



BACHELOR'S THESIS

THE ANOMALOUS MAGNETIC MOMENT OF THE MUON IN THE Z' -MODEL

Submitted by

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Contents

1	Abstract	1
2	Introduction	2
2.1	The magnetic moment in classical and quantum mechanics	2
2.2	The anomalous magnetic moment	2
2.3	Derivation of the Dirac equation and first calculations for g	4
2.4	Tree-level contribution	8
2.5	Form factors	10
3	One-loop contribution in QED	12
3.1	Relations and simplifications	13
3.2	Integral evaluation and final result	17
4	One-loop contribution of arbitrary vector bosons	19
4.1	Gauge boson contribution	22
4.2	Goldstone boson contribution	24
4.3	Combined final contribution to $(g - 2)_\mu$	26
5	Flavor conserving and violating BSM extensions	27
5.1	$L_\mu - L_\tau$ extension	27
5.2	Dark Photon extension	29
5.3	Lepton flavor violating Z' extension	33
6	Conclusion	36
7	Appendix	37
A1.1	Properties of γ matrices	37
A1.2	The Gordon identity	38
A1.3	Properties of σ_{ij}	38
A1.4	Feynman parameters	39
A1.4.1	$(1 - z) \rightarrow u$ integral substitution	40
A1.5	Generic integrals	40
A1.6	Feynman rules	42
A1.7	Coupling constants c_V, c_A and C_V, C_A relations	43
	Bibliography	45

1 Abstract

This thesis investigates the anomalous magnetic moment of the muon, $a_\mu = (g-2)/2$, with focus on SM (SM) contributions and possible beyond SM (BSM) explanations. Starting from the Dirac equation and tree-level scattering, the classical result $g = 2$ was derived. The one-loop QED vertex correction was then evaluated explicitly using Feynman parametrization and Wick rotation, reproducing Schwinger's known result $a_\mu^{\text{QED}} = \alpha/(2\pi)$.

The calculation was then extended to a new massive vector boson X_μ , yielding analytical expressions for both its physical and Goldstone contributions. The combined result shows a dependence on the boson mass M_X and couplings C_V, C_A , enabling constraints from the prevailing deviation between experiment and theory $\Delta a_\mu = (262 \pm 45) \times 10^{-11}$ [1] [2].

Three BSM extensions were examined: the flavor-conserving $L_\mu - L_\tau$ model, a kinetically mixed dark photon, and a lepton-flavor violating Z' . Parameter scans demonstrate that only restricted pairs of coupling and mass can account for the anomaly while remaining consistent with experimental bounds. In particular, light flavor violating Z' bosons with asymmetric couplings to muons and taus remain viable candidates.

The thesis confirms the SM QED prediction, derives the general one-loop Z' boson correction, and identifies BSM extensions capable of reconciling theory with experiment.

2 Introduction

2.1 The magnetic moment in classical and quantum mechanics

Before performing calculations on the anomalous magnetic moment of a muon, it is advisable to start dealing with classical mechanics, to get an impression of what the magnetic moment is. This section is based on [3, pp. 195–197] and [4, p. 53].

In classical mechanics, the definition of the magnetic moment of an electric current density distribution $\vec{j}(\vec{r})$ is given by

$$\vec{\mu} = \frac{1}{2} \int d^3x [\vec{r} \times \vec{j}(\vec{r})]. \quad (2.1)$$

Assuming the electric current is made up of a single particle of mass m and an electric charge q at the position $\vec{r}_0(t)$ with velocity $\vec{v}(t)$, we are able to define the electric current density distribution as follows

$$\vec{j}(\vec{r}) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_0(t)). \quad (2.2)$$

Substituting eq. (2.2) into eq. (2.1) results in

$$\vec{\mu} = \frac{1}{2} q \int d^3x [\vec{r} \times \vec{v}(t)] \delta(\vec{r} - \vec{r}_0(t)). \quad (2.3)$$

Due to the δ -function, the integral disappears and the final expression is

$$\vec{\mu} = \frac{1}{2} q [\vec{r}_0(t) \times \vec{v}(t)] = \frac{1}{2m} q [\vec{r}_0(t) \times \vec{p}(t)] = \frac{q}{2m} \cdot \vec{L}, \quad (2.4)$$

where one can immediately see the relation between the orbital angular momentum \vec{L} and the magnetic momentum $\vec{\mu}$. The proportionality constant $\frac{q}{2m}$ is the so-called gyromagnetic ratio.

In quantum mechanics, the particle does not only have an orbital angular momentum, but also the intrinsic contribution to $\vec{\mu}$ by its spin \vec{S} . As the spin of a particle has no classical analogue, it is a pure quantity of quantum mechanics and does not contribute to $\vec{\mu}$ in the same way as \vec{L} does. In fact, the magnetic moment created by the particles spin is given by

$$\vec{\mu}_s = g \frac{q}{2m} \cdot \vec{S}, \quad (2.5)$$

with g being the so-called *Landé Factor*. Its value amounts to $g \gtrsim 2$ and can be calculated quite precisely in the SM.

2.2 The anomalous magnetic moment

The deviation from 2 is the so-called anomalous magnetic moment

$$a_l = \frac{g - 2}{2}. \quad (2.6)$$

In quantum field theory (QFT), higher order of perturbation theory can be used to calculate the value of g to many decimal digits. There are many experiments designed to measure the anomalous magnetic moment of fermions with very high precision of order $\sim 10^{-14}$. Therefore, a_l is a suitable quantity to test the SM, the currently best and most precise model of QFT describing particles and their interactions. We are especially interested in the contribution to a_l from quantum electrodynamics (QED), the most precise theory of all theories in physics, which describes electrodynamics of particles like leptons (e, μ, τ). The theory introduces photons (γ) as the mediators of the electromagnetic force. The anomalous magnetic moment exists for all kinds of particles like charged leptons, baryons and mesons. However, there is one particle, the muon μ , that shows a deviation between experimentally measured value and theoretically predicted value at decimal digits of around 10^{-8} .

Latest experimental data (world average) [1] yield

$$a_\mu^{\text{exp}} = 1165920715(145) \times 10^{-12}, \quad (2.7)$$

and the SM contribution derived with e^+e^- -collider data on hadronic contributions [2]

$$a_\mu^{\text{SM}} = 116591810(43) \times 10^{-11}. \quad (2.8)$$

The current deviation therefore is

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (262 \pm 45) \times 10^{-11}, \quad (2.9)$$

a 5.8σ deviation. As a convention, a 5σ deviation in particle physics is associated with potential new physics.

The theoretically predicted value is made up of three contributions

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{hadron}}. \quad (2.10)$$

QED is involved, because the muon has a charge $-e$. The a_μ^{hadron} contribution can be derived from e^+e^- -collider data or with lattice quantum chromodynamics (QCD). We will not go into details of hadronic interactions, but it must be mentioned that latest SM calculations with lattice QCD can settle the deviation between SM prediction and experimental data [5].

There are also relevant electroweak interactions with W , Z and Higgs bosons. The a_μ^{EW} part cannot be neglected, unlike in the case of electrons, because of the mass proportions between both particles ($m_\mu \approx 200 \cdot m_e$). The QED contribution to a_e for an electron is very dominant. The muon is much heavier than the electron and therefore electroweak contributions are much more significant. As the SM cannot explain the deviation for the muon in eq. (2.9), this might hint that there is physics beyond the SM.

In addition, the prevailing deviation cannot be settled by QED, as the uncertainty of QED on a_μ is much smaller than the current uncertainty in eq. (2.9). Moreover, QED predictions for other observables are very precise and don't hint new physics. Therefore, the deviation might be explained by a new gauge boson.

In this thesis, we want to take a look at a new hypothetical massive vector boson X_μ , whose contribution to a_μ might settle the prevailing deviation between theoretically calculated values for a_μ and measured ones. At first, we will perform analytical calculations to extract the first order correction to g in QED. This calculation was first performed by Julian Schwinger in 1947 [6] and yields $a_\mu = \frac{\alpha}{2\pi}$. In addition, we will calculate the contribution on one-loop level from the new massive gauge boson X_μ with arbitrary coupling constants. This new boson will be the foundation for BSM physics discussed in this thesis and we will identify possible parameter pairs for m_X and the coupling strengths g' to leptons in different BSM extensions.

We are going to introduce three different models that aim to explain the $(g-2)_\mu$ anomaly and go even beyond that up to Dark Matter physics. However, in this thesis we will not go into any details on Dark Matter physics.

We will see that there are already many experiments designed to search for new particles that give constraints on the parameter space for the new X_μ gauge boson.

Before we do all that, it is essential to introduce the theory and go through the necessary derivations for later calculations.

2.3 Derivation of the Dirac equation and first calculations for g

The introduction of required equations is taken from [7, Chapter 3, 5]. As a convention we introduce $\hbar = c = 1$, which from now on will be applied on all following calculations. Exceptions are marked specifically.

In special relativity, the prevailing connection between energy and momentum is given by

$$E^2 = \vec{p}^2 c^2 + m^2 c^4. \quad (2.11)$$

After performing the quantum mechanical substitutions for E and \vec{p} , given in eq. (2.12)

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad \vec{p} \rightarrow -i\hbar \vec{\nabla}, \quad (2.12)$$

one can derive a relativistic Hamilton equation for a wavefunction ψ , which is also-called the *Klein-Gordon equation*

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \right) \psi = m^2 \psi. \quad (2.13)$$

As eq. (2.13) is lacking Lorentz covariance, we introduce ∂^μ as the combined derivative of time and space

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \quad \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right). \quad (2.14)$$

Implementing eq. (2.14) in eq. (2.13) results in the Lorentz covariant equation

$$(\square^2 + m^2)\psi = 0, \quad (2.15)$$

where $\square^2 = \partial_\mu \partial^\mu$. eq. (2.15) describes a free spinless particle. As the spin for fermions is $\frac{1}{2}$, we have to look for another equation for both the particle and its associated antiparticle. Dirac's attempt to linearize eq. (2.11) in $\frac{\partial}{\partial t}$ and $\vec{\nabla}$, brought a solution to describe particles with non-zero spin. Dirac's approach to derive the linear form of eq. (2.11) starts with the equation

$$H\psi = E\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi, \quad (2.16)$$

whose form is similar to eq. (2.11). The new introduced quantities $\vec{\alpha}$ and β are unknown at first, but can be derived from eq. (2.11), as the relativistic energy-momentum relation must be fulfilled. To derive constraints, we can multiply eq. (2.16) with H and compare the resulting terms:

$$H^2\psi = E^2\psi = (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)\psi \quad (2.17)$$

$$\Rightarrow (\alpha_i^2 p_i^2 + (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) p_i m + \beta^2 m^2)\psi = (\vec{p}^2 + m^2)\psi \quad (2.18)$$

From eq. (2.18), one can easily derive the requirements for $\vec{\alpha}$ and β , that must be fulfilled. These are

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \alpha_i^2 = \beta^2 = 1, \quad (2.19)$$

where $\{A, B\}$ is the anticommutator of A and B . Apparently, α_i and β cannot be scalars, as the anti-commutator is 0. The solution of the lowest dimension for both the α_i 's and β is of dimension 4 and has the following structure:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (2.20)$$

Here, $\vec{\sigma}$ are the Pauli matrices ($\sigma_1, \sigma_2, \sigma_3$) and $\mathbf{1}$ is the identity in 2 dimensions. As one can easily see the 4 dimensional structure of α_i and β , the dimension of the wavefunction ψ must also be 4. We therefore exchange ψ with Ψ , and introduce ψ_i ($i = 0, 1, 2, 3$) as the components of Ψ , a 4 dimensional

spinor. To obtain the covariant form of eq. (2.16), we multiply β from the left and rearrange the terms. With applying the conditions from eq. (2.19), we obtain

$$i\beta \frac{\partial}{\partial t} \Psi = -i\beta \cdot \vec{\alpha} \cdot \vec{\nabla} \Psi + m\Psi \quad (2.21)$$

$$i\gamma^\mu \partial_\mu \Psi = m\Psi. \quad (2.22)$$

Finally, we derived the covariant form of the Dirac equation from eq. (2.21)

$$(i\gamma^\mu \partial_\mu - m) \Psi = 0, \quad (2.23)$$

where the γ^μ are the so-called Dirac matrices. They are defined by

$$\gamma^\mu = (\beta, \beta\vec{\alpha}), \quad (2.24)$$

which is the Pauli-Dirac representation of the Dirac matrices. There are different ways of writing the γ^μ , e.g the Weyl representation, but we will stay with the already introduced representation as it simplifies later performed calculations.

To identify the covariant form of the electric current density distribution j^μ , we start by introducing the adjoint representation of eq. (2.23):

$$\begin{aligned} \left(i\gamma^0 \frac{\partial}{\partial t} \Psi\right)^\dagger + \left(i\gamma^k \frac{\partial}{\partial x^k} \Psi\right)^\dagger - (m\Psi)^\dagger &= 0 \\ -i \frac{\partial}{\partial t} \Psi^\dagger \gamma^{0\dagger} - i \frac{\partial}{\partial x^k} \Psi^\dagger \gamma^{k\dagger} - \Psi^\dagger m &= 0 \\ \xrightarrow{\text{A1.4,A1.7}} -i \frac{\partial}{\partial t} \Psi^\dagger \gamma^0 - i \frac{\partial}{\partial x^k} \Psi^\dagger (\gamma^0 \gamma^k \gamma^0) - \Psi^\dagger m &= 0 \end{aligned} \quad (2.25)$$

Multiplying eq. (2.25) from the right with γ^0 yields

$$\begin{aligned} -i \frac{\partial}{\partial t} (\Psi^\dagger \gamma^0) \gamma^0 - i \frac{\partial}{\partial x^k} (\Psi^\dagger \gamma^0) \gamma^k \gamma^0 \gamma^0 - m (\Psi^\dagger \gamma^0) &= 0 \\ \xrightarrow{\text{A1.5}} -i \frac{\partial}{\partial t} \bar{\Psi} \gamma^0 - i \frac{\partial}{\partial x^k} \bar{\Psi} \gamma^k - m \bar{\Psi} &= 0, \end{aligned} \quad (2.26)$$

where we define $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$ as the adjoint spinor of Ψ . From eq. (2.26) the resulting expression therefore is

$$i\partial_\mu \bar{\Psi} \gamma^\mu + m\bar{\Psi} = 0. \quad (2.27)$$

Multiplying eq. (2.27) from the right with Ψ and eq. (2.23) with $\bar{\Psi}$ from the left and adding together both results gives

$$\begin{aligned} \bar{\Psi} i\gamma^\mu \partial_\mu \Psi + i\partial_\mu (\bar{\Psi}) \gamma^\mu \Psi &= 0 \\ \partial_\mu (\bar{\Psi} \gamma^\mu \Psi) &= 0 \\ \partial_\mu j^\mu &= 0, \end{aligned}$$

where $j^\mu \equiv \bar{\Psi} \gamma^\mu \Psi$ is the charge current density, which satisfies the continuity equation. In later sections we will see, that the charge current density is an important part of integrals for calculating the scattering amplitudes for fermions, from an initial state to a final state.

As we are about to describe fermions with an electric charge of $-e$, we can multiply j^μ with $-e$, and call the resulting expression $j^\mu \equiv -e\bar{\Psi}\gamma^\mu\Psi$ the electron (or muon) current density.

Taking a look at eq. (2.16) again, we can rewrite Ψ as a vector with two components, Ψ_a and Ψ_b , which themselves are two component vectors. This allows eq. (2.16) to become a well-organized matrix equation:

$$E \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}. \quad (2.28)$$

A proper solution for Ψ , that satisfies eq. (2.23), is given by

$$\Psi = u(\vec{p})e^{-ipx}, \quad (2.29)$$

with the 4 component spinor $u(\vec{p})$. This representation of Ψ also allows us to simplify eq. (2.23)

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)u(\vec{p})e^{-ipx} &= 0 \\ (\gamma^\mu p_\mu - m)u(\vec{p})e^{-ipx} &= 0 \\ (\not{p} - m)u(\vec{p}) &= 0, \end{aligned} \quad (2.30)$$

where we introduced the slash notation with $\not{p} = \gamma^\mu p_\mu = \gamma_\mu p^\mu$. A similar result can be derived for $\bar{u}(\vec{p}) = u(\vec{p})\gamma^0$. To obtain this result we need the adjoint form of eq. (2.30).

$$\begin{aligned} u(\vec{p})^\dagger(\gamma^{\mu\dagger}p_\mu - m) &= 0 \\ \xrightarrow{\cdot\gamma^0} u(\vec{p})^\dagger(\gamma^{\mu\dagger}\gamma^0p_\mu - m\gamma^0) &= 0 \\ \xrightarrow{\text{A1.7}} \bar{u}(\vec{p})(\not{p} - m) &= 0 \end{aligned} \quad (2.31)$$

Substituting eq. (2.29) into eq. (2.28) yields

$$E \begin{pmatrix} u_a \\ u_b \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} u_a \\ u_b \end{pmatrix}. \quad (2.32)$$

This system of linear equations has 4 solutions in total, with two solutions including a negative energy E for antiparticles, and the other 2 solutions with a positive energy for normal particles. What can be derived from eq. (2.32) is that in the non-relativistic limit ($v \ll c$), u_a is the dominant part of Ψ , which allows one to only consider Ψ_a in the non-relativistic limit.

To show the ratio between the two spinors u_a and u_b in this case, we go back to standard units, to derive a dimensionless ratio dependent to v and c between both spinors. In the non-relativistic limit $v \ll c$, one can apply $\gamma \approx 1$, which is the Lorentz factor from special relativity.

Therefore, from eq. (2.32) we can derive

$$\left(\frac{\vec{\sigma} \cdot \vec{p}c}{E + mc^2} \right) u_a = \left(\frac{\gamma m v c \cdot \vec{\sigma} \cdot \vec{n}}{\gamma m c^2 + m c^2} \right) u_a \approx \left(\frac{v \cdot \vec{\sigma} \cdot \vec{n}}{2c} \right) u_a = \vec{\sigma} \cdot \vec{n} \cdot \frac{1}{2} \frac{v}{c} u_a = u_b, \quad (2.33)$$

which shows that u_a is dominant regarding u_b by a factor of $\frac{v}{c}$. The vector \vec{n} is an arbitrary vector pointing in the direction of the particle's momentum.

With eq. (2.28), we have a system of 2 equations that can be used to derive the Schrödinger-Pauli equation and the g factor.

For doing that, we introduce an electromagnetic field $A^\mu = (A^0, \vec{A})$ through $p^\mu \rightarrow p^\mu + eA^\mu$. The idea is to use the non-relativistic limit for deriving a solution for Ψ_a . From eq. (2.28) the following expression can be derived:

$$\begin{aligned} \frac{1}{E+m+eA^0} \left(\vec{\sigma} \cdot \vec{p} + e\vec{\sigma} \cdot \vec{A} \right)^2 \Psi_a &= (E-m+eA^0)\Psi_a \\ \frac{1}{2m} \left(\sigma_1(p^1+eA^1) + \sigma_2(p^2+eA^2) + \sigma_3(p^3+eA^3) \right)^2 \Psi_a &= (E-m+eA^0)\Psi_a. \end{aligned} \quad (2.34)$$

In eq. (2.34) we assume $|eA^0| \ll m$ and apply the non-relativistic limit through $\frac{1}{E+m+eA^0} \approx \frac{1}{2m}$. Performing the square results in

$$\frac{1}{2m} \left((\vec{p} + e\vec{A})^2 + \sum_{i=1}^3 \sum_{j \neq i} \sigma_i \sigma_j (p^i + eA^i)(p^j + eA^j) \right) \Psi_a = (E - m + eA^0)\Psi_a. \quad (2.35)$$

In spatial coordinates we can interpret p^k to be of the form $-i\frac{\partial}{\partial x^k}$. Because of $[A^i, A^j] = [p^i, p^j] = 0$, we can rewrite eq. (2.35):

$$\frac{1}{2m} \left((\vec{p} + e\vec{A})^2 + e \sum_{i=1}^3 \sum_{j \neq i} \sigma_i \sigma_j (p^i A^j + A^i p^j) \right) \Psi_a = (E - m + eA^0)\Psi_a. \quad (2.36)$$

The two sums result in a coupling between $\vec{\sigma}$ and the magnetic field \vec{B} , which is shown in A1.24. Therefore, what follows from eq. (2.36) is the *Schrödinger-Pauli equation*

$$\left(\frac{1}{2m} (\vec{p} + e\vec{A})^2 + \underbrace{\frac{e}{2m} \vec{\sigma} \cdot \vec{B}}_{H_M} - eA^0 \right) \Psi_a = E_{NR} \Psi_a. \quad (2.37)$$

In general, the interaction between the spin of a fermion and a magnetic field \vec{B} is given by

$$H_M = -\vec{\mu}_s \cdot \vec{B} = g \frac{e}{4m} \vec{\sigma} \cdot \vec{B}. \quad (2.38)$$

Comparing eq. (2.38) with H_M from eq. (2.37) immediately shows that $g = 2$. This result is a triumph of the Dirac equation in eq. (2.23). In the next step, we want to derive the same value from perturbation theory at tree level.

2.4 Tree-level contribution

This chapter is based on [7, pp. 117–119]. We want to take a look at the scattering amplitude of a muon from state Ψ_i to Ψ_f in an electromagnetic potential A^μ . To do so, we at first substitute the potential A^μ via the gauge covariant derivative ∂^μ into eq. (2.23).

$$[\gamma_\mu(i\partial^\mu + eA^\mu) - m]\Psi = 0 \quad (2.39)$$

What follows is a separation into

$$(\gamma_\mu i\partial^\mu - m)\Psi = -e\gamma_\mu A^\mu \Psi \equiv \gamma^0 V \Psi, \quad (2.40)$$

where the last expression is just convention. It is useful to keep a similar form compared to the Schrödinger-Pauli equation. As we cannot calculate the exact energies for Ψ in eq. (2.40), we will make use of first order perturbation theory, to calculate the amplitude T_{fi} for scattering a fermion with charge $-e$ from state Ψ_i (initial state) to Ψ_f (final state)

$$T_{fi} = -i \int_{\mathbb{R}^4} \Psi_f^\dagger(x) V(x) \Psi_i(x) d^4x. \quad (2.41)$$

In eq. (2.40) we find $V(x) = -e\gamma^0 \gamma_\mu A^\mu(x)$. Remembering that $\bar{\Psi} = \Psi^\dagger \gamma^0$ turns eq. (2.41) into

$$T_{fi} = ie \int_{\mathbb{R}^4} \bar{\Psi}_f \gamma_\mu A^\mu \Psi_i d^4x, \quad (2.42)$$

where one is able to identify the electric current density distribution j_μ . Introducing j_μ turns eq. (2.42) into

$$T_{fi} = -i \int_{\mathbb{R}^4} j_\mu^{fi} A^\mu d^4x. \quad (2.43)$$

Before further evaluating the integral, we want to take a look at the relevant lowest order Feynman diagram, which links the scattering amplitude iM^μ to the electric current density distribution j_μ^{fi} .

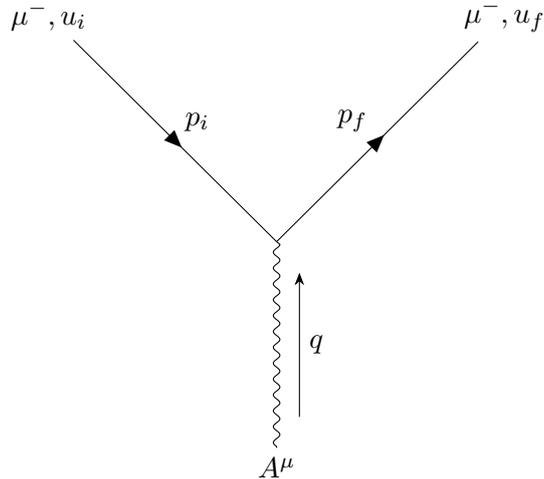


Figure 1: Lowest-order interaction Feynman diagram for an incoming muon with spinor u_i and an outgoing muon with spinor u_f with the electromagnetic potential A^μ .

Applying Feynman rules (see A1.6) on fig. 1 results in the amplitude

$$iM^\mu = -ie\bar{u}_f(\vec{p}_f)\gamma^\mu u_i(\vec{p}_i) \quad (2.44)$$

$$M_\mu e^{i(p_f - p_i)x} = j_\mu^{fi}, \quad (2.45)$$

which is directly linked to j_μ^i . To separate the electric and magnetic interaction between μ^- and A^μ , we employ the Gordon identity (see A1.17). Furthermore, we will assume that A^μ is time independent and therefore we have energy conservation. Thus, the integral over dx^0 results in a delta expression.

$$M^\mu = \frac{-e}{2m} \cdot \bar{u}_f(\vec{p}_f) [(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu] u_i(\vec{p}) \quad (2.46)$$

Plugging in eq. (2.45) into eq. (2.43) and considering energy conservation yields

$$\begin{aligned} T_{fi} &= -2\pi i \delta(E_f - E_i) \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f ((p_f + p_i)_\mu + i\sigma_{\mu\nu}(p_f - p_i)^\nu) \Psi_i A^\mu d^3x \\ &= -2\pi i \delta(E_f - E_i) \left[\int_{\mathbb{R}^3} \bar{\Psi}_f \frac{-e(p_f + p_i)_\mu}{2m} \Psi_i A^\mu d^3x + \int_{\mathbb{R}^3} \bar{\Psi}_f \frac{-e i \sigma_{\mu\nu} (p_f - p_i)^\nu}{2m} \Psi_i A^\mu d^3x \right] \\ &= -2\pi i \delta(E_f - E_i) [I_{\text{el}} + I_{\text{magn}}]. \end{aligned} \quad (2.47)$$

We will ignore I_{el} and concentrate on rewriting I_{magn} to extract the coupling between spin and A^μ .

$$\begin{aligned} I_{\text{magn}} &= \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{\mu\nu} (p_f - p_i)^\nu \Psi_i A^\mu d^3x \\ \xrightarrow{(p_f - p_i)^0 = 0} &= \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{\mu k} (p_f - p_i)^k \Psi_i A^\mu d^3x \end{aligned} \quad (2.48)$$

$$\begin{aligned} \xrightarrow{A^0 = \phi} &= \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{0k} (p_f - p_i)^k \Psi_i \phi d^3x + \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{jk} (p_f - p_i)^k \Psi_i A^j d^3x \\ &= I'_{\text{el}} + \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{jk} (p_f - p_i)^k \Psi_i A^j d^3x \end{aligned} \quad (2.49)$$

After the first evaluation of I_{magn} , we are able to extract another contribution to the scattering amplitude, that comes from pure electrical interactions. Therefore, we will also ignore this integral. Due to energy conservation, we are able to exchange ν with k , which is an index that only takes the values 1, 2, 3. The same happens after extracting the electric interaction, where we exchange μ with j . For further evaluations, we introduce $q = p_f - p_i$ and continue working on the second integral in eq. (2.49)

$$I'_{\text{magn}} = \int_{\mathbb{R}^3} \frac{-e}{2m} \bar{\Psi}_f i \sigma_{jk} q^k \Psi_i A^j d^3x. \quad (2.50)$$

It is mandatory to take a closer look at the structure of σ_{jk} . It can be shown (see A1.23), that this tensor can be written in terms of the Pauli spin matrices σ_l . Using this relation yields

$$I'_{\text{magn}} = \int_{\mathbb{R}^3} \frac{-ie}{2m} \bar{\Psi}_f \sigma_l \epsilon_{ljk} q^k \mathbb{1}_2 \Psi_i A^j d^3x = \int_{\mathbb{R}^3} \frac{e}{2m} \bar{u}_f(\vec{p}_f) \sigma_l \epsilon_{ljk} \partial^k (e^{iqx}) \mathbb{1}_2 u_i(\vec{p}_i) A^j d^3x. \quad (2.51)$$

What we made use of in eq. (2.51) is the spatial coordinate representation of q^k which is a derivative $-i\partial^k$ acting on the fields $\bar{\Psi}_f$ and Ψ_i . We can apply the product rule to find a different expression for eq. (2.51)

$$I'_{\text{magn}} = \int_{\mathbb{R}^3} \frac{e}{2m} \bar{u}_f(\vec{p}_f) \sigma_l \epsilon_{ljk} \partial^k (e^{iqx} A^j) \mathbb{1}_2 u_i(\vec{p}_i) d^3x - \int_{\mathbb{R}^3} \frac{e}{2m} \bar{\Psi}_f \sigma_l \epsilon_{ljk} \partial^k A^j \mathbb{1}_2 \Psi_i d^3x. \quad (2.52)$$

One is able to employ the gaussian divergence theorem to rewrite the first integral in eq. (2.52)

$$\int_{\mathbb{R}^3} \frac{e}{2m} \bar{u}_f(\vec{p}_f) \sigma_l \epsilon_{ljk} \partial^k (e^{iqx} A^j) \mathbb{1}_2 u_i(\vec{p}_i) d^3x = \oint_{\partial\mathbb{R}^3} \frac{e}{2m} \bar{u}_f(\vec{p}_f) (e^{iqx} \vec{A} \times \vec{\sigma}) \mathbb{1}_2 u_i(\vec{p}_i) d^2x, \quad (2.53)$$

and show that after applying

$$\vec{\sigma} \cdot (\vec{\nabla} \times e^{iqx} \vec{A}) = \vec{\sigma} \cdot (\vec{\nabla} \times e^{iqx} \vec{A}) - e^{iqx} \vec{A} \cdot (\vec{\nabla} \times \vec{\sigma}) = \vec{\nabla} \cdot (e^{iqx} \vec{A} \times \vec{\sigma}),$$

the integral vanishes with assuming that the field \vec{A} vanishes on $\partial\mathbb{R}^3$. Therefore, we can simply omit the integral and only consider the second integral in eq. (2.52).

$$\begin{aligned}
\frac{\epsilon_{ljk} = -\epsilon_{lkj}}{\longrightarrow} \quad I'_{\text{magn}} &= \int_{\mathbb{R}^3} \frac{e}{2m} \bar{\Psi}_f \sigma_l \epsilon_{lkj} \partial^k A^j \mathbb{1}_4 \Psi_i d^3x \\
\frac{\epsilon_{lkj} \partial^k A^j = B^l}{\longrightarrow} &= \int_{\mathbb{R}^3} \bar{\Psi}_f \left(\frac{e}{2m} \vec{\sigma} \cdot \vec{B} \right) \mathbb{1}_4 \Psi_i d^3x \\
\frac{\Psi \approx \Psi^A}{\longrightarrow} &= \int_{\mathbb{R}^3} \bar{\Psi}_f^A \left(\frac{e}{2m} \vec{\sigma} \cdot \vec{B} \right) \Psi_i^A d^3x
\end{aligned} \tag{2.54}$$

In eq. (2.54) we used the non-relativistic limit, which allows one to only consider the first two components of Ψ . This is a consequence that follows from eq. (2.33). The derived integral I'_{magn} includes the magnetic interaction between A^μ and the spin of the particle. The final expression for eq. (2.47) therefore is

$$T_{fi} = -2\pi i \delta(E_f - E_i) \left[\int_{\mathbb{R}^3} \bar{\Psi}_f^A \left(\frac{e}{2m} \vec{\sigma} \cdot \vec{B} \right) \Psi_i^A d^3x + (I_{el} + I'_{el}) \right], \tag{2.55}$$

where the integrals for the electrical interaction do not include any couplings between the spin and the electromagnetic potential A^μ . As a consequence, also the tree level contribution yields $g = 2$, after comparing I'_{magn} with eq. (2.38).

In the next step, we will take a look at the general vertex function Γ^μ and derive the form factors F_1 and F_2 .

2.5 Form factors

After deriving $g = 2$ from the Dirac equation and the tree-level contribution, our next step is showing that $g = 2$ also follows from the interaction of a muon with an electromagnetic field A^μ in QED. This section is based on [8, pp. 185–186]. At first, we don't want to specify the interaction and take a look at the most general form of interactions. The Feynman diagram in fig. 2 contains a blob, which represents all possible interactions between μ^- and A^μ . All interactions are combined in the vertex function Γ^μ , which in general can be any function of p_i^μ, p_f^μ or γ^μ .

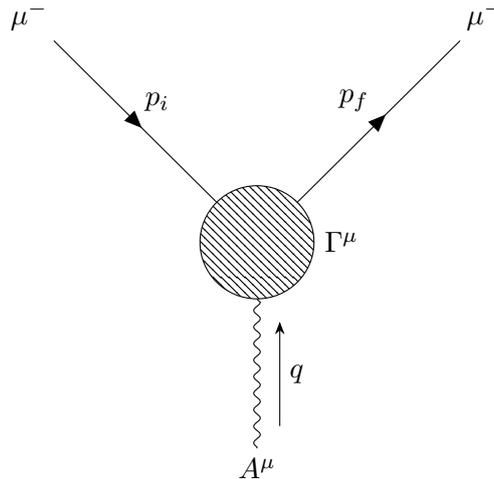


Figure 2: The general interaction of an incoming muon with momentum p_i and an outgoing muon with momentum p_f . Both particles are connected via the generalized interaction with the photon γ^μ from A^μ . The interaction is not specified here and is represented by a blob, representing a general vertex function called Γ^μ .

Figure 2 can be translated to the scattering amplitude

$$\mathcal{M} = J^\mu A_\mu = -ie\bar{u}(p_f)\Gamma^\mu(p_f, p_i)u(p_i)A_\mu. \quad (2.56)$$

We introduce the general form of Γ^μ with

$$\Gamma^\mu = A \cdot \gamma^\mu + B \cdot (p_f + p_i)^\mu + C \cdot (p_f - p_i)^\mu, \quad (2.57)$$

where A, B and C can be made up of Dirac matrices dotted into p_i or p_f , as long as they satisfy Lorentz invariance. Due to the relations in eq. (2.30) and eq. (2.31), we are able to express A, B and C in terms of ordinary numbers without loss of generality. The only scalar that A, B and C can be dependent to is $q^2 = -2p_f p_i + 2m^2$. An opportunity to obtain a clearer impression of how Γ^μ looks like, can be derived from the Ward identity [8]

$$q_\mu \Gamma^\mu = 0, \quad (2.58)$$

which is a consequence from the continuity equation $\partial_\mu j^\mu = 0$. Taking a closer look at $q_\mu \Gamma^\mu$, we can conclude with $q_\mu = (p_f - p_i)_\mu$

$$\begin{aligned} q_\mu \Gamma^\mu &= q_\mu \gamma^\mu \cdot A + (p_f - p_i)_\mu (p_f + p_i)^\mu \cdot B + q^2 \cdot C \\ 0 &= \not{q} \cdot A + (p_f^2 - p_i^2) \cdot B + q^2 \cdot C \\ 0 &= q^2 \cdot C, \end{aligned} \quad (2.59)$$

which implies $C = 0$. The first term vanishes due to $\bar{u}(p_f)\not{q}u(p_i) = 0$ and the second term due to $p_f^2 = p_i^2 = m^2$. With eq. (2.59), eq. (2.57) becomes

$$\Gamma^\mu = \gamma^\mu \cdot A + (p_f + p_i)^\mu \cdot B. \quad (2.60)$$

To further evaluate this expression, we use the Gordon identity (see A1.17) to exchange the second term with

$$2mB \left(\frac{(p_f + p_i)^\mu}{2m} \right) = 2mB \left(\gamma^\mu - \frac{i\sigma^{\mu\nu}(q)_\nu}{2m} \right). \quad (2.61)$$

Plugging in eq. (2.61) into eq. (2.60) results in

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu(A + 2mB) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(-2mB) \\ &= \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2), \end{aligned} \quad (2.62)$$

where we name F_1 and F_2 *form factors*, which are unknown functions of q^2 . We already derived that $F_1(0) = 1$, simply by comparing eq. (2.62) and eq. (2.44). In the non-relativistic limit, the connection between g and the form factors is given by

$$g = 2(F_1(0) + F_2(0)). \quad (2.63)$$

This implies that any anomaly of the g factor is carried by the magnetic form factor F_2 , as

$$a_f = \frac{g - 2}{2} = F_2(0). \quad (2.64)$$

Equation (2.64) is the anomalous magnetic moment, and we are going to calculate its value in the SM and in BSM extensions. To proceed, we need QFT and Feynman diagrams, that can be translated into mathematical expressions. For the evaluation of all Feynman diagrams in this thesis, we will stick to the Feynman rules in appendix A1.6.

3 One-loop contribution in QED

In the previous sections, we employed the Dirac equation and the lowest order perturbation theory to derive $g = 2$. As already mentioned, experiments show $g \gtrsim 2$, which we want to derive in QED.

We will examine the vertex correction, or one-loop contribution, where an additional photon links the two muon lines. This leads to two virtual fermions F with unambiguous momenta k_1 and k_2 . Thus, it is inevitable to perform an integral over one of the unambiguous momenta. We choose k_1 as the integration variable, which is linked to k_2 and q through momenta conservation. The following calculation is retraced from [8, pp. 189–196].

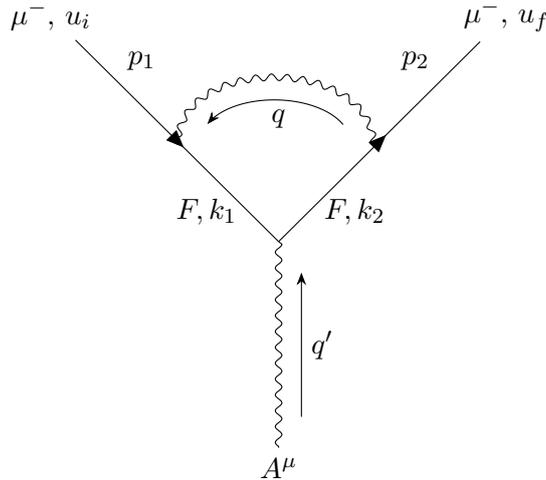


Figure 3: One-loop vertex correction diagram for the muon–photon interaction. An incoming muon with momentum p_1 and spinor u_i scatters into an outgoing muon with momentum p_2 and spinor u_f , exchanging an internal fermion F with loop momenta k_1, k_2 and a virtual photon with loop momentum q . The external photon with momentum q' couples through the vertex A^μ .

The Feynman diagram in fig. 3 is made up of the tree level diagram and the one-loop contribution. This implies a possible split up of both contributions. Considering the blob again in fig. 2, we are able to write $\Gamma^\sigma = \gamma^\sigma + \delta\Gamma^\sigma$, where $\delta\Gamma^\sigma$ considers the one-loop contribution from the virtual photon and carries the F_2 contribution. Applying Feynman rules yields for $\delta\Gamma^\sigma$:

$$\begin{aligned} \bar{u}_f(p_2)\delta\Gamma^\sigma u_i(p_1) &= \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) (-ie\gamma^\nu) \frac{i(\not{k}_2 + m)}{k_2^2 - m^2 + i\epsilon} (\gamma^\sigma) \frac{i(\not{k}_1 + m)}{k_1^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} u_i(p_1) \\ &= -ie^2 \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[\frac{\gamma^\nu (\not{k}_2 + m) \gamma^\sigma (\not{k}_1 + m) \gamma_\nu}{(k_2^2 - m^2 + i\epsilon)(k_1^2 - m^2 + i\epsilon)(q^2 + i\epsilon)} \right] u_i(p_1). \end{aligned} \quad (3.1)$$

Note that in $\delta\Gamma^\sigma$ we dropped the ie in front of γ^σ , as it is already considered within the interaction in the γ^σ term in $\gamma^\sigma + \delta\Gamma^\sigma$. We are only interested in inner loop effects and the general correction to the vertex structure, which justifies the omission of the factor ie . Furthermore, we will set $m_\mu = m$, as there are no different types of masses. To simplify the equations we also omit all vector arrows from now on.

In the next steps, the main goal will be to find simplifications, until the integral appears in an integrable form, and the form factor F_2 explicitly appears. To achieve this, we will separately deal with the denominator and numerator. We will start with the numerator where the multiple appearing γ matrices can be brought into a more manageable form. To achieve this, we can simply make use of A1.2 and A1.1 multiple times.

3.1 Relations and simplifications

The numerator can be simplified in the following way:

$$\begin{aligned}
\gamma^\nu(\not{k}_2 + m)\gamma^\sigma(\not{k}_1 + m)\gamma_\nu &= (\gamma^\nu\gamma^c k_{2,c}\gamma^\sigma + \gamma^\nu m\gamma^\sigma)(\gamma^i k_{1,i}\gamma_\nu + m\gamma_\nu) \\
&= k_{2,c}k_{1,i}\gamma^\nu\gamma^c\gamma^\sigma\gamma^i\gamma_\nu + mk_{2,c}\gamma^\nu\gamma^c\gamma^\sigma\gamma_\nu + mk_{1,i}\gamma^\nu\gamma^\sigma\gamma^i\gamma_\nu + m^2\gamma^\nu\gamma^\sigma\gamma_\nu \\
&= k_{2,c}k_{1,i}(2g^{\nu c} - \gamma^c\gamma^\nu)\gamma^\sigma(2g_\nu^i - \gamma_\nu\gamma^i) + mk_{2,c}(2g^{\nu c} - \gamma^c\gamma^\nu)\gamma^\sigma\gamma_\nu \\
&\quad + mk_{1,i}\gamma^\nu\gamma^\sigma(2g_\nu^i - \gamma_\nu\gamma^i) + m^2\gamma^\nu\gamma^\sigma\gamma_\nu \\
&= k_{2,c}k_{1,i}[4g^{\nu c}g_\nu^i\gamma^\sigma - 2g^{\nu c}\gamma^\sigma\gamma_\nu\gamma^i - 2g_\nu^i\gamma^c\gamma^\nu\gamma^\sigma + \gamma^c\gamma^\nu\gamma^\sigma\gamma_\nu\gamma^i] \\
&\quad + mk_{2,c}[2g^{\nu c}\gamma^\sigma\gamma_\nu - \gamma^c\gamma^\nu\gamma^\sigma\gamma_\nu] + mk_{1,i}[2g_\nu^i\gamma^\nu\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma_\nu\gamma^i] + m^2[\gamma^\nu\gamma^\sigma\gamma_\nu] \\
&= k_{2,c}k_{1,i}[4g^{ic}\gamma^\sigma - 2\gamma^\sigma\gamma^c\gamma^i - 2\gamma^c\gamma^i\gamma^\sigma - 2\gamma^c\gamma^\sigma\gamma^i] \\
&\quad + mk_{2,c}[2\gamma^\sigma\gamma^c + 2\gamma^c\gamma^\sigma] + k_{1,i}[2\gamma^i\gamma^\sigma + 2\gamma^\sigma\gamma^i] - 2m^2\gamma^\sigma \\
&= k_{2,c}k_{1,i}[4g^{ic}\gamma^\sigma - 2\gamma^\sigma\gamma^c\gamma^i - 4g^{i\sigma}\gamma^c] + 4mk_{2,c}g^{\sigma c} + 4mk_{1,i}g^{i\sigma} - 2m^2\gamma^\sigma \\
&= k_{2,c}k_{1,i}[4g^{ic}\gamma^\sigma - 4g^{i\sigma}\gamma^c - 4g^{ci}\gamma^\sigma + 4g^{\sigma i}\gamma^c - 2\gamma^i\gamma^\sigma\gamma^c] \\
&\quad + 4m(k_1 + k_2)^\sigma - 2m^2\gamma^\sigma \\
&= -2\not{k}_1\gamma^\sigma\not{k}_2 + 4m(k_1 + k_2)^\sigma - 2m^2\gamma^\sigma.
\end{aligned} \tag{3.2}$$

Plugging in eq. (3.2) into eq. (3.1) results in

$$\bar{u}_f(p_2)\delta\Gamma^\sigma u_i(p_1) = 2ie^2 \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[\frac{\not{k}_1\gamma^\sigma\not{k}_2 - 2m(k_1 + k_2)^\sigma + m^2\gamma^\sigma}{(k_2^2 - m^2 + i\epsilon)(k_1^2 - m^2 + i\epsilon)(q^2 + i\epsilon)} \right] u_i(p_1). \tag{3.3}$$

Now we will simplify the denominator. We will make use of Feynman parameters, that have multiple advantages for evaluating the integral. In general, one is able to use the following identity [8]:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 dx_2 \cdots dx_n \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \tag{3.4}$$

Comparing the denominator in eq. (3.3) with the denominator in the given formula, one is able to identify the exact same structure. Therefore, we will set $n = 3$, plug in the 3 coefficients

$$A = (k_2^2 - m^2 + i\epsilon)$$

$$B = (k_1^2 - m^2 + i\epsilon)$$

$$C = (q^2 + i\epsilon),$$

and obtain the additional form (proved in eq. (A1.25))

$$\frac{1}{ABC} = \int_0^1 dx dy dz \frac{2 \cdot \delta(x + y + z - 1)}{[xA + yB + zC]^3} = \int_0^1 dx dy dz \frac{2 \cdot \delta(x + y + z - 1)}{D^3}, \tag{3.5}$$

with

$$D = x(k_1^2 - m^2) + y(k_2^2 - m^2) + z(k_1 - p_1)^2 + (x + y + z)i\epsilon. \tag{3.6}$$

In eq. (3.6) we already made use of momenta conservation to exchange q with $k_1 - p_1$. The parameters x, y and z are Feynman parameters, that fulfill the relation $x + y + z = 1$.

To obtain a more tractable form of the denominator, we can apply some algebra and shift the momentum k_1 to a new variable $l = k_1 + yq' - zp_1$. To proceed, we only need momenta conservation (here especially $k_2 = k_1 + q'$) and the identity $x + y + z = 1$

$$\begin{aligned}
D &= x(k_1^2 - m^2) + y((k_1 + q')^2 - m^2) + z(k_1 - p_1)^2 + i\epsilon \\
&= k_1^2 x - m^2 x + k_1^2 y + 2k_1 q' y + yq'^2 - m^2 y + zk_1^2 - 2k_1 p_1 z + p_1^2 z + i\epsilon \\
&= k_1^2(x + y + z) - m^2(x + y) + 2k_1(q'y - p_1 z) + yq'^2 + p_1^2 z + i\epsilon \\
&= k_1^2 + 2k_1(q'y - p_1 z) + yq'^2 + p_1^2 z + i\epsilon - (x + y)m^2 \\
&= \underbrace{(k_1 + q'y - p_1 z)^2}_{l^2} - q'^2 y^2 - p_1^2 z^2 + yq'^2 + p_1^2 z - (x + y)m^2 + 2yzq'p_1 + i\epsilon \\
&= l^2 - q'^2 y(1 - y) + p_1^2 z(1 - z) - (1 - z)m^2 + 2yzq'p_1 + i\epsilon \\
&= l^2 + \underbrace{xyq'^2 - (1 - z)^2 m^2}_{-\Delta} + i\epsilon + yz(q'^2 + 2q'p_1) \\
&= l^2 - \Delta + i\epsilon.
\end{aligned} \tag{3.7}$$

In eq. (3.7) we completed the square for $(k_1 + q'y - p_1 z)^2$ to introduce the already mentioned shifted momentum l . Moreover, we employed $q'^2 + 2q'p_1 = 0$. This is quickly derived through momenta conservation

$$q'^2 + 2q'p_1 = (q' + p_1)^2 - p_1^2 = p_2^2 - p_1^2 = 0. \tag{3.9}$$

The new denominator eventually turns into

$$\frac{1}{(k_2^2 - m^2 + i\epsilon)(k_1^2 - m^2 + i\epsilon)(q^2 + i\epsilon)} = \int_0^1 dx dy dz \frac{2 \cdot \delta(x + y + z - 1)}{[l^2 - \Delta + i\epsilon]^3}. \tag{3.10}$$

After introducing the new momentum l , we also have to substitute it into the numerator. We therefore exchange k_1 with $l + zp_1 - yq'$ and k_2 with $l + zp_1 + (1 - y)q'$. This exchange yields

$$\text{numerator} = \underbrace{(l + zp_1 - yq')\gamma^\sigma(l + zp_1 + (1 - y)q')}_{\text{I}} + m^2\gamma^\sigma - 2m(2l + 2zp_1 + (1 - 2y)q')^\sigma, \tag{3.11}$$

where the first main task will be evaluating the expression I. Our goal will be to get rid of all l and l^μ terms.

$$\begin{aligned}
\text{I} &= l\gamma^\sigma(l + zp_1 + (1 - y)q') + (zp_1 - yq')\gamma^\sigma l + (zp_1 - yq')\gamma^\sigma(zp_1 + (1 - y)q') \\
&= \underbrace{l\gamma^\sigma l}_{(1)} + z\underbrace{(l\gamma^\sigma p_1 + p_1\gamma^\sigma l)}_{(2)} - y\underbrace{(l\gamma^\sigma q' + q'\gamma^\sigma l)}_{(3)} + l\gamma^\sigma q' + (zp_1 - yq')\gamma^\sigma(zp_1 + (1 - y)q')
\end{aligned} \tag{3.12}$$

One important relation to evaluate the terms (1) to (3) is the identity

$$\gamma^a \gamma^b \gamma^c = 2(g^{bc}\gamma^a - g^{ac}\gamma^b + g^{ab}\gamma^c) - \gamma^c \gamma^b \gamma^a, \tag{3.13}$$

whose proof can be found in A1.3.

By expanding, one can show:

$$(1) = \not{l}\gamma^\sigma \not{l} = l_\mu l_\nu [2(g^{\sigma\nu}\gamma^\mu - g^{\mu\nu}\gamma^\sigma + g^{\mu\sigma}\gamma^\nu) - \gamma^\nu\gamma^\sigma\gamma^\mu] = 2(l^\sigma \not{l} - l^2\gamma^\sigma + l^\sigma \not{l}) - \not{l}\gamma^\sigma \not{l} = 2l^\sigma \not{l} - l^2\gamma^\sigma \quad (3.14)$$

$$(2) = \not{l}\gamma^\sigma \not{p}_1 + \not{p}_1\gamma^\sigma \not{l} = \not{l}\gamma^\sigma \not{p}_1 + p_{1,\mu}l_\nu [2(g^{\sigma\nu}\gamma^\mu - g^{\mu\nu}\gamma^\sigma + g^{\mu\sigma}\gamma^\nu) - \gamma^\nu\gamma^\sigma\gamma^\mu] = 2(l^\sigma \not{p}_1 - p_1 \cdot l + \not{l}p_1^\sigma) \quad (3.15)$$

$$(3) = \not{l}\gamma^\sigma \not{q}' + \not{q}'\gamma^\sigma \not{l} = \not{l}\gamma^\sigma \not{q}' + q'_{\mu}l_\nu [2(g^{\sigma\nu}\gamma^\mu - g^{\mu\nu}\gamma^\sigma + g^{\mu\sigma}\gamma^\nu) - \gamma^\nu\gamma^\sigma\gamma^\mu] = 2(\not{q}'l^\sigma - q' \cdot l + \not{l}q'^\sigma) \quad (3.16)$$

With these different forms, eq. (3.12) becomes

$$I = 2l^\sigma \not{l} - l^2\gamma^\sigma + 2(l^\sigma \not{p}_1 - p_1 \cdot l + \not{l}p_1^\sigma) + 2(\not{q}'l^\sigma - q' \cdot l + \not{l}q'^\sigma) + \not{l}\gamma^\sigma \not{q}' + (z\not{p}_1 - y\not{q}')\gamma^\sigma (z\not{p}_1 + (1-y)\not{q}'). \quad (3.17)$$

To proceed, it is inevitable to consider the symmetry of the original momenta integral in eq. (3.1). We integrate over the complete \mathbb{R}^4 , which implies that linear contributions of the form l^ν must vanish due to the symmetry of the denominator. In general, we can define the two following relations

$$\int_{\mathbb{R}^4} \frac{d^4l}{(2\pi)^4} \frac{l^\nu}{D^3} = 0 \quad (3.18)$$

$$\int_{\mathbb{R}^4} \frac{d^4l}{(2\pi)^4} \frac{l^\nu l^\mu}{D^3} = \int_{\mathbb{R}^4} \frac{d^4l}{(2\pi)^4} \frac{\frac{1}{4}g^{\nu\mu}l^2}{D^3}, \quad (3.19)$$

that allow us to omit all of the terms inside I that are proportional to l^σ , $\not{l} = l_\nu\gamma^\nu$ or a product of the form $l \cdot q = l_\nu q^\nu$. Applying these relations results in

$$\begin{aligned} I &= 2l^\sigma l_\mu \gamma^\mu - l^2\gamma^\sigma + (z\not{p}_1 - y\not{q}')\gamma^\sigma (z\not{p}_1 + (1-y)\not{q}') \\ &= -\frac{1}{2}l^2\gamma^\sigma + (z\not{p}_1 - y\not{q}')\gamma^\sigma (z\not{p}_1 + (1-y)\not{q}'). \end{aligned} \quad (3.20)$$

Plugging back in I into the numerator gives the first interim result

$$\text{numerator} = -\frac{1}{2}l^2\gamma^\sigma + \underbrace{(z\not{p}_1 - y\not{q}')\gamma^\sigma (z\not{p}_1 + (1-y)\not{q}')}_{I'} + m^2\gamma^\sigma - 2m[2l + 2zp_1 + (1-2y)q']^\sigma. \quad (3.21)$$

Next, we further evaluate I' . In doing so, it is essential to apply momenta conservation (especially $p_1 + q' = p_2$) again and use the relation $x + y + z = 1$.

$$\begin{aligned} I' &= (z\not{p}_1 - y\not{q}')\gamma^\sigma (z\not{p}_1 + (1-y)\not{q}') \\ &= z^2 \underbrace{\not{p}_1\gamma^\sigma \not{p}_1}_{(1)} + z(1-y) \underbrace{\not{p}_1\gamma^\sigma \not{q}'}_{(2)} - yz \underbrace{\not{q}'\gamma^\sigma \not{p}_1}_{(3)} - y(1-y) \underbrace{\not{q}'\gamma^\sigma \not{q}'}_{(4)} \end{aligned} \quad (3.22)$$

To rearrange (1) up to (4), we will make use of A1.3 again multiple times. It is also useful to apply the Dirac equation that yields $\bar{u}_f(p_2)\not{p}_2 = \bar{u}_f(p_2) \cdot m$, $\not{p}_1 u_i(p_1) = m \cdot u_i(p_1)$ and $\bar{u}_f(p_2)\not{q}' u_i(p_1) = 0$.

$$(1) = \not{p}_1\gamma^\sigma \not{p}_1 = 2mp_1^\sigma - m^2\gamma^\sigma$$

$$(2) = \not{p}_1\gamma^\sigma \not{q}' = \not{p}_2\gamma^\sigma \not{q}' - \not{q}'\gamma^\sigma \not{q}' = m\gamma^\sigma \not{q}' + q'^2\gamma^\sigma = m(2p_2^\sigma - m\gamma^\sigma) - m^2\gamma^\sigma + q'^2\gamma^\sigma \quad (3.23)$$

$$= 2mp_2^\sigma - 2m^2\gamma^\sigma + q'^2\gamma^\sigma \quad (3.24)$$

$$(3) = \not{q}'\gamma^\sigma \not{p}_1 = \not{p}_2\gamma^\sigma \not{p}_1 - \not{p}_1\gamma^\sigma m = m^2\gamma^\sigma - m(2p_1^\sigma - \gamma^\sigma m) = 2m^2\gamma^\sigma - 2mp_1^\sigma \quad (3.25)$$

$$(4) = \not{q}'\gamma^\sigma \not{q}' = -q'^2\gamma^\sigma \quad (3.26)$$

Plugging back in all the obtained expressions results in a relatively long expression for the numerator:

$$\begin{aligned}
\text{numerator} &= -\frac{1}{2}l^2\gamma^\sigma + z^2(2mp_1^\sigma - m^2\gamma^\sigma) + z(1-y)(2mp_2^\sigma - 2m^2\gamma^\sigma + q'^2\gamma^\sigma) - yz(2m^2\gamma^\sigma - 2mp_1^\sigma) \\
&\quad + y(1-y)q'^2\gamma^\sigma + m^2\gamma^\sigma - 2m(1-2y)q'^\sigma - 4mzp_1^\sigma \\
&= \gamma^\sigma[-\frac{1}{2}l^2 + m^2(1-2z-z^2) + q'^2(1-x)(1-y)] \\
&\quad + \underbrace{2mp_1^\sigma(z^2 + yz - 2z) + 2mp_2^\sigma(z - yz) - 2mq'^\sigma(1-2y)}_{(1)}
\end{aligned} \tag{3.27}$$

Term (1) is still not in a desired form, as for extracting the form factor F_2 we have to introduce the tensor $\sigma^{\sigma\nu}$. It is therefore necessary to find an expression within (1) that is proportional to $(p_1 + p_2)^\sigma$.

$$(1) = 2mp_1^\sigma(z^2 + yz - 2z) + 2mp_2^\sigma(z - yz) - 2mq'^\sigma(1 - 2y) \tag{3.28}$$

$$= 2mp_1^\sigma(z^2 - z) - 2mzp_1^\sigma + 2mzp_2^\sigma - 2mq'^\sigma(1 - 2y + yz) \tag{3.29}$$

$$= 2mzp_1^\sigma(z - 1) - 2mq'^\sigma(1 - 2y + yz - z) \tag{3.30}$$

$$= (p_1 + p_2)^\sigma mz(z - 1) - 2mq'^\sigma(1 - 2y + yz - z + \frac{z}{2}(z - 1)) \tag{3.31}$$

In eq. (3.28) we set

$$2mp_1^\sigma(z^2 - 2z + yz) = 2mP_1^\sigma(z^2 - z) + 2mp_2^\sigma yz - 2mq'^\sigma yz, \tag{3.32}$$

whereas in eq. (3.29) we used momenta conservation to rearrange

$$2mp_1^\sigma(z^2 - 2z) = 2mp_1^\sigma(z^2 - z) - 2mp_1^\sigma z = 2mp_1^\sigma(z^2 - z) - 2mp_2^\sigma z - 2mq'^\sigma z. \tag{3.33}$$

In the last step in eq. (3.30), we split $2mp_1^\sigma$ and employ $p_1 = p_2 - q'$. After that, we will find a term proportional to $(p_1 + p_2)^\sigma$. Before we plug in our result into the numerator, we at first simplify the bracket after q'^σ with the relation $x + y + z = 1$

$$\begin{aligned}
-2(1 - 2y + yz - z + \frac{z}{2}(z - 1)) &= -2 + 4y - 2yz + 2z - z(z - 1) \\
&= -2 + 4y - 2yz + 3z - z^2 \\
&= -2x + 2y - 2yz + z - z^2 \\
&= -2x + 2y - 2yz + xz + yz \\
&= -2x + 2y + xz - yz \\
&= (x - y)(z - 2).
\end{aligned}$$

After all rewriting, the numerator turns into

$$\text{numerator} = \gamma^\sigma[-\frac{1}{2}l^2 + (1 - 2z - z^2)m^2 + (1 - x)(1 - y)q'^2] + (p_1 + p_2)^\sigma mz(z - 1) + mq'^\sigma(z - 2)(x - y). \tag{3.34}$$

Considering the symmetry of the denominator under an exchange $x \leftrightarrow y$ and the antisymmetry under $x \leftrightarrow y$ in $q'^\sigma(x - y)$ forces the q'^σ term to vanish after performing the Feynman parameter integral. We are now able to use the Gordon identity (A1.17) to find a term proportional to $\sigma^{\sigma\nu}$. We exchange

$$(p_1 + p_2)^\sigma = 2m\gamma^\sigma - i\sigma^{\sigma\nu}q'_\nu. \tag{3.35}$$

This exchange yields

$$\text{numerator} = \gamma^\sigma[-\frac{1}{2}l^2 + (1 - x)(1 - y)q'^2 + (1 - 4z + z^2)m^2] + i\sigma^{\sigma\nu}q'_\nu mz(1 - z) \tag{3.36}$$

and transforms the integral into the following form:

$$\begin{aligned} \bar{u}_f(p_2)\delta\Gamma^\sigma u_i(p_1) &= 2ie^2 \int_{\mathbb{R}^4} \frac{d^4l}{(2\pi)^4} \int_0^1 dx dy dz 2\delta(x+y+z-1) \\ &\times \bar{u}_f(p_2) \left[\frac{\gamma^\sigma [-\frac{1}{2}l^2 + (1-x)(1-y)q'^2 + (1-4z+z^2)m^2] + i\sigma^{\sigma\nu} q'_\nu m z(1-z)}{[l^2 - \Delta + i\epsilon]^3} \right] u_i(p_1). \end{aligned} \quad (3.37)$$

The $\sigma^{\sigma\nu}$ term can be expanded with $2m$ to

$$\frac{i\sigma^{\sigma\nu} q'_\nu}{2m} \cdot \frac{2m^2 z(1-z)}{[l^2 - \Delta + i\epsilon]^3}, \quad (3.38)$$

for an easier comparison with eq. (2.62). In this final expression for the whole integral, one is able to identify two types of momenta integrals:

$$I_1 = \int_{\mathbb{R}^4} d^4l \frac{l^2}{[l^2 - \Delta + i\epsilon]^3} \quad (3.39)$$

$$I_2 = \int_{\mathbb{R}^4} d^4l \frac{1}{[l^2 - \Delta + i\epsilon]^3} \quad (3.40)$$

3.2 Integral evaluation and final result

The first integral I_1 is divergent in any case and is not relevant for the F_2 form factor. We will ignore it and focus on the second integral I_2 . Note that due to the Minkowski signature $(+, -, -, -)$ of Minkowski space, we are not able to perform a 4-dimensional Euclidean integral over l . We therefore have to find a way to substitute l^0 , so that the signature turns into $(-, -, -, -)$. A proper substitution is

$$l^0 \equiv i l_E^0; \quad \vec{l} = \vec{l}_E. \quad (3.41)$$

The function $l^2 - \Delta + i\epsilon$ owns two singularities

$$l^2 - \Delta + i\epsilon = (l^0)^2 - |\vec{l}|^2 - \Delta + i\epsilon = 0 \Rightarrow l_{1,2}^0 = \pm \sqrt{|\vec{l}|^2 + \Delta - i\epsilon} \approx \pm \sqrt{|\vec{l}|^2 + \Delta} \mp i\epsilon, \quad (3.42)$$

that are located slightly above and below the axis of real numbers in the complex plane, as ϵ is a small number.

ϵ is used to shift the singularities from the real axis into the complex plane and therefore allows the so-called *Wick Rotation*.

The *Wick Rotation* is used to shift the integration path from l^0 onto the imaginary axis. This allows a switch in the prefactor of l^0 in the integral and justifies a 4-dimensional integration in Euclidean space. The idea of the *Wick Rotation* is depicted in fig. 4. We exchange the integral over l^0 with a contour integral over the real axis, the imaginary axis and two quarter circles in the complex plane, that we call C . This integral vanishes due to the Cauchy theorem, as it does not include any singularities within its integration domain.

$$\int_{\mathbb{R}} dl^0 + \int_{i\mathbb{R}} dl^0 + \int_C dl^0 = 0 \quad (3.43)$$

The contribution of the arcs, written as the integral over C , vanishes for $l^0 \rightarrow \infty$ and thus we can conclude

$$\int_{-\infty}^{\infty} dl^0 = \int_{-i\infty}^{i\infty} dl^0 = i \int_{-\infty}^{\infty} dl_E^0. \quad (3.44)$$

With this shift from the real axis onto the complex axis, we finally have a Euclidean metric, that allows evaluating the integral. After performing the *Wick Rotation*, we find the integral I_2 to be of the form

$$I_2 = -i \int d^4l_E \frac{1}{[l_E^2 + \Delta - i\epsilon]^3}. \quad (3.45)$$

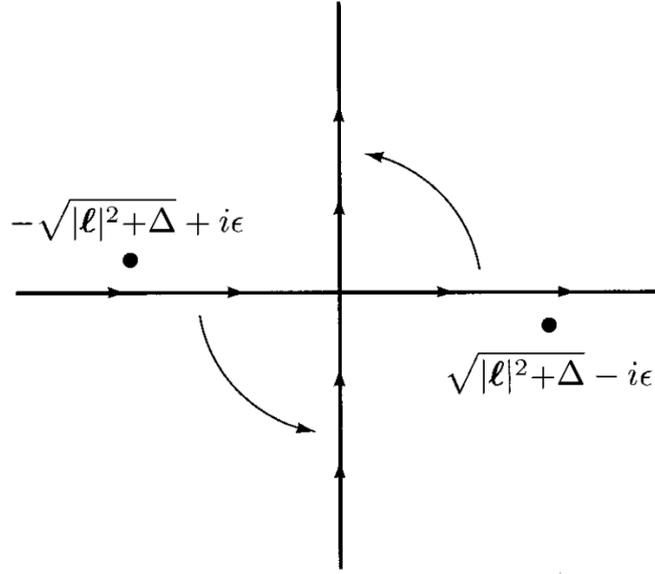


Figure 4: Wick Rotation in the complex plane. Taken from [8].

The further evaluation is performed in A1.5 and the result is given in A1.42. For the case discussed here, we plug in $D = 4$ and $n = 3$. Note that for this choice of parameters, the Γ function is well defined and the integral is convergent:

$$I_2 = -i\pi^2(\Delta - i\epsilon)^{-1} \frac{\Gamma(1)}{\Gamma(3)} = \frac{-i\pi^2}{\Delta - i\epsilon} \frac{1}{2}. \quad (3.46)$$

In the Limit $\epsilon \rightarrow 0$ we obtain

$$I_2 = \frac{-i\pi^2}{2} \frac{1}{\Delta}. \quad (3.47)$$

With this result we can go back to eq. (3.37) and extract the coefficient $F_2(q'^2)$. We can do so by comparing the coefficient in eq. (3.38), which is following from eq. (3.37), with those in eq. (2.62). We find

$$F_2(q'^2) = \frac{2ie^2}{(2\pi)^4} \int_0^1 dx dy dz 2\delta(x + y + z - 1) \frac{-i\pi^2}{2} \frac{1}{\Delta} \frac{2m^2 z(1-z)}{-xyq'^2 + (1-z)^2 m^2} \quad (3.48)$$

For the final integration we set $e = \sqrt{4\pi\alpha}$ and apply the non-relativistic limit with $q'^2 \rightarrow 0$. This yields

$$\begin{aligned} F_2(q'^2 \rightarrow 0) &= \frac{\alpha}{\pi} \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{z}{1-z} \\ &= \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z} \\ &= \frac{\alpha}{\pi} \int_0^1 dz z \\ &= \frac{\alpha}{2\pi} \approx 0.00116. \end{aligned} \quad (3.49)$$

What we derived is the anomalous magnetic moment of fermions in QED on one-loop level. The g -factor using eq. (2.63) therefore amounts to:

$$g = 2(F_1(0) + F_2(0)) = 2 + \frac{\alpha}{\pi} \approx 2.00232 \quad (3.50)$$

This result was first calculated by Julian Schwinger in 1947 [6].

4 One-loop contribution of arbitrary vector bosons

The SM contribution to the anomalous magnetic moment of fermions on one-loop level was successfully calculated in the previous section. Nevertheless, performing higher order perturbation theory shows a significant deviation between the calculated and measured anomalous magnetic moment of the muon. Thus, we want to extend the SM. The aim is to find another massive vector boson X_μ , that contributes to $(g-2)_\mu$ and settles the anomaly. For performing parameter scans on different BSM extensions, the general contribution to $(g-2)_\mu$ from the new vector boson has to be calculated. The calculation is performed in the Feynman gauge.

First, we exchange the photon loop in section 3 with a general gauge boson loop, whose Lagrangian is given by [9] as

$$\mathcal{L}_{int,X} = \bar{\mu}[C_V\gamma^\mu + C_A\gamma^\mu\gamma^5]FX_\mu + \text{h.c.} \quad (4.1)$$

The hermitian conjugated part is given by

$$\begin{aligned} X_\mu^\dagger F^\dagger [C_V\gamma^\mu + C_A\gamma^\mu\gamma^5]^\dagger \bar{\mu}^\dagger &= X_\mu^* F^\dagger [C_V^*\gamma^{\mu\dagger} + C_A^*\gamma^{5\dagger}\gamma^{\mu\dagger}]\gamma^0\mu \\ &= X_\mu^* F^\dagger [C_V^*\gamma^0\gamma^\mu\gamma^0 + C_A^*\gamma^{5\dagger}\gamma^0\gamma^\mu\gamma^0]\gamma^0\mu \\ &= X_\mu^* \bar{F} [C_V^*\gamma^\mu + C_A^*\gamma^\mu\gamma^5]\mu, \end{aligned} \quad (4.2)$$

which we can plug in into eq. (4.1) to receive the total Lagrangian

$$\mathcal{L}_{int,X} = \bar{\mu}[C_V\gamma^\mu + C_A\gamma^\mu\gamma^5]FX_\mu + X_\mu^* \bar{F} [C_V^*\gamma^\mu + C_A^*\gamma^\mu\gamma^5]\mu. \quad (4.3)$$

The gauge boson propagator is given in [10] in the R_ξ gauges as

$$\Pi_{\mu\nu}^X = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 - \xi M_X^2} (1 - \xi) \right) \cdot \frac{-i}{q^2 - M_X^2}. \quad (4.4)$$

There are different gauges that can be chosen for the introduced gauge parameter ξ . The contribution to the anomalous magnetic moment is gauge independent, however, certain gauges like the unitary gauge ($\xi \rightarrow 0$) handle divergencies poorly, so that they explicitly appear and are not easily canceled. As an advantage of the unitary gauge, there are no unphysical scalar contributions that need to be added. The choice for the presented calculations in this thesis is the Feynman gauge ($\xi \rightarrow 1$). There are less divergencies appearing and the integral evaluation overall is simpler. Nevertheless, we have to consider unphysical scalar contributions from a Goldstone boson G_0 , which is associated with the new massive gauge boson X_μ . Goldstone bosons are a consequence of broken gauge symmetries, which we do not want to discuss.

The propagator for G_0 is given in [10] with

$$\Pi^{G_0} = \frac{i}{q^2 - \xi M_X^2}, \quad (4.5)$$

and the corresponding Lagrangian is

$$\mathcal{L}_{int,G} = \bar{\mu}[c_V + c_A\gamma^5]FG_0 + \text{h.c.} \quad (4.6)$$

The calculation of the hermitian conjugated part is analogous to the calculation in eq. (4.2). We obtain

$$\begin{aligned} G_0^* F^\dagger [c_V^* + c_A^*\gamma^{5\dagger}](\mu^\dagger\gamma^0)^\dagger &= G_0^* F^\dagger [c_V^* + c_A^*\gamma^5]\gamma^0\mu \\ &= G_0^* \bar{F} [c_V^* - c_A^*\gamma^5]\mu, \end{aligned} \quad (4.7)$$

and for the total Lagrangian

$$\mathcal{L}_{int,G} = \bar{\mu}[c_V + c_A\gamma^5]FG_0 + G_0^* \bar{F} [c_V^* - c_A^*\gamma^5]\mu. \quad (4.8)$$

To derive eq. (4.2) and eq. (4.7), one has to consider A1.5, A1.9 and A1.11.

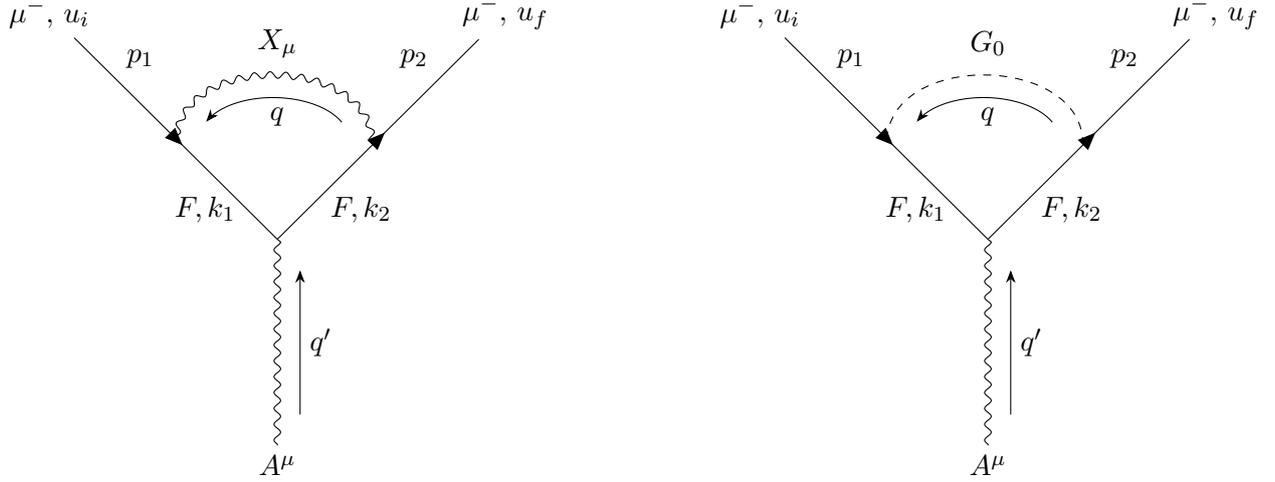
a) Physical X_μ boson contribution.b) Unphysical G_0 scalar contribution.

Figure 5: The combined one-loop diagram for a new arbitrary gauge boson in the Feynman gauge. In this gauge, one also has to take into account the unphysical contribution by a scalar, induced by the Goldstone boson G_0 . Both diagrams must be combined in the end to obtain the actual contribution to F_2 .

Now we apply the Feynman gauge $\xi \rightarrow 1$ on the propagators and find them in the following form:

$$\Pi_{\mu\nu}^X = \frac{-ig_{\mu\nu}}{q^2 - M_X^2 + i\epsilon} \quad (X_\mu)_{\xi \rightarrow 1} \quad (4.9)$$

$$\Pi^{G_0} = \frac{i}{q^2 - M_X^2 + i\epsilon} \quad (G_0)_{\xi \rightarrow 1} \quad (4.10)$$

For the final vertex $\Gamma^\sigma = \gamma^\sigma + \delta\Gamma^\sigma$, both diagrams in fig. 11 must be added together. For the combined contribution we find

$$ieQ_\mu \bar{u}_f(p_2) \delta\Gamma^\sigma u_i(p_1) = ieQ_\mu \bar{u}_f(p_2) [\delta\Gamma_X^\sigma + \delta\Gamma_{G_0}^\sigma] u_i(p_1), \quad (4.11)$$

whereas the separated contributions are given by:

$$\begin{aligned} \bar{u}_f(p_2) \delta\Gamma_X^\sigma u_i(p_1) &= \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[i\gamma^\nu (C_V + C_A \gamma^5) \frac{i(\not{k}_2 + m_F)}{k_2^2 - m_F^2 + i\epsilon} \left(\frac{Q_F}{Q_\mu} \gamma^\sigma \right) \right. \\ &\quad \left. \times \frac{i(\not{k}_1 + m_F)}{k_1^2 - m_F^2 + i\epsilon} i\gamma^\mu (C_V^* + C_A^* \gamma^5) \frac{-ig_{\mu\nu}}{q^2 - M_X^2 + i\epsilon} \right] u_i(p_1) \quad (4.12) \end{aligned}$$

$$\begin{aligned} \bar{u}_f(p_2) \delta\Gamma_{G_0}^\sigma u_i(p_1) &= \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[i(c_V + c_A \gamma^5) \frac{i(\not{k}_2 + m_F)}{k_2^2 - m_F^2 + i\epsilon} \left(\frac{Q_F}{Q_\mu} \gamma^\sigma \right) \right. \\ &\quad \left. \times \frac{i(\not{k}_1 + m_F)}{k_1^2 - m_F^2 + i\epsilon} i(c_V^* - c_A^* \gamma^5) \frac{i}{q^2 - M_X^2 + i\epsilon} \right] u_i(p_1). \quad (4.13) \end{aligned}$$

In both integrals Q_F stands for the charge of the inner fermion and $Q_\mu = -1$ for the charge of the outer muon. Note that the coupling coefficients c_V and c_A from G_0 still have to be brought into a relation to the arbitrary constants C_V and C_A from X_μ . The relation follows from the requirement of ξ gauge independence and is derived in appendix A1.7.

As one can easily see from eq. (4.12) and eq. (4.13), the denominator in both integrals is the same. Therefore, we can already introduce Feynman parameters and shift the integration momentum. Before, we will make use of the same trick as in eq. (3.10) and rewrite the denominator. We set

$$\frac{1}{(k_2^2 - m_F^2 + i\epsilon')(k_1^2 - m_F^2 + i\epsilon')(q^2 - M_X^2 + i\epsilon')} \rightarrow \frac{1}{D^3}, \quad (4.14)$$

with

$$\begin{aligned} D &= x(k_1^2 - m_F^2) + y((k_1 + q')^2 - m_F^2) + z((k_1 - p_1)^2 - M_X^2) + (x + y + z)i\epsilon \\ &= k_1^2 x - m_F^2 x + k_1^2 y + 2k_1 q' y + y q'^2 - m_F^2 y + z k_1^2 - 2k_1 p_1 z + p_1^2 z - z M_X^2 + i\epsilon \\ &= k_1^2 (x + y + z) - m_F^2 (x + y) + 2k_1 (q' y - p_1 z) + y q'^2 + p_1^2 z - z M_X^2 + i\epsilon \\ &= k_1^2 + 2k_1 (q' y - p_1 z) + y q'^2 + p_1^2 z - z M_X^2 + i\epsilon - (x + y) m_F^2 \\ &= \underbrace{(k_1 + q' y - p_1 z)^2}_{l^2} + y q'^2 (1 - y) + z p_1^2 (1 - z) + 2y z q' p_1 - m_F^2 (x + y) - z M_X^2 + i\epsilon \\ &= l^2 + \underbrace{xy q'^2 + z m_\mu^2 (1 - z) - m_F^2 (1 - z) - z M_X^2}_{-\Delta} + i\epsilon \\ &= l^2 + \underbrace{xy q'^2 - m_\mu^2 u^2 + u(m_\mu^2 - m_F^2) - (1 - u) M_X^2}_{-\Delta} + i\epsilon \\ &= l^2 - \Delta + i\epsilon. \end{aligned} \quad (4.15)$$

As many expressions will carry a factor of $(1 - z)$, we also perform a substitution $(1 - z) \rightarrow u$, which simplifies the equations. The integral over all Feynman parameters then changes into a concise and clear integral of the following form:

$$\int_0^1 dx dy dz \delta(x + y + z - 1) f(x + y) \longrightarrow \int_0^1 du u f(u) \quad (4.17)$$

The relation is proven in A1.4.1.

Before starting with the integral evaluation, it is advisable to give the momenta conservation relations first. What follows from both diagrams in fig. 11 are the following relations:

$$p_1 + q = k_1 \quad (4.18)$$

$$p_2 + q = k_2 \quad (4.19)$$

$$k_1 + q' = k_2 \quad (4.20)$$

$$q' + p_1 = p_2. \quad (4.21)$$

The introduced substitution to shift the momenta from k_1 to l follows from eq. (4.15):

$$k_1 = l - y q' + z p_1 \quad (4.22)$$

$$k_2 = l + (1 - y) q' + z p_1. \quad (4.23)$$

In the end, we are interested in the non-relativistic limit $q'^2 \rightarrow 0$ again, which applied to Δ yields

$$\Delta(q'^2 \rightarrow 0) = u^2 m_\mu^2 + M_X^2 (1 - u) + u(m_F^2 - m_\mu^2). \quad (4.24)$$

In the following sections, we will separately calculate the contributions to $(g - 2)_\mu$ from X_μ and G_0 .

4.1 Gauge boson contribution

At first, we will calculate the physical contribution from the new arbitrary X_μ boson. The corresponding integral in eq. (4.12) can be written in the following way:

$$\bar{u}_f(p_2)\delta\Gamma_X^\sigma u_i(p_1) = Q_{Fi} \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[\frac{\gamma^\nu (C_V + C_A \gamma^5)(\not{k}_2 + m_F)\gamma^\sigma(\not{k}_1 + m_F)\gamma^\mu (C_V^* + C_A^* \gamma^5)g_{\mu\nu}}{(k_2^2 - m_F^2 + i\epsilon')(k_1^2 - m_F^2 + i\epsilon')(q^2 - M_X^2 + i\epsilon')} \right] u_f(p_1). \quad (4.25)$$

The numerator can be transformed into the following compact form

$$\gamma^\nu (C_V + C_A \gamma^5)(\not{k}_2 + m_F)\gamma^\sigma(\not{k}_1 + m_F)\gamma^\mu (C_V^* + C_A^* \gamma^5)g_{\mu\nu} = A \cdot g_{\mu\nu}. \quad (4.26)$$

Solving the integral and finding the relevant expression $(p_1 + p_2)^\sigma$ for the Gordon Identity needs a simplification of the numerator. Before multiplying $g_{\mu\nu}$ and changing indices, we expand the bracket expression A and sort all contributions. Expanding and sorting the brackets yields

$$A = |C_V|^2 [\gamma^\nu \not{k}_2 \gamma^\sigma \not{k}_1 \gamma^\mu + m_F (\gamma^\nu \not{k}_2 \gamma^\sigma \gamma^\mu + \gamma^\nu \gamma^\sigma \not{k}_1 \gamma^\mu) + m_F^2 \gamma^\nu \gamma^\sigma \gamma^\mu] \quad (4.27)$$

$$+ |C_A|^2 [\gamma^\nu \not{k}_2 \gamma^\sigma \not{k}_1 \gamma^\mu - m_F (\gamma^\nu \not{k}_2 \gamma^\sigma \gamma^\mu + \gamma^\nu \gamma^\sigma \not{k}_1 \gamma^\mu) + m_F^2 \gamma^\nu \gamma^\sigma \gamma^\mu] \quad (4.28)$$

+ irrelevant.

Relevant contributions are proportional to $|C_{V,A}|^2$, as all other contributions are proportional to γ^5 and do not contribute to F_2 (γ^5 expressions would contribute to F_3 , a higher order form factor). Therefore, the irrelevant part is neglected from now on.

Multiplying the metric tensor $g_{\mu\nu}$ and pulling down the μ and changing it to ν gives

$$\text{num}_X = A \cdot g_{\mu\nu} = |C_V|^2 [\gamma^\nu \not{k}_2 \gamma^\sigma \not{k}_1 \gamma_\nu + m_F (\gamma^\nu \not{k}_2 \gamma^\sigma \gamma_\nu + \gamma^\nu \gamma^\sigma \not{k}_1 \gamma_\nu) + m_F^2 \gamma^\nu \gamma^\sigma \gamma_\nu] \quad (4.29)$$

$$+ |C_A|^2 [\gamma^\nu \not{k}_2 \gamma^\sigma \not{k}_1 \gamma_\nu - m_F (\gamma^\nu \not{k}_2 \gamma^\sigma \gamma_\nu + \gamma^\nu \gamma^\sigma \not{k}_1 \gamma_\nu) + m_F^2 \gamma^\nu \gamma^\sigma \gamma_\nu]. \quad (4.30)$$

One can easily see that the contributions proportional to $|C_V^2|$ and $|C_A^2|$ are almost equal and only differ in the prefactor of m_F . For further simplifications, we will consider a combined expression, covering both coupling constants, that can be rearranged into an already known term

$$\gamma^\nu \not{k}_2 \gamma^\sigma \not{k}_1 \gamma_\nu \pm m_F (\gamma^\nu \not{k}_2 \gamma^\sigma \gamma_\nu + \gamma^\nu \gamma^\sigma \not{k}_1 \gamma_\nu) + m_F^2 \gamma^\nu \gamma^\sigma \gamma_\nu = \gamma^\nu (\not{k}_2 \pm m_F) \gamma^\sigma (\not{k}_1 \pm m_F) \gamma_\nu. \quad (4.31)$$

This expression is already known, and the simplification can be taken from eq. (3.2). Therefore, we find

$$\gamma^\nu (\not{k}_2 \pm m_F) \gamma^\sigma (\not{k}_1 \pm m_F) \gamma_\nu = -2\not{k}_1 \gamma^\sigma \not{k}_2 \pm 4m_F (k_1 + k_2)^\sigma - 2m_F^2 \gamma^\sigma. \quad (4.32)$$

This is the final expression from where on the goal is to identify the relevant part for the F_2 form factor.

The contribution from $\not{k}_1 \gamma^\sigma \not{k}_2$ is already known from section 3:

$$\not{k}_1 \gamma^\sigma \not{k}_2 = (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [-2m_\mu^2 z(z+1)] \quad (4.33)$$

$$= (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [-2m_\mu^2 (u-1)(u-2)]. \quad (4.34)$$

The contribution from $k_1^\sigma + k_2^\sigma$ yields

$$k_1^\sigma + k_2^\sigma = 2k_1^\sigma + q'^\sigma \quad (4.35)$$

$$= 2(l^\sigma - yq'^\sigma + zp_1^\sigma) + q'^\sigma \quad (4.36)$$

$$= (1 - 2y)q'^\sigma + 2zp_1^\sigma = (1 - 2y)q'^\sigma + zp_1^\sigma + z(p_2^\sigma - q'^\sigma) \quad (4.37)$$

$$= (x - y)q'^\sigma + (p_1^\sigma + p_2^\sigma)z \quad (4.38)$$

$$= \frac{i\sigma^{\sigma\nu}q'_\nu}{2m_\mu} [-2m_\mu z] \quad (4.39)$$

$$= \frac{i\sigma^{\sigma\nu}q'_\nu}{2m_\mu} [2m_\mu(u - 1)], \quad (4.40)$$

and therefore the total contribution to F_2 from num_X amounts to

$$\text{num}_X = \frac{i\sigma^{\sigma\nu}q'_\nu}{2m_\mu} |C_{V,A}|^2 [4m_\mu^2(u - 1)(u - 2) \pm 8m_\mu m_F(u - 1)]. \quad (4.41)$$

Note that the given contribution actually is made up of two parts, proportional to $|C_V|^2$ and $|C_A|^2$, with a change in the prefactor of m_F . In eq. (4.41), the $+$ is associated with $|C_V|^2$ contribution and the $-$ with $|C_A|^2$.

Plugging in the contributions into eq. (4.25) and performing the momenta shift with introducing the integral over the Feynman parameters accordingly to eq. (4.17) gives

$$F_{2,X}(q'^2 \rightarrow 0) = Q_F i |C_{V,A}|^2 \int_0^1 du \cdot 2u \int_{\mathbb{R}^4} \frac{d^4 l}{(2\pi)^4} \frac{4m_\mu^2(u - 1)(u - 2) \pm 8m_\mu m_F(u - 1)}{[l^2 - \Delta + i\epsilon]^3}. \quad (4.42)$$

Note that the factor 2 in front of u comes from the fraction $\frac{2}{D^3}$ and is a result of the Feynman parameter introduction. The momenta integral needed is already known:

$$\int_{\mathbb{R}^4} \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta + i\epsilon]^3} = \frac{-i}{32\pi^2} \frac{1}{\Delta}. \quad (4.43)$$

Therefore, we obtain

$$F_{2,X}(q'^2 \rightarrow 0) = \frac{Q_F}{32\pi^2} |C_{V,A}|^2 \int_0^1 du \cdot 2u \frac{4m_\mu^2(u - 1)(u - 2) \pm 8m_\mu m_F(u - 1)}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \quad (4.44)$$

$$= \frac{-Q_F m_\mu^2}{4\pi^2} |C_{V,A}|^2 \int_0^1 du \cdot u \frac{(1 - u)(u - 2) \pm \frac{2m_F}{m_\mu}(1 - u)}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \quad (4.45)$$

$$= \frac{-Q_F m_\mu^2}{4\pi^2} |C_{V,A}|^2 \int_0^1 du \frac{(u - u^2)(u - 2) \pm \frac{2m_F}{m_\mu}(u - u^2)}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \quad (4.46)$$

$$= \frac{-Q_F m_\mu^2}{4\pi^2} |C_{V,A}|^2 \int_0^1 du \frac{(u - u^2) \left(u + \frac{\pm 2m_F}{m_\mu} - 2 \right)}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)}. \quad (4.47)$$

Writing out the separated contributions from both coupling constants yields for the physical contribution to $(g - 2)_\mu$ from X_μ :

$$F_{2,X}(0) = \frac{-Q_F m_\mu^2}{4\pi^2} \int_0^1 du \frac{|C_V|^2(u - u^2) \left(u + \frac{2m_F}{m_\mu} - 2 \right) + |C_A|^2(u - u^2) \left(u - \frac{2m_F}{m_\mu} - 2 \right)}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \quad (4.48)$$

In the next step, we will perform the same calculation for the Goldstone boson G_0 .

4.2 Goldstone boson contribution

The integral related to the contribution from the Goldstone boson G_0 is given in eq. (4.13). As before, we will first simplify the numerator. The integral is given with

$$\bar{u}_f(p_2)\delta\Gamma_{G_0}^\sigma u_i(p_1) = -Q_F i \int_{\mathbb{R}^4} \frac{d^4 k_1}{(2\pi)^4} \bar{u}_f(p_2) \left[\frac{(c_V + c_A \gamma^5)(\not{k}_2 + m_F)\gamma^\sigma(\not{k}_1 + m_F)(c_V^* - c_A^* \gamma^5)}{(k_2^2 - m_F^2 + i\epsilon')(k_1^2 - m_F^2 + i\epsilon')(q^2 - M_X^2 + i\epsilon')} \right] u_f(p_1). \quad (4.49)$$

Expanding the numerator yields

$$\begin{aligned} \text{num}_{G_0} &= (c_V \not{k}_2 + c_V m_F + c_A \gamma^5 \not{k}_2 + c_A m_F \gamma^5) \gamma^\sigma (c_V^* \not{k}_1 - c_A^* \not{k}_1 \gamma^5 + m_F c_V^* - c_A^* m_F \gamma^5) \\ &= |c_V|^2 [\not{k}_2 \gamma^\sigma \not{k}_1 + m_F (\not{k}_2 \gamma^\sigma + \gamma^\sigma \not{k}_1) + m_F^2 \gamma^\sigma] \\ &\quad + |c_A|^2 [\not{k}_2 \gamma^\sigma \not{k}_1 - m_F (\not{k}_2 \gamma^\sigma + \gamma^\sigma \not{k}_1) + m_F^2 \gamma^\sigma] \\ &\quad + \text{irrelevant}. \end{aligned} \quad (4.50)$$

The contributions from $|c_V|^2$ and $|c_A|^2$ again only differ in the prefactor of m_F . To identify the contribution to F_2 , both expressions $\not{k}_2 \gamma^\sigma \not{k}_1$ and $\not{k}_2 \gamma^\sigma + \gamma^\sigma \not{k}_1$ must be expanded.

$$\not{k}_2 \gamma^\sigma \not{k}_1 = (p_2 + q) \gamma^\sigma (p_1 + q) \quad (4.51)$$

$$\begin{aligned} &= m_\mu^2 \gamma^\sigma + m_\mu (q \gamma^\sigma + \gamma^\sigma q) + q \gamma^\sigma q \\ &= (\dots) \gamma^\sigma + 2m_\mu q^\sigma + 2q^\sigma q \\ &= (\dots) \gamma^\sigma + 2m_\mu (l^\sigma - y q'^\sigma + (z-1)p_1^\sigma) + 2(l^\sigma - y q'^\sigma + (z-1)p_1^\sigma)(l - y q' + (z-1)p_1) \\ &= (\dots) \gamma^\sigma + 2m_\mu (y q'^\sigma + (z-1)p_1^\sigma) + 2(l^\sigma l - y(z-1)m_\mu q'^\sigma + m_\mu (z-1)^2 p_1^\sigma) \\ &= (\dots) \gamma^\sigma - 2m_\mu q'^\sigma (y + y(z-1)) + 2m_\mu z(z-1)p_1^\sigma \\ &= (\dots) \gamma^\sigma - 2yzm_\mu q'^\sigma + 2m_\mu z(z-1)p_1^\sigma \\ &= (\dots) \gamma^\sigma - 2yzm_\mu q'^\sigma + m_\mu z(z-1)p_1^\sigma + m_\mu z(z-1)(p_2^\sigma - q'^\sigma) \\ &= (\dots) \gamma^\sigma + (p_1^\sigma + p_2^\sigma)[m_\mu z(z-1)] + q'^\sigma [m_\mu z(2y - x - y)] \\ &= (\dots) \gamma^\sigma + (p_1^\sigma + p_2^\sigma)[m_\mu z(z-1)] - m_\mu q'^\sigma z(y - x) \\ &= (\dots) \gamma^\sigma + (p_1^\sigma + p_2^\sigma)[m_\mu z(z-1)] \\ &= (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [2m_\mu^2 z(1-z)] \end{aligned} \quad (4.52)$$

$$= (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [2m_\mu^2 u(1-u)] \quad (4.53)$$

All expressions that do not contribute to F_2 are collected inside the term $(\dots)\gamma^\sigma$.

$$\begin{aligned}
\gamma^\sigma k_1 + k_2 \gamma^\sigma &= \gamma^\sigma (l - y q' + z p_1) + (l + (1 - y) q' + u p_1) \\
&= \gamma^\sigma (-y(p_2 - m_\mu) + z m_\mu) + ((1 - y)(m_\mu - p_1) + z p_1) \gamma^\sigma \\
&= \gamma^\sigma (y + z) m_\mu - y \gamma^\sigma p_2 + (1 - y) m_\mu \gamma^\sigma - x p_1 \gamma^\sigma \\
&= (\dots) \gamma^\sigma - 2y p_2^\sigma - 2x p_1^\sigma \\
&= (\dots) \gamma^\sigma - y p_2^\sigma - y(q'^\sigma + p_1^\sigma) - x p_1^\sigma - x(p_2^\sigma - q'^\sigma) \\
&= (\dots) \gamma^\sigma - (x + y)(p_1^\sigma + p_2^\sigma) + q'^\sigma (x - y) \\
&= (\dots) \gamma^\sigma + (p_1^\sigma + p_2^\sigma)(z - 1) \\
&= (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [2m_\mu(1 - z)] \tag{4.54}
\end{aligned}$$

$$= (\dots) \gamma^\sigma + \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} [2m_\mu u] \tag{4.55}$$

With both contributions expanded and the contribution to F_2 extracted, we can go back to eq. (4.50) and plug in the results from eq. (4.53) and eq. (4.55). The relevant contribution amounts to

$$\text{num}_{G_0} = \frac{i\sigma^{\sigma\nu} q'_\nu}{2m_\mu} |c_{V,A}|^2 [2m_\mu^2 u(1 - u) \pm 2m_\mu m_F u]. \tag{4.56}$$

Plugging in num_{G_0} into eq. (4.49) and again introducing the Feynman parameter integral yields

$$F_{2,G_0}(q'^2 \rightarrow 0) = -Q_F i |c_{V,A}|^2 \int_0^1 du \cdot 2u \int_{\mathbb{R}^4} \frac{d^4 l}{(2\pi)^4} \frac{2m_\mu^2 u(1 - u) \pm 2m_\mu m_F u}{[l^2 - \Delta + i\epsilon]^3}. \tag{4.57}$$

The momenta integral is known from eq. (4.43). Plugging in the result and simplifying the expression results in

$$F_{2,G_0}(q'^2 \rightarrow 0) = \frac{-Q_F}{16\pi^2} |c_{V,A}|^2 \int_0^1 du \cdot u \frac{2m_\mu^2 u(1 - u) \pm 2m_\mu m_F u}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \tag{4.58}$$

$$= \frac{-Q_F}{8\pi^2} |c_{V,A}|^2 \int_0^1 du \frac{u^2(m_\mu^2 \pm m_\mu m_F) - u^3 m_\mu^2}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \tag{4.59}$$

$$= \frac{Q_F m_\mu^2}{4\pi^2} \frac{1}{2} |c_{V,A}|^2 \int_0^1 du \frac{u^3 - u^2(1 \pm \frac{m_F}{m_\mu})}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)}. \tag{4.60}$$

Separating the contributions from both coupling constants again yields for the contribution to $(g - 2)_\mu$ from G_0 :

$$F_{2,G_0}(0) = \frac{Q_F m_\mu^2}{4\pi^2} \frac{1}{2} \int_0^1 du \frac{|c_V|^2 [u^3 - u^2(1 + \frac{m_F}{m_\mu})] + |c_A|^2 [u^3 - u^2(1 - \frac{m_F}{m_\mu})]}{u^2 m_\mu^2 + M_X^2(1 - u) + u(m_F^2 - m_\mu^2)} \tag{4.61}$$

Before we can combine both contributions from X_μ and G_0 , we must identify the relation between the coupling constants c_V, c_A and C_V, C_A .

4.3 Combined final contribution to $(g-2)_\mu$

The relation between the coupling constants of G_0 and X_μ is calculated in A1.7 and yields

$$|c_V|^2 = \frac{(m_\mu - m_F)^2}{M_X^2} |C_V|^2 \quad |c_A|^2 = \frac{(m_\mu + m_F)^2}{M_X^2} |C_A|^2. \quad (4.62)$$

As our result shall be dependent to the coupling constants C_V and C_A , we employ eq. (4.62) and plug in the relations into eq. (4.61). We obtain

$$F_{2,G_0}(0) = \frac{Q_F m_\mu^2}{4\pi^2} \frac{1}{2M_X^2} \int_0^1 du \frac{|C_V|^2 (m_\mu - m_F)^2 [u^3 - u^2(1 + \frac{m_F}{m_\mu})] + |C_A|^2 (m_\mu + m_F)^2 [u^3 - u^2(1 - \frac{m_F}{m_\mu})]}{u^2 m_\mu^2 + M_X^2(1-u) + u(m_F^2 - m_\mu^2)}. \quad (4.63)$$

The numerator can be rearranged into a different form by simply expanding the brackets and simplifying afterwards:

$$(m_\mu \mp m_F)^2 (u^3 - u^2(1 \pm \frac{m_F}{m_\mu})) = u^3 (m_\mu \mp m_F)^2 - u^2 (m_\mu \mp m_F)^2 (1 \pm \frac{m_F}{m_\mu}) \quad (4.64)$$

$$= u^3 (m_\mu \mp m_F)^2 - u^2 (m_\mu^2 \mp 2m_\mu m_F + m_F^2) (1 \pm \frac{m_F}{m_\mu}) \quad (4.65)$$

$$= u^3 (m_\mu \mp m_F)^2 - u^2 \left[m_\mu^2 \pm m_\mu m_F \mp 2m_\mu m_F - 2m_F^2 + m_F^2 \pm \frac{m_F^3}{m_\mu} \right] \quad (4.66)$$

$$= u^3 (m_\mu \mp m_F)^2 - u^2 \left[m_\mu^2 \mp m_\mu m_F - m_F^2 \pm \frac{m_F^3}{m_\mu} \right] \quad (4.67)$$

$$= u^3 (m_\mu \mp m_F)^2 + u^2 (m_F^2 - m_\mu^2) \left(1 \mp \frac{m_F}{m_\mu} \right). \quad (4.68)$$

Plugging back in the expression yields

$$F_{2,G_0}(0) = \frac{Q_F m_\mu^2}{4\pi^2} \frac{1}{2M_X^2} \int_0^1 du \left[\frac{|C_V|^2 \left[u^3 (m_\mu - m_F)^2 + u^2 (m_F^2 - m_\mu^2) \left(1 - \frac{m_F}{m_\mu} \right) \right]}{u^2 m_\mu^2 + M_X^2(1-u) + u(m_F^2 - m_\mu^2)} + \frac{|C_A|^2 \left[u^3 (m_\mu + m_F)^2 + u^2 (m_F^2 - m_\mu^2) \left(1 + \frac{m_F}{m_\mu} \right) \right]}{u^2 m_\mu^2 + M_X^2(1-u) + u(m_F^2 - m_\mu^2)} \right]. \quad (4.69)$$

Combining this result with eq. (4.48) results in the final contribution to $(g-2)_\mu$ from the new gauge boson X_μ in the non-relativistic limit:

$$F_2(0) = \frac{-Q_F m_\mu^2}{4\pi^2} \int_0^1 du \left[|C_V|^2 \left\{ (u - u^2) \left(u + \frac{2m_F}{m_\mu} - 2 \right) - \frac{1}{2M_X^2} \left(u^3 (m_\mu - m_F)^2 + u^2 (m_F^2 - m_\mu^2) \left(1 - \frac{m_F}{m_\mu} \right) \right) \right\} + |C_A|^2 \{m_F \rightarrow -m_F\} \right] \times \{u^2 m_\mu^2 + M_X^2(1-u) + u(m_F^2 - m_\mu^2)\}^{-1} \quad (4.70)$$

The result is confirmed by [9, eq.(3)]. In the given reference, the contribution is calculated in the unitary gauge $\xi \rightarrow 0$. All calculations here are performed in the Feynman gauge $\xi \rightarrow 1$ and therefore gauge invariance of the contribution to $(g-2)_\mu$ is confirmed as well.

5 Flavor conserving and violating BSM extensions

In the previous section, we calculated the contribution to the anomalous magnetic moment of the muon from an arbitrary gauge boson X_μ . We will use it to approach the prevailing discrepancy between the experimentally measured value a_μ^{exp} and the SM prediction.

The goal is to perform parameter scans on different models on a new introduced gauge boson, that could settle the deviation. In general, there are two approaches to construct a model. One is a flavor conserving model and one is a flavor violating one.

5.1 $L_\mu - L_\tau$ extension

In the SM, there are three families of leptons: the electron (e), the muon (μ), and the tau (τ), each with its own lepton number L_e, L_μ, L_τ . These lepton numbers are conserved separately in SM interactions, meaning that electrons, muons and taus are counted independently.

The $L_\mu - L_\tau$ model is a minimal extension of the SM that introduces a new $U(1)$ gauge symmetry based on the difference between the muon and tau lepton number. The associated gauge boson is a new massive boson called Z' . It exclusively couples to taus, muons and their associated neutrinos with an arbitrary coupling strength of g' . The relevant interaction Lagrangian of this model for the Z' boson coupling to μ and τ is given by

$$\mathcal{L}_{\text{int}} \supset \bar{\mu}[C_V\gamma^\mu + C_A\gamma^\mu\gamma^5]FX_\mu \quad \longrightarrow \quad \mathcal{L}_{\text{int}} \supset g'\bar{\mu}\gamma^\nu\mu Z'_\nu - g'\bar{\tau}\gamma^\nu\tau Z'_\nu. \quad (5.1)$$

In our case, we are only interested in interactions with muons. Therefore, we simply neglect the τ part and obtain

$$\mathcal{L}_{\text{int}}^a \supset g'\bar{\mu}\gamma^\nu\mu Z'_\nu. \quad (5.2)$$

We renamed the general gauge boson from X to Z' so that it is in accordance with reference articles. The coupling constant g' and the mass of the gauge boson are arbitrary parameters. From $\mathcal{L}_{\text{int}}^a$ we can conclude $m_F = m_\mu$ and $C_V = g'$. As the $U(1)_{L_\mu - L_\tau}$ group is abelian, the charge of right-handed and left-handed muons is the same (+1). Therefore, there is no axial coupling and we set $C_A = 0$. Plugging in the model assumptions into eq. (4.70), gives for the contribution to the anomalous magnetic moment

$$a_\mu^{L_\mu - L_\tau} = \frac{g'^2}{4\pi} \frac{1}{2\pi} \int_0^1 dx \frac{2m_\mu^2 x^2 (1-x)}{m_\mu^2 x^2 + m_{Z'}^2 (1-x)}, \quad (5.3)$$

which is confirmed by eq.(14) in [11]. The contribution can now be scanned with different pairs of g' and $m_{Z'}$. The parameter scan is shown in fig. 6 and includes multiple excluded regions. There are several experiments that constrain parameter combinations of g' and $m_{Z'}$. In fig. 6, the red area excludes all gauge bosons with a mass $m_{Z'}$ less than approximately 10 MeV. ΔN_{eff} is the difference between the experimentally measured neutrino-like degrees of freedom and it's SM prediction. If the mass of Z' is very small, it would stay relativistic for a very long time during the radiation-dominated era of the universe and would contribute significantly to the radiation density. For a small mass $m_{Z'}$, we would therefore measure a different cosmic microwave background than we actually do.

The grey area in fig. 6 is excluded by *CCFR* (Chicago Columbia Fermilab Rochester) data [12], where neutrino trident events were counted. In the Coulomb field of an atomic nucleus, a ν_μ neutrino can interact with the nucleus via exchanging a boson that can produce a $\mu^+\mu^-$ pair

$$\nu_\mu + N \rightarrow \nu_\mu + \mu^+ + \mu^- + N. \quad (5.4)$$

The leading order process of neutrino trident processes from eq. (5.4) is depicted in fig. 7.

For certain combinations of $m_{Z'}$ and the coupling constant g' , the amount of counted events would be higher than measured. Furthermore, the measured data is in accordance with SM predictions. As one can see in fig. 6, the *CCFR* data excludes almost all parameter combinations for the scanned area that would explain the $(g-2)_\mu$ anomaly.

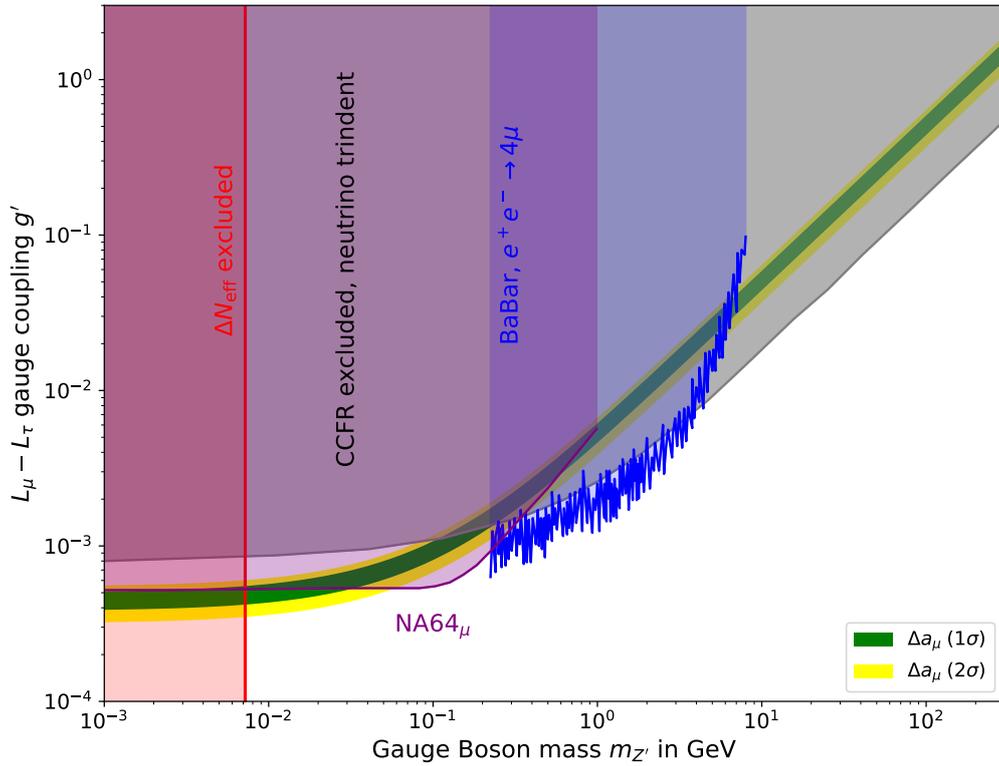


Figure 6: Parameter scan for the flavor conservative $U(1)_{L_\mu-L_\tau}$ model. Possible parameter combinations of g' and $m_{Z'}$ that would settle the $(g-2)_\mu$ anomaly are marked in green and yellow (up to 2σ -interval). The red area is excluded due to the measured neutrino-like degrees of freedom [11]. The gray area shows combinations that are excluded by the *CCFR* experiment [12] and in blue, one finds an excluded area by the *BaBar* experiment [13]. In addition, measurements at the *CERN* give constraints from the *NA64* $-\mu$ experiment [14] in purple. Short explanations for all experiments are given in the text.

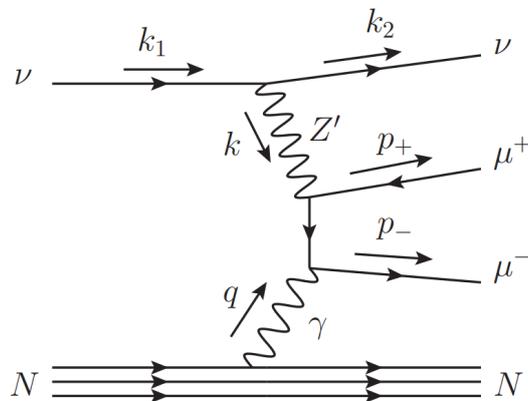


Figure 7: Leading order neutrino trident process, taken from [12].

As an additional constraint the *BaBar* experiment [13] measured 4-muon events from colliding e^- and e^+

$$e^+e^- \rightarrow \mu^- \mu^+ Z' \rightarrow \mu^- \mu^+ \mu^- \mu^+ \quad \text{with} \quad Z' \rightarrow \mu^- \mu^+, \quad (5.5)$$

and identified the invariant mass spectrum of the produced muons and anti-muons. One characteristic of the spectrum was that there were no peaks observed that would predict a new particle. Moreover, the measured data are in accordance with the SM. The comparison between theoretically predicted mass spectrum and the actually measured one, gives constraints on the cross section σ for the process in eq. (5.5). As $\sigma \propto g'^2(m_{Z'})$, we can derive constraints on the parameters for Z' . The blue area in fig. 6 shows the constraints that were derived with a 90 % confidence level.

The NA64 – μ experiment [14] uses a beam with around $\sim 2 \cdot 10^{10}$ muons (from the MP2 beam at *CERN*) and an active target's nucleus N . The muon beam is made up of muons with 160 GeV that can interact with the target's nucleus via a bremsstrahlung-like reaction. During this reaction, a light Z' boson can be produced via

$$\mu + N \rightarrow \mu + Z' + N \quad (5.6)$$

and therefore carries away a bit of the energy of the incoming muon. Thus, the muon must be of less energy after the reaction. The boson decays invisibly or propagates downstream. The scattered muon is detected and its momenta and energy is measured. The amount of muons being scattered and the invariant mass spectrum is predicted with Monte-Carlo simulations that give the constraints for a hypothetical boson Z' , after comparison with the actual measured spectrum.

Even though the $L_\mu - L_\tau$ model would be able to explain the $(g-2)_\mu$ anomaly and would also fit to problems regarding dark matter, most of its parameter space in fig. 6 is constrained. Only a small area around $m_{Z'} \approx 10$ MeV and $g' \approx 0.0005$ is left.

5.2 Dark Photon extension

The second model introduces a dark sector with an additional photon A' (Z') (also-called dark photon), that is coupling to dark sector states with a charge in the $U(1)_D$ extension, but can also interact with the SM. The interaction causes a mixing term in the Lagrangian with the mixing parameter ϵ . The model introduces two dark sector states that own fermion properties. A' couples to both these states and to the dark sector current. Taking a look at the general Lagrangian, one can identify A_μ as the SM photon, \mathcal{J}_{EM} the electromagnetic current in the SM and \mathcal{J}_D the dark sector current. The Lagrangian is given by [15] and the relevant part can be rearranged to

$$\mathcal{L}_D \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\epsilon}{2}F_{\mu\nu}F'^{\mu\nu} + eA_\mu\mathcal{J}_{\text{EM}}^\mu + g_D A'_\mu\mathcal{J}_D^\mu, \quad (5.7)$$

where $F_{\mu\nu}F'^{\mu\nu}$ describes the kinetic mixing term and therefore is non-diagonal. To resolve this issue, we transform the photon field A_μ

$$A_\mu \longrightarrow A_\mu + \epsilon A'_\mu, \quad (5.8)$$

and cancel out the mixing term in the kinetic part in the Lagrangian. Introducing the transformation yields

$$F_{\mu\nu} = \partial_\mu A_\nu - A_\nu \partial_\mu \longrightarrow \partial_\mu (A_\nu + \epsilon A'_\nu) - (A_\mu + \epsilon A'_\mu) \partial_\nu, \quad (5.9)$$

and therefore we find

$$\partial_\mu (A_\nu + \epsilon A'_\nu) - (A_\mu + \epsilon A'_\mu) \partial_\nu = \partial_\mu A_\nu - A_\mu \partial_\nu + \epsilon(\partial_\mu A'_\nu - A'_\mu \partial_\nu) \quad (5.10)$$

$$= F_{\mu\nu} + \epsilon F'_{\mu\nu} \quad (5.11)$$

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} + \epsilon F'_{\mu\nu}. \quad (5.12)$$

Plugging in the transformed magnetic strength tensor into the kinetic part of eq. (5.7) yields

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(F_{\mu\nu} + \epsilon F'_{\mu\nu})(F^{\mu\nu} + \epsilon F'^{\mu\nu}) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\epsilon}{2}(F_{\mu\nu} + \epsilon F'_{\mu\nu})F'^{\mu\nu} \quad (5.13)$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\epsilon}{4}(F_{\mu\nu}F'^{\mu\nu} + F'_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\epsilon}{2}F_{\mu\nu}F'^{\mu\nu} + \mathcal{O}(\epsilon^2) \quad (5.14)$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\epsilon}{4}(2F_{\mu\nu}F'^{\mu\nu}) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\epsilon}{2}F_{\mu\nu}F'^{\mu\nu} + \mathcal{O}(\epsilon^2) \quad (5.15)$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \mathcal{O}(\epsilon^2) \quad (5.16)$$

$$\approx -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}. \quad (5.17)$$

As ϵ is a very small coupling parameter ($\sim 10^{-4}$), we neglect all the $\mathcal{O}(\epsilon^2)$ terms. Introducing the field transformation for A_μ therefore yields for the Lagrangian in eq. (5.7):

$$L'_D \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + e(A_\mu + \epsilon A'_\mu)\mathcal{J}_{\text{EM}}^\mu + g_D A'_\mu \mathcal{J}_D^\mu \quad (5.18)$$

$$\supset \mathcal{L}_{\text{kin}} + eA_\mu \mathcal{J}_{\text{EM}}^\mu + A'_\mu (\epsilon e \mathcal{J}_{\text{EM}}^\mu + g_D \mathcal{J}_D^\mu). \quad (5.19)$$

With the transformation, the interaction between leptons and the new introduced dark sector boson A'_μ is given by

$$\mathcal{L}_{\text{int}}^b \supset \epsilon e A'_\mu \mathcal{J}_{\text{EM}}^\mu = \epsilon e \bar{F} Q_F \gamma^\mu F A'_\mu \stackrel{F \rightarrow \mu}{=} -\epsilon e \bar{\mu} \gamma^\mu \mu A'_\mu \quad (5.20)$$

Comparing $\mathcal{L}_{\text{int}}^b$ with eq. (4.1) yields

$$C_V = -\epsilon e \quad C_A = 0 \quad m_F = m_\mu. \quad (5.21)$$

The contribution to the anomalous magnetic moment of the muon from the introduced A'_μ boson therefore is

$$a_\mu^D = \frac{\epsilon^2 e^2}{4\pi} \frac{1}{2\pi} \int_0^1 dx \frac{2m_\mu^2 x^2 (1-x)}{m_\mu^2 x^2 + m_{Z'}(1-x)} = \frac{\alpha \epsilon^2}{2\pi} \int_0^1 dx \frac{2m_\mu^2 x^2 (1-x)}{m_\mu^2 x^2 + m_{Z'}(1-x)}, \quad (5.22)$$

with the fine structure constant α . In eq. (5.22) we set $m_{Z'} = m_{A'}$. All constraints depicted in the parameter scan in fig. 8 are independent from dark sector state properties. The upper region for $\epsilon \gtrsim 0.03$ is excluded by LEP data [19]. The constraint is based on well known SM observables, regarding neutral current interactions like *Bhabha* scattering, and $U(1)_D$ model calculations with the introduced Z' boson. The confidence level for the given constraint is 95 %.

Also data from *BaBar* [18] restrict a relatively large area in the parameter space. The examined decay for the constraints in green in fig. 8 is

$$e^+ + e^- \rightarrow \gamma + A' \quad A' \rightarrow e^+ e^-, \mu^+ \mu^-. \quad (5.23)$$

The way of receiving constraints is the same as for the *BaBar* experiment [13] regarding the $L_\mu - L_\tau$ model. Again, there were no peaks found in the invariant mass spectrum, that would predict a new particle. The measurements fit well to the SM predictions and result in constraints with a 90% confidence level.

The *KLOE* experiments [20][21][22] include 3 decays in total and deriving the parameter constraints works equally to deriving the *BaBar* constraints. *KLOE* shoots electrons onto a fixed target and is searching for certain decays around the target.

The first decay [22] is

$$\phi \rightarrow A' + \eta \quad A' \rightarrow e^+ + e^-, \quad (5.24)$$

where a ϕ meson decay is followed by the decay of A' into $e^+ e^-$.

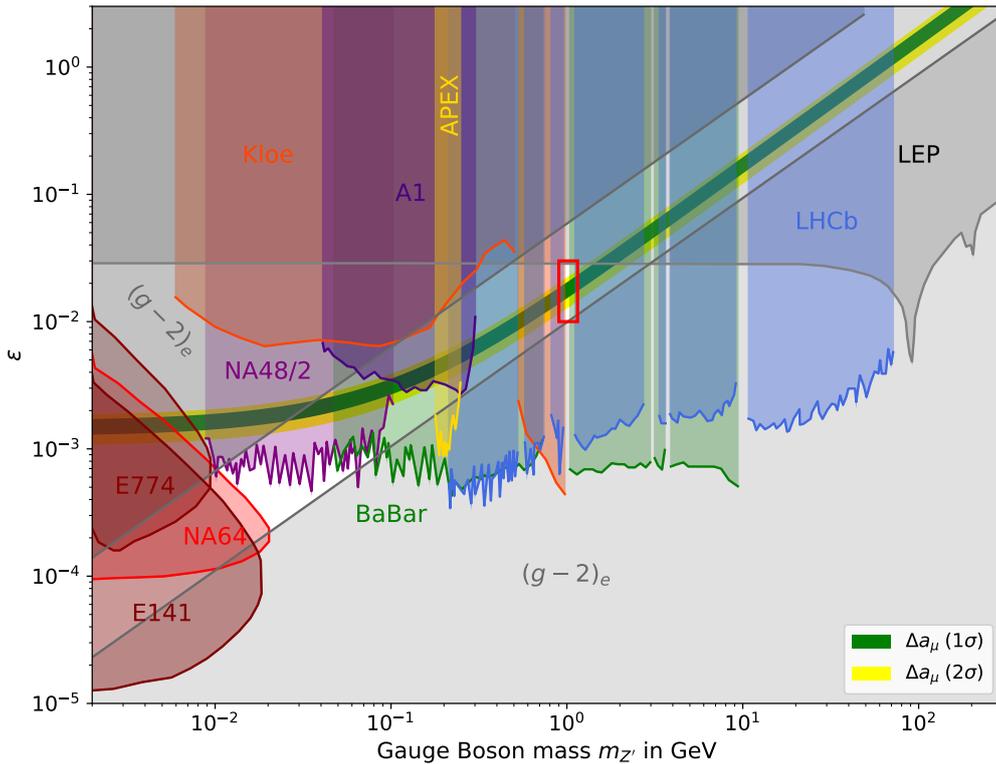


Figure 8: Parameter scan for the flavor conservative $U(1)_D$ model. Possible parameter combinations of ϵ and $m_{Z'}$ that would settle the $(g-2)_\mu$ anomaly are marked in green and yellow (up to 2σ -interval). There are multiple constraints given from different experiments: NA48/2 [16], NA64 [17], BaBar [18], LEP [19], KLOE [20][21][22], E774 [23], E141 [24], A1 [25], APEX [26] and LHCb [27]. Short descriptions for all performed experiments are given in the text. The red box shows the last possible parameter region, that could settle the $(g-2)_\mu$ anomaly.

The second decay [21]

$$e^+ + e^- \rightarrow A' + \gamma \quad A' \rightarrow \pi^+ + \pi^-, \quad (5.25)$$

results in two pions, that can be measured around a cylindrical volume around the electron beam.

The third decay [20] is a prompt decay

$$A' \rightarrow e^+ + e^-, \quad (5.26)$$

that was measured inside a cylindrical decay volume.

Again, from the measured invariant mass spectra, one is able to derive upper limits on the cross section σ for any given process for the new dark photon A' .

The NA48/2 experiment [16] was performed in 2003-2004 at CERN and the decay of pions π^0 was examined. The relevant decay is

$$\pi^0 \rightarrow \gamma + A' \quad A' \rightarrow e^+ + e^-. \quad (5.27)$$

The recorded invariant mass spectrum didn't show any peaks which would predict a new particle A' , and is in harmony with the SM. Constraints again are derived from the constraints on the cross section σ , associated with the decay in eq. (5.27).

The NA64 experiment [17] at CERN provides additional constraints for relatively light A' with $m_{A'} \lesssim 23$ MeV. Excited ${}^8\text{Be}^*$ (excited beryllium nucleus) decays show an excess of e^+e^- pairs, that can't be explained by the SM. Therefore, there is a hint for a new particle that decays into e^+e^- . A possible decay that could lead to the excess is

$$e^- + N \rightarrow e^- + N + A'(X) \quad A'(X) \rightarrow e^+ + e^-. \quad (5.28)$$

X is another boson that the experiment was searching for with a mass of 16.7 MeV. With an electron beam of around 5.4×10^{10} electrons shot on target, there was no significant peak found in the invariant mass spectrum.

The *E774* [23] and *E141* [24] experiments were performed in 1989 and 1991 and one tried to find new particles with a 275 GeV electron beam that was shot on a tungsten beam dump. Behind the target, neutral particles like A' can decay into an e^+e^- pair and can be measured through scintillators behind the dump. The experiment didn't show an excess of e^+e^- pairs. The relevant scattering and decay is shown in eq. (5.29). Thus, the results are in harmony with the SM, leading to constraints for a new particle.

$$e^- + N \rightarrow A' \quad A' \rightarrow e^- + e^+ \quad (5.29)$$

The idea of examining e^+e^- pair production in $e^- + N$ scattering, see eq. (5.29) again, is also the fundament of the experiments *A1* [25] and *APEX* [26]. In both experiments, there was no evidence found for a new gauge boson.

A lot of constraints also were derived from the *LHCb* [27] experiment in 2018. 13 TeV pp collisions were used to cause the following decay:

$$A' \rightarrow \mu^+ + \mu^-. \quad (5.30)$$

The experiment was designed to search for prompt A' decays and also long lived A' , but there was no evidence found for new particles.

The last constraint in dark grey in fig. 8 comes from the anomalous magnetic moment of the electron. As for the muon, there is a small deviation between the experimentally measured value [28] and the theoretically predicted value [29] of $(g-2)_e$. It amounts to

$$a_e(\text{RB}) = a_e^{\text{exp}} - a_e^{\text{SM}} = (34 \pm 16) \times 10^{-14}. \quad (5.31)$$

RB stands for the element *rubidium*, which was used to measure the fine structure constant α .

Technically, the same calculations as in previous sections can be performed for the electron. A parameter scan for the electron in the given model leads to strong constraints beyond the 2σ -interval of a_e .

Apparently, what we can derive from fig. 8 is that the dark photon extension will probably stay a theoretical model without real analogue. Its parameter space is largely constrained by multiple experiment data. Nevertheless, the few remaining parameter pairs can be identified very precisely in a region of $m_{Z'} \approx 1$ GeV and $\epsilon \approx 0.015$. With these information, experiments could be adjusted to search for new particles in the mentioned parameter regions with high precision and sensitivity.

The shape of the graphs in fig. 6 and fig. 8 that represent the 1σ and 2σ deviation from Δa_μ can be explained considering two scenarios. First, assuming $m_{Z'} \ll m_\mu \approx 0.1$ GeV, then the integrands in eq. (5.3) and eq. (5.22) would converge against the expression $2(1-x)$, which would cause the integral to become constant. As a consequence, the couplings g' and ϵ must stay constant as well, resulting in horizontally extending bands for $m_{Z'} \rightarrow 0$.

For the case $m_{Z'} \gg m_\mu$, the denominators become very large and the integral value relatively small. As a_μ is a constant, the coupling constants g' and ϵ must grow to compensate the decreasing integral value, causing the two σ bands to also extend vertically.

5.3 Lepton flavor violating Z' extension

As one can see from fig. 6 and fig. 8, the allowed parameters for a hypothetical Z' are well constrained by multiple experiments. Flavor conserving models appear to be strongly restricted. Therefore, we want to examine a flavor violating model. In fact, some experiments also hint, that lepton flavor violation (LFV) is dominant in certain decays related to muons and taus.

We will work along a LFV model that introduces a leptophilic Z' boson. As an important property, we assume this new boson to only have non-diagonal couplings between taus and muons. Moreover, the coupling can be chirally asymmetric. After taking a look at the Lagrangian, we will explore two different combinations on the left- and right-handed coupling constants. The model is described by the following Lagrangian (taken from [30])

$$\mathcal{L} \supset g'_L(\bar{\mu}\gamma^\alpha P_L\tau + \bar{\nu}_\mu\gamma^\alpha P_L\nu_\tau)Z'_\alpha + g'_R(\bar{\mu}\gamma^\alpha P_R\tau)Z'_\alpha, \quad (5.32)$$

where we neglect the neutrino part, as we are only interested in τ and μ interactions. Thus, we have

$$\mathcal{L}_{\text{int}} \supset \bar{\mu}[C_V\gamma^\mu + C_A\gamma^\mu\gamma^5]FX_\mu \quad \longrightarrow \quad \mathcal{L}_{\text{int}}^f \supset g'_L(\bar{\mu}\gamma^\alpha P_L\tau)Z'_\alpha + g'_R(\bar{\mu}\gamma^\alpha P_R\tau)Z'_\alpha, \quad (5.33)$$

where P_L and P_R are left- and right-handed projection operators. They are defined with

$$P_L = \frac{1 - \gamma^5}{2} \quad P_R = \frac{1 + \gamma^5}{2}. \quad (5.34)$$

A muon interacting with the introduced Z' boson would result in a tau. Therefore, $m_F = m_\tau$ and a comparison between eq. (4.1) and eq. (5.33) yields

$$C_V = \frac{g'_L + g'_R}{2} \quad C_A = \frac{g'_R - g'_L}{2}. \quad (5.35)$$

Plugging in the assumptions into eq. (4.70) and rewriting the $\frac{1}{M_X^2}$ term leads to

$$-\frac{1}{2M_X^2}(\dots) = -\frac{x^2}{2M_X^2} \left(x(m_F - m_\mu)^2 + (m_F^2 - m_\mu^2) \left(1 - \frac{m_F}{m_\mu} \right) \right) \quad (5.36)$$

$$= -\frac{x^2}{2M_X^2} \left(x(m_F - m_\mu)^2 + (m_F + m_\mu)(m_F - m_\mu) \left(1 - \frac{m_F}{m_\mu} \right) \right) \quad (5.37)$$

$$= -\frac{x^2(m_F - m_\mu)^2}{2M_X^2} \left(x + \frac{(m_F + m_\mu)(m_F - m_\mu)}{(m_F - m_\mu)^2} \left(1 - \frac{m_F}{m_\mu} \right) \right) \quad (5.38)$$

$$= -\frac{x^2(m_F - m_\mu)^2}{2M_X^2} \left(x + \frac{(m_F + m_\mu)}{(m_F - m_\mu)} \left(\frac{m_\mu - m_F}{m_\mu} \right) \right) \quad (5.39)$$

$$= -\frac{x^2(m_F - m_\mu)^2}{2M_X^2} \left(x - \frac{m_F}{m_\mu} - 1 \right), \quad (5.40)$$

$$(5.41)$$

and for the final contribution

$$a_\mu = \frac{m_\mu^2}{4\pi^2} \frac{|g'_L + g'_R|^2}{4} \int_0^1 dx \left((x - x^2) \left(x + \frac{2m_\tau}{m_\mu} - 2 \right) - \frac{x^2(m_\tau - m_\mu)^2}{2m_{Z'}^2} \left(x - \frac{m_\tau}{m_\mu} - 1 \right) \right) \quad (5.42)$$

$$+ \frac{m_\mu^2}{4\pi^2} \frac{|g'_R - g'_L|^2}{4} \int_0^1 dx \left((x - x^2) \left(x - \frac{2m_\tau}{m_\mu} - 2 \right) - \frac{x^2(m_\tau + m_\mu)^2}{2m_{Z'}^2} \left(x + \frac{m_\tau}{m_\mu} - 1 \right) \right). \quad (5.43)$$

We are interested in two specific scenarios. The first one assumes $g'_L = g'_R$ (model f1), while the second one assumes $g'_L = \frac{g'_R}{10}$ (model f2). Plugging in these relations results in

$$a_\mu^{f1} = \frac{m_\mu^2 g_R'^2}{4\pi^2} \int_0^1 dx (x - x^2) \left(x + \frac{2m_\tau}{m_\mu} - 2 \right) - \frac{x^2 (m_\tau - m_\mu)^2}{2m_{Z'}^2} \left(x - \frac{m_\tau}{m_\mu} - 1 \right) \quad (5.44)$$

$$a_\mu^{f2} = \frac{m_\mu^2 g_R'^2}{4\pi^2} \frac{121}{400} \int_0^1 dx (x - x^2) \left(x + \frac{2m_\tau}{m_\mu} - 2 \right) - \frac{x^2 (m_\tau - m_\mu)^2}{2m_{Z'}^2} \left(x - \frac{m_\tau}{m_\mu} - 1 \right) \quad (5.45)$$

$$+ \frac{m_\mu^2 g_R'^2}{4\pi^2} \frac{81}{400} \int_0^1 dx (x - x^2) \left(x - \frac{2m_\tau}{m_\mu} - 2 \right) - \frac{x^2 (m_\tau + m_\mu)^2}{2m_{Z'}^2} \left(x + \frac{m_\tau}{m_\mu} - 1 \right). \quad (5.46)$$

What is evident at first glance in fig. 9 is that a symmetric coupling cannot explain the $(g-2)_\mu$ anomaly, as the whole suitable parameter space that would settle the anomaly is constrained by τ decays. Therefore, the right-handed chirally asymmetric coupling must be preferred. The parameter scan for this case is depicted in fig. 10.

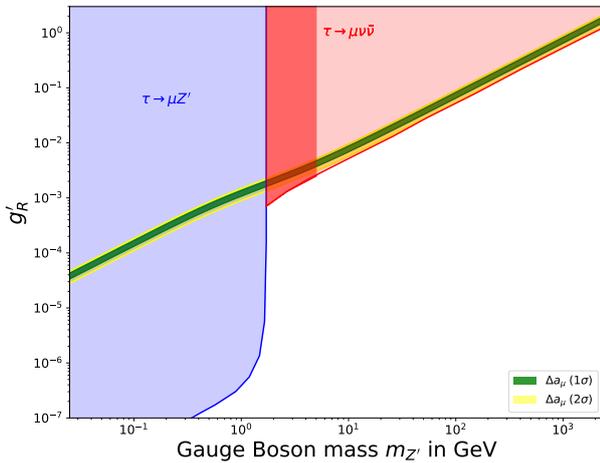


Figure 9: Chirally symmetric coupling with $g'_R = g'_L$. Both constraints are derived from tau decays and comparisons with SM predictions.

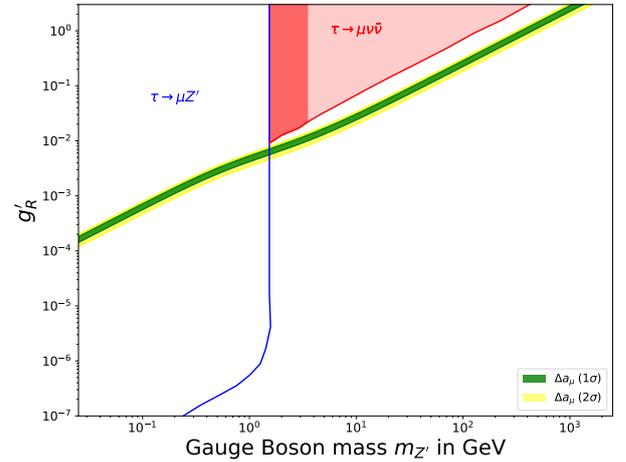


Figure 10: Chirally asymmetric coupling with $\frac{g'_R}{10} = g'_L$. Both constraints are derived from tau decays and comparisons with SM predictions.

The shift from the red constrain in fig. 10 is a result of the asymmetric coupling, that can be derived from the first part of the Lagrangian in eq. (5.32). In this part the left-handed coupling allows τ to decay into $\mu Z'$ and Z' into two neutrinos. Neutrino interactions are only involved in left-handed interactions and as we assume $g_R = 10 \cdot g_L$, we must shift the red constraint up by a factor of 10.

For the flavor violating model, there are two main sorts of tau decays, mediated by a tree level exchange of a W boson in the SM, that set constraints on the parameters g'_R and $m_{Z'}$. In fig. 9 and fig. 10, the constraint in blue is derived from the decay

$$\tau \rightarrow \mu + Z', \quad (5.47)$$

which is the dominant constraint for $m_{Z'} < m_\tau - m_\mu$. For this case of a relatively light Z' , the decay

$$\tau \rightarrow \mu^- + Z' \quad Z' \rightarrow \nu + \bar{\nu} \quad (5.48)$$

kinematically is possible and sets strong constraints on the parameters for $m_{Z'} < m_\tau - m_\mu$.

The produced Z' can only decay into neutrinos in this case and thus missing energy events exclude a large area of possible parameter pairs. An experiment that derived the given constraint is the *ARGUS* experiment [31], that examined $\tau \rightarrow \mu^- + \phi$ decays, with ϕ being an arbitrary unobservable particle.

In red we find the dominant constraint for $m_{Z'} \gtrsim 2$ GeV to be derived from the two decays

$$\tau \rightarrow \mu + \nu_\tau + \bar{\nu}_\mu \quad \text{and} \quad \tau \rightarrow \mu + \nu_\mu + \bar{\nu}_\tau. \quad (5.49)$$

Both decays in eq. (5.49) are described by the short form $\tau \rightarrow \mu\nu\bar{\nu}$. The constraints are derived by a comparison of SM predictions for the branching ratio [30]

$$R_{\mu e}^{\text{SM}} = \frac{\text{BR}(\tau \rightarrow \mu\nu_\tau\bar{\nu}_\mu)_{\text{SM}}}{\text{BR}(\tau \rightarrow e\nu_\tau\bar{\nu}_e)_{\text{SM}}} = 0.972559 \pm 0.000005, \quad (5.50)$$

and experimental data (provided by another experiment at BaBar [32])

$$R_{\mu e}^{\text{PDG}} = 0.979 \pm 0.004. \quad (5.51)$$

The deviation between both ratios is

$$\frac{R_{\mu e}^{\text{PDG}}}{R_{\mu e}^{\text{SM}}} - 1 = 0.0066 \pm 0.0041, \quad (5.52)$$

and sets the constraints for the parameter space. A new Z' boson would contribute to the decays in eq. (5.49). It would increase the ratio $R_{\mu e}$ in eq. (5.50) and therefore, the deviation from experimental data is the basis for constraints. The correction to $R_{\mu e}^{\text{SM}}$ from Z' can be calculated with neglecting the muon mass m_μ for $m_{Z'} \gg m_\tau$, but has to be taken into account for $m_{Z'} \sim m_\tau$. In fact, for $m_{Z'} \sim m_\tau$ there is a significant change in the muon energy spectrum for $\tau \rightarrow \mu\nu_\tau\bar{\nu}_\mu$ due to softer muons than in the SM. A possible impact of the spectrum change on the decays is not taken into account. As the change is significant for $m_{Z'} \sim m_\tau$, there are dark red areas in fig. 9 and fig. 10, to imply possible deviations on constraints in the marked area.

The lepton flavor violating Z' model could be a possible model, that can explain the $(g-2)_\mu$ anomaly. However, one has to assume a chirally asymmetric coupling $g'_L = \frac{g'_R}{10}$, as the symmetric coupling is completely excluded by τ decay experiments. Nevertheless, with this kind of coupling, there is a relatively large area for possible parameter pairs of $m_{Z'}$ and g'_R that could hint a new boson, as one can obtain from fig. 10.

6 Conclusion

In this thesis, the anomalous magnetic moment of the muon was examined, beginning from the fundamental theoretical description by the Dirac equation, up to possible extensions of the SM. Starting with the Dirac equation, it was shown that the Schrödinger Pauli equation yields the gyromagnetic factor $g = 2$, a result that reflects the connection between spin and magnetic interactions in relativistic quantum mechanics. Furthermore, $g = 2$ was derived from the tree-level interaction. Moving to quantum electrodynamics, the one-loop correction to a_l was derived, reproducing Schwinger's result of $\frac{\alpha}{2\pi}$ which represents the leading contribution to the anomalous magnetic moment.

The analysis was extended employing BSM physics and introducing a new massive gauge boson X_μ , whose couplings C_V and C_A give additional one-loop contributions to a_μ . The calculation was performed in the Feynman gauge $\xi \rightarrow 1$ under consideration of the Goldstone boson G_0 that is associated to X_μ . The calculation was successfully performed, reproducing the result from [9], a known paper featuring analytical results to a_μ in different scenarios.

The analytical contribution to a_μ from X_μ allowed the exploration of specific beyond SM scenarios, including the $L_\mu - L_\tau$ extension, the Dark Photon model and Lepton-flavor violating Z' interactions. Each of these extensions provides viable parameter regions, where the observed deviation between experiment and SM prediction can potentially be explained. Regarding the Dark Photon model, one can see at first glance in fig. 8 that the parameter space is strongly constrained by multiple experiments. However, the suitable parameters that are left can be identified quite precisely, opening up the opportunity to perform high-sensitivity measurements in the given parameter regions.

The same can be concluded for the $L_\mu - L_\tau$ extension, as the parameter space also is excluded to a relatively small area. It is advisable to perform adjusted experiments for both models. As they both still remain with suitable parameter pairs, they can still be proven correct.

The flavor violating Z' model with chirally asymmetric couplings owns the largest parameter space of all models discussed in this thesis, that could explain the prevailing deviation.

It remains to be seen whether any of the extensions discussed will prove to be correct. So far, there are no experimental data that would confirm any of the scenarios. The SM cannot explain everything we measure yet, which is a clear indicator for physics beyond the SM. Further measurements are required, and the search for possible indications must continue.

The discussion on the muon $g - 2$ remains an open challenge, particularly regarding whether hadronic contributions should be derived from e^+e^- -collider data or lattice QCD calculations. This dialogue will need to continue as both approaches evolve and improve.

7 Appendix

A1.1 Properties of γ matrices

In this first section (based on [7]), the multiple properties of the γ matrices will be introduced to simplify calculations. To do so, we will use the Pauli Dirac representation $\gamma^\mu = (\beta, \vec{\alpha})$. At first, it is advisable to take a look at the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (\text{A1.1})$$

where $g^{\mu\nu}$ is the metric tensor of Minkowski Space. γ matrices obey the Clifford-Algebra. Using A1.1 allows rearrangements of the following expressions:

$$\begin{aligned} \gamma^\mu \gamma^\nu \gamma_\mu &= (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) \gamma_\mu \\ &= 2\gamma^\nu - \gamma^\nu \gamma^\mu \gamma_\mu \\ &= 2\gamma^\nu - \gamma^\nu \cdot 4 \\ &= -2\gamma^\nu \end{aligned} \quad (\text{A1.2})$$

$$\begin{aligned} \gamma^a \gamma^b \gamma^c &= (2g^{ab} - \gamma^b \gamma^a) \gamma^c \\ &= 2g^{ab} \gamma^c - \gamma^b \gamma^a \gamma^c \\ &= 2g^{ab} \gamma^c - \gamma^b (2g^{ac} - \gamma^c \gamma^a) \\ &= 2g^{ab} \gamma^c - 2g^{ac} \gamma^b + \gamma^b \gamma^c \gamma^a \\ &= 2g^{ab} \gamma^c - 2g^{ac} \gamma^b + (2g^{bc} - \gamma^c \gamma^b) \gamma^a \\ &= 2(g^{ab} \gamma^c - g^{ac} \gamma^b + g^{bc} \gamma^a) - \gamma^c \gamma^b \gamma^a. \end{aligned} \quad (\text{A1.3})$$

As γ^0 is hermitian and $\gamma^0 = \beta$, one can show the following relations:

$$\gamma^{0\dagger} = \gamma^0 \quad (\text{A1.4})$$

$$(\gamma^0)^2 = \mathbf{1}_4 \quad (\text{A1.5})$$

$$\gamma^{k\dagger} = (\beta \vec{\alpha}^k)^\dagger = \vec{\alpha}^{k\dagger} \beta = \vec{\alpha}^k \beta = -\beta \vec{\alpha}^k = -\gamma^k \quad (\text{A1.6})$$

$$\gamma^{k\dagger} = (\beta \vec{\alpha}^k)^\dagger = \vec{\alpha}^{k\dagger} \beta = \vec{\alpha}^k \beta = \beta^2 \vec{\alpha}^k \beta = \gamma^0 \gamma^k \gamma^0. \quad (\text{A1.7})$$

Introducing the new gauge boson X_μ requires the definition of the γ^5 matrix

$$\gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4, \quad (\text{A1.8})$$

that satisfies:

$$\gamma^{5\dagger} = \gamma^5 \quad (\text{A1.9})$$

$$(\gamma^5)^2 = \mathbf{1}_4 \quad (\text{A1.10})$$

$$\{\gamma^5, \gamma^\mu\} = 0. \quad (\text{A1.11})$$

A1.2 The Gordon identity

The goal of this section is to find another expression for $\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})$, which is split up into the electric and magnetic interaction of \bar{u} and u with a photon γ^μ . Therefore, we will need eq. (2.23). We will start by writing down both Dirac equations for $\bar{u}(\vec{p}')$ and $u(\vec{p})$

$$(\gamma^\mu p_\mu - m)u(\vec{p}) = 0 \quad (\text{A1.12})$$

$$\bar{u}(\vec{p}')(\gamma^\mu p_\mu - m) = 0. \quad (\text{A1.13})$$

In the next step, we multiply eq. (A1.13) from the right with $\gamma^\nu u(\vec{p})$ and eq. (A1.12) from the left with $\bar{u}(\vec{p}')\gamma^\nu$. The following two equations then are

$$\bar{u}(\vec{p}')\gamma^\nu(mu(\vec{p})) = \bar{u}(\vec{p}')\gamma^\nu\gamma^\mu p_\mu u(\vec{p})$$

$$(m\bar{u}(\vec{p}'))\gamma^\nu u(\vec{p}) = \bar{u}(\vec{p}')\gamma^\mu\gamma^\nu p'_\mu u(\vec{p}),$$

where the left side of both equations is equal. Therefore, we can add them together and divide by $\frac{1}{2m}$. This results in

$$\bar{u}(\vec{p}')\gamma^\nu u(\vec{p}) = \frac{1}{2m}\bar{u}(\vec{p}')(\gamma^\nu\gamma^\mu p_\mu + p'_\mu\gamma^\mu\gamma^\nu)u(\vec{p}). \quad (\text{A1.14})$$

To further evaluate eq. (A1.14), we define the antisymmetric tensor $\sigma^{\mu\nu}$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu], \quad (\text{A1.15})$$

which allows us to rewrite $\gamma^\mu\gamma^\nu$ and $\gamma^\nu\gamma^\mu$ with help of eq. (A1.1) in the following way

$$\gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu} \quad \gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}. \quad (\text{A1.16})$$

Plugging in eq. (A1.16) into eq. (A1.14) results in the Gordon identity:

$$\bar{u}(\vec{p}')\gamma^\nu u(\vec{p}) = \bar{u}(\vec{p}')\left(\frac{(p+p')^\nu}{2m} + \frac{i\sigma^{\nu\mu}(p'-p)_\mu}{2m}\right)u(\vec{p}). \quad (\text{A1.17})$$

A1.3 Properties of σ_{ij}

Let us take a look at the antisymmetric tensor σ_{ij} , after changing from greek letters μ and ν , which run from 0 to 3, to latin letters i and j that run from 1 to 3. The definition of $\sigma_{\mu\nu}$ was already introduced in eq. (A1.15) and we write the γ matrices in the Pauli-Dirac representation.

$$\begin{aligned} \sigma_{ij} &= \frac{i}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i) \\ &= \frac{i}{2}(\beta\alpha_i\beta\alpha_j - \beta\alpha_j\beta\alpha_i) \\ &= \frac{i}{2}(-\alpha_i\beta\beta\alpha_j + \alpha_j\beta\beta\alpha_i) \end{aligned} \quad (\text{A1.18})$$

$$= \frac{i}{2}(\alpha_j\alpha_i - \alpha_i\alpha_j) \quad (\text{A1.19})$$

In eq. (A1.18) we made use of the anticommutator relation in eq. (A1.1) for β and α_k . After that, we simply replaced $\beta\beta$ with $\mathbb{1}$. The two multiplications of matrices α_i and α_j result in multiplications between two spin matrices

$$\alpha_j\alpha_i - \alpha_i\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} = (\sigma_j\sigma_i - \sigma_i\sigma_j)\mathbb{1}_2. \quad (\text{A1.20})$$

$\mathbb{1}_2$ is the two dimensional identity and σ_j are the 3 Pauli spin matrices.

To further evaluate eq. (A1.20), we will make use of the algebraic identity

$$\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \epsilon_{ijk} \sigma_k, \quad (\text{A1.21})$$

which results in an expression for eq. (A1.20) of the form

$$\alpha_j \alpha_i - \alpha_i \alpha_j = 2i \epsilon_{ijk} \sigma_k \mathbb{1}_2. \quad (\text{A1.22})$$

Plugging in eq. (A1.22) into eq. (A1.19) gives the final expression for σ_{ij}

$$\sigma_{ij} = -\epsilon_{ijk} \sigma_k \mathbb{1}_2. \quad (\text{A1.23})$$

In the introduction, we have to show that a coupling between the magnetic field \vec{B} and the spin matrices $\vec{\sigma}$ is hidden inside the two sums in eq. (2.36). To reveal the coupling, we can make use of A1.21:

$$\begin{aligned} \sum_{i=1}^3 \sum_{j \neq i} \sigma_i \sigma_j (p^i A^j + A^i p^j) &= i \sum_{i=1}^3 \sum_{j \neq i} \sum_{k=1}^3 \epsilon_{ijk} \sigma_k (p^i A^j + A^i p^j) \\ &\xrightarrow{\text{P.R.}} i \sum_{i=1}^3 \sum_{j \neq i} \sum_{k=1}^3 \epsilon_{ijk} \sigma_k (p^i (A^j) + A^j p^i + A^i p^j) \\ &\xrightarrow{\epsilon_{ijk} \text{ antisymmetric}} i \sum_{i=1}^3 \sum_{j \neq i} \sum_{k=1}^3 \epsilon_{ijk} \sigma_k p^i A^j \\ &\xrightarrow{p^i \rightarrow -i \frac{\partial}{\partial x^i}} = \epsilon_{ijk} \sigma_k \frac{\partial}{\partial x^i} A^j \\ &= \sigma_k \epsilon_{kij} \frac{\partial}{\partial x^i} A^j \\ &= \sigma_k \cdot (\vec{\nabla} \cdot \vec{A})_k = \vec{\sigma} \cdot \vec{B} \end{aligned} \quad (\text{A1.24})$$

A1.4 Feynman parameters

In this short section, we want to prove the identity that allows evaluating one-loop integrals in a relatively simple way, with exchanging the denominator. The method introduces Feynman parameters, which are subsequently integrated with respect to, yielding the original denominator. We only prove the case $n = 3$.

$$\begin{aligned} \int_0^1 dx dy dz \frac{2 \cdot \delta(x+y+z-1)}{[xA+yB+zC]^3} &= 2 \int_0^1 dx dy \frac{1}{[x(A-C)+y(B-C)+C]^3} \Big|_{z=1-x-y} \\ &= 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[x(A-C)+y(B-C)+C]^3} \Big|_{\text{set boundaries}} \\ &= \frac{1}{C-B} \int_0^1 dx \left[\frac{1}{[x(A-B)+B]^2} - \frac{1}{[x(A-C)+C]^2} \right] \\ &= \frac{1}{C-B} \left[\frac{1}{B-A} \left(\frac{1}{A} - \frac{1}{B} \right) + \frac{1}{A-C} \left(\frac{1}{A} - \frac{1}{C} \right) \right] \\ &= \frac{1}{C-B} \left(\frac{1}{AB} - \frac{1}{AC} \right) \\ &= \frac{1}{ABC} \end{aligned} \quad (\text{A1.25})$$

A1.4.1 $(1 - z) \rightarrow u$ integral substitution

This integral shift and its proof are retraced from [10].

$$\int_0^1 dx dy dz \delta(x + y + z - 1) f(x + y) = \int_0^1 dx \int_0^{1-x} dy f(x + y) \quad (\text{A1.26})$$

Introducing substitutions $u = x + y$ and $v = x - y$ yields

$$\int_0^1 dx \int_0^{1-x} dy f(x + y) \longrightarrow \int_0^1 du \int_{-u}^u dv f(u) |J|, \quad (\text{A1.27})$$

where $|J|$ is the Jacobian determinant. It is given by

$$J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -\frac{1}{2}. \quad (\text{A1.28})$$

Plugging in the determinant and integrating over v gives the final integral

$$\int_0^1 du \int_{-u}^u dv f(u) |J| = \int_0^1 du 2u f(u) \frac{1}{2} = \int_0^1 du u f(u). \quad (\text{A1.29})$$

A1.5 Generic integrals

This chapter is based on [33, pp. 26–28].

The integrals appearing in section 3 and section 4 are of the general form

$$I_n(\Delta) = \int d^D l \frac{1}{(l^2 - \Delta + i\epsilon)^n}, \quad (\text{A1.30})$$

whose value can be calculated and brought into the form of a compact equation. The first steps are already done in section 3, where we performed a *Wick Rotation* to transform the integral into the following form:

$$I_n(\Delta) = (-1)^n i \int d^D l_E \frac{1}{(l_E^2 + \Delta - i\epsilon)^n}. \quad (\text{A1.31})$$

The integral owns Euclidean geometry and is spherically symmetric. Therefore, one can use spherical coordinates in D dimensions to evaluate the integral. At first, we must calculate the solid angle of a sphere in D dimensions. To define the solid angle, we will use

$$\begin{aligned} (\sqrt{\pi})^D &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^D \\ &= \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_D e^{-\sum_{i=1}^D x_i^2}. \end{aligned} \quad (\text{A1.32})$$

In eq. (A1.32) we can use spherical symmetry to exchange the integral with an integral over all angles and the radius. This yields

$$(\sqrt{\pi})^D = \int d\Omega_{D-1} \int_0^{\infty} dx^{D-1} e^{-x^2}. \quad (\text{A1.33})$$

With the substitution $u = x^2$, the integral turns into

$$(\sqrt{\pi})^D = \int d\Omega_{D-1} \cdot \frac{1}{2} \int_0^{\infty} dx^2 (x^2)^{D/2-1} e^{-x^2} = \frac{1}{2} \Omega_{D-1} \cdot \Gamma(D/2), \quad (\text{A1.34})$$

which can be used to define the solid angle with

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \quad (\text{A1.35})$$

In the next step, the shift to spherical coordinates is performed. We can exchange

$$\int dl_E \longrightarrow \int d\Omega_D \int_0^\infty dl_E^2 \frac{1}{2} (l_E^2)^{D/2-1}, \quad (\text{A1.36})$$

and therefore find for the integral in eq. (A1.31)

$$I_n(\Delta) = \frac{(-1)^n i}{2} \int d\Omega_D \int_0^\infty dl_E^2 \frac{l_E^2}{(l_E^2 + \Delta - i\epsilon)^n} = \frac{(-1)^n i}{2} \Omega_D \int_0^\infty dl_E^2 \frac{l_E^2}{(l_E^2 + \Delta - i\epsilon)^n}. \quad (\text{A1.37})$$

The solid angle in D dimensions is given by eq. (A1.35) and can be replaced with the calculated expression. Furthermore, we can exchange l_E^2 with x , to turn the integral into a one dimensional integral

$$I_n(\Delta) = (-1)^n i \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dx x^{D/2-1} \frac{x}{(x + \Delta - i\epsilon)^n}. \quad (\text{A1.38})$$

To evaluate the integral, we apply the following substitution:

$$y = \frac{\Delta - i\epsilon}{x + \Delta - i\epsilon}, \quad dy = -\frac{\Delta - i\epsilon}{(x + \Delta - i\epsilon)^2} dx \quad (\text{A1.39})$$

Plugging in the substitution gives

$$I_n(\Delta) = (-1)^n i \frac{\pi^{D/2}}{\Gamma(D/2)} (\Delta - i\epsilon)^{D/2-n} \int_0^1 dx dy (1-y)^{D/2-1} y^{n-D/2-1} \quad (\text{A1.40})$$

$$I_n(\Delta) = (-1)^n i \frac{\pi^{D/2}}{\Gamma(D/2)} (\Delta - i\epsilon)^{D/2-n} \frac{\Gamma(D/2)\Gamma(n-D/2)}{\Gamma(n)}. \quad (\text{A1.41})$$

The final integral for the generic integral therefore is

$$I_n(\Delta) = (-1)^n i \pi^{D/2} (\Delta - i\epsilon)^{D/2-n} \frac{\Gamma(n-D/2)}{\Gamma(n)}. \quad (\text{A1.42})$$

A1.6 Feynman rules

Evaluating the discussed Feynman diagrams in this thesis was performed with the following Feynman rules. For vertex factors we applied:

Fermion–Vector boson vertex		$=i\gamma^\mu(C_V + C_A\gamma^5)$
Fermion–Goldstone scalar vertex		$=i(c_V + c_A\gamma^5)$
Fermion–Photon vertex		$=ieQ_f\gamma^\mu$

The following propagators in the R_ξ (Feynman gauge $\xi \rightarrow 1$) are necessary for one-loop contributions in this thesis:

Photon propagator		$\frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$
Vector boson propagator		$\frac{-ig_{\mu\nu}}{q^2 - M^2 + i\epsilon}$
Goldstone scalar propagator		$\frac{i}{q^2 - M^2 + i\epsilon}$
Fermion propagator		$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

For ingoing and outgoing fermions, the associated spinors are:

ingoing fermion $\longrightarrow \bullet$ $u(p)$

outgoing fermion $\bullet \longrightarrow$ $\bar{u}(p)$

A1.7 Coupling constants c_V, c_A and C_V, C_A relations

The arbitrary massive gauge boson X_μ has an associated Goldstone boson G_0 , whose coupling to fermions is not known yet. Nevertheless, the choice of c_V and c_A is not arbitrary, as any process must satisfy ξ gauge independence. Therefore, there is a relation between the given coupling constants that has to be derived. A relatively easy way to show the relation is by calculating the scattering amplitude in the t -channel between an incoming muon μ and fermion f and two outgoing particles of the same composition. All fermions and muons are on shell particles, that satisfy the Dirac equation. This section is based on [10, chapter 2.2.5].

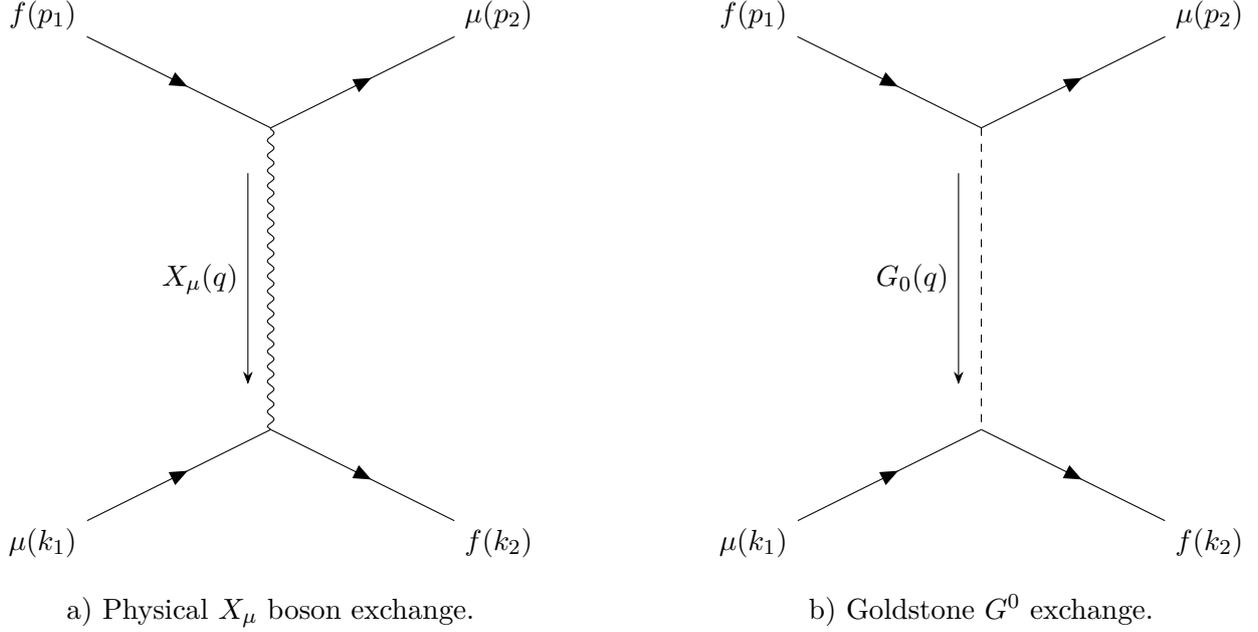


Figure 11: t -channel scattering of a muon and a fermion with exchange of (a) the physical gauge boson X_μ and (b) the associated Goldstone boson G^0 .

Using Feynman rules, the total scattering amplitude in the R_ξ gauge is

$$\begin{aligned}
 i\mathcal{M} &= [\bar{\mu}(p_2)i\gamma^\nu(C_V^* + C_A^*\gamma^5)f(p_1)][\bar{f}(k_2)i\gamma^\mu(C_V + C_A\gamma^5)\mu(k_1)] \\
 &\quad \times \left[\frac{-i}{q^2 - M_X^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_X^2} \right) - \frac{i}{q^2 - \xi M_X^2} \left(\frac{q_\mu q_\nu}{M_X^2} \right) \right] \\
 &\quad + [\bar{\mu}(p_2)i(c_V^* - c_A^*\gamma^5)f(p_1)][\bar{f}(k_2)i(c_V + c_A\gamma^5)\mu(k_1)] \left[\frac{i}{q^2 - \xi M_X^2} \right]. \quad (\text{A1.43})
 \end{aligned}$$

We rearranged the X_μ propagator from eq. (4.4) with partial fraction decomposition

$$\frac{1 - \xi}{(q^2 - \xi M_X^2)(q^2 - M_X^2)} = \frac{1}{M_X^2} \left(\frac{1}{q^2 - M_X^2} - \frac{1}{q^2 - \xi M_X^2} \right). \quad (\text{A1.44})$$

With this trick, we can extract the part that is dependent to the gauge parameter ξ . For gauge independence, it must vanish. From this requirement, we can figure out the relation between the coupling constants.

Comparing the contributions in eq. (A1.43), one finds the ξ dependent part to amount to

$$\begin{aligned} & \frac{i}{q^2 - \xi M_X^2} \left[[\bar{\mu}(p_2) i (c_V^* + c_A^* \gamma^5) f(p_1)] [\bar{f}(k_2) i (c_V + c_A \gamma^5) \mu(k_1)] \right. \\ & \quad \left. - [\bar{\mu}(p_2) i \frac{\not{q}}{M_X} (C_V^* + C_A^* \gamma^5) f(p_1)] [\bar{f}(k_2) i \frac{\not{q}}{M_X} (C_V + C_A \gamma^5) \mu(k_1)] \right]. \end{aligned} \quad (\text{A1.45})$$

For its canceling, the following conditions are required:

$$\bar{f}(k_2) i (c_V + c_A \gamma^5) \mu(k_1) = \bar{f}(k_2) i \frac{\not{q}}{M_X} (C_V + C_A \gamma^5) \mu(k_1) \quad (\text{A1.46})$$

$$\bar{\mu}(p_2) i (c_V^* - c_A^* \gamma^5) f(p_1) = \bar{\mu}(p_2) i \frac{\not{q}}{M_X} (C_V^* + C_A^* \gamma^5) f(p_1) \quad (\text{A1.47})$$

Both conditions are equivalent. We will take a closer look at eq. (A1.46) first. Note that $q = k_1 - k_2$ and that the muons and fermions are on-shell particles. With this property, we can apply the Dirac equation on the outer spinors.

$$\bar{f}(k_2) (c_V + c_A \gamma^5) \mu(k_1) = \bar{f}(k_2) \frac{\not{k}_1 - \not{k}_2}{M_X} C_V \mu(k_1) + \bar{f}(k_2) \frac{\not{k}_1 - \not{k}_2}{M_X} C_A \gamma^5 \mu(k_1) \quad (\text{A1.48})$$

$$= \bar{f}(k_2) \frac{m_\mu - m_F}{M_X} C_V \mu(k_1) - \bar{f}(k_2) \frac{m_\mu + m_F}{M_X} C_A \mu(k_1) \quad (\text{A1.49})$$

$$= \bar{f}(k_2) \left(\frac{m_\mu - m_F}{M_X} C_V - \frac{m_\mu + m_F}{M_X} C_A \gamma^5 \right) \mu(k_1) \quad (\text{A1.50})$$

In eq. (A1.49) it is important to switch $\not{k}_1 \gamma^5 = -\gamma^5 \not{k}_1$ before applying the Dirac equation. From comparing the coefficients in eq. (A1.50), the relation between the coupling constants is evident:

$$c_V = \frac{m_\mu - m_F}{M_X} C_V \quad c_A = -\frac{m_\mu + m_F}{M_X} C_A \quad (\text{A1.51})$$

A similar result can be obtained through eq. (A1.47) with $q = p_1 - p_2$:

$$\bar{\mu}(p_2) (c_V^* - c_A^* \gamma^5) f(p_1) = \bar{\mu}(p_2) \frac{\not{p}_1 - \not{p}_2}{M_X} C_V^* f(p_1) + \bar{\mu}(p_2) \frac{\not{p}_1 - \not{p}_2}{M_X} C_A^* \gamma^5 f(p_1) \quad (\text{A1.52})$$

$$= \bar{\mu}(p_2) \frac{m_F - m_\mu}{M_X} C_V^* f(p_1) - \bar{\mu}(p_2) \frac{m_F + m_\mu}{M_X} C_A^* \gamma^5 f(p_1) \quad (\text{A1.53})$$

$$= \bar{\mu}(p_2) \left(\frac{m_F - m_\mu}{M_X} C_V^* - \frac{m_F + m_\mu}{M_X} C_A^* \gamma^5 \right) f(p_1) \quad (\text{A1.54})$$

Comparing again and taking the complex conjugate yields

$$c_V = -\frac{m_\mu - m_F}{M_X} C_V \quad c_A = \frac{m_\mu + m_F}{M_X} C_A. \quad (\text{A1.55})$$

Comparing eq. (A1.51) and eq. (A1.55) shows, that both conditions that have to be fulfilled, result in different prefactors. In fact, choosing eq. (A1.51) or eq. (A1.55) is just a question of convention. Both relations are correct, as the prefactors will cancel out after performing the square over $c_{V,A}$. The final relation between the square of the coupling constants that is necessary to combine the results from eq. (4.48) and eq. (4.61) is

$$|c_V|^2 = \frac{(m_\mu - m_F)^2}{M_X^2} |C_V|^2 \quad |c_A|^2 = \frac{(m_\mu + m_F)^2}{M_X^2} |C_A|^2. \quad (\text{A1.56})$$

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