Casimir Effect

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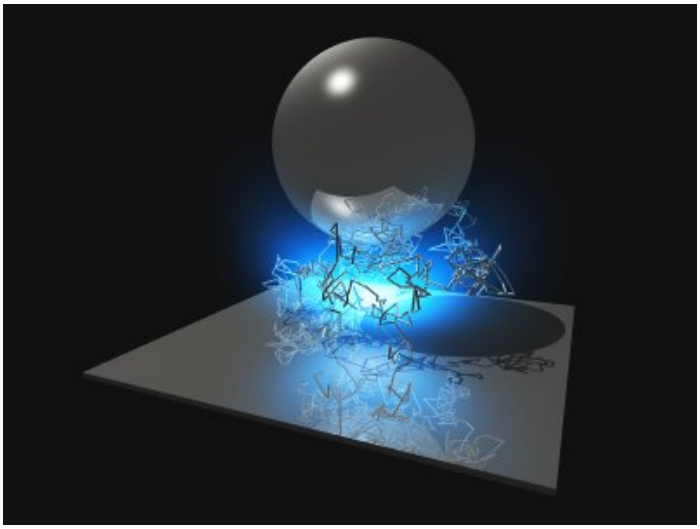


Abstract

Bachelor Thesis from the Bachelor Degree accomplished in the *Institüt für Theoretische Physik* of *Westfälische-Wilhelms Universität* at Münster.

Contents

1 Introduction

What is the vacuum? This notion has been always really hard to define. It is not the absence of anything. In classical physics, the definition of the vacuum is based on the pressure of the system: for a pressure of 10^{-8} Pa, physicists speak about *ultra-vacuum*. We can also consider that a system which is not subjected to forces is empty. Classically, we can consider the vacuum system as a system with a extremely low temperature, closed to absolute zero. In classical physics it is considered as without particles.

So we can do the following think experience: what happen if we put two conducting non-charged plates parallel? We are up to answer: nothing at all. Actually, and we can be really surprised of it, there is an attractive force between these two plates! How this effect, called *Casimir Effect* can be explained? This effect is named from the Dutch physicist Hendrik Casimir who did its first prediction (with the help of Dirk Polder, another Dutch physicist).

One century ago, the advent of the quantum theory, especially the Quantum Field Theory (QFT) showed us it was totally relevant to consider the vacuum has its own energy. How that is possible? According to the quantum theory, all the fields, in particular electromagnetic fields, have fluctuations. They have all the time oscillations around an average value. It is due to creation and annihilation of virtual particles. We speak about vacuum fluctuations. So in reality the vacuum is never empty.

In QFT, the fields are quantised (because that solves all the problems of Klein-Gordon and Dirac equations). The fields are then not considered as wave functions, but as physical systems with an infinite number of degrees of freedom.

Since the development of the Quantum Electro Dynamics (QED), a particle becomes in quantum language normal modes of excitation of fields. Any particle can exists if it satisfies the Heisenberg relations. Thus the quantum vacuum can have effects. One of them is Casimir effect, and i will speak about it all along this Bachelor thesis.

2 Theoretical Background

2.1 The concept of the vacuum energy

The interesting point put on spotlights in Casimir effect is the following: the physical vacuum energy of a quantised field must be calculated in taking into account the external constraints. Hence, the vacuum energy of a quantised field is the difference between the zero-point energy corresponding to the vacuum energy with constraints and the zero-point energy corresponding to the free-vacuum configuration:

$$E_{vacuum} = E_{vacuum\ with\ constraints} - E_{free\ vacuum} \tag{1}$$

The key of the Casimir calculation is to define the "true" field Hamiltonian. In classical physics the Hamiltonian (for an harmonic oscillator) is:

$$H = \hbar\omega \left(n + \frac{1}{2}\right), n \in \mathbb{N} \tag{2}$$

In QFT, the Hamiltonian takes the form:

$$\hat{H} = \frac{1}{2}\sum_k \omega_k (a_k^\dagger a_k + a_k a_k^\dagger) = \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right), \tag{3}$$

where the ω_k are the eigenvalues of the Klein-Gordon operator, $a_k^\dagger$ and a_k which are respectively the operators of creation and annihilation, satisfy the commutation rules:

$$[a(k), a^\dagger(k')] = \delta_{kk'}, [a(k), a(k')] = 0, [a^\dagger(k), a^\dagger(k)] = 0. \tag{4}$$

This Hamiltonian is corresponding to an infinite number of harmonic oscillators with creation and annihilation operators. We can define the number $n_k = a_k^\dagger a_k$, thus:

$$\hat{H} = \sum_k \omega_k \left(n_k + \frac{1}{2}\right) \tag{5}$$

The vacuum state is defined such as:

$$a_k|0\rangle = 0, \tag{6}$$

and in calculate the vacuum expectation value we obtain:

$$E_0 =< 0 \mid \hat{H} \mid 0 > . \tag{7}$$

The half of the sum of all eigenfrequencies:

$$E_0 = \frac{1}{2}\sum_k \omega_k \tag{8}$$

is in general a **divergent** quantity. That corresponds to the zero-point energy, which is the energy of the ground state. The vacuum has energy because of quantum fluctuations and this energy is divergent. The fluctuations are due to a lot of virtual particles which constantly appear and disappear in the vacuum.

2.2 Van Der Waals and London theories

2.2.1 Definitions

We will in this thesis in particular speak about an great article with a lot of repercussions: *The influence of Retardation of the London-van der Waals forces* published by Casimir and Polder in 1947 (it was followed by an other article by Casimir, we will go back on it later). To do it properly, we have first of all to introduce what is the theories of the two physicists named in the article.

To understand it well, we have to explain what is a dipole. There is actually a non-uniform distribution of positives and negatives charges in an atom: so in a molecule that creates electric dipole moments, which is a measure of the separation of positive and negative electrical charges, i.e. a measure of the charge system's overall polarity. The presence of electric dipoles means there is the presence of an inherent electric field, which can have three forms:

- permanent: this is the case in which two atoms in a molecule have different electro-negativity. One atom becomes more positive, the other more negative because they do not attract in the electrons in the same way. The molecules with permanent dipole moment are called polar molecules.
- instantaneous: that happens when by chance the electrons are more concentrated in a place than in another. So this phenomenon is temporal.
- induced: in this case, one molecule with a permanent dipole repels the electrons of another molecule. That induced a dipole moment in this last molecule. A molecule is **polarised** when it carries an induced dipole. An example of induced dipole is an electron cloud who sustain a deformation if we apply an external field (in case of Rayleigh diffusion for example).

We called *forces of Van der Waals* weak electric interactions between atoms, molecules, or between a molecule and a crystal. It can have three origins:

- orientation: between two permanent dipoles (studied by the physicist Keesom);
- induction: between a permanent dipole and a induced dipole (studied by Debye);
- dispersion: between two instantaneously induced dipoles (studied by London).

The ones involved in the attraction of the plates correspond in this classification to the third group.

The *London forces* exists because the electronic density of atoms or molecules is **probabilist**: there is a lot of chances in any moment for this density not being uniform through the atom (or molecule). That creates a weak dipole moment. The dipole moment change really quickly all along the time. So, each inhomogeneous distribution creates an induced dipole moment who can interact with the ones of neighbours atoms or molecules and consequently a force appears between the molecules. In case of polar entity, this interaction stays weak in comparison with hydrogen bond, ionic interaction or Keesom (orientation) interaction. Nonetheless, in case of neutral molecules, it is the only one intermolecular attractive force at large distance. In this thesis we are interested exclusively in atoms and neutral molecules because we work on two non-charged plates. The energy created by such a force (without any retardation effect) between two molecules or atoms may be expressed as:

$$E_{London} = -\frac{3}{4}\frac{h\nu\alpha_1\alpha_2}{(4\pi\epsilon_0)^2 r^6}, \tag{9}$$

where ν is the absorption frequency, α is the electric polarisability (i.e the ability to be polarised for a charge distribution like the electron cloud of an atom or molecule). Thus we may see in deriving that the force falls in power 7.

2.2.2 The retardation of these forces

On 1947, the Dutch scientists Hamaker, Verwey and Overbeek did a work about colloidal particles: in their theory, the attraction between several particles of this type is described by the London-van der Waals forces. But they found a problem: at large distances, the result of the attraction force was not the one expected, i.e. in R^{-7} (where R is the radius). Indeed, from the moment where the particles are closed to each other by a distance comparable to the wave lengths of the atoms, there is an influence of **retardation**. Casimir and Polder were then some of the firsts who really treated this problem by the help of QED.

This attraction is delayed because the light has a finite velocity. We can call the forces of attraction *long-range dispersion van der Waals forces.*

2.3 Perturbation theory

In the case in which we cannot solve exactly a problem, we used the called *approximated methods*, which have a fundamental importance to obtain predictions of quantum mechanics. We can use a numerical method, but a realistic analytical one is better to understand really the physics features of the problem. We can use the perturbation theory when the studied problem is not so much different to a problem which we now how to solve exactly. If we apply a perturbation on a system, that will modify the Hamiltonian of it and modify its eigenvalues (i.e. will make a distortion of the spectrum of energy) and its associated eigenstates. If the effect of this perturbation is not so important, we will estimate this deformation in using a parameter λ, enough small to be able to be expanded in powers of λ. The smallness of this parameter is the basis of the perturbation method of **Rayleigh-Schrödinger**. We will treat only the perturbations not only depending on time (static perturbations).

2.3.1 Perturbation theory of Rayleigh-Schrödinger for a non-degenerate level

Let us consider a problem described by an Hamiltonian $\hat{H}$ such as:

$$\hat{H} = \hat{H}_0 + \hat{H}_p, \tag{10}$$

where $\hat{H}_0$ is the Hamiltonian for which we know the exact solution (non-perturbed Hamiltonian), $\hat{H}_p$ is the Hamiltonian who described a **perturbation**. The perturbation is considered as small if the effect on the eigenvalues and states is small. We introduce the adimensional parameter λ such as:

$$\hat{H}_p = \lambda \hat{W} \text{ with } [\lambda] = 1 \text{ and } \lambda \ll 1 \tag{11}$$

If the matrix elements of $\hat{W}$ are of the same order than the ones of $\hat{H}_0$, as a consequence λ is sufficiently small for $\hat{H}_p$ bring only small perturbations to the eigenvalues and states of $\hat{H}_0$. We are considering the case in which the spectrum of $\hat{H}_0$ is discrete. We define:

$$\hat{H}_0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \tag{12}$$

The equation to solve for the perturbed system is:

$$\hat{H}|\psi_n\rangle = (\hat{H}_0 + \lambda\hat{W}) = E_n|\psi_n\rangle \tag{13}$$

The goal is to determine E_n and ψ_n from E_n^0 and ψ_n^0. The eigenvalues E_n^0 can be degenerate (the degree of degeneracy of the energy level E_n^0 is called g_n).

2.3.2 Method of perturbative development of eigenstates and -values

The goal of this method is to develop the E_n and the $|\psi_n\rangle$ in powers of λ:

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda|\psi_n^1\rangle + \lambda^2|\psi_n^2\rangle + ..., \tag{14}$$

$$E_n = E_n^0 + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \tag{15}$$

The equation (13) becomes after substitution:

$$(\hat{H}_0+\lambda\hat{W})(|\psi_n^0\rangle+\lambda|\psi_n^1\rangle+\lambda^2|\psi_n^2\rangle+...) = (E_n^0+\lambda E_n^{(1)}+\lambda^2 E_n^{(2)}+...)(|\psi_n^0\rangle+\lambda|\psi_n^1\rangle+\lambda^2|\psi_n^2\rangle+...) \tag{16}$$

In saying the Hilbert space of the states of the problem has to be the same, we can use the same representation and develop the states $|\psi_n^i\rangle$ on the basis of the eigenstates of $\hat{H}_0$:

$$|\psi_n^i\rangle = \sum_p \sum_{g=1}^{g_p} C_{p,g}^i |\psi_p^{0,g}\rangle, \tag{17}$$

where g_p is the degeneracy degree of the level of energy E_p^0 of $\hat{H}_0$. Let us now assume the calculation in the order (2) is sufficient to build a correct modelling of the studied system. We can extrapolate the method in bigger orders. Let us identify the powers of λ with the help of the equation (16):

$$\hat{H}_0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \tag{18}$$

$$\lambda\hat{H}_0|\psi_n^1\rangle + \lambda\hat{W}|\psi_n^0\rangle = \lambda E_n^{(1)}|\psi_n^0\rangle + E_n^0\lambda|\psi_n^1\rangle \tag{19}$$

$$\lambda^2\hat{H}_0|\psi_n^2\rangle + \lambda^2\hat{W}|\psi_n^1\rangle = E_n^0\lambda^2|\psi_n^2\rangle + \lambda^2 E_n^{(1)}|\psi_n^1\rangle + \lambda^2 E_n^{(2)}|\psi_n^0\rangle \tag{20}$$

So:

$$\hat{H}_0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \tag{21}$$

$$\hat{H}_0|\psi_n^1\rangle + \hat{W}|\psi_n^0\rangle = E_n^{(1)}|\psi_n^0\rangle + E_n^0|\psi_n^1\rangle \tag{22}$$

$$\hat{H}_0|\psi_n^2\rangle + \hat{W}|\psi_n^1\rangle = E_n^0|\psi_n^2\rangle + E_n^{(1)}|\psi_n^1\rangle + E_n^{(2)}|\psi_n^0\rangle \tag{23}$$

Normalisation:

The start point is the spectrum of the Hamiltonian $\hat{H}_0$, the eigenstates are known and for the normalised and orthogonal states we have:

$$\langle\psi_p^{0,i}|\psi_n^{0,j}\rangle = \delta_{np}\delta_{ij}, \tag{24}$$

where $i = 1, ..., g_p$ and $j = 1, ..., g_n$ where g_n and g_p are the respective degeneracies of the levels E_n^0 and E_p^0.

Let us calculate the square of $|\psi_n\rangle$ ((14)):

$$\langle\psi_n|\psi_n\rangle = \langle\psi_n^0|\psi_n^0\rangle + \lambda\langle\psi_n^0|\psi_n^1\rangle + \lambda\langle\psi_n^1|\psi_n^0\rangle + \lambda^2\langle\psi_n^1|\psi_n^1\rangle + \lambda^2\langle\psi_n^0|\psi_n^2\rangle + \lambda^2\langle\psi_n^2|\psi_n^0\rangle$$

$$= \langle\psi_n^0|\psi_n^0\rangle + 2\lambda\Re\langle\psi_n^0|\psi_n^1\rangle + 2\lambda^2\Re\langle\psi_n^0|\psi_n^2\rangle + \lambda^2\langle\psi_n^1|\psi_n^1\rangle$$

We impose the normalisation of the $|\psi_n\rangle$, we obtain in the second order:

$$\Re(\langle\psi_n^0|\psi_n^1\rangle) = 0$$
$$2\Re(\langle\psi_n^0|\psi_n^2\rangle) + \langle\psi_n^1|\psi_n^1\rangle = 0$$

The resolution of this equation constrains the $C_{p,g}^i$ components of the equation (17) and yet on the corrections $|\psi_n^i\rangle$ of order (i) of the different states.

2.3.3 Resolution by order

We consider we already solved exactly the equation in the order (0). We put in the equation (22) the known solution E_n^0 and $|\psi_n^0\rangle$ to obtain $E_n^{(1)}$ and $|\psi_n^1\rangle$:

$$\begin{cases} |\psi_n\rangle = |\psi_n^0\rangle + \lambda|\psi_n^1\rangle \\ E_n = E_n^0 + \lambda E_n^{(1)} \end{cases} \tag{25}$$

We now inject these expressions in the equation (23) to complete the solution until the order (2):

$$\begin{cases} |\psi_n\rangle = |\psi_n^0\rangle + \lambda|\psi_n^1\rangle + \lambda^2|\psi_n^2\rangle \\ E_n = E_n^0 + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \end{cases} \tag{26}$$

Notation: The correction of the level of energy E_n^0 is noted ΔE_n. If for example we do the calculation in the order (2):

$$\Delta E_n = E_n - E_n^0 = \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \tag{27}$$

2.3.4 Perturbation of a non-degenerated level

In the first order we have for the eigenvalues:

$$E_n = E_n^0 + \lambda E_n^{(1)} = E_n^0 + \lambda \langle \psi_n^0 | \hat{W} | \psi_n^0 \rangle \tag{28}$$

So:

$$\Delta E_n^1 = \lambda \langle \psi_n^0 | \hat{W} | \psi_n^0 \rangle \tag{29}$$

The associated correction in the first order correspondent to the non-degenerate energy level E_n^0 is given by:

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda |\psi_n^1\rangle, \text{ with } |\psi_n^1\rangle = \sum_{p \neq n} \sum_{i=1}^{g_p} \frac{\langle \psi_p^{0,i} | \hat{W} | \psi_n^0 \rangle}{E_n^0 - E_p^0} |\psi_p^{o,i}\rangle \tag{30}$$

In the second order we have:

$$E_n = E_n^0 + \lambda \langle \psi_n^0 | \hat{W} | \psi_n^0 \rangle + \lambda^2 \sum_{p \neq n} \sum_{i=1}^{g_p} \frac{| \langle \psi_p^{0,i} | \hat{W} | \psi_n^0 \rangle |^2}{E_n^0 - E_p^0} \tag{31}$$

$$\Delta E_n^{(2)} = \lambda \langle \psi_n^0 | \hat{W} | \psi_n^0 \rangle + \lambda^2 \sum_{p \neq n} \sum_{i=1}^{g_p} \frac{| \langle \psi_p^{0,i} | \hat{W} | \psi_n^0 \rangle |^2}{E_n^0 - E_p^0} \tag{32}$$

2.4 Method of regularisation

In physics, especially in quantum field theory (associated with the process of renormalisation), the regularisation is a method to deal with divergent integrals or sums in introducing a cut-off. In the case of Casimir calculation, we will cut the high frequencies because at very high frequencies, the conductor material of the plate is no longer perfect, but dielectric and transparent to radiation. In this case, the boundary conditions are no longer applicable, and we have to introduce a regularised expression, as we will see later. After regularisation, the renormalisation is a way to go on limit in modifying the original Lagrangian of the system. In this subsection, we will work on detail on a special way of regularisation used in the case of Casimir effect, the zeta-function regularisation.

2.4.1 The zeta function

We want to evaluate the determinants of differential operators. Let us consider an operator $\hat{H}$ with the eigenvalues λ_n. Let us assume the characteristic polynomial of $\hat{H}$ is divided into its zeroes.
Thus the determinant of the operator $\hat{H}$ can be expressed as:

$$\det \hat{H} = \prod_n \lambda_n \tag{33}$$

As the λ_n are increasing, this sum is clearly divergent. We have to regularise it and for doing it, we shall first introduce the associated zeta function:

$$\zeta(s) = \sum_n \lambda_n^{-s} = \sum_n \frac{1}{\lambda_n^s} \tag{34}$$

We have:

$$\zeta(s) = \sum_n \frac{1}{\lambda_n^s} = \sum_n \exp\left(\log \frac{1}{\lambda_n^s}\right) = \sum_n e^{-s \log \lambda_n} \tag{35}$$

So:

$$\zeta'(s) = \frac{d\zeta}{ds} = -\sum_n \log(\lambda_n) e^{-s \log \lambda_n} \tag{36}$$

Thus in $s = 0$:

$$\zeta'(0) = -\sum_n \log(\lambda_n) \tag{37}$$

Link with the determinant:

$$\exp(-\zeta'(0)) = \exp(-\sum_n \log \lambda_n) = \prod_n \exp(\log \lambda_n) = \prod_n \lambda_n \tag{38}$$

So:

$$\det \hat{H} = \exp(-\zeta'(0)) \tag{39}$$

2.4.2 Path integrals

Stephen Hawking studied in 1977 the procedure of zeta-function regularisation in his paper *Zeta Function Regularization of Path Integrals in Curved Spacetime*. In this, he defines the partition function of a canonical ensemble (with the Boltzmann constant $k_B = 1$) as:

$$Z = \int d[g]d[\phi] \exp iI[g, \phi], \tag{40}$$

which is a **path integral**. That signifies we integer on functions rather on number. It was introduced by Richard Feynman in 1942. The path integral representation gives the quantum amplitude to go from point x to point y as an integral over all paths. It gives hence all the possible probabilities for a path, instead of integer on only one path as in classical integration theory. An example of path integral is the action of a system.

In the Hawking's article, $d[g]$ is a measure on a space of metrics g, $d[\phi]$ is a measure on the space of matter fields ϕ and $I[g, \phi]$ is the action. After, he choices two fields ϕ_0 and g_0 who satisfy classical field equations and boundary or periodicity conditions. It is possible to define:

$$\begin{array}{rcl} g & = & g_0 + \tilde{g} \\ \phi & = & \phi_0 + \tilde{\phi} \end{array} \tag{41}$$

where $\tilde{g}$ and $\tilde{\phi}$ are the respective fluctuations of g and ϕ. Hawking expands afterwards the action in a Taylor series:

$$I[g, \phi] = I[g_0, \phi_0] + I_2[\tilde{g}] + I_2[\tilde{\phi}], \tag{42}$$

in the second order. We can take the natural logarithm of Z:

$$\begin{array}{rcl} \log Z & = & \log \int d[g]d[\phi] \exp iI[g_0, \phi_0] + \log \int d[\tilde{g}] \exp iI_2[\tilde{g}] + \log \int d[\tilde{\phi}] \exp iI_2[\tilde{\phi}] \\ \log Z & = & iI[g_0, \phi_0] + \log \int d[\tilde{g}] \exp iI_2[\tilde{g}] + \log \int d[\tilde{\phi}] \exp iI_2[\tilde{\phi}] \end{array} \tag{43}$$

He assumes by quantum gravity postulates that:

$$I_2[\tilde{\phi}] = -\int \frac{1}{2} \tilde{\phi} A \tilde{\phi} (-g_0)^{1/2} d^4x, \tag{44}$$

where A is second order differential order and a composition of g_0 and ϕ_0. We have in an idem way:

$$I_2[\tilde{g}] = -\int \frac{1}{2} \tilde{g} A \tilde{g} (-g_0)^{1/2} d^4x. \tag{45}$$

The author takes now the case in which the background metric g_0 is Euclidean (i.e. real and positive), the operator A is so real and self-adjoint. The relation between eigenstates and eigenvalues (which are real by definition of self-adjoint) is hence:

$$A\phi_n = \lambda_n \phi_n \tag{46}$$

It is possible to normalise the eigenvectors:

$$\int \phi_n \phi_m (g_0)^{1/2} d^4x = \delta_{nm} \tag{47}$$

And the fluctuations are given by:

$$\tilde{\phi} = \sum_n a_n \phi_n \tag{48}$$

And the measure can be expressed in function of the a_n and a normalisation constant μ:

$$d[\phi] = \prod_n \mu da_n \tag{49}$$

There is so, in using (44):

$$\begin{aligned} Z[\tilde{\phi}] &= \int d[\phi] \exp(iI_2[\tilde{\phi}]) \\ &= \int \prod_n \mu da_n \exp\left[-\tfrac{1}{2} \int \sum_n a_n \phi_n A \sum_n a_n \phi_n (-g_0)^{1/2} d^4x\right], \end{aligned} \tag{50}$$

and with (46) and (47):

$$\begin{aligned} Z[\tilde{\phi}] &= \int \prod_n \mu da_n \exp\left[-\tfrac{i}{2} \int \sum_n a_n \phi_n \sum_n a_n \lambda_n \phi_n (-g_0)^{1/2} d^4x\right] \\ &= \tfrac{1}{2}\mu \prod_n \int da_n e^{-\lambda_n a_n^2} (\text{by interversion product-integral} \end{aligned} \tag{51}$$

By the definition of a Gauss integral (see the appendix (A)) we have:

$$\int_{-\infty}^{+\infty} e^{-\lambda_n a_n^2} dx = \sqrt{\frac{\pi}{\lambda_n}} \tag{52}$$

Thus:

$$Z[\tilde{\phi}] = \frac{1}{2}\mu \pi^{1/2} \prod_n \lambda_n^{1/2} \tag{53}$$

As in (33) we have $\det A = \prod_n \lambda_n$ and at last:

$$Z[\tilde{\phi}] = (\det(4\mu^{-2}\pi^{-1}A))^{-1/2} \tag{54}$$

3 Calculations

3.1 Retardation of London-Van der Waals forces

The starting point of the calculation of Casimir effect is the article published by Casimir and Polder in 1947: *The influence of Retardation of the London-van der Waals forces*. This article is made of two parts: in the first, they study the interaction between a neutral atom with a perfectly conducting plane and in the second, they look the interaction between two atoms. For the first, they are considering a cubic box in the space, of length L. By the boundary conditions of the Maxwell equations we have:

$$\begin{aligned} E_x(\mathbf{k},\lambda) &= e_x(\mathbf{k},\lambda)\cos k_1 x \sin k_2 y \sin k_3 z.C_e \\ E_y(\mathbf{k},\lambda) &= e_y(\mathbf{k},\lambda)\sin k_1 x \cos k_2 y \sin k_3 z.C_e \\ E_z(\mathbf{k},\lambda) &= e_z(\mathbf{k},\lambda)\sin k_1 x \sin k_2 y \cos k_3 z.C_e, \end{aligned} \tag{55}$$

where C_e is the normalisation factor and $\mathbf{k}$ is the wave vector.

As we are studying a plane wave in the vacuum, the vector $\mathbf{E}$ of the electric field is perpendicular to the wave vector $\mathbf{k}$ and its direction is constant. So $\mathbf{E}$ is evolving in a plane and $\mathbf{k}$ is normal to its plane. Consequently there is two parameters which are sufficient to completely define $\mathbf{E}$ i.e. the direction of polarisation (which can be elliptic or circular in changing these parameters). For example, if $\mathbf{k}$ is on the z-axis, $\mathbf{E}$ is defined by E_x and E_y. Finally, there exists two vectors $\mathbf{e}$ which correspond to the two directions of polarisation for each wave vector $\mathbf{k}$.

We define the vector potential of the electromagnetic field in the box:

$$\mathbf{A} = \sum_{k,\lambda} (A_{k,\lambda} e^{-i\omega t} + A^{\dagger}_{k,\lambda} e^{i\omega t}) \mathbf{E}(k,\lambda) \tag{56}$$

The operator G of the interaction between a neutral atom and the radiation field is given by:

$$G = \sum_i \left[-\frac{e}{mc}(\mathbf{p}_j \mathbf{A}) + \frac{e^2}{2mc^2} A^2 \right], \tag{57}$$

where the summation is over all the electrons in the atom and $\mathbf{p}_j$ is the operator of the momentum of an electron. We do the perturbation calculation in the lowest level in which the perturbation energy is not zero. As A has no diagonal elements, there is no first-order perturbation proportional to e. Consequently, for the terms proportional to e we use second-order perturbation, and for the ones proportional to e^2 we use the first-order. Casimir and Polder do the calculation for the case in which the atom is situated at a very large distance from the walls of the box and for the case in which it is at a short distance from one of the

walls. Finally for the interaction between a neutral atom and the radiation field (which takes the variation of the electromagnetic field inside the atom) we have the total perturbation energy given by:

$$\Delta_d E = \Delta_1 E + \Delta_2 E \tag{58}$$

Afterwards, the authors take into account the electrostatic interaction whose the perturbation energy is $\Delta_e E$. Between a dipole q^x at $x = R$ and a conducting wall at $x = 0$ there is the electrostatic energy:

$$\epsilon_x = -\frac{(q^x)^2}{8R^3} \tag{59}$$

This energy is in the case of a dipole q^y or q^z is:

$$\epsilon_{y,z} = -\frac{(q^{y,z})^2}{16R^3} \tag{60}$$

After some calculations, they thus find the total interaction between the atom and the wall as:

$$\Delta_t E = \Delta_d E + \Delta_e E \tag{61}$$

They by the way assume that the angular momentum J is for 0 for the zero state. That means the matrix elements of $\mathbf{p}$ exist only between this state and the threefold degenerate states with $J = 1$. The three waves functions corresponding to these states are choose to have the same transformation properties under a rotation as x,y, and z. Finally they have:

$$\Delta_t E = -\frac{2}{\pi}\sum_n \int_0^\infty \frac{k_n u^2 du}{u^2 + k_n^2}\frac{e^{-2uR}}{2R} \times \left[2 \mid q^x_{0;n} \mid^2 \left(\frac{2}{2uR} + \frac{2}{4u^2R^2}\right) + (\mid q^y_{0;n} \mid^2 + \mid q^z_{0;n} \mid^2)\left(1 + \frac{2}{2uR} + \frac{2}{4u^2R^2}\right)\right] \tag{62}$$

In very small distances ($R \to 0$) we have:

$$\frac{e^{-2uR}}{2R} \underset{0}{\sim} \frac{1}{2R} \text{ and } \left(1 + \frac{2}{2uR} + \frac{2}{4u^2R^2}\right) \underset{0}{\sim} \frac{1}{4u^2R^2} \tag{63}$$

From which we obtain:

$$\Delta_r E(R \to 0) = -\frac{1}{16R^3}\sum_n (2 \mid q^x_{0;n} \mid^2 \mid q^y_{0;n} \mid^2 + \mid q^z_{0;n} \mid^2), \tag{64}$$

which is equal to the value of London energy. For a very large R (R larger than all $\lambda_n = 2\pi/k_n$) we have:

$$\Delta_t E(R \to \infty) = -\sum_n \frac{\mid q^x_{0;n} \mid^2 \mid q^y_{0;n} \mid^2 + \mid q^z_{0;n} \mid^2}{4\pi k_n R^4} \tag{65}$$

In term of polarisability:

$$\Delta_t E(R \to \infty) = -\frac{\hbar c}{8\pi R^4}(\alpha_x + \alpha_y + \alpha_z) \tag{66}$$

In addition, we have the relation:

$$\sum \mid q^x_{0;n} \mid^2 = \sum \mid q^y_{0;n} \mid^2 = \sum \mid q^z_{0;n} \mid^2 = \sum \mid q_{0;n} \mid^2, \tag{67}$$

where the summation extends over the three states with $J = 1$, belonging to one degenerate level, which will be indicated from now by one symbol n. We can now write the equation (62) as:

$$\Delta_t E = -\frac{2}{\pi}\sum_n \int_0^\infty \frac{k_n u^2 du}{u^2 + k_n^2}\frac{e^{-2uR}}{2R} \times \mid q_{0;n} \mid^2 \left(1 + \frac{2}{2uR} + \frac{2}{4u^2R^2}\right), \tag{68}$$

where each term of the sum over n represents the contribution of all three states with $J = 1$ belonging to one degenerate level. As the total London interaction energy (without retardation effect) is equal to $-3\hbar c\alpha/8\pi R^3$, we may see with all this calculation and especially the results (64) and (66) that for short distances (shorter than the atomic wavelength) the correction factor due to the retardation is unitary. For large distances in comparison with this wavelength, the London interaction energy is proportional to R^{-1}. At large distance, the interaction between a perfectly conducting plate and an atom or molecule in the limit of large distances is thus given by:

$$\Delta E = -\frac{3}{8\pi}\hbar c\frac{\alpha}{R^4} \tag{69}$$

In the second part of the article, Casimir and Polder study the interaction between two particles (two neutral atoms in their case). The calculations are much more complicated, but at the end the obtain for large distances R:

$$\Delta E = -\frac{23}{4\pi}\hbar c\frac{\alpha_1\alpha_2}{R^7} \tag{70}$$

3.2 Casimir Calculation

Before any calculation, we are able to anticipate the dimension of the result. We assume we will calculate a pressure P, therefore:

$$P \propto \text{J.m}^{-3}$$

We have also:

$$\hbar \propto \text{J.s} \ ; \ c \propto \text{m.s}^{-1} \ ; \ L \propto \text{m}$$

Thus:

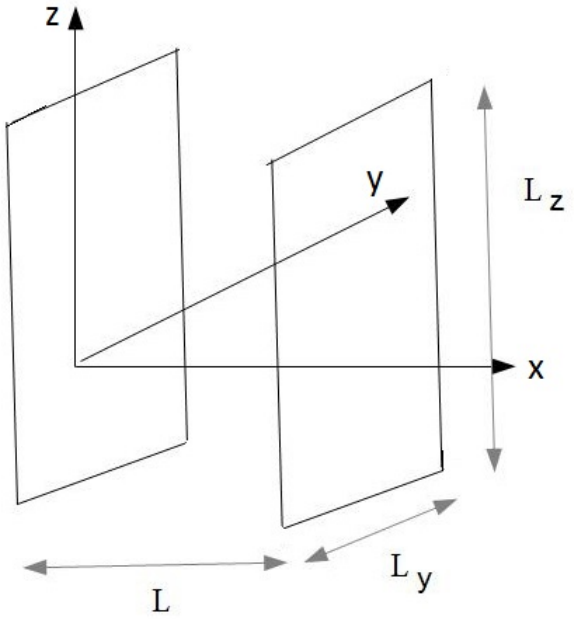
$$P \propto \hbar.\text{s}^{-1}.\text{m}^{-3}$$
$$P \propto \hbar.c.\text{m}^{-4}$$

And finally

$$\boxed{P \propto \frac{\hbar c}{L^4}}$$

We are considering two parallel **non-charged and conducting** plates separated by a distance L. This is described by the Figure 1.

Figure 1: The configuration of the plates



Between the two plates, the stationary modes of the electromagnetic field are described by the wave vector (k_x, k_y, k_z) where $\mathbf{k} = (k_y, k_z)$ is parallel to the plates. By the boundary conditions of electromagnetism we have:

$$k_x = \frac{\pi n}{L} \tag{71}$$

The waves between the plates are plane waves of the form:

$$\psi(\mathbf{r}, t) = Ae^{i\mathbf{k}.\mathbf{r}-\omega t}, \tag{72}$$

where A is a constant. We fix periodical limits (which are fictive because not viable for infinite plates). We have so on the y component:

$$\psi(0, t) = \psi(L_y, t) \tag{73}$$

Consequently, in using (72) we have:

$$\begin{array}{rccc} & e^{ik_y.0-\omega t} & = & e^{ik_y.L_y} \\ \Leftrightarrow & 1 & = & e^{ik_y.L_y} \\ \Leftrightarrow & e^{2in_y\pi} & = & e^{ik_y.L_y}, \end{array} \tag{74}$$

where $n_y \in \mathbb{Z}$. So finally we have:

$$k_y = \frac{2n_y\pi}{L_y} \text{ and idem } k_z = \frac{2n_z\pi}{L_z} \tag{75}$$

This mode (k_x, k_y, k_z), which we called $(n, \mathbf{k})$ has the following angular velocity:

$$\omega = \omega_n(\mathbf{k}) = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\frac{\pi^2 n^2}{L^2} + k^2} \tag{76}$$

For each k_x, k_y, k_z there is two standing waves (two different directions of polarisation), as we already explained in 3.1. Each mode is consequently degenerate twice, but the mode $n = 0$.
We will do first the calculation for a **zero temperature**.
The electromagnetic energy of the cavity formed by the two plates is the sum on the modes of the point-zero energies (i.e. the energies of the ground state of each mode):

$$E_0 = \sum_{n,\mathbf{k}} \epsilon_0[\omega_n(\mathbf{k})] \text{ , with } \epsilon_0 = \frac{1}{2}\hbar\omega \tag{77}$$

This sum is *divergent*. It is due to the fact we considered the plates as perfectly conducting. But for very high frequencies the plate becomes dielectric and transparent to the radiation. So we have to cut the very high frequencies in introducing a *cut-off*: this process is called *regularisation*. We obtain the new sum:

$$E_0 = \sum_{n,\mathbf{k}} \epsilon_0[\omega_n(\mathbf{k})]\chi\left(\frac{\omega_n(\mathbf{k})}{\omega_c}\right), \tag{78}$$

where χ is the cut-off function,defined such as $\chi(0) = 1$ and to be regular when it is closed to the origin, and ω_c is the cut-off frequency. That means we cut all the frequencies $\omega_n > \omega_c$. Indeed, χ is 0 when $\frac{\omega_n}{\omega_c} \to \infty$, which means when $\omega_n >> \omega_c$. So it cuts all the *problematic* terms for which we have $\omega_n > \omega_c$ and **the sum is consequently no longer divergent**. For example we can take:

$$\chi\left(\frac{\omega_n}{\omega_c}\right) = Ae^{-\frac{\omega_n}{\omega_c}}, \tag{79}$$

where A is a constant.

We are considering big plates, thus we can replace the sum by an integral. We can treat n and $\mathbf{k}$ separately:

$$\sum_{n,\mathbf{k}} = \sum_n \sum_{\mathbf{k}}, \tag{80}$$

because n describes the problem in the x-direction and $\mathbf{k}$ in the y and z directions. The statistical physics theory gives us, in case of a box of length L in dimension d:

$$\sum_{\mathbf{k}} \to \left(\frac{L}{2\pi}\right)^d \int d^d\mathbf{k} \tag{81}$$

In the case of 2 dimensions (the plates are surfaces) with plates of surface $A = L_x.L_y$ that becomes:

$$\sum_{\mathbf{k}} \to \frac{A}{(2\pi)^2} \int d^2\mathbf{k} \tag{82}$$

However, we have to treat also the part related to n in the sum. The first sum of (80) is discrete and describe the transverse part of electromagnetic modes. It is given by:

$$\sum_n \to 2\sum_{n=0}^{\infty}{}', \tag{83}$$

The multiplication by 2 is due to the double-degeneracy of the modes and the prima signifies we take into account the level 0 is non-degenerate (i.e. this value has a weigh 1/2, so 1 when it's multiplied by 2). Let us introduce $\epsilon(\omega) = \epsilon_0(\omega)\chi(\omega/\omega_c)$. Finally the equation (78) becomes:

$$E_0 = 2\frac{A}{(2\pi)^2}\sum_{n=0}^{\infty}{}', \int_{\mathbb{R}^2} d^2\mathbf{k}\epsilon[\omega_n(\mathbf{k})] \tag{84}$$

Let us develop the integral in spherical coordinates, if $k =\mid \mathbf{k} \mid$:

$$\int_{\mathbb{R}^2} d^2\mathbf{k}\epsilon[\omega_n(\mathbf{k})] = \int_{\mathbb{R}^2} kdkd\varphi\epsilon[\omega_n(\mathbf{k})] = \int_0^{2\pi} d\varphi \int_0^{\infty} kdk\epsilon[\omega_n(\mathbf{k})] = \int_0^{\infty} 2\pi kdk\epsilon[\omega_n(\mathbf{k})] \tag{85}$$

Otherwise we have $\omega_n^2 = c^2\left(\frac{\pi^2 n^2}{L^2} + \mathbf{k}^2\right)$ from which, if we fix n, we obtain the differential $\omega d\omega = c^2 kdk$ and thus:

$$\int_{\mathbb{R}^2} d^2\mathbf{k}\epsilon[\omega_n(\mathbf{k})] = \int_{\omega_n(0)}^{\infty} \frac{2\pi}{c^2}\omega d\omega\epsilon(\omega). \tag{86}$$

Ultimately we obtain for the zero-point energy:

$$E_0(L) = \frac{A}{\pi c^2}\sum_{n=0}^{\infty}{}', \int_{\omega_n}^{\infty} d\omega\omega\epsilon_0(\omega)\chi\left(\frac{\omega}{\omega_c}\right), \tag{87}$$

where $\omega_n \equiv \omega_n(0) = \pi cn/L$. By definition of a potential energy, the force which created this energy is given by:

$$X_0 = -\frac{\partial E_0}{\partial L} = -\frac{A}{\pi c^2}\sum_{n=0}^{\infty}{}', \frac{1}{L}\omega_n^2\epsilon_0(\omega_n)\chi\left(\frac{\omega_n}{\omega_c}\right) \tag{88}$$

If we fix the function $g(n) = n^3\chi\left(\frac{\omega_n}{\omega_c}\right)$:

$$X_0 = -A\frac{\pi^2\hbar c}{2L^4}\sum_{n=0}^{\infty}{}' g(n) \tag{89}$$

We can find the equivalent of this in the case in which L is really big. We take the continuous limit in replacing the sum by an integral:

$$X_0^{\infty} = -A\frac{\pi^2\hbar c}{2L^4}\int_0^{\infty} dng(n) \tag{90}$$

This force is the inverse of the **force exerced by the infinite electromagnetic vacuum outside of the plates**, which has to be taken into account if we want to calculate the total point-zero energy. We can now calculate the resulting force $F = X_0 - X_0^{\infty}$:

$$F_{vacuum} = -A\frac{\pi^2\hbar c}{2L^4}\left[\sum_{n=0}^{\infty}{}'g(n) - \int_0^{\infty} dng(n)\right] \tag{91}$$

To calculate this we can use the Euler-Maclaurin formula:

$$\sum_{n=0}^{\infty}{}'g(n) - \int_0^{\infty} dng(n) = -\frac{1}{12}g'(0) + \frac{1}{6!}g'''(0) + O(g^{(5)}(0)) \tag{92}$$

In computing the derivatives:

$$g'(0) = 0 \ , \ g'''(0) = 6\chi(0) = 6 \tag{93}$$

Hence (92) becomes:

$$\sum_{n=0}^{\infty}{}'g(n) - \int_0^{\infty} dng(n) = -\frac{1}{5!} + O\left(\frac{1}{\omega_c^2}\right) = -\frac{1}{120} + O\left(\frac{1}{\omega_c^2}\right) \tag{94}$$

We can finally calculate the total point-zero force at zero-temperature in using (91)and (94):

$$F_{vacuum} = -A\frac{\pi^2\hbar c}{240L^4} \tag{95}$$

The corresponding pressure is obtain in dividing by the surface:

$$P = -\frac{\pi^2}{240}\frac{\hbar c}{L^4}, \tag{96}$$

which is congruent with our estimation in the very beginning of this part. A is thus equal to $A = -\pi^2/240$. It is universal, that means it is independent of the nature of the conducting plates. We can now ultimately using the definition of the force as a derivative of energy to find the **point-zero energy** in this configuration:

$$F_{vacuum} = -\frac{\partial\mathcal{E}_0}{\partial L}, \text{ hence } \mathcal{E}_0 = -A\frac{\pi^2}{720}\frac{\hbar c}{L^3} \tag{97}$$

3.3 Temperature effects

We can know consider we are no longer in the zero-temperature, but now in a temperature T. There is at this temperature some photons between the two plates, and they will follow the distribution of the black body radiation. Each mode $(n, \mathbf{k})$ is thus described by a quantum Hamiltonian with the frequency $\omega_n(\mathbf{k})$. The levels of energy of this Hamiltonian are $\epsilon_m = \hbar\omega(m + \frac{1}{2})$ where m is the number of photons. Each of them has an energy $\hbar\omega = h\nu$. Let is consider only one mode. The free energy, i.e. the energy in a physical system that can be converted to do work is given by:

$$f(\omega) = \epsilon_0(\omega) + f_T(\omega) \tag{98}$$

with:

$$f_T(\omega) = \frac{\varphi(\beta\hbar\omega)}{\beta} \ , \ \beta = \frac{1}{k_B T}. \tag{99}$$

φ is a simple function, $f_T(\omega)$ is the free energy due to the thermal radiation and statistical physics define the partition function Z (which describes the repartition of probabilities in micro-states) of an harmonic oscillator as:

$$Z = \sum_{m=0}^{\infty} e^{-\beta\epsilon_m} = \sum_{m=0}^{\infty} e^{-\beta\hbar\omega(m+\frac{1}{2})} = e^{-\frac{1}{2}\beta\hbar\omega}\frac{1}{1 - e^{-\beta\hbar\omega}} \tag{100}$$

By definition the free energy is:

$$f(\omega) = -\frac{1}{\beta}\ln Z = \frac{1}{2}\hbar\omega + \frac{1}{\beta}\ln\left(1 - e^{-\beta\hbar\omega}\right), \tag{101}$$

ergo:

$$\varphi(x) = \ln\left(1 - e^{-x}\right) \tag{102}$$

As we defined the electromagnetic energy between the two plates in the (77), we can define the total thermal energy by such a sum on the modes:

$$\mathcal{F}_{\mathcal{T}} = \sum_{(n,\mathbf{k})} f_T[\omega_n(\mathbf{k})] \tag{103}$$

We can write this convergent sum (because it is not associated to the vacuum) as, in using (82) and (83):

$$\mathcal{F}_{\mathcal{T}} = \sum_{(n,\mathbf{k})} \frac{\varphi[\beta\hbar\omega(\mathbf{k})]}{\beta} = 2\frac{A}{(2\pi)^2}\sum_{n=0}^{\infty}{}' \int_{\mathbb{R}^2} d^2\mathbf{k}\frac{\varphi[\beta\hbar\omega(\mathbf{k})]}{\beta} \tag{104}$$

And finally with (85):

$$\mathcal{F}_{\mathcal{T}}(L) = 2\frac{A}{(2\pi)^2}\sum_{n=0}^{\infty}{}' \int_{\omega_n(0)}^{+\infty} \frac{2\pi}{c^2}\omega d\omega \varphi[\beta\hbar\omega(\mathbf{k})] \tag{105}$$

Let us introduce $x = \beta\hbar\omega$ and $u_n = \beta\hbar\omega_n(0)$ and we obtain:

$$\mathcal{F}_{\mathcal{T}}(L) = 2\frac{A}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{n=0}^{\infty}{}' \int_{u_n}^{\infty} x dx \varphi(x) \tag{106}$$

We introduce the parameter ψ such as:

$$\psi(u) = \int_{u}^{\infty} x dx \varphi(x) \tag{107}$$

We use the adimensional parameter α to describe the u_n:

$$\alpha = \frac{\beta\pi\hbar c}{L} \ , \ u_n = n\alpha \tag{108}$$

Now let us consider two consecutive modes, with parallel wave zero vectors: $\mathbf{k} = \mathbf{0}$. The difference of energy between two photons of this mode is

$$\hbar\delta\omega = \hbar(\omega_{n+1}(\mathbf{0}) - \omega_n(\mathbf{0})) = \frac{(n+1-n)\hbar\pi c}{L} = \frac{\hbar\pi c}{L} = \alpha k_B T \tag{109}$$

The modes are no longer discrete if this difference is little in comparison with the thermal energy $k_B T$, i.e if $\alpha < 1$. We are thus in the limit of large distances or high temperatures (which are equivalent). So if $\alpha << 1$ the modes are not discrete but they are continuous: we can take the limit to replace the precedent sum in an integral. To do it, we have first to multiply by α:

$$\alpha\mathcal{F}_{\mathcal{T}}{}^{\infty} \sim \lim_{\alpha\to 0}\alpha\mathcal{F}_{\mathcal{T}} = 2\frac{A}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\int_0^{\infty} du\psi(u) \tag{110}$$

By integration by parts we have:

$$\int_0^{\infty} u\psi'(u)du = [u.\psi(u]_0^{\infty} - \int_0^{\infty} u'\psi(u)du \tag{111}$$

But we have $u' = 0$ and ψ vanish in infinity so:

$$\int_0^\infty \psi(u)du = -\int_0^\infty dx x^2 \varphi(x) = -\int_0^\infty dx x^2 \ln(1-e^{-x}) = -2!\zeta(2+2) = -2\zeta(4) = -\frac{\pi^4}{45} \tag{112}$$

By dividing by α we obtain at last:

$$\mathcal{F_T}^\infty = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\left(-\frac{2}{\alpha}\frac{\pi^4}{45}\right) = -\frac{AL}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}. \tag{113}$$

This is the energy of a black-body in a volume $V = AL$. The total energy due to thermal radiation is hence:

$$\mathcal{F}_{therm} = \mathcal{F_T} - \mathcal{F_T}^\infty = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\left[\sum_{n=0}^{\infty}\psi(\alpha n) + \frac{2}{\alpha}\zeta(4)\right] \tag{114}$$

In the thermal equilibrium, the radiation **bewteen** the plates apply a pressure on it. The thermodynamics gives us $P_T = -\frac{\partial \mathcal{F_T}}{\partial V}$. As we have $dV = AdL$ we have a pressure $P_T = -\frac{1}{A}\frac{\partial \mathcal{F_T}}{\partial L}$ and thus the force due to inside thermal radiation apply on the plates is:

$$F_2 = -\frac{\partial \mathcal{F_T}}{\partial L} \tag{115}$$

We will calculate it with (106):

$$\mathcal{F_T}(L) = 2\frac{A}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{n=0}^{\infty}{}'\psi(u_n) \tag{116}$$

We have by the Leibniz's rule and $u_n = \frac{\beta\hbar n\pi c}{L}$:

$$-\frac{\partial\psi(u_n)}{\partial L} = -\frac{\partial u_n}{\partial L}\psi'(u_n) = \frac{\partial u_n}{\partial L} = -\frac{1}{L^2}.\beta\hbar n\pi c\varphi(u_n)u_n = -\frac{u_n^2}{L}\varphi(u_n) \tag{117}$$

Finally:

$$F_2 = -2\frac{A}{2\pi\beta}\frac{1}{(\beta\hbar c)^2}\sum_{n=0}^{\infty}{}'u_n^2\varphi(u_n) \tag{118}$$

As $(1-e^{-x})<1$ we have $\varphi(x)<0$. Consequently the force is positive: the force is repulsive. This force is corresponding to the repulsion force due to thermal radiation when the plates are separated by a finite distance. If $L\to\infty$:

$$F_2^{\infty}=\frac{A}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45}=-\frac{\partial F_T^{\infty}}{\partial L} \tag{119}$$

We found again the result of (113). The total force due to thermal radiation is finally:

$$F_{therm}=F_2-F_2^{\infty}=-\frac{\partial \mathcal{F}_{therm}}{\partial L} \tag{120}$$

The expression obtained in (118) is not so easy to manipulate. We can do a Taylor-expansion of this expression in low temperature (i.e. short distance), where α is now bigger than 1 (on the contrary of before). We have:

$$\varphi(u_n)=\ln(1-e^{-u_n})=\ln(1-e^{-n\beta\hbar\pi c/L}) \tag{121}$$

As $u_n=n\alpha$, u_n is bigger than 1 to. For x big we have $\varphi(x)\sim -e^{-x}$ and by the way for $n=0$, $\varphi(u_n)=0$, consequently $u_n^2\varphi(u_n)\sim -n^2\alpha^2e^{-n\alpha}$ in case of low temperature so:

$$F_2=-\frac{A}{\pi\beta\hbar L}\frac{1}{(\beta\hbar c)^2}[\alpha^2\varphi(\alpha)+4\alpha^2\varphi(2\alpha)...] \tag{122}$$

And by definition of $\alpha^2=(\beta\hbar\pi c/L)^2$, we can expand it in the first order to obtain the expression for low temperature of the force applied on a plate by the radiation which exists between the plates:

$$F_2=+\frac{A\pi}{\beta L^3}[e^{-\alpha}+O(e^{-2\alpha})]\ ,\ \alpha>>1 \tag{123}$$

At last, the total force due to a thermal radiation is:

$$F_{therm}=F_2-F_2^{\infty}=+\frac{A}{\pi\beta\hbar L}\frac{1}{(\beta\hbar c)^2}[\alpha^2e^{-\alpha}+O(e^{-2\alpha})]-\frac{A}{\beta}\frac{1}{(\beta\hbar c)^3}\frac{\pi^2}{45} \tag{124}$$

$$F_{therm}=-\frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\frac{1}{L}\left[\frac{\pi^2}{45\alpha}-\alpha^2[e^{-\alpha}+O(e^{-2\alpha})]\right] \tag{125}$$

3.4 Total free energy

We can now calculate the total free energy $\mathcal{E}$, sum of the vacuum energy and the free thermal energy, thus:

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{F}_{therm} \tag{126}$$

We can write $\mathcal{F}_{therm}$ of (114) in function of α:

$$\mathcal{F}_{therm} = \frac{A}{\pi\beta}\frac{1}{(\beta\hbar c)^2}\left[\sum_{n=0}^{\infty}{}'\psi(\alpha n) + \frac{2}{\alpha}\zeta(4)\right] = A\frac{\pi^2\hbar c}{L^3}\mathcal{G}(\alpha) \tag{127}$$

$$\mathcal{G}(\alpha) = \frac{L^3}{\beta^3\hbar^3c^3\pi^3}\left[\sum_{n=0}^{\infty}{}'\psi(\alpha n) + \frac{\pi^4}{45\alpha}\right] = \frac{1}{\alpha^3}\left[\sum_{n=0}^{\infty}{}'\psi(\alpha n) + \frac{\pi^4}{45\alpha}\right] \tag{128}$$

By (97) we have:

$$\mathcal{E} = A\frac{\pi^2\hbar c}{L^3}\left[-\frac{1}{720} + \mathcal{G}(\alpha)\right] \tag{129}$$

At low temperature (i.e. for $T << 1$, so $\alpha >> 1$), in first order in α we have:

$$\mathcal{G}(\alpha) = \frac{1}{\alpha^3}\left[\frac{1}{2}\psi(0) + \psi(\alpha) + \frac{2}{\alpha}\zeta(4)\right] \tag{130}$$

By definition of psi in (107):

$$\psi(0) = \int_0^\infty x dx \varphi(x) = \int_0^\infty x dx \ln(1 - e^{-x}) = -\zeta(3) \tag{131}$$

At low temperature, (156) gives us:

$$\psi(\alpha) = -(\alpha + 1)[e^{-\alpha} + O(e^{-2\alpha})] \tag{132}$$

At last, for low temperature ($\alpha >> 1$):

$$\mathcal{E} = A\frac{\pi^2\hbar c}{L^3}\left[-\frac{1}{720} + \frac{1}{\alpha^3}\left(-\frac{1}{2}\zeta(3) + \frac{2}{\alpha}\zeta(4) - (\alpha+1)[e^{-\alpha} + O(e^{-2\alpha})]\right)\right] \tag{133}$$

3.5 Some remarks and conclusion of calculation

Let us remind ourselves that the formula of the Casimir pressure, at zero-temperature has a universal limit for perfect conductors:

$$P = -\frac{\pi^2}{240}\frac{\hbar c}{L^4} \tag{134}$$

We can finally calculate the total force applied on the plates, due to for one part to the vacuum energy, and for the other to the thermal radiation at low temperature:

$$F_{tot} = F_{therm} + F_{vacuum} = -A\frac{\pi^2}{240}\frac{\hbar c}{L^4} - \frac{\pi^2}{45\beta}\frac{1}{(\beta\hbar c)^3} + \frac{1}{\beta}\frac{\pi}{L^3}e^{-\alpha} + O(e^{-2\alpha}), \tag{135}$$

which gives, by unit area, the pressure:

$$P_{tot} = -\frac{\pi^2}{240}\frac{\hbar c}{L^4} - \frac{\pi^2}{45\beta}\frac{1}{(\beta\hbar c)^3} + \frac{1}{\beta}\frac{\pi}{L^3}e^{-\alpha} + O(e^{-2\alpha}) \tag{136}$$

The two first terms are corresponding respectively to the Casimir pressure and to the black-body pressure. The last is a low temperature correction (which traduce the discrete feature of the modes). These two terms are dominant (the correction due to the modes is exponentially little). We may thus calculate $\gamma = \frac{F_{therm}}{F_{vacuum}}$. In neglecting the last term:

$$\gamma \simeq -\frac{F^{\infty}}{F_{vacuum}} = \frac{240}{45}\frac{L^4}{(\beta\hbar c)^4} = \frac{1}{3}\left(\frac{2\pi}{\alpha}\right)^4 \tag{137}$$

For $L = 500\text{nm}$, $\alpha = 48$, we find $\gamma = 0.98 \times 10^{-4}$. Hence even for a ordinary temperature, the force due to vacuum fluctuations is much bigger than the one of the black-hole. The situation is the same than in the zero-temperature.

3.6 Calculation by regularisation

As we see in 2.4 we have, in postulating that in our case the operator A is the Hamiltonian of the system $\hat{H}$:

$$\begin{aligned} Z[\tilde{\phi}] &= (4\mu^{-2}\pi^{-1}\det(\hat{H}))^{-1/2} \\ \det\hat{H} &= \exp(-\zeta'(0)) \end{aligned} \tag{138}$$

Hence:

$$\log Z[\tilde{\psi}] = \frac{1}{2}\log\left(\frac{1}{4}\pi\mu^2\right)\zeta(0) + \frac{1}{2}\zeta'(0) \tag{139}$$

We will now apply that in case of Casimir configuration at temperature $T = \beta^{-1}$ (in assuming $k_B = 1$ to simplify). The eigenvalues (by periodic boundary conditions) are:

$$\lambda_n = \left(\frac{2\pi n}{\beta}\right)^2 + k^2 \tag{140}$$

We use know exactly the same reasoning than the one which gave us (87):

$$\zeta(s) = \sum_n \lambda_n^{-s} = \frac{4\pi V}{(2\pi)^3}\left[\int k^{2-2s} + 2\sum_{n=1}^{\infty}\int k^2 dk (4\pi^2\beta^{-2}n^2 + k^2)^{-s}\right] \tag{141}$$

The first term corresponds to the term $n = 0$ of the sum of the second-term (we shall remind this first term has the half of the weight of each term of the sum, that is why we have to deal separately with it. The second term can be integrated by parts to give:

$$-\frac{8\pi V}{(2\pi)^3}\sum_{n=1}^{\infty}\int dk (4\pi^2\beta^{-2}n^2 + k^2)^{-s+1}(2-2s)^{-1}(\cosh y)^{-2s+3} \tag{142}$$

In expressing $k = \frac{2\pi n}{\beta}\sinh(y)$:

$$\begin{aligned} &-\tfrac{8\pi V}{(2\pi)^3}\sum_{n=1}^{\infty}\int dy (2\pi\beta^{-1}n)^{-2s+3}(2-2s)^{-1}(\cosh(y))^{-2s+3} \\ &= -\tfrac{8\pi V}{(2\pi)^3}(2\pi\beta^{-1})^{3-2s}\zeta_r(2s-3)(2-2s)^{-1}\tfrac{1}{2}\tfrac{\Gamma(1/2)\Gamma(s-3/2)}{\Gamma(s-1)} \end{aligned} \tag{143}$$

where ζ_R is the usual Riemann zeta function, and Γ is the gamma-function (see A). We finally have in $s = 0$:

$$\zeta'(0) = -2\pi V\beta^{-3}\zeta_R(-3)(-1)\Gamma(1/2)\Gamma(-3/2) \tag{144}$$

In using (159) and (154), at last:

$$\begin{aligned} \zeta'(0) &= 2\pi V T^3\left(-\tfrac{B_4}{4}\right)\sqrt{\pi}\tfrac{4\sqrt{\pi}}{3} \\ &= 2\pi^2 V T^3 \tfrac{1}{120}\tfrac{4}{3} \end{aligned} \tag{145}$$

$$\zeta'(0) = \frac{\pi^2}{45}VT^3 \tag{146}$$

As we have $\zeta(0) = 0$ we obtain for the partition function (139):

$$\log Z = \frac{\pi^2 V T^3}{90} \tag{147}$$

By definition we also have:

$$E = -\frac{d}{d\beta} \log Z = \frac{\pi^2}{30} V T^4 \tag{148}$$

And one of the thermodynamic identities is given by:

$$dE = TdS - PdV \tag{149}$$

$$P = \frac{dE}{dV} + T\frac{dS}{dV} = \frac{\pi^2}{90} T^4 \tag{150}$$

To obtain the pressure in the case of Casimir effect, we sum on all the fields which are zero on the plates, and we obtain:

$$\log Z = \frac{\pi^2 A \tau}{720 L^3}, \tag{151}$$

where τ is an interval of imaginary time, from which we deduce:

$$P = -\frac{\pi^2}{240}\frac{1}{L^4} F = -A\frac{\pi^2}{240}\frac{1}{L^4} \tag{152}$$

which are the results we already found in 3.2, in imposing $\hbar = c = 1$.

4 Conclusion

We thus calculate the pressure sustained by two non-charged plates in the vacuum, including the effects of temperature. We also did by zeta-function regularisation.

Since Casimir did the calculation, there has been several experimental verifications. The first was done by Marcus Spaarnay, and just showed there was no discrepancy between his experimental result and Casimir's theory. Overbeek and Van Bokland did another in 1978, which had a precision around 25%. At the end of the 80's, Umar Mohideen and his colleagues of California university verified Casimir effect with a precision about 1%. At last but not least, a conglomerate of scientists from Hong Kong University of Science and Technology, University of Florida, Harvard University, Massachusetts Institute of Technology, and Oak Ridge National Laboratory have for the first time demonstrated a compact integrated silicon chip that can measure the Casimir force.

An interesting point to emphazise on it is Casimir effect, which was unexpected and fascinating since the beginning, became even more interesting when the years passed. It appears that, unlike van der Waals forces -which are always attractive-, the ones associated to Casimir effect can be either attractive or repulsive. It's depending on the nature on the field and geometrical features, as the dimension of the space-time or the geometrical boundary conditions. That made the Casimir effect really mysterious, and really interesting.

References

[1] Modern Language Association of America. *MLA Handbook for Writers of Research Papers*. New York: The Modern Language Association of America, 2009. Print.

[2] http://www.professores.uff.br/gusso/casimir-effect.jpg

[3] http://www.larecherche.fr/savoirs/physique/force-qui-vient-du-vide-01-06-2004-88969

[4] http://www.astrosurf.com/luxorion/quantique-particules3.htm

[5] Birrell, Nicholas David, and Paul Charles William Davies. *Quantum fields in curved space*. No. 7. Cambridge university press, 1984.

[6] Bertrand Duplantier ; *Introduction à l'effet Casimir, sḿinaire Poincaré* (Paris, 9 mars 2002).

[7] Boyer, *Quantum zero-point energy and long-range forces*, T.H. Annals Phys. 56 (1970) 474-503

[8] H. B. G. Casimir, D. Polder, *The Influence of Retardation on the London-van der Waals Forces* Phys. Rev. (1978) 73, 360

[9] Elizalde, E., and A. Romeo. *Essentials of the Casimir effect and its computation.* Am. J. Phys 59.8 (1991): 711-719.

A Formula sheet

Riemann zeta function

$$\zeta_R(p) = \sum_{n=1}^{\infty} n^{-p} \tag{153}$$

We can express the negative values of this function of the Bernoulli number B_p:

$$\zeta_R(-p) = -\frac{B_{p+1}}{p+1}. \tag{154}$$

Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \text{ where } \alpha \in \mathbb{R}^+ \tag{155}$$

$\mathbf{I}_p$ integrals:

$$I_p = -\int_0^{+\infty} x^p dx \ln(1 - e^{-x}) = -p!\zeta_R(p+2), p \in \mathbb{N}, \tag{156}$$

where ζ_R is the usual Riemann zeta function.

Gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \tag{157}$$

This function has poles (0,-1,-2,-3...) and we can evaluate the residues of these poles by the following formula, for a pole in n:

$$\frac{(-1)^n}{n!} \tag{158}$$

On the second hand we have:

$$\begin{gathered} \Gamma(z+1) = z\Gamma(z), \\ \Gamma(1) = 1 \text{ and } \Gamma(1/2) = \sqrt{\pi}. \end{gathered} \tag{159}$$