2m_{\tilde{\chi}}^2} + \frac{1}{2(m_{\tilde{\chi}}^2 - m_f^2)} - \frac{4}{4m_{\tilde{\chi}}^2 - m_h^2} \right) |\vec{p}_i|^2 + \mathcal{O}\left(|\vec{p}_i|^4\right) \right], \tag{8.3}$$

$$A(^3P_0): \quad -2c_2\delta_{\lambda_i,0}\delta_{\lambda_f,0} \frac{(c_{11} + c_{12})\sqrt{m_{\tilde{\chi}}^2 - m_f^2}}{4m_{\tilde{\chi}}^2 - m_h^2} \times \\ \times \left[|\vec{p}_i| + \left(\frac{1}{2(m_{\tilde{\chi}}^2 - m_f^2)} - \frac{4}{4m_{\tilde{\chi}}^2 - m_h^2} \right) |\vec{p}_i|^3 + \mathcal{O}\left(|\vec{p}_i|^5\right) \right]. \tag{8.4}$$

An interesting observation we can make by looking at this result is that the s-wave only contributes in the case of parity violation. Since the total spin

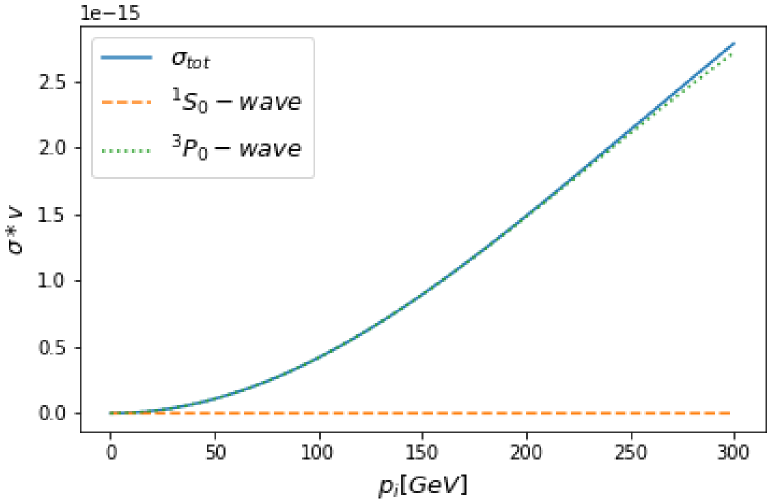

Figure 2: σv and the contributions of each partial wave for Neutralino annihilation's with a Higgs boson in s-channel and two outgoing bottom quarks.

can only have the values 0, 1 we see that there are no higher orbital angular momenta that can contribute to this amplitude. The results in section 5.2 show that for even orbital angular momenta the amplitude is an even function ($f(-x) = f(x)$) depending on $|\vec{p}_i|$ and for odd angular momenta the amplitude is an odd function ($f(-x) = -f(x)$). So in this case there is a way of obtaining the amplitudes $A(^{2s+1}l_j)$ without using the Taylor expansion:

$$A(^1S_0) : \quad \frac{\mathcal{M}_{fi}(|\vec{p}_i|) + \mathcal{M}_{fi}(-|\vec{p}_i|)}{2}, \tag{8.5}$$

$$A(^3P_0) : \quad \frac{\mathcal{M}_{fi}(|\vec{p}_i|) - \mathcal{M}_{fi}(-|\vec{p}_i|)}{2}. \tag{8.6}$$

Figure 2 shows a plot of the cross section and the contributions of each partial wave. In both of the cases we will look at, one can easily see by looking at the amplitudes, that for unpolarized initial particles no interference terms arise. So we can plot the cross section and the contributions of each partial wave.

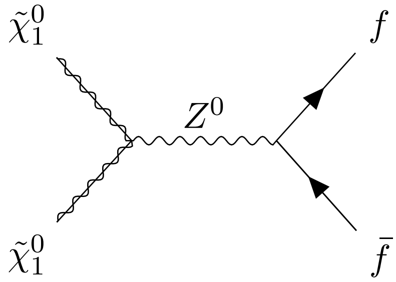

Figure 3: Feynman diagram for Neutralino annihilation into SM-fermions with Z^0 in s-channel.

8.3 Neutralino annihilation into SM-fermions via Z^0

Figure 3 shows the Feynman diagram for Neutralino annihilation into SM-fermions with Z^0 in s-channel. The invariant amplitude is

$$\begin{aligned}\mathcal{M}_{fi} =& \frac{1}{s-m_Z^2}\bar{u}(\vec{p}_f,\lambda_c)\gamma^\nu(c_{21}P_R+c_{22}P_L)v(-\vec{p}_f,\lambda_d)\left(\eta_{\nu\mu}-\frac{P_\nu P_\mu}{m_Z^2}\right)\times\\ &\times\bar{v}((\vec{p}_i,\lambda_a)\gamma^\mu(c_{11}P_R+c_{12}P_L)u(-\vec{p}_i,\lambda_b) \qquad (8.7)\\ =& -\frac{(c_{11}-c_{12})(c_{21}-c_{22})}{s-m_Z^2}4\lambda_a\lambda_c\delta_{\lambda_i,0}\delta_{\lambda_f,0}\left(1-\frac{s}{m_Z^2}\right)m_{\tilde{\chi}}m_f\\ &+\Bigg[(c_{11}-c_{12})(c_{21}-c_{22})\sqrt{2}\lambda_i|\lambda_f||\vec{p}_i||\vec{p}_f|\\ &-(c_{11}+c_{12})(c_{21}-c_{22})|\lambda_f||\vec{p}_f|(\sqrt{2}|\lambda_i|E_i+\delta_{\lambda_i,0}m_{\tilde{\chi}})\\ &+(c_{11}-c_{12})(c_{21}+c_{22})\sqrt{2}\lambda_i|\vec{p}_i|(\lambda_f E_f-\delta_{\lambda_f,0}m_f)\\ &+(c_{11}+c_{12})(c_{21}+c_{22})(\lambda_f E_f-\delta_{\lambda_f,0}m_f)(\sqrt{2}|\lambda_i|E_i+\delta_{\lambda_i,0}m_{\tilde{\chi}})\Bigg]\times\\ &\times\frac{\sqrt{3}}{s-m_Z^2}(\tfrac{1}{2},\lambda_d,1\lambda_f|\tfrac{1}{2},\lambda_c)D^{(1)}_{\lambda_i,\lambda_f}(\Omega)^*. \qquad (8.8)\end{aligned}$$

Again using the Taylor expansion for $|\vec{p}_i| \ll m_{\tilde{\chi}}$ we can determine the amplitudes $A(^{(2s+1)}l_j)$:

$$A(^1S_0): \quad (c_{11}-c_{12})(c_{21}-c_{22})4\lambda_a\lambda_c\delta_{\lambda_i,0}\delta_{\lambda_f,0}\frac{m_{\tilde{\chi}}m_f}{m_Z^2}, \tag{8.9}$$

$$A(^3P_0): \quad 0, \tag{8.10}$$

$$A(^1P_1): \quad 0, \tag{8.11}$$

$$\begin{aligned} A(^3P_1):& \\ &\frac{1}{4m_{\tilde{\chi}}^2-m_Z^2}(c_{11}-c_{12})\sqrt{3}(\tfrac{1}{2},\lambda_d,1\lambda_f|\tfrac{1}{2},\lambda_c)\sqrt{2}\lambda_i\times \\ &\times\Bigg\{|\vec{p}_i|\left[(c_{21}-c_{22})|\lambda_f|\sqrt{m_{\tilde{\chi}}^2-m_f^2}+(c_{21}+c_{22})\left(\lambda_f m_{\tilde{\chi}}-\delta_{\lambda_f,0}m_f\right)\right] \\ &+|\vec{p}_i|^3\Bigg[(c_{21}-c_{22})|\lambda_f|\sqrt{m_{\tilde{\chi}}^2-m_f^2}\left(\frac{1}{2(m_{\tilde{\chi}}^2-m_f^2)}-\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right) \\ &+(c_{21}+c_{22})\left(\lambda_f m_{\tilde{\chi}}\left(\frac{1}{2m_{\tilde{\chi}}^2}-\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right)+\delta_{\lambda_f,0}m_f\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right)\Bigg]\Bigg\} \end{aligned} \tag{8.12}$$

$$\begin{aligned} A(^3S_1)+A(^3D_1):& \\ &\frac{m_{\tilde{\chi}}}{4m_{\tilde{\chi}}^2-m_Z^2}(c_{11}+c_{12})\sqrt{3}(\tfrac{1}{2},\lambda_d,1\lambda_f|\tfrac{1}{2},\lambda_c) \\ &\Bigg\{-(c_{21}-c_{22})|\lambda_f|\sqrt{m_{\tilde{\chi}}^2-m_f^2}(\delta_{\lambda_i,0}+\sqrt{2}|\lambda_i|) \\ &+(c_{21}+c_{22})\left[\lambda_f m_{\tilde{\chi}}(\delta_{\lambda_i,0}+\sqrt{2}|\lambda_i|)-\delta_{\lambda_f,0}m_f(\delta_{\lambda_i,0}+\sqrt{2}|\lambda_i|)\right] \\ &+|\vec{p}_i|^2\Bigg[-(c_{21}-c_{22})|\lambda_f|\Bigg(\sqrt{2}|\lambda_i|\sqrt{m_{\tilde{\chi}}^2-m_f^2}\frac{1}{2m_{\tilde{\chi}}^2} \\ &\quad+(\delta_{\lambda_i,0}+\sqrt{2}|\lambda_i|)\sqrt{m_{\tilde{\chi}}^2-m_f^2}\left(\frac{1}{2(m_{\tilde{\chi}}^2-m_f^2)}-\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right)\Bigg) \\ &+(c_{21}+c_{22})\Bigg((\lambda_f\delta_{\lambda_i,0}m_{\tilde{\chi}}-\delta_{\lambda_f,0}\sqrt{2}|\lambda_i|m_f)\left(\frac{1}{2m_{\tilde{\chi}}^2}-\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right) \\ &\quad+\lambda_f\sqrt{2}|\lambda_i|m_{\tilde{\chi}}\left(\frac{1}{m_{\tilde{\chi}}^2}-\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\right)+\delta_{\lambda_f,0}\delta_{\lambda_i,0}m_f\frac{4}{4m_{\tilde{\chi}}^2-m_Z^2}\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{8.13}$$

Note that for $s=0$ we demand $\lambda_i=\lambda_f=0$. For the $A(^3S_1)+A(^3D_1)$ amplitude, the constant part belongs to the s-wave whereas the momentum

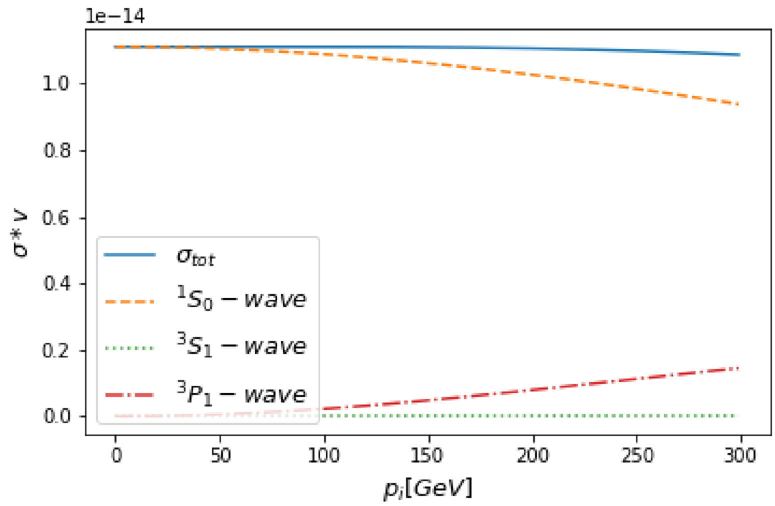

Figure 4: σv and the contributions of each partial wave for Neutralino annihilation's with a Z^0 in s-channel and two outgoing bottom quarks.

dependent part is a mixture of the s-wave and the d-wave. Figure 4 shows a plot of the cross section and the contributions of each partial wave.

9 Conclusion

In this thesis we derive a formalism to partial wave expand the invariant amplitude in the case of two initial fermions. The amplitude can be expanded by Wigner D-functions and each term can be matched with a total angular momentum. To expand the amplitude for orbital angular momenta we derive the momentum dependence of the partial waves of the initial wave. This shows us that we can expand the amplitude for small momenta and then the constant part corresponds to the s-wave and the part linear in the momentum corresponds to the p-wave etc. In the end we apply this formalism to the MSSM and partial wave expand the invariant amplitudes of Neutralino annihilation's via Higgs and Z^0 bosons into fermions pairs.

A Normalisation

We want to determine N_j. Therefore we start with eq. (4.9)

$$|\Omega, \lambda_1, \lambda_2\rangle = \sum_{j,m} N_j D^{(j)}_{m\lambda}(\Omega) \, |j, m, \lambda_1, \lambda_2\rangle \, . \tag{A.1}$$

Using eq. (4.10) we immediately see

$$\langle j, m, \lambda_1, \lambda_2 | \Omega, \lambda_1', \lambda_2'\rangle = N_j D^{(j)}_{m,\lambda}(\Omega) \delta_{\lambda_1',\lambda_1} \delta_{\lambda_2',\lambda_2}. \tag{A.2}$$

Now we will write down our states of definite angular momentum using the completeness relation for $|\Omega, \lambda_1, \lambda_2\rangle$

$$|j, m, \lambda_1, \lambda_2\rangle = \sum_{\lambda_1',\lambda_2'} \int d\Omega \, |\Omega, \lambda_1', \lambda_2'\rangle \, \langle \Omega, \lambda_1', \lambda_2' | j, m, \lambda_1, \lambda_2\rangle \tag{A.3}$$

$$= \int d\Omega N_j D^{(j)}_{m,\lambda}(\Omega)^* \, |\Omega, \lambda_1, \lambda_2\rangle \, . \tag{A.4}$$

Using eqs. (4.5) and (B.1) we obtain

$$\langle j', m', \lambda_1', \lambda_2' | j, m, \lambda_1, \lambda_2\rangle = N_j^2 \frac{4\pi}{2j+1} \delta_{j',j} \delta_{m',m} \delta_{\lambda',\lambda} \delta_{\lambda_1',\lambda_1} \delta_{\lambda_2',\lambda_2}. \tag{A.5}$$

Compare this with eq. (4.10) and we see

$$N_j = \sqrt{\frac{(2j+1)}{4\pi}}. \tag{A.6}$$

B Useful relations

B.1 Winger D-function

We list a few useful relation regarding the Wigner D-function.

$$\int d\Omega D^{(j)}_{m_1',m_1}(\Omega)^* D^{(j')}_{m_2',m_2}(\Omega) = \frac{4\pi}{2j+1} \delta_{j',j} \delta_{m_1,m_2} \delta_{m_1',m_2'} \tag{B.1}$$

$$D^{(j)}_{m,m'}(0,0,0) = \delta_{m,m'} \tag{B.2}$$

The spherical harmonics are related to the D-function via[4]

$$D^{(l)}_{m,0}(\Omega)^* = \sqrt{\frac{4\pi}{2l+1}} Y_{l,m}(\Omega). \tag{B.3}$$

The D-functions satisfy the following coupling rule

$$D^{(j_1)}_{m_1,m'_1} D^{(j_2)}_{m_2,m'_2} = \sum_{j_3,m_3,m'_3} (j_1,m_1,j_2,m_2|j_3,m_3)(j_1,m'_1,j_2,m'_2|j_3,m'_3) D^{(j_3)}_{m_3,m'_3} \tag{B.4}$$

or equivalently

$$D^{(j_1)}_{m_1,m'_1} D^{*(j_3)}_{m_3,m'_3} = \sum_{j_2,m_2,m'_2} \frac{2j_2+1}{2j_3+1}(j_1,m_1,j_2,m_2|j_3,m_3)\times \\ \times(j_1,m'_1,j_2,m'_2|j_3,m'_3) D^{*(j_2)}_{m_2,m'_2}. \tag{B.5}$$

The Wigner D-functions have the symmetry[4]

$$D^{*(j)}_{m,m'} = (-1)^{m-m'} D^{(j)}_{-m,-m'} \tag{B.6}$$

and transform under parity transformations as follows

$$D^{(j)}_{m,m'}(\pi+\phi,\pi-\theta,0) = e^{ij\pi} D^{(j)}_{m,-m'}(\phi,\theta,0). \tag{B.7}$$

B.2 Spherical harmonics

Spherical harmonics are defined in such a way that they are the position (or momentum) space representation of the orbital angular momentum

$$\langle \Omega | l, m_l \rangle . \tag{B.8}$$

It follows from eqs. (B.3) and (B.4) that

$$Y_{l_1,m_{l_1}}(\Omega) Y_{l_2,m_{l_2}}(\Omega) = \sum_{l',m'_l} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l'+1)}} (l_1,0,l_2,0|l',0)\times \\ \times (l_1,m_{l_1},l_2,m_{l_2}|l',m'_l) Y_{l',m'_l}(\Omega). \tag{B.9}$$

The first few spherical harmonics are given by

$$Y_{0,0}(\Omega) = \sqrt{\frac{1}{4\pi}}, \tag{B.10}$$

$$Y_{1,1}(\Omega) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{+i\varphi}, \tag{B.11}$$

$$Y_{1,0}(\Omega) = \sqrt{\frac{3}{4\pi}} \cos\theta, \tag{B.12}$$

$$Y_{1,-1}(\Omega) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi}. \tag{B.13}$$

$j=$	$m_2=\frac{1}{2}$	$m_2=-\frac{1}{2}$
$j_1+\frac{1}{2}$	$\left[\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$	$\left[\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$
$j_1-\frac{1}{2}$	$-\left[\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$	$\left[\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$

Table 3: Explicit results for $(j_1, m-m_2, \frac{1}{2}, m_2|j, m)$.[8]

$j=$	$m_2=1$	$m_2=0$	$m_2=-1$
j_1+1	$\left[\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right]^{\frac{1}{2}}$
j_1	$-\left[\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{m^2}{j_1(j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}\right]^{\frac{1}{2}}$
j_1-1	$\left[\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}\right]^{\frac{1}{2}}$

Table 4: Explicit results for $(j_1, m-m_2, 1, m_2|j, m)$.[8]

B.3 Clebsch Gordan coefficients

Clebsch Gordan coefficients arise when coupling two angular momenta (see section 3.1). They are chosen to be real. For Clebsch Gordan coefficients such as $(j, m|j_1, m_1, j_2, m_2)$ the coefficients only differ from zero if $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $m = m_1 + m_2$. Additionally any value of j must differ from $j_1 + j_2$ by an integer.
Since the Clebsch Gordan coefficients are real it follows

$$(j_1, m_1, j_2, m_2|j, m) = (j, m|j_1, m_1, j_2, m_2)^\dagger = (j, m|j_1, m_1, j_2, m_2). \quad \text{(B.14)}$$

The following orthogonality relations can be easily show by using the orthogonality and completeness relation for angular momentum states:

$$\sum_{j,m}(j_1, m_1, j_2, m_2|j, m)(j, m|j_1', m_1', j_2', m_2') = \delta_{j_1,j_1'}\delta_{j_2,j_2'}\delta_{m_1,m_1'}\delta_{m_2,m_2'}. \quad \text{(B.15)}$$

A few special cases can be found in tables 3 and 4.

C Spherical Bessel functions

The differential equation

$$\left(\frac{1}{\rho}\frac{d^2}{d\rho^2}\rho - \frac{l(l+1)}{\rho^2} + 1\right) f_l(\rho) = 0 \quad \text{(C.1)}$$

is solved by the spherical Bessel functions $j_l(\rho)$ and the spherical Neumann functions $n_l(\rho)$. The defining differential equation arises in the derivation of the partial wave expansion. This partial waves must be normalizable. Therefore we will look at the asymptotic behavior for $\rho \to 0$:

$$j_l(\rho \to 0) \propto \rho^l, \tag{C.2}$$

$$n_l(\rho \to 0) \propto \frac{1}{\rho^{l+1}}. \tag{C.3}$$

We see that the spherical Neumann functions are not normalizable.Another observation we make using this asymptotic behavior is, the spherical harmonics are linearly independent. There is a power series expansion of the spherical Bessel function:

$$j_l(\rho) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+l+\frac{3}{2})} \left(\frac{\rho}{2}\right)^{2n+l}. \tag{C.4}$$

$\Gamma(z)$ is the Gamma function.
A plane wave can be expanded into spherical Bessel functions as follows

$$e^{i\vec{k}\vec{r}} = \sum_{n,m_n} 4\pi i^n j_n(kr) Y_{n,m_n}(\hat{r}) Y_{n,m_n}(\hat{k})^*, \tag{C.5}$$

with $r := |\vec{r}|$ and $k := |\vec{k}|$. From this we can easily derive

$$\begin{aligned} e^{i\vec{k}(\vec{r_1}-\vec{r_2})} &= \sum_{n,m_n} 4\pi i^n j_n(k|\vec{r_1}-\vec{r_2}|) Y_{n,m_n}\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|}\right) Y_{n,m_n}(\hat{k})^* \\ &= \sum_{\substack{n_1,m_{n_1}\\ n_2,m_{n_2}}} (4\pi)^2 i^{n_1+n_2} j_{n_1}(kr_1) Y_{n_1,m_{n_1}}(\hat{r_1}) Y_{n_1,m_{n_1}}(\hat{k})^* \times \\ &\quad \times j_{n_2}(kr_2) Y_{n_2,m_{n_2}}(\hat{-r_2}) Y_{n_2,m_{n_2}}(\hat{k})^* \\ &= \sum_{\substack{n_1,m_{n_1}\\ n_2,m_{n_2}}} (4\pi)^2 i^{n_1+n_2} j_{n_1}(kr_1) Y_{n_1,m_{n_1}}(\hat{r_1}) j_{n_2}(kr_2) Y_{n_2,m_{n_2}}(\hat{-r_2}) \times \\ &\quad \times \sum_{n',n'_m} \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2n'+1)}} (n_1,0,n_2,0|n',0) \times \\ &\quad \times (n_1,m_{n_1},n_2,m_{n_2}|n',m'_n) Y_{n',m'_n}(\hat{k})^*. \end{aligned} \tag{C.6}$$

Since $Y_{n,m_n}(\hat{k})^*$ are linearly independent we see that $n' = n$ and therefore n is related to n_1 and n_2 via angular momentum coupling.

D Radial and angular part of the Dirac equation

In section 5 we have used a few relations without derivation. This relations shall be proven here.
Using the Grassmann identity of the cross product

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{D.1}$$

we can write

$$\begin{aligned} \vec{\nabla} &= \hat{r}(\hat{r} \cdot \vec{\nabla}) - \hat{r} \times (\hat{r} \times \vec{\nabla}) \\ &= \hat{r}\frac{\partial}{\partial r} - i\frac{\hat{r}}{r} \times \vec{L}. \end{aligned} \tag{D.2}$$

In the differential equation for the spinor the operator $\vec{\sigma} \cdot \vec{\nabla}$ arises. With eq. (D.2) we obtain

$$\vec{\sigma} \cdot \vec{\nabla} = \vec{\sigma} \cdot \hat{r}\frac{\partial}{\partial r} - i\vec{\sigma} \cdot \frac{\hat{r}}{r} \times \vec{L}. \tag{D.3}$$

Next we use that, for any vectors whose components commute with the Pauli matrices [1]

$$A_i\sigma_i B_j\sigma_j = A_i B_j \delta_{ij}\mathbb{1} + i\sigma_k\epsilon_{i,j,k}A_i B_j = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}) \tag{D.4}$$

to rewrite the second term of eq. (D.3). The final result is

$$\vec{\sigma} \cdot \vec{\nabla} = \vec{\sigma} \cdot \hat{r}\left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma} \cdot \vec{L}\right). \tag{D.5}$$

This derivation can be found in [8].
Next we want to derive eq. (5.19). First we write $\xi_{\varkappa,m}(\Omega_r)$ in terms of spherical harmonics:

$$\xi_{\varkappa,m}(\Omega_r) = \langle\Omega_r|j,m,l,s\rangle = \langle\Omega_r| \sum_{m_l,m_s} (l,m_l,s,m_s|j,m)\,|l,m_l,s,m_s\rangle \tag{D.6}$$

$$= \sum_{m_l,m_s} (l,m_l,s,m_s|j,m) Y_{l,m_l}(\Omega_r)\,|s,m_s\rangle\,. \tag{D.7}$$

[1] Note

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{i,j,k}\sigma_k$$

Note that

$$\hat{r}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \qquad \text{for} \quad |\tfrac{1}{2}, m_s\rangle = \begin{pmatrix} \delta_{m_s,\frac{1}{2}} \\ \delta_{m_s,-\frac{1}{2}} \end{pmatrix}. \tag{D.8}$$

It follows

$$\begin{aligned}
\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) &= \sum_{m_l,m_s} (l,m_l,s,m_s|j,m)Y_{l,m_l}(\Omega_r)\Big[\sin\theta e^{-i\varphi}\,|s,m_s+1\rangle \\
&\qquad + \sin\theta e^{i\varphi}\,|s,m_s-1\rangle + 2m_s\cos\theta\,|s,m_s\rangle\Big] \qquad\qquad (\text{D.9}) \\
&= \sum_{m_l,m_s}\sqrt{\frac{4\pi}{3}}Y_{l,m_l}(\Omega_r)\,|s,m_s\rangle\Big[\sqrt{2}(l,m_l,s,m_s-1|j,m)Y_{1,-1}(\Omega_r) \\
&\quad - \sqrt{2}(l,m_l,s,m_s+1|j,m)Y_{1,1}(\Omega_r) \\
&\quad + 2m_s(l,m_l,s,m_s|j,m)Y_{1,0}(\Omega_r)\Big] \qquad\qquad (\text{D.10})
\end{aligned}$$

where we used eqs. (B.11) to (B.13). Now we use eq. (B.9) to couple the spherical harmonics:

$$\begin{aligned}
\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) = \sum_{m_l,m_s}\sum_{l',m_l'} &\sqrt{\frac{2l+1}{2l'+1}}(1,0,l,0|l',0)Y_{l',m_l'}(\Omega_r)\,|s,m_s\rangle \times \\
&\times \Big[\sqrt{2}(l,m_l,s,m_s-1|j,m)(1,-1,l,m_l|l',m_l') \\
&- \sqrt{2}(l,m_l,s,m_s+1|j,m)(1,1,l,m_l|l',m_l') \\
&+ 2m_s(l,m_l,s,m_s|j,m)(1,0,l,m_l|l',m_l')\Big]. \qquad (\text{D.11})
\end{aligned}$$

With table 4 the first Clebsch Gordan coefficient can be written as

$$(1,0,l,0|l',0) = \sqrt{\frac{l+1}{2l+1}}\delta_{l',l+1} - \sqrt{\frac{l}{2l+1}}\delta_{l',l-1}. \tag{D.12}$$

Let us now look at the spinor component with $m_s = 1/2$. The other case can be done analogously. Inserting the other Clebsch Gordan coefficients from tables 3 and 4 as well we obtain

$$\begin{aligned}\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) =& \sum_{l'}\sqrt{\frac{2l+1}{2l'+1}}(1,0,l,0|l',0)Y_{l',m_l'}(\Omega_r)\times \\ & \times\Big[\sqrt{2}(l,m+\tfrac{1}{2},s,-\tfrac{1}{2}|j,m)(1,-1,l,m+\tfrac{1}{2}|l',m-\tfrac{1}{2}) \\ & +(l,m-\tfrac{1}{2},s,\tfrac{1}{2}|j,m)(1,0,l,m-\tfrac{1}{2}|l',m-\tfrac{1}{2})\Big] \qquad \text{(D.13)}\\ =& \sum_{l'}\sqrt{\frac{1}{2l'+1}}Y_{l',m_l'}(\Omega_r)\Bigg\{\sqrt{l+1}\delta_{l',l+1}\times \\ & \times\left[\sqrt{2}\left(\frac{l\mp m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}\left(\frac{(l-m+\frac{1}{2})(l-m+\frac{3}{2})}{(2l+1)(2l+2)}\right)^{\frac{1}{2}}\right. \\ & \left.\pm\left(\frac{l\pm m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}\left(\frac{(l-m+\frac{3}{2})(l+m+\frac{1}{2})}{(2l+1)(l+1)}\right)^{\frac{1}{2}}\right] \\ & +\sqrt{l}\delta_{l',l-1}\left[\sqrt{2}\left(\frac{l\mp m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}\left(\frac{(l+m+\frac{1}{2})(l+m-\frac{1}{2})}{2l(2l+1)}\right)^{\frac{1}{2}}\right. \\ & \left.\mp\left(\frac{l\pm m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}\left(\frac{(l-m+\frac{1}{2})(l+m-\frac{1}{2})}{l(2l+1)}\right)^{\frac{1}{2}}\right]\Bigg\}. \qquad \text{(D.14)}\end{aligned}$$

The $\pm$ denotes the two possible values of the total angular momentum $j = l \pm \frac{1}{2}$. The result of carrying out the multiplication inside the brackets and writing the remaining part in term of Clebsch Gordan coefficients (table 3) again is

$$\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) = -\sum_{l'}Y_{l',m_l'}(\Omega_r)(l'm-\tfrac{1}{2},s,\tfrac{1}{2}|j,m)\begin{cases}\delta_{l',l+1} & \text{for} \quad j=l+\frac{1}{2}\\ \delta_{l',l-1} & \text{for} \quad j=l-\frac{1}{2}\end{cases}. \qquad \text{(D.15)}$$

This is the first spinor component of $\langle\Omega_r|j,m,l',s\rangle$ with $l' = l\pm1$ for $j = l\pm\frac{1}{2}$. Considering both spinor components again and eq. (5.16) our final result is

$$\hat{r}\cdot\vec{\sigma}\xi_{\varkappa,m}(\Omega_r) = -\xi_{-\varkappa,m}(\Omega_r). \qquad \text{(D.16)}$$

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I hereby confirm that this thesis on “Partial wave analysis of dark matter annihilation’s” is solely my own work and that I have used no sources or aids other than the ones stated. All passages in my thesis for which other sources, including electronic media, have been used, be it direct quotes or content references, have been acknowledged as such and the sources cited.

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