

Samuel Ruben Blümer

› Gravitational Waves from
Type-I Cosmic Strings

2026



Gravitational Waves from Type-I Cosmic Strings

BACHELOR THESIS

submitted to the

Universität Münster

Institut für Theoretische Physik

by

Samuel Ruben Blümer

September 2026

First examiner:

Jun.-Prof. Dr. Kai Schmitz

Second examiner:

Prof. Dr. Michael Klasen

Day of submission:

Monday 7th September, 2026

Abstract

During phase transitions in the early universe, cosmic defects, objects that preserve the old state of the universe, can form and emit gravitational waves, which contribute to a stochastic gravitational wave background (SGWB). This SGWB is being studied to discover physics beyond the Standard Model. This thesis deals with a one-dimensional cosmic defect, cosmic strings, that can arise from a $U(1)$ spontaneous symmetry breaking and form a network. Cosmic strings, which contribute to the SGWB, consist of a scalar and a gauge field core, and depending on the ratio of the widths of these cores, we divide them into type-I and type-II cosmic strings. While type-I strings attract other strings, type-II strings repel them, so the evolution of the networks depends on the type of the strings. This thesis aims to compare the gravitational wave spectrum from type-I and type-II cosmic strings from the perspective of the future gravitational wave observatory, LISA. In this thesis, it is demonstrated that it is possible to distinguish between the signals that might be measured by LISA. The results show that the difference between the signals from the two types is detectable, and the shape of the signal differs.

Declaration of Academic Integrity

I hereby confirm that this thesis, entitled *Gravitational Waves from Type-I Cosmic Strings*, is solely my own work and that I have used no sources or aids other than the ones stated. All passages in my thesis for which other sources, including electronic media, have been used, be it direct quotes or content references, have been acknowledged as such and the sources cited. I am aware that plagiarism is considered an act of deception which can result in sanction in accordance with the examination regulations.

I consent having my thesis cross-checked with other texts to identify possible similarities and to having it stored in a database for this purpose.

I confirm that I have not submitted the following thesis in part or whole as an examination paper before.

Münster, 7th September 2026

Samuel Ruben Blümer

Contents

1	Introduction	1
2	Theoretical Background	5
2.1	Gravitational Waves	5
2.1.1	Einstein's Field Equation	5
2.1.2	The Quadrupole Formula	6
2.2	When the Symmetry Breaks	9
2.2.1	Symmetries in Physics	9
2.2.2	Spontaneous Symmetry Breaking	11
2.2.3	Phase Transition	14
2.3	Cosmic Strings In The Sky With Loops	17
2.3.1	Formation of a Cosmic String	17
2.3.2	Superconductors as a Pendant to Cosmic Strings	20
2.3.3	The Evolution of Cosmic String Network	23
2.3.4	Gravitational Wave Spectrum from Cosmic Strings	26
3	Methods	31
3.1	Computation of both Spectra	31
3.2	LISA: Laser Interferometer Space Antenna	35
3.2.1	Introduction to LISA	36
3.2.2	The Signal-to-Noise ratio	38
3.2.3	LISA's Noise	40
4	Results	43
4.1	Type-I versus Type-II Signals	43
4.1.1	Example: red star	45
4.1.2	Example: blue star	47

4.2	The New SNR Formula	50
5	Conclusion	53
6	Acknowledgments	55
A	The Moments of Gravitational Waves	57
B	LISA's Singal-to-Noise Ratio	59
B.1	The spectral density	59
B.2	The auto-correlation statistic	60
B.3	The response function of the detector	62
C	Additional Plots	63
C.1	SNR for GWB signals from type-I and type-II cosmic strings	63
C.2	Power-law approximations of the spectra	64
C.3	Power-law integrated sensitivity curves	66
	Bibliography	67

Introduction

1

One of the greatest achievements in physics of the last few decades may be the development of the Standard Model of particle physics. It provides a comprehensive overview of the fundamental particles and their interactions. By employing the local gauge symmetry description of the Standard Model $SU(3)_C \times SU(2)_L \times U(1)_Y$ these particle interactions can be systematically derived. Over the years, the Standard Model has been tested with remarkable precision in terrestrial laboratories.

However, open questions remain, such as “What is dark matter?” or “Can the forces be unified?”. Additionally, certain observations are inconsistent with the Standard Model, such as the neutrino mass. In order to solve these questions and problems, several theories have been proposed. For example, Grand Unified Theories (GUTs) aim to unify the strong, weak, and electromagnetic interactions. It has been discovered that the coupling constants of these interactions depend on the energy scales [1]. A GUT states that at high energies $\sim 10^{16}$ GeV, these three coupling constants converge to a single value. Above this energy scale, the symmetry group of the Standard Model is embedded within a larger symmetry group. For lower energies, the single coupling constant splits into the three distinct constants, and spontaneous symmetry breaking reduces the larger symmetry group to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ group of the Standard Model. [2].

Unfortunately, providing evidence for such a theory is challenging due to the technological and physical limitations of our laboratories. The primary tools that are used to test the Standard Model are particle colliders. However, they face certain constraints, such as their physical size and accessible beam energy [3]. In fact, it is highly unlikely that a terrestrial laboratory will ever reach the energy scales that are required to test a GUT.

Luckily, there exists a certain place in space or, better, a certain time where these energy scales were reached. According to the Big Bang theory, the universe began in an extremely hot and dense state and expanded rapidly, causing the temperature

and density to decrease over time until today. Thus, the very early universe, within the first $\sim 10^{-35}$ seconds after the Big Bang, matches the desired energy scales predicted by GUTs, offering a unique and ideal setting to validate these theories [4]. One problem remains: How can we observe the early universe? The earliest point in time we can look back to using electromagnetic radiation is ~ 380.000 years after the Big Bang, when the energy of the universe was significantly smaller than the energy scales required to test GUTs. This specific electromagnetic radiation became famous under the name cosmic microwave background (CMB). However, there exists a messenger that is able to propagate more freely through space and time and offers insights into the early universe: gravitational waves [5]. As predicted by Albert Einstein [6, 7], gravitational waves – ripples in the fabric of spacetime – are emitted by time-varying mass quadrupoles (or octupoles, hexadecapoles, ...) and travel through spacetime without being absorbed by matter as electromagnetic waves are. Analogous to the CMB, a stochastic gravitational wave background (SGWB) from early-universe physical processes is expected to persist to this day [8]. It is widely assumed that phase transitions, similar to those in a ferromagnet, also occurred in the early universe. As the universe cooled and dropped below a critical temperature, spontaneous symmetry breaking occurred. For example, the symmetry group of a GUT could break down to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry group of the Standard Model. The Kibble-Zurek mechanism explains how such a process can lead to objects, which are known as cosmological defects. These objects preserve the old state of the universe and contribute to the SGWB [9]. Thus, observing the SGWB might be the most promising way to uncover new physics, physics beyond the Standard Model. Over the past decades, it has become clear that particle physics can advance cosmology, and vice versa.

One promising candidate for a source of the SGWB is cosmic strings, a type of cosmic defect. A cosmic string is a one-dimensional topological defect that preserves the early universe's symmetric state within its core. It is expected that cosmic strings form networks, which contribute to the SGWB. Cosmic strings have a condensed matter analogue: vortices in superconductors. Just as superconductors are classified into type-I and type-II superconductors, cosmic strings can also be classified into type-I and type-II cosmic strings. While type-II cosmic strings have been studied extensively in the literature [10], type-I cosmic strings have received comparatively less attention.

This thesis explains the fundamentals of gravitational waves and spontaneous symmetry breaking during phase transitions. It then describes cosmic strings and the evolution of their networks and presents the formula for the gravitational wave spectrum from type-I and type-II cosmic strings. Using these spectra, the thesis explores whether the future space-based gravitational wave observatory, LISA, can distinguish between the two signals. For that, we use LISA's Signal-to-Noise ratio (SNR) and compare the two spectra in the phase space, which is spanned by the logarithmic derivative of the gravitational wave spectrum and its amplitude. This introduction is inspired by [\[5\]](#).

Theoretical Background

This thesis compares gravitational wave spectra from cosmic strings. For that, we need knowledge about gravitational waves and cosmic strings. In this chapter, we discuss the very basics of general relativity to derive Einstein’s quadrupole formula, which is used to calculate the emitted power via gravitational waves. Then, we study the phenomenon of symmetry breaking, which, during a phase transition, leads to the formation of cosmic strings. This process is studied in more detail in the second section of this chapter. Finally, the formulas for the gravitational wave spectra from type-I and type-II cosmic strings are presented in this chapter’s last section.

2.1 Gravitational Waves

In the year 1916, Einstein published a theory that should be a relativistic generalisation of Newton’s theory of gravity: “Die Grundlagen der allgemeinen Relativitätstheorie” [6]. In this publication, he presents the field equation, which should be named after him. In the same year, Einstein found wave solutions of his field equations and, thereby, predicted gravitational waves [11, 7]. The goal of this chapter is to show the gravitational wave spectrum from cosmic strings. For that, we will briefly discuss the necessities of general relativity in this section. The presented information is based on the papers [12, 13], and the book [14]. It should be mentioned that throughout this thesis we use the Einstein summation convention and Greek indices $\mu, \nu, \dots$ run over 0,1,2,3, whereas Latin indices $i, j, \dots$ run over 1,2,3 unless stated otherwise.

2.1.1 Einstein’s Field Equation

The first thing we do is a generalisation of the Minkowski metric $\eta_{\mu\nu}$, which is the metric used in special relativity, so that curvatures of spacetime can be considered:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu ,$$

where $g_{\mu\nu}$ is the metric tensor, a 4×4 symmetric matrix. Based on the weak equivalence principle, which states that all falling objects are accelerated identically regardless of shape or internal structure [15], we assume that a particle's movement under the influence of gravity is described by a spacetime geodesic, a path between two points that is locally a straight line. Therefore, gravity is based on a feature of its own instead of features of the interacting object. This feature is the spacetime curvature. Moreover, we use the strong equivalence principle: "A frame of reference in free fall in a gravitational field is locally equivalent to an inertial frame (without gravity)" [13], which ensures that locally special relativity is a special case of general relativity. The field equations connect the energy-mass distribution to the curvature of spacetime as follows:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (2.1)$$

$T_{\mu\nu}$ is known as the energy – momentum tensor, and describes the energy/mass and momentum distribution and is the source of gravity. $T_{\mu\nu}$ is also symmetric. T^{00} can be interpreted as the energy density and T^{0i} as the momentum density. As an example, the energy-momentum tensor of an ideal fluid is given as [16]:

$$T^{\alpha\beta} = M^{\alpha\beta} + P^{\alpha\beta} = \left(\rho + \frac{P}{c^2} \right) u^\alpha u^\beta - \eta^{\alpha\beta} P, \quad (2.2)$$

where ρ is the energy density, u^α four-velocity, P isotropic pressure, $M^{\alpha\beta} = \rho u^\alpha u^\beta$ the energy part and $P^{\alpha\beta} = P(\frac{u^\alpha u^\beta}{c^2} - \eta^{\alpha\beta})$ the pressure part.

$G_{\mu\nu} = G_{\mu\nu}(g_{\mu\nu}) = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor and describes the curvature of spacetime caused by $T_{\mu\nu}$. $R_{\mu\nu}$ is named the Ricci tensor and is a trace of the Riemann curvature tensor $R^\mu_{\nu\rho\sigma}$. R is named the Ricci scalar and is the trace of the Ricci tensor. To put it in the words of John Archibald Wheeler, as cited in [13]: "mass tells spacetime how to curve, and spacetime tells mass how to move.":

$$(\text{spacetime curvature}) = \frac{8\pi G}{c^4} (\text{mass} - \text{energy distribution}).$$

2.1.2 The Quadrupole Formula

Similar to an electromagnetic wave, which is a travelling electric field and a travelling magnetic field perpendicular to the wave's propagation direction, a gravitational

wave is a travelling gravitational field that stretches and squeezes space perpendicular to the wave's propagation direction [8]. We assume that the gravitational waves exist in a flat spacetime described by the Minkowski metric $\eta_{\mu\nu}$. Therefore, we express the gravitational wave as a small perturbation $h_{\mu\nu}$, such that the metric is given as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \text{ where } |h_{\mu\nu}| \ll 1 .$$

Thus, the Riemann curvature tensor in first order of $h_{\mu\nu}$ is given as:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}) .$$

We also use the trace-reversed metric perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad , \text{ where } h = \eta^{\mu\nu} h_{\mu\nu} ,$$

so that the field equations (2.1) become:

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu} .$$

Applying the Lorentz gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ gives us:

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} . \quad (2.3)$$

In vacuum, one can apply the transverse-traceless gauge (TT gauge) [14]:

$$h^{0\mu} = 0; \quad h_i^i = 0; \quad \text{and} \quad \partial^j h_{ij} = 0 ,$$

to show that the metric perturbation has two degrees of freedom and use it to solve equation (2.3) outside a source:

$$\square \bar{h}_{\mu\nu} = 0 \implies h_{ab}^{\text{TT}}(t, z) = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{ab} \cos\left[\omega\left(t - \frac{z}{c}\right)\right] , \quad (2.4)$$

where $h_\times$ and h_+ are called “cross” and “plus” polarisation. This is the solution for a wave travelling along the z-axis, so that $a, b \in \{1, 2\}$ are indices in the transverse (x, y) plane.

Anyway, for the derivation of the quadrupole formula, we use equation (2.3) that can be solved by Green's function $G(x - x')$:

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x'),$$

where $\square_x G(x - x') = \delta^4(x - x')$. From electrodynamics the retarded Green's function $G(x - x') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta(x_{\text{ret}}^0 - x'^0)$ is known, such that:

$$h_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right).$$

We can apply the TT-gauge outside the source, for which we introduce the operator $\Lambda_{ij,kl}$, which satisfies $h_{ij}^{\text{TT}} = \Lambda_{ij,kl} h_{kl}$, such that we obtain:

$$h_{ij}^{\text{TT}} = \frac{4G}{c^4} \Lambda_{ij,kl}(\mathbf{n}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right),$$

where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$. The radius of the source should be $d \ll r := |\mathbf{x}|$, so that we approximate the denominator $|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \mathbf{n} + \mathcal{O}(d^2/r)$, where $\mathbf{x}' \cdot \mathbf{n}$ is negligible when $r \rightarrow \infty$, so far from the source. This gives us:

$$h_{ij}^{\text{TT}} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\mathbf{n}) \int d^3x' T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right).$$

Next, we approximate the energy-momentum tensor by studying its Fourier transform:

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right) = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t-r/c+\mathbf{x}' \cdot \hat{\mathbf{n}}/c)+i\mathbf{k} \cdot \mathbf{x}'}, \quad (2.5)$$

where $d^4k = \frac{1}{c} d\omega d^3k$. For that, we use the low-velocity expansion:

- the typical velocity inside the source $\omega_S d$ is smaller than c ,
- the radiation's frequency is of order $\omega_S \implies \omega \sim \omega_S \sim \frac{v}{d}$,
- we introduce the reduced wavelength: $\bar{\lambda} = \frac{c}{\omega} \implies \bar{\lambda} \sim \frac{c}{v} d$ and
- we consider a non-relativistic scenario: $v \ll c$ such that $\bar{\lambda} \gg d$.

That leads to this approximation of the e-function, where we use $\frac{\omega}{c} \vec{x}' \cdot \hat{\vec{n}} \lesssim \frac{\omega s d}{c} \ll 1$:

$$e^{-i\omega(t-r/c+\vec{x}' \cdot \hat{\vec{n}}/c)} = e^{-i\omega(t-r/c)} \left[1 - \frac{i\omega}{c} x^i n^i + \frac{1}{2} \left(-\frac{i\omega}{c} \right)^2 x^i x'^j n^i n^j + \dots \right]. \quad (2.6)$$

One plugs this approximation into the formula of the energy-momentum's Fourier transform and defines several moments (more details shown in [14] or the Appendix A) to show that:

$$\left[h_{ij}^{\text{TT}}(t, \vec{x}) \right]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \ddot{Q}_{ij}^{\text{TT}}(t - r/c),$$

where $Q_{ij} = \int d^3r \rho(t, \vec{x}) (x^i x^j - \frac{1}{3} r^2 \delta^{ij})$ is the quadrupole moment. Using the formula for radiated power gives the desired quadrupole formula

$$\frac{dP}{dA} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle \Rightarrow P = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle \quad (2.7)$$

Since gravity does not have a negative charge, the quadrupole term is the first term of interest in the multipole expansion.

2.2 When the Symmetry Breaks

A cosmic string is an example of a cosmic defect. Besides cosmic strings, there are other examples of cosmic defects, such as domain walls, monopoles, and textures, as listed in [17]. All of these cosmic defects arise during a phase transition that transforms the system from a symmetric state to a state of lower symmetry, thereby breaking the symmetry. This section summarises the concept of symmetry breaking during a phase transition. As we will see in the next section 2.3, this is the process that can lead to the formation of cosmic strings. This section is based on the books [1, 18, 19], and the papers [20, 21], unless otherwise stated.

2.2.1 Symmetries in Physics

Before the breaking of a symmetry is explained, it is crucial to highlight the concept of symmetries in physics. A system or a law is invariant or symmetric under a certain transformation, such as rotation, translation, or reflection, if this transformation does not change the system's properties or the law. In other words, "[s]ymmetry is defined as invariance under a transformation or class of transformations" [18, p. 21].

One example of a symmetric system is a ferromagnet. Above its Curie temperature, the magnetic moments of the ferromagnet are not aligned; over long ranges they point independently in all directions. The system's Lagrangian has no preferred direction, and the ground state of the system is therefore invariant under any rotation. Symmetries are a powerful tool for theoretical physicists. Noether's theorem, for instance, shows that for each continuous symmetry an associated conserved quantity exists. For example, invariance under time translation implies conservation of energy. These conserved quantities are used to describe nature because of their conservation property.

As defined above, a symmetry can be described using a set of transformations. Mathematically, these transformations can be described as maps, which form groups. The mathematical framework used to study symmetries is therefore called group theory. A group is a set G of elements g with a binary operation, $\circ : G \times G \rightarrow G$ that is associative. The set G contains a unique identity element e , an inverse element g^{-1} for all elements g in G , and is closed under $\circ$: $g \circ g' \in G$ for all $g, g' \in G$. It can be shown from a physics perspective that a set of symmetry transformations satisfies all these properties. A simple example of these concepts is the group of rotations in two dimensions:

Consider the one-dimensional sphere S^1 in the Euclidean vector space $\mathbb{R}^2$ with radius r , so the set $S^1 = \{(a \ b)^T \in \mathbb{R}^2 : a^2 + b^2 = r^2\}$. This set S^1 , which is just a circle, is invariant under all rotations by an arbitrary angle. In group theory, this set of rotations is denoted as the group $SO(2) \equiv SO_2(\mathbb{R}) \subset \text{Aut}(\mathbb{R}^2)$, the group of the well-known 2×2 rotation matrices $\mathcal{R}_\vartheta$, where SO stands for special orthogonal, and $\text{Aut}(\mathbb{R}^2)$ denotes the set of automorphisms of $\mathbb{R}^2$. “Orthogonal” reflects the fact that these transformations preserve the length of vectors and angles between them, while “special” restricts the group to transformations without reflections [22]. The usual matrix multiplication is the operation of $SO(2)$, which is equivalent to applying rotations successively. Since two successively applied matrices can be expressed by one matrix, where the angle of both matrices is added modulo 2π , the set $SO(2)$ is closed under its operation. Moreover, it does not matter in which order one applies the rotations, which means that the operation is associative. A rotation by $\vartheta = 0$ does not change any vector of S^1 ; therefore, the identity 2×2 matrix is the unique identity element of $SO(2)$. For every rotation by an angle ϑ , we can apply another rotation $\vartheta' = 2\pi - \vartheta$, so that the first rotation is undone; the inverse of the

rotation matrix by an angle of ϑ is the rotation matrix by an angle of ϑ' . Since the rotation angle may take any real value modulo 2π , $SO(2)$ is a continuous symmetry group. A square, by contrast, is only invariant under four discrete rotations and therefore symmetric under a discrete symmetry group. One notices that while the set S^1 is invariant under these transformations, a single element of this set is clearly not.

In the same way, we can consider a circle in the one-dimensional unitary space $\mathbb{C}$: $A = \{r e^{i\beta}; 0 \leq \beta < 2\pi\}$. The rotation of an element then becomes a multiplication by a phase factor $e^{i\vartheta}$. In this case, the symmetry group is called $U(1) = \{e^{i\vartheta}; \vartheta \in \mathbb{R}\}$, the unitary group. Again, “unitary” points out that the length of a vector and the angle between two vectors stay unchanged under a transformation of this group. This symmetry group $U(1)$ will be the one symmetry group of interest in the thesis, because a breaking of this symmetry group gives rise to cosmic strings. Using the right group homomorphism between $SO(2)$ and $U(1)$, one can show that these groups are isomorphic.

2.2.2 Spontaneous Symmetry Breaking

As a first example of symmetry breaking, we come back to the ferromagnet. Above its Curie temperature, the ground state of the system is invariant under rotations. However, when the ferromagnet is cooled below this temperature, the magnetic moments spontaneously choose a single direction among all other possible directions and align with it. Although the Lagrangian remains invariant under rotations, the ground state is no longer invariant under rotations; the symmetry is broken. This leads to magnetisation of the material in the absence of an external magnetic field.

For cosmic defects, we assume a scalar field $\phi(x_\mu)$ as a function of spacetime that permeates all of spacetime. For cosmic strings, it will be a complex scalar field $\phi(x_\mu) \in \mathbb{C}$. Instead of the Lagrangian $L(q_i, \dot{q}_i)$ that is used in classical mechanics and particle physics to describe the behaviour of particles and depends on a set of generalised coordinates q_i and $\dot{q}_i$, we use the Lagrangian density $\mathcal{L}(\phi(x_\mu), \partial_\mu \phi(x_\mu))$ to describe our system, which is the natural formulation used in Quantum Field Theory. Thus, the treatment of a discrete system of particles becomes a continuous system using a field, where the particles are described as excitations of this field

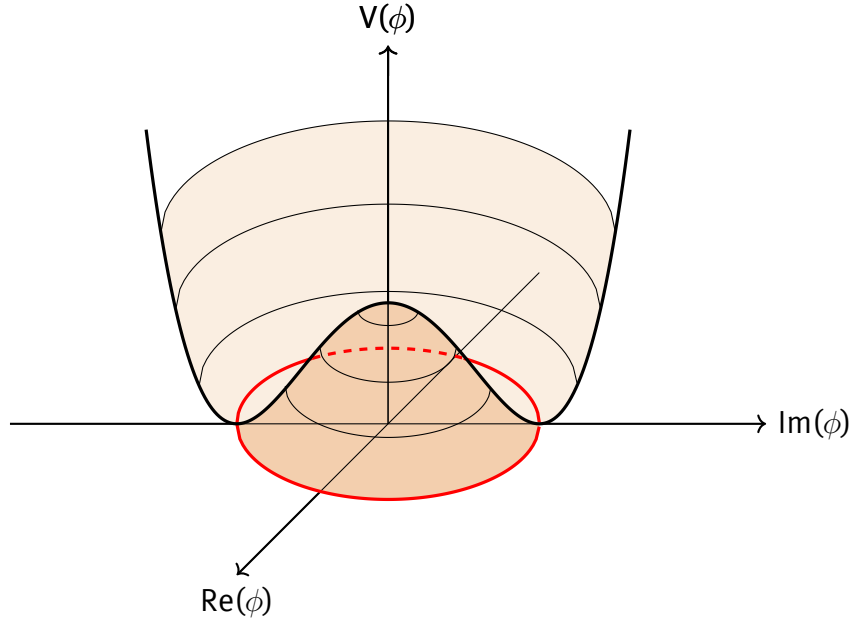


Figure 2.1: A sketch of the Mexican hat potential. The red circle marks the set of the new degenerate vacuum states. This sketch is based on the shown Mexican hat potential in [19].

ϕ that fulfil the field equations. Both formulations are connected via the following equation:

$$L(q_i, \dot{q}_i) = \int dx^3 \mathcal{L}.$$

In general, there are two types of transformations: global and local transformations. While for the former the transformation does not depend on spacetime, for the latter it does. Consider, as an example, the Lagrangian of a complex scalar field ϕ :

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi), \quad (2.8)$$

where $V(\phi) = \frac{1}{4}\lambda(\phi^*\phi - \eta^2)^2$ is called the Mexican hat potential and shown in figure 2.1. It is assumed that η , termed the vacuum expectation value (VEV), i.e. the value of ϕ at a minimum of the potential, and λ are real and greater than zero. Under the global U(1) transformation:

$$\phi \rightarrow \phi' = \phi e^{i\alpha}, \quad (2.9)$$

where α does not depend on spacetime, we see that the potential $V(\phi)$ and the Lagrangian density $\mathcal{L}$ are invariant, but, as long as $\phi \neq 0$ and $\alpha \neq n \cdot 2\pi$, the complex

scalar field is not invariant.

The symmetric state at the local maximum of the potential $\phi = 0$ is called the old or false vacuum state and is unstable. This Mexican hat potential has a set of infinitely many degenerate minima states, marked in red in figure 2.1, called the new or true vacuum states. Each element ϕ_0 of this set satisfies $|\phi_0|^2 = \eta^2$. In order to lower the system's energy, the complex scalar field settles into one of these true vacuum states from this set, so that it is non-invariant under the transformation and, therefore, the state is not symmetric.

However, Goldstone's theorem states that "if there is [a] continuous [global] symmetry transformation under which the Lagrangian is invariant, then either the vacuum state is also invariant under the transformation, or there must exist spinless particles of zero mass" [23]. By applying perturbation theory to this model, one can show that this Lagrangian (2.8) describes one massive and one massless particle. The spinless, massless boson is named the Goldstone boson. Although cosmic strings can be generated in this system, it is not of interest for this thesis because global cosmic strings, named after the global transformation, can lose energy more effectively via particle radiation than via gravitational wave radiation. Thus, the gravitational wave radiation is not strong enough to be measured.

This can be fixed by assuming a local U(1) transformation:

$$\phi \rightarrow \phi e^{i\alpha(x)}, \quad (2.10)$$

where $\alpha(x)$ depends on spacetime. We see that we get an additional term from the derivative term of the Lagrangian (2.8) when the transformation is applied:

$$\partial^\mu(\phi e^{i\alpha(x)}) = ((\partial^\mu \phi) + i\phi(\partial^\mu \alpha(x)))e^{i\alpha(x)}.$$

To keep the Lagrangian symmetric under the local transformation, we introduce a vector field $A_\mu(x)$, named the gauge field. That is the reason why this transformation is also referred to as a gauge transformation. In some sense, this gauge field absorbs the Goldstone boson that causes the mentioned problems, and since the gauge boson becomes massive, the problem is solved. The gauge field transforms as:

$$A_\mu(x) \rightarrow A_\mu(x) - q^{-1}\partial_\mu \alpha(x), \quad (2.11)$$

where q is the gauge coupling constant. Moreover, we use the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$, so that the Lagrangian becomes:

$$\mathcal{L} = D_\mu^* \phi^* D^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.12)$$

where the $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ term is the kinetic term of the gauge field. $V(\phi)$ is the already introduced Mexican hat potential. Therefore, the Lagrangian (2.12) is invariant under the local transformation (2.10) if we include the transformation of the gauge field (2.11). Since we are dealing with the same potential as before, the ground state of the scalar field is non-invariant; the symmetry is broken.

Before spontaneous symmetry breaking, the complex scalar field has two degrees of freedom that may be interpreted, for example, as the phase, so the motion along the circle of true vacuum states, and as the radius, motion in the radial direction. While the motion along the circle corresponds to a massless Goldstone boson, the motion in the radial direction corresponds to a massive boson. The gauge field also has two degrees of freedom. As for photons, it has two transverse polarisations. Using the local transformation (2.10), we can choose $\alpha(x)$ such that the phase of the scalar field vanishes at every point in spacetime. After the symmetry breaking, the gauge boson becomes massive and, therefore, has an additional longitudinal polarisation. Thus, the scalar field has one degree of freedom, and the gauge field has three.

2.2.3 Phase Transition

Let us return to the symmetry breaking of a ferromagnet one final time. For temperatures above the Curie temperature, thermal fluctuations dominate over the interaction between neighbouring magnetic moments, such that the magnetic moments of the ferromagnet are correlated only over short distances. As the temperature cools to the Curie temperature, this exchange interaction between magnetic moments becomes more dominant, and to lower the system's energy, the magnetic moments adjust their directions to align with those of neighbouring magnetic moments. Below the Curie temperature, this process is finished, so that the ferromagnet consists of domains in which the magnetic moments point in the same direction. The ground state loses its symmetry [24]. This process at temperatures of T_c is called a phase transition.

A similar mechanism occurs for a phase transition that forms cosmic defects. In a simple model, the temperature dependence of the potential looks like:

$$V_T(|\phi|) = -\frac{1}{2}m^2(T)|\phi|^2 + \frac{\lambda}{4}|\phi|^4 + \frac{\lambda}{4}\eta^4, \quad (2.13)$$

where $m^2(T) = m_0^2(1 - \frac{T^2}{T_c^2})$, $T_c^2 = \frac{24m_0^2}{\lambda}$ and m_0 is the mass of the scalar boson. As the Universe expands and cools down, the shape of the potential changes. The behaviour is illustrated by three limiting cases:

$$m^2(T) \begin{cases} < 0, & \text{for } T \gg T_c, \\ = 0, & \text{for } T = T_c, \\ \approx m_0^2 & \text{for } T \ll T_c. \end{cases} \quad (2.14)$$

Therefore, in the early universe, where the temperature was high, the potential had a parabolic shape. Thus, the potential possesses a single minimum, so that the ground state of the scalar field is at $\phi = 0$ and invariant under a U(1) transformation. However, the universe cools down, and the parabolic-shaped potential continuously deforms into the Mexican hat potential. This means that the single minimum at $\phi = 0$ continuously shifts into the degenerate minima of the Mexican-hat potential. During this process, the complex scalar field smoothly rolls down to one of these true vacuum states, thereby spontaneously breaking the U(1) symmetry. The choice of the new vacuum state is determined by thermal and quantum fluctuations and, therefore, varies from one region of space to another. We can characterise these patches with a correlation length ξ beyond which the scalar field becomes uncorrelated. The correlation length of the scalar field is limited by the causal horizon d_H , so that $\xi < d_H$ holds. This feature of patches in space with an uncorrelated scalar field is known as the Kibble mechanism and is one important ingredient for the formation of cosmic defects. If neighbouring regions take on different values of the scalar field, the scalar field tends to adapt at the intersection. Under certain conditions, this can force the scalar field back to its old vacuum state, creating a boundary with a large amount of energy. This is called a cosmic defect. We see the details for cosmic strings in the next section 2.3.

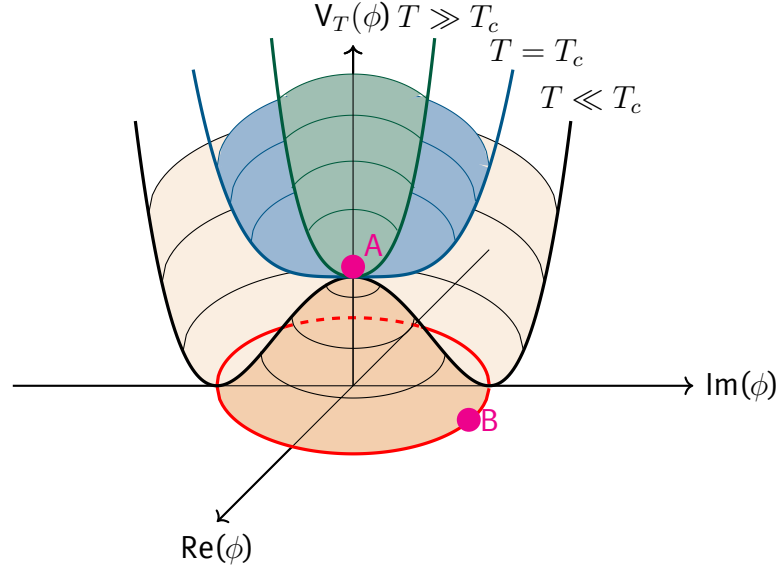


Figure 2.2: Illustration of a second-order phase transition. When the temperature T is above the critical temperature T_c , the potential has a parabolic form (green). As the temperature decreases, the shape transforms to the blue colored potential and then to the Mexican hat potential (orange). The scalar field rolls in one of the new vacuum states, starting from A and ending at B, for example. This sketch is based on the shown Mexican hat potential in [19] and inspired by [21].

This type of phase transition, denoted second-order, is characterised by the smooth change of the scalar field as soon as the temperature is below the critical temperature. Another possibility is that the potential develops a potential barrier as the universe cools down, so that the scalar field remains trapped at its origin $|\phi| = 0$ in a classical physics sense. In that case, the potential can look like this:

$$V(|\phi|, T) = (-m'^2 + T^2)|\phi|^2 - T|\phi|^3 + \lambda|\phi|^4,$$

where for $m' < T < 2m'\sqrt{\lambda}/\sqrt{4\lambda - 1}$ the ground state of the scalar field is not $\phi = 0$. Because of quantum mechanical tunnelling effects, the scalar field can overcome this potential barrier and settle into one of the new vacuum states. This leads to the creation of bubbles containing true vacuum in a sea of the old vacuum. These bubbles grow and collide, which can also generate gravitational waves [25], but is not of interest in this thesis. The scalar field is uncorrelated between these bubbles as well. So if they grow and collide, they may form boundaries at the

place of intersection that contain a lot of energy, as it is described for the second-order phase transition. Unlike a second-order phase transition, during a first-order phase transition, which is the term for this phase transition, the process does not occur as soon as the temperature is below the critical temperature. The scalar field therefore remains in the old vacuum until bubbles of the true vacuum nucleate through quantum tunnelling and expand.

2.3 Cosmic Strings In The Sky With Loops

We have seen that during a phase transition, regions in space form that contain different true vacuum states. This section explains how these patches can form the cosmic defect of interest, the cosmic string. As we will see, a certain condition must be satisfied for the creation of a cosmic string. Moreover, the details of cosmic strings and the difference between type-I and type-II are shown. Finally, the formulas for the gravitational wave spectrum of both types are explained. This section is based on the books [19, 26], and the papers [10, 21], unless otherwise stated.

2.3.1 Formation of a Cosmic String

The idea of the formation of a cosmic string through a first-order phase transition is shown in figure 2.3. Bubbles with different true vacuum states are formed and nucleate. Let us consider a closed path that surrounds a location where a cosmic string might be formed. If we go along this path, we can see how the scalar field changes from a true vacuum state to the old one if we pass through the bubble wall. As the bubbles grow, they expel the old vacuum state, such that along the whole closed path the scalar field has settled into true vacuum states. However, since the true vacuum states inside the bubbles differ, the scalar field still varies smoothly along the path. The condition for a cosmic string is that the scalar field's phase ϑ changes exactly by $\Delta\vartheta = n \cdot 2\pi$ as we walk once along this path. Distant from the string, the scalar field is given by $\phi \approx \eta e^{i\vartheta}$, where η is the VEV. If we now contract the closed path, at some point ϑ can not be well-defined, because the change of the scalar field's phase is discontinuous. This is resolved by a continuous change of the scalar field to $|\phi| = 0$, so that a core is created in which the scalar field is at its old vacuum state. A sketch of this is shown in figure 2.4.

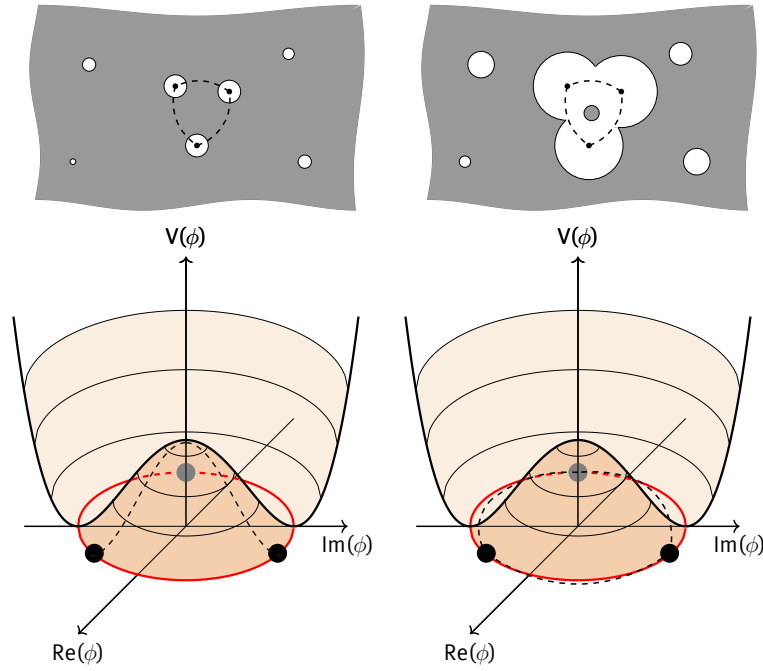


Figure 2.3: Illustration of a cosmic string formation during a first-order phase transition. The upper panels show the physical space, while the lower show the Mexican hat potential and the vacuum manifold. Grey marks the sea of the old vacuum state, and white marks the bubbles with a new vacuum state. The choice of the true vacuum states inside the three bigger bubbles is marked with black points on the Mexican hat potential. The dashed black line shows how the state changes on the vacuum manifold while we move through physical space. This sketch is based on the figure shown in [27].

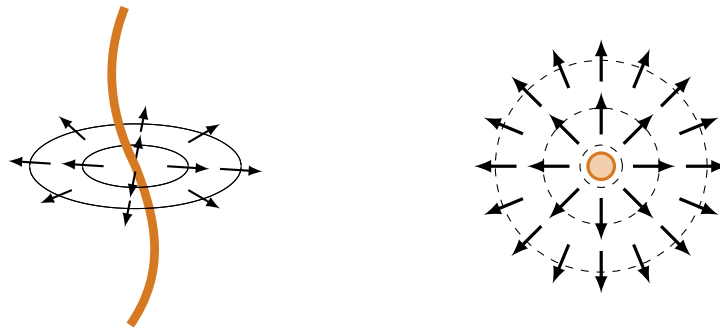


Figure 2.4: A sketch of the condition for a cosmic string. The left panel shows a part of a cosmic string in physical three-dimensional space, and the right panel shows a two-dimensional slice of the string, together with the scalar field demonstrated as arrows. This figure is based on a figure shown in [19].

For a second-order phase transition, we need to keep in mind that no bubbles are formed, because no potential barrier exists. The scalar field settles at a true vacuum state as soon as the temperature is below the critical temperature. Nevertheless, cosmic strings under the same condition can form, since there also exist patches with different scalar field states. This also tells us that cosmic strings must not have any loose ends, so they are either infinitely long or closed loops.

Mathematically, the condition is as follows: We have a group $G = U(1)$ of transformations under which the Lagrangian and the scalar field state are invariant before the symmetry breaking. Then, the spontaneous symmetry breaking occurs; the scalar field settles into one of the true vacuum states and is no longer invariant under G but under a smaller group of transformations H . The difference between G and H is that transformations of H not only keep the Lagrangian unchanged but also the scalar field state. Transformations of $G \setminus H$ change the scalar field's state but not the energy. The set of the system's ground states, the red-marked circle in figure 2.1, forms the vacuum manifold $\mathcal{M}$. In algebraic terms, it is the coset $\mathcal{M} = G/H$. The condition for cosmic strings is the existence of non-contractible loops in the vacuum manifold $\mathcal{M}$, which means that the first homotopy group of $\mathcal{M}$ is not trivial: $\pi_1(\mathcal{M}) \neq \mathbb{1}$, where $\mathbb{1}$ is the trivial group.

We can use the knowledge about the scalar field's phase change to calculate the magnetic flux in a string. Far from the string, the scalar field will settle into a true vacuum state; the scalar field can be approximated as a true vacuum state $\phi \approx \eta e^{i\alpha}$, where $\alpha = n\vartheta$, n is the winding number of the string, and ϑ is the cylindrical angle. Thus, the scalar field is at a minimum of the potential: $V(\phi) \approx 0$. The energy tries to minimise, so that we also get $(\partial_\mu + iqA_\mu)\phi \rightarrow 0$, which implies $A_\mu = \frac{1}{q}\partial_\mu\alpha$. The magnetic flux is given as:

$$\Phi_B = \int \underbrace{\mathbf{B}}_{\nabla \times \mathbf{A}} d\mathbf{S} \stackrel{\text{Stoke's Theorem}}{=} \oint \mathbf{A} d\mathbf{l} = \frac{-1}{q} \int_0^{2\pi} \frac{\partial \alpha}{\partial \vartheta} d\vartheta = \frac{-2\pi n}{q},$$

where $\mathbf{S}$ is a cross-section of the cosmic string and $\mathbf{l}$ is the boundary of the cross-section. This shows us that there is a quantised magnetic flux in the string.

Furthermore, we can approximate the energy per unit length μ , the tension, of a cosmic string. A cosmic string consists of two cores, a scalar core from the scalar

field and a gauge core from the gauge field. The widths δ of these cores can be approximated using the Compton wavelengths:

$$\delta_\phi \sim m_\phi^{-1} \quad \delta_A \sim m_A^{-1}$$

where the masses $m_\phi = \sqrt{\lambda}\eta$ and $m_A = q\eta$ can be determined by applying perturbation theory to the Lagrangian (2.12). Inside the cosmic string, the scalar field is zero, as we discussed. So the energy per unit length of the scalar field is given by the energy of one point in the string times the slice's area of the scalar core:

$$V(0) \sim \lambda\eta^4 \implies \mu \sim \lambda\eta^4\delta_\phi^2 \sim \eta^2.$$

The gauge field's energy per unit length is given by the energy density of the magnetic field B times the slice's area of the gauge core:

$$\mu \sim B^2\delta_A^2 \sim \eta^2.$$

Thus the total energy per unit length of a cosmic string is $\mu \sim \eta^2$. A GUT local string, where $\eta \sim 10^{16}$ GeV, has an energy per unit length of $\mu \sim 10^{22}$ g cm⁻¹ as a hint of their cosmological significance [19].

The difference between a type-I and a type-II cosmic string is the ratio of scalar and gauge mass $\beta = (\frac{m_\phi}{m_A})^2$. If $\beta < 1$, then the scalar core is wider than the gauge core, and it is a type-I cosmic string. If $\beta > 1$, then the gauge core is wider than the scalar core, and it is a type-II cosmic string.

2.3.2 Superconductors as a Pendant to Cosmic Strings

Before we derive the formulas for the gravitational wave spectra, we have a brief look at where the names type-I and type-II come from. For that, we briefly turn to condensed matter theory, in which superconductors are also separated into two classes: type-I and type-II superconductors. This subsection presents a short introduction to superconductors and explains the differences between type-I and type-II superconductors, such that they can be compared to type-I and type-II cosmic strings at the end. The information is taken from the books [28] and [24].

Besides the feature of superconductivity, which alone does not uniquely characterise superconductors, superconductors exhibit the Meissner-Ochsenfeld effect. The Meissner-Ochsenfeld effect states that a superconductor expels an externally applied magnetic field from its interior below a critical temperature T_c and below a critical magnetic field B_c . The distinctive feature of superconductors is that they expel the magnetic field independently of the order of processes: The superconductor can be cooled down before the external magnetic field is applied, or vice versa. Therefore, a superconductor is not only a perfect conductor but also a perfectly diamagnetic material, meaning its magnetisation satisfies $\mathbf{M} = -\mathbf{H}_a$, where $\mathbf{H}_a = \frac{1}{\mu_0}\mathbf{B}_a$ is the externally applied field. A sketch of the Meissner-Ochsenfeld effect is shown in figure 2.5. Below the critical temperature, an attractive force between electrons binds two electrons with opposite spin into one Cooper pair. Thus, instead of two fermions, we have one boson, and all Cooper pairs can occupy the same state. This leads to superconductivity.

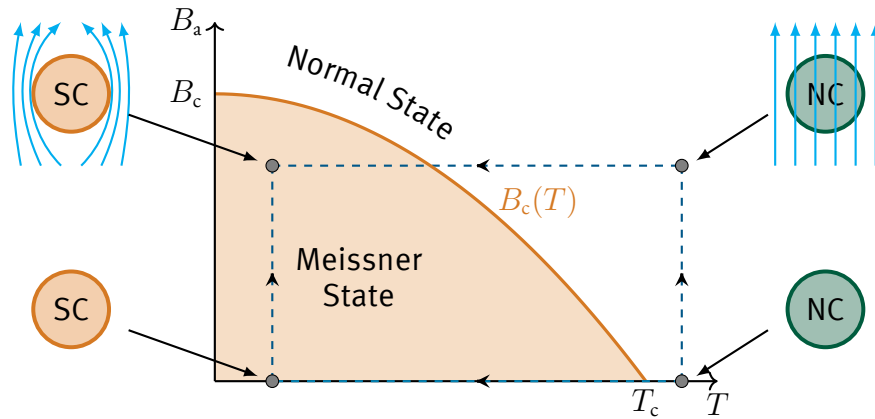


Figure 2.5: Illustration of the Meissner-Ochsenfeld effect. For a superconductor, it does not matter if first the temperature is cooled down and then the magnetic field is applied or the other way around; in both cases, it will expel the magnetic field. The critical magnetic field is given by $B_c(T) = B_{c,0}(1 - (T/T_c)^2)$. This figure is based on a figure shown in [24].

However, that is only the complete story for type-I superconductors, but not for type-II superconductors. Type-II superconductors have an additional state, called the mixed or vortex state, above the Meissner state, as demonstrated in figure 2.6. When the superconductor is in this state, it does not expel the applied magnetic field completely, so that the magnetic field can penetrate the superconductor. As a result, a periodic pattern, an Abrikosov vortex lattice, of normal cores inside the

superconductor is created. These normal cores are surrounded by vortices of the superconducting current and allow the magnetic field to penetrate the superconductor. The magnetic flux inside a superconductor is quantised $\Phi = \frac{nh}{2e}$, and when it penetrates the superconductor, n magnetic flux tubes are formed. Superconductors have two characteristic distances in the Ginzburg-Landau theory: the penetration depths λ and the coherence length ξ . The former describes the distance over which the magnetic field penetrates the superconductor; the latter is roughly the size of a Cooper pair. We introduce the Ginzburg-Landau parameter $\kappa = \frac{\lambda}{\xi}$ to distinguish between type-I and type-II superconductors. If $\kappa < 1/\sqrt{2}$ then it is a type-I superconductor; otherwise, it is a type-II superconductor.

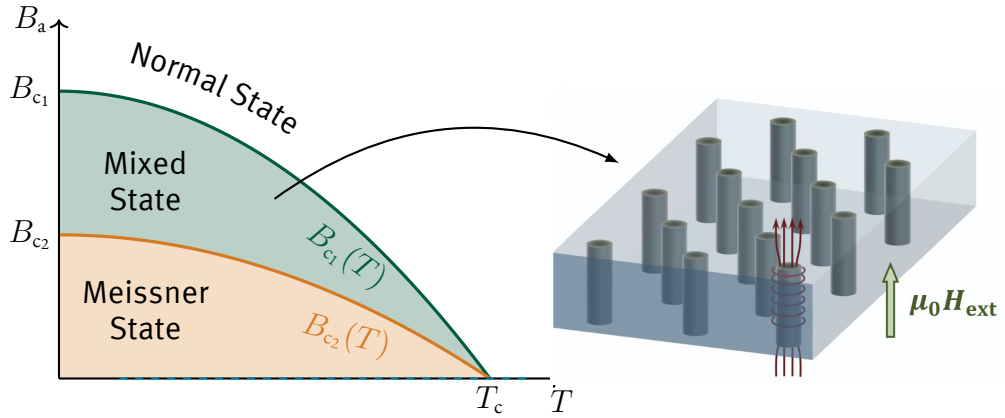


Figure 2.6: The Meissner and mixed states of a type-II superconductor. In the mixed state, the magnetic field penetrates the superconductor, creating an Abrikosov lattice of quantised magnetic flux tubes within the material. This figure of the Abrikosov pattern, as well as the idea of this complete figure is taken from [24].

Let us come back to cosmic strings and think about in what manner the scalar and gauge fields act. To minimise the energy, the area in which the scalar field is not in a true vacuum state and the scalar field changes over space has to be as small as possible. Thus, the scalar field acts in an attractive manner. The gauge field acts in a repulsive manner, since the overlap of two parallel magnetic fluxes increases the energy. In type-I cosmic strings, the magnetic flux is confined inside the core, so that type-I cosmic strings attract each other due to the wider scalar core. When two type-I cosmic strings intersect at one point, they can stick together and zip up along the strings, as demonstrated in figure 2.7. So, strings with higher windings are formed. This leads to higher string tensions, which influence the gravitational

wave spectrum. Type-II cosmic strings repel each other because the magnetic flux is not confined. Therefore, type-II cosmic strings with higher winding numbers are unstable, and they will split into strings with winding number $n = 1$. In summary [21]:

$$\beta = \begin{cases} < 1 & \text{n-vortex stable (B flux lines confined in core) - type-I} \\ > 1 & \text{Unstable: splitting into } n \text{ vortices of flux } \frac{2\pi}{q} \text{ - type-II.} \end{cases}$$

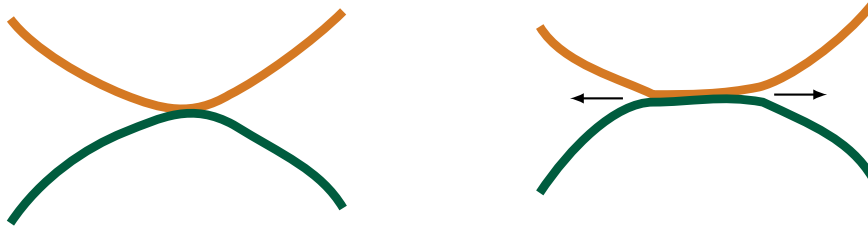


Figure 2.7: A sketch of type-I strings zipping up. Two (or more) type-I cosmic strings intersect at some point. Due to the attractive force of the scalar field, they stick together and zip up along the strings. This sketch is based on a figure shown in [26].

2.3.3 The Evolution of Cosmic String Network

We have seen how cosmic strings form. The formation of a cosmic string occurs several times in the universe, so that a cosmic string network is established, which generates a stochastic gravitational-wave background. With the purpose of calculating the emitted gravitational waves from this network, we have to study it briefly. The first thing we have to ensure is that the energy density of the network does not decrease more slowly than the critical energy density.

Domain walls, another cosmic defect, for example, have in the usual description an energy density that decreases more slowly than the critical energy density, so that they would dominate the universe. They arise from spontaneous symmetry breaking of a discrete $\mathbb{Z}_2$ symmetry, where the scalar field is real, and the potential is a double-well potential. Thus, in this model two true vacuum states exist: $\pm\eta$. Therefore, domains with different VEV's appear, and around the boundaries, domain walls form. By introducing a shift of the potential value between the two minima ($V(+\eta) \neq V(-\eta)$), a domain wall becomes unstable and collides. Therefore, the energy density no longer decreases more slowly than the critical energy density [29].



Figure 2.8: Sketches of possible loop formations. It is based on a figure shown in [26].

For cosmic strings, we allow the formation of string loops. This means that strings can interact with other strings or interact with themselves, thereby chopping off loops. The interaction between cosmic strings is also called intercommutation. Examples are given in figure 2.8. Therefore, the network of local cosmic strings produces string loops continuously that oscillate due to their high tension and lose energy via gravitational waves. These are the gravitational waves we are interested in, because gravitational radiation from the long string network is expected to be much weaker than that emitted by oscillating loops. Through this loss of energy, the energy density of the network is never larger than the critical energy density, as we will see in this subsection. In order to understand the behaviour of the string network's energy density, we need to study its dynamics using the so-called VOS model.

Since we are dealing with strings, we can approximate these as thin one-dimensional objects propagating through spacetime. Thus, we describe them using a two-dimensional world sheet $x^\mu = x^\mu(\zeta^a)$, where $a \in \{0, 1\}$. The zeroth coordinate is time-like, the first is space-like. We obtain the following metric:

$$ds^2 = \underbrace{g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b}}_{\gamma_{ab}} d\zeta^a d\zeta^b,$$

where γ_{ab} is the two dimensional world sheet metric. The Lagrangian density can be approximated using general spacetime transformations, arbitrary reparametrisation of the world sheet and dimensional analysis of the Lagrangian density as

$\mathcal{L} = -\mu + \alpha\kappa + \beta\frac{\kappa^2}{\mu} + \dots$, where $\kappa \ll \mu$ is the curvature of the string and α and β are numerical coefficients. Hence, the action of the string is given as:

$$S = \int \mathcal{L} \sqrt{-\gamma} d^2\zeta = -\mu \int \sqrt{-\gamma} d^2\zeta,$$

where this action is called the Nambu-Goto action and $\gamma = \det(\gamma_{ab})$. Using this action, we first calculate the energy-momentum tensor $T^{\mu\nu}$ and then the energy:

$$T^{\mu\nu} \sqrt{-g} = -2 \frac{\delta S}{\delta g_{\mu\nu}} \implies E = \int d^3x T^{00}(x) = \mu a \int \epsilon d\zeta^1,$$

where $\epsilon = \sqrt{x'^2/(1 - \dot{x}^2)}$. To study the evolution of the string network over time, we take the derivative with respect to time, and instead of the energy, we focus on the energy density $\rho \propto E/a^3$, where a is the scale factor. This gives us the following differential equation:

$$\frac{d\rho}{dt} + 2H(1 + v^2)\rho = 0, \quad (2.15)$$

where v is the root-mean-square velocity of the strings

$$v^2 = \langle \dot{x}^2 \rangle = \frac{\int \dot{\mathbf{x}}^2 \epsilon d\zeta^1}{\int \epsilon d\zeta^1}.$$

For further results on the large-scale dynamics of the network, we apply the velocity-dependent one-scale (VOS) model, originally shown in [30] and [31]. The VOS model assumes that the string network of infinite strings can be described by one scale length L , the correlation length beyond which the string's direction becomes uncorrelated, which is set equal to the average interstring distance. This is used to show that the network approaches a scaling regime in which the evolution is self-similar; L stays constant relative to the horizon size. For that, we focus on the infinite strings, marking the parameters with ∞ as a subscript. The energy density is, therefore, $\rho_\infty = \mu/L^2$ and equation (2.15) becomes:

$$\frac{d\rho_\infty}{dt} + 2H(1 + v_\infty^2)\rho_\infty + \left(\frac{d\rho_\infty}{dt}\right)_{\text{to loops}} = 0, \quad (2.16)$$

where the $(d\rho_\infty/dt)_{\text{to loops}}$ term takes the energy loss due to loop creation into account, and the term in the middle is the energy loss due to expansion. The string network always loses energy and never gains energy from loops because we assume that

loops cannot form infinite strings. In [32], all simulated local U(1) string-string collisions with relative approach velocities up to $v/c = 0.9$ are found to intercommute. So, we assume an intercommutation probability $P = 1$, which gives us:

$$\left(\frac{d\rho_\infty}{dt} \right)_{\text{to loops}} = \frac{\tilde{c}\rho_\infty v_\infty}{L}, \quad (2.17)$$

where $\tilde{c}$ is a phenomenological parameter, named the loop-chopping efficiency, that indicates the efficiency of the loop-chopping process. This equation is shown and explained in [30]. In [33], it is stated that for Nambu-Goto strings, the VOS model is in good agreement with network simulations for a loop-chopping efficiency of $\tilde{c} = 0.23 \pm 0.04$. Using $\rho_\infty = \mu/L^2$, equation (2.16) looks like this in terms of L :

$$\frac{dL}{dt} - (1 + v_\infty^2)HL - \frac{\tilde{c}v_\infty}{2} = 0. \quad (2.18)$$

In [10], it is shown how to obtain the differential equation for v_∞ :

$$\frac{dv_\infty}{dt} = (1 - v_\infty^2) \left[\frac{k}{L} - 2Hv_\infty \right], \quad (2.19)$$

where k , called the momentum parameter, is related to the small-scale structure. A more detailed analysis can be found in [31]. So the parameters L and v_∞ describe the evolution of the long string network. If the scale factor obeys $a(t) \propto t^\beta$, $\beta = \text{const.}$ and $\beta \in (0, 1)$, then we have the attractor solution $L \propto t$ and $v_\infty = \text{const.}$, [31]. This implies that the cosmic string energy density is given as:

$$\rho_\infty = \mu \times \frac{\text{length}}{\text{volume}} = \mu \frac{L}{L^3} \sim t^{-2},$$

and therefore does not decrease more slowly than the critical energy density of the universe $\rho_{\text{crit}} \propto t^{-2}$. We can introduce the parameter $\xi(t)$, such that $L = \xi(t)t$. In [10], values of v_∞ and ξ are determined for the matter domination era and the radiation domination era.

2.3.4 Gravitational Wave Spectrum from Cosmic Strings

Finally, we can discuss the gravitational wave spectra. We start with the gravitational wave emission from one cosmic string loop. Then, the number of string loops is

approximated to obtain the complete spectrum. The gravitational wave emission of one loop is given by [19]:

$$P = \Gamma G \mu^2 \quad (2.20)$$

where G is the gravitational constant and $\Gamma \simeq 50$ a constant [34]. As explained in a talk by Jun.-Prof. Dr. Kai Schmitz at the Chung-Ang university, which were recorded and uploaded [35], we can use the quadrupole formula (2.7) and an estimation of the loop's quadrupole: The length of a loop is l , so the quadrupole moment is $Q \sim M l^2$, where $M = \mu l$ is the mass of a loop, and $\omega \sim 1/l$ is the oscillation frequency of a loop. Hence, the emitted energy of loops via gravitational waves per unit time is given by:

$$\dot{E}_{\text{GW}} = \frac{G}{5c^5} \langle \ddot{Q}_{ij}, \ddot{Q}_{ij} \rangle \sim G Q^2 \omega^6 \sim G \mu^2.$$

The energy of a loop is given by $E = \mu l$, and since we know how much energy the loop loses over time, we can calculate the length l of the string for t

$$l = l' - \Gamma G \mu (t - t'), \quad (2.21)$$

where the loop had the length l' at time t' . Moreover, we can split the gravitational power by frequencies/harmonics of the loop's length:

$$f_k = \frac{2k}{l}, \quad (2.22)$$

where k is an integer. Thus, we can calculate the gravitational power (2.20) per frequency mode k :

$$\frac{dE_k}{dt} = \Gamma_k G \mu^2 \quad \text{with} \quad \Gamma_k = \frac{\Gamma}{\zeta(q)} k^{-q}, \quad (2.23)$$

where $\zeta(q)$ is the Riemann zeta function and q is the spectral index that accounts for the shape of the loops and small-scale structure that dominate the emission. For this thesis, it is expected that the radiation comes from cusps, for which $q = 4/3$ holds.

The last thing we need to know is the number of loops, or, better yet, the number density of loops, $n(l, t)$. For that, we introduce the loop production function $f(l, t)$, where $f(l, t)dl$ is the number density of non-self-intersecting string loops with the

length between l and $l + dl$ produced per time unit [36]. We calculate the loop number density by using the loop production function:

$$n(l, t) = \int_{t_{\text{ini}}}^t f(l'(l, t, t'), t') \left(\frac{a(t')}{a(t)} \right)^3 dt', \quad (2.24)$$

where t_{ini} is the time at which loop production becomes important and the scale factor term accounts for the dilution of loops by expansion of the universe. The energy that the string network loses through loop production should be the same as the energy of the string loop, which gives us:

$$\frac{1}{\gamma} \left(\frac{d\rho_{\infty}}{dt} \right)_{\text{to loops}} = \mu \int_0^{L(t)} l f(l, t) dl, \quad (2.25)$$

where the Lorentz factor $\gamma = \sqrt{2}$ accounts for the center-of-mass velocity of the loops and $\alpha \simeq 0.1/\xi(t)$. Originally motivated by [37], we can assume that all loops at all times are created with the same fraction of the VOS parameter L : $l_{\star}(t) = \alpha L(t)$, where the subscript $\star$ denotes the birth of the string loop. Applying this delta distribution of the loop lengths to equation (2.25) and the expression of the energy loss to loops (2.17) gives us:

$$f(l, t) = \mathcal{F} \frac{\tilde{c}v_{\infty}(t)}{\sqrt{2}lL^3(t)} \delta(l - \alpha L(t)), \quad (2.26)$$

where $\mathcal{F} \sim 0.1$ takes the error of the loop length's delta distribution into account [36]. Using equation (2.24), we obtain:

$$n(l, t) = \Theta(t - t_{\star}) \Theta(t_{\star} - t_{\text{ini}}) \mathcal{F} \frac{\tilde{c}v_{\infty}(t)}{\sqrt{2}\xi(t)^4 t_{\star}^4} \frac{1}{|\Gamma G\mu + \alpha\xi(t_{\star}) + \alpha \frac{d\xi}{dt}(t_{\star})|} \left(\frac{a(t_{\star})}{a(t)} \right)^3 \quad (2.27)$$

where the Heaviside functions ensure that only loops are taken into account that are both already produced and produced after t_{ini} . $t_{\star}$, the time of a loop's birth, is the time at which the loop with length l at time t had the length $L(t_{\star}) = \alpha\xi(t_{\star})t_{\star}$ according to equation (2.21), so the solution of:

$$l - \Gamma G\mu(t_{\star} - t) - \alpha\xi(t_{\star})t_{\star} = 0.$$

Now, we can put all things together. Equation (2.23) can be expressed as power per unit frequency:

$$\frac{dP_k^{\text{GW}}}{df}(l, f) = G\mu^2 l \Gamma_k \delta(lf - 2k) \Rightarrow \frac{dP^{\text{GW}}}{df}(l, f) = G\mu^2 l \sum_{k=1}^{\infty} \Gamma_k \delta(lf - 2k).$$

So the energy density of gravitational waves emitted at t' from string loops per unit time and frequency is the power per unit frequency of a loop with length l times the number density $n(l, t')$ of loops with length l , integrated over all possible loop lengths:

$$\frac{d\rho_{\text{GW}}}{dt' df'}(t', f') = \int_0^{\alpha L(t')} \frac{dP^{\text{GW}}}{df'}(l, f') n(l, t') dl.$$

For the complete spectrum that we measure today, we have to integrate this expression over time starting at t_{ini} until today, t_0 . However, we have to keep in mind that the radiation redshifts like $\propto a^{-4}$ and the frequency like $\propto a^{-1}$ due to the expansion of the universe:

$$\frac{d\rho_{\text{GW}}}{df}(f) = G\mu^2 \int_{t_{\text{ini}}}^{t_0} dt' \left(\frac{a(t')}{a(t_0)} \right)^3 \int_0^{\alpha L(t')} dl n(l, t') l \sum_{k=1}^{\infty} \Gamma_k \delta(l(f a(t_0)/a(t')) - 2k),$$

where $\delta(l(f a(t_0)/a(t')) - 2k) = \frac{1}{(f a(t_0)/a(t'))} \delta(l - \frac{2k}{f a(t_0)/a(t')})$. Instead of the energy density, we study the ratio of the energy density and the critical energy density and take a logarithmic frequency scale:

$$\begin{aligned} \Omega_{\text{GW}}(f) &= \frac{d\rho_{\text{GW}}(f)}{\rho_c d\ln(f)} \\ &= \frac{8\pi G^2 \mu^2 f}{3H_0^2} \frac{1}{f} \int_{t_{\text{ini}}}^{t_0} dt' \left(\frac{a(t')}{a(t_0)} \right)^4 \int_0^{\alpha L(t')} dl n(l, t') l \sum_{k=1}^{\infty} \Gamma_k \delta\left(l - \frac{2k}{f a(t_0)/a(t')}\right), \end{aligned}$$

where the critical energy density is $\rho_c = \frac{3H_0^2}{8\pi G}$. Solving the integral over the loop's length gives the spectrum from type-II cosmic strings:

$$\Omega_{\text{GW}}^{(\text{II})}(f; G\mu) = \frac{16\pi}{3f} \left(\frac{G\mu}{H_0} \right)^2 \int_{t_{\text{ini}}}^{t_0} dt' \left(\frac{a(t')}{a(t_0)} \right)^5 \sum_{k=1}^{\infty} k \Gamma_k n\left(\frac{2ka(t')}{fa(t_0)}, t'\right). \quad (2.28)$$

For type-I cosmic strings, we need to keep in mind that they have higher winding numbers due to the zippering effect between strings and, therefore, higher string

tensions μ . As executed in [38] and [39] and originally derived in [40], we express the string tension as a function of the string's winding number N_ω :

$$\mu(N_\omega) = 2\pi v_\phi^2 c_1 (1 + c_2 \ln N_\omega),$$

where c_1 and c_2 are numerical coefficients as a function of $\beta^{-1} = \left(\frac{m_A}{m_\phi}\right)^2 = \left(\frac{gv_\phi}{m_\phi}\right)^2$, the ratio of the boson and scalar mass:

$$c_1 \approx \frac{4.2}{\ln(\beta^{-1})} + \frac{14}{\ln^2(\beta^{-1})} \quad c_2 \approx \frac{2.6}{\ln(\beta^{-1})} + \frac{57}{\ln^2(\beta^{-1})}$$

and v_ϕ is the VEV of the complex scalar field and g is the gauge coupling constant. The higher string tension is then taken into account in the spectrum by multiplying the factor:

$$\mathfrak{a} := 1 + c_2 \ln(N_\omega^{\max}(\mathfrak{t}_*)) \quad , \text{ where } N_\omega^{\max}(t) = \left(\frac{t}{M_p/(m_\phi v_\phi)} \right)^{0.22}.$$

$N_\omega^{\max}$ is the maximal winding number of the strings when they enter the scaling regime, and M_p is the Planck mass. Thus, the formula for the spectrum from type-I cosmic strings is:

$$\Omega_{\text{GW}}^{(I)}(f; m_\phi, \nu_\phi) = \frac{16\pi}{3f} \left(\frac{G\mu_0}{H_0} \right)^2 \int_{t_{\text{ini}}}^{t_0} dt' \left(\frac{a(t')}{a(t_0)} \right)^5 \mathfrak{a}^2 \sum_{k=1}^{\infty} k \Gamma_k n\left(\frac{2ka(t')}{fa(t_0)}, t'\right), \quad (2.29)$$

where $\mu_0 = \mu(N_\omega = 1)$.

For type-II cosmic strings, $t_{\text{ini}} \propto \frac{t_{\text{pl}}}{(G\mu)^2}$, the time when the friction era of long cosmic strings ends and string loops are formed, is a good approximation for the initial time. For type-I cosmic strings, $t_{\text{ini}} = \frac{1}{\alpha} \frac{m_\phi^{-1}}{(G\mu)^2}$ is the time gravitational waves dominate energy emission. Throughout this thesis, we work with $h^2 \Omega_{\text{GW}}$ instead of Ω_{GW} . This is because the Hubble parameter H_0 itself is uncertain, whereas $H_0/h = 100 \frac{\text{km}}{\text{s Mpc}}$ [41].

Methods

In the last chapter, we learned what a cosmic string is and how they arise, what the differences between type-I and type-II are and saw the formulas for the spectra. Now, we shall recall that the aim of this thesis is to compare the spectra of type-I and type-II cosmic strings from the perspective of LISA, a future space-based gravitational wave observatory [42]. To complete this task, we need computations of the spectra and a certain measure for the similarity between both spectra, which will be the Signal-to-Noise ratio (SNR). This chapter is dedicated to explaining how we obtain code implementations of the spectrum formulas (2.29) and (2.28), and briefly introducing the Laser Interferometer Space Antenna LISA as well as the calculation for the SNR.

3.1 Computation of both Spectra

As we have seen in section 2.3.4, the gravitational wave spectra from type-II cosmic strings as well as from type-I cosmic strings depend on the scale factor $a(t)$. Furthermore, for more accurate results, we will not assume, as suggested in subsection 2.3.3, a fixed value of the VOS parameters ξ and v_∞ once for the matter-dominated era and once for the radiation-dominated era, respectively, but do a numerical computation of both parameters. In the following, the numerical computations of the scale factor $a(t)$, ξ and v_∞ used for the spectra are shown. Moreover, an expression for the spectra is shown that depends only on the fundamental mode, which saves a lot of computation time.

Before we discuss the computation of the scale factor $a(t)$, we briefly summarise what the scale factor is and how it is derived. Unless stated otherwise, this part follows [43].

The main assumption for the derivation of the scale factor is called the Cosmological Principle, which states: “[...] the universe is spatially homogeneous and isotropic”

[44, p. 409]. This means that the universe looks the same independent of the observer's position and the direction the observer looks.

Therefore, the space has to have a spatially constant curvature $K(t)$, which leads to the line element in terms of the spherical coordinates t, R, ϑ, φ :

$$ds^2 = c^2 dt^2 - \left(\frac{dR^2}{1 - KR^2} + R^2 d\Omega \right),$$

where $R(t)$ and $K(t)$ are functions of time and $d\Omega = d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$. We introduce the comoving coordinate r , where the physical distance is related by $R(t) = a(t)r$:

$$ds^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega \right),$$

where $\kappa = a^2(t)K(t)$. This metric is named after H.P. Robertson and A.G. Walker. We see that two objects with a fixed comoving distance r_0 have a relative velocity:

$$\frac{dR(t)}{dt} = \dot{a}(t)r_0 = \frac{\dot{a}}{a}R = HR,$$

where $H = \dot{a}/a$ is called the Hubble parameter. To determine the scale factor, we have to solve the field equations (2.1). For that, we apply the Cosmological Principle to the covariant energy-momentum tensor $T^{\mu\nu}$, which is then given by the covariant energy-momentum tensor of a perfect fluid: $T^{\mu\nu} = \text{diag}(\rho(t), -p(t), -p(t), -p(t))$, where $\rho(t)$ is the energy density and $p(t)$ is the pressure. We get this expression from equation (2.2) for $u^0 = c$ and $u^i = 0$. By applying the conservation of energy, we obtain the two Friedmann equations:

$$\begin{aligned} H^2(a) &= \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2}, \\ \left(\frac{\ddot{a}}{a} \right) &= \frac{-4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right). \end{aligned}$$

The first Friedmann equation allows us to determine the critical energy density $\rho_c(a) = 3m_{\text{Pl}}^2 H^2(a)$, where $m_{\text{Pl}} = (8\pi G)^{-\frac{1}{2}}$, the energy density of a spatially flat universe $\kappa = 0$. Furthermore, we introduce the density parameter $\Omega = \rho/\rho_c$, such that the first Friedmann equation for a flat universe becomes:

$$1 = \Omega(t) = \Omega_r(t) + \Omega_m(t) + \Omega_\Lambda(t).$$

This is used to compute the Hubble parameter:

$$H(a)^2 = H_0^2 \left[\Omega_r^{(0)} \left(\frac{a_0}{a} \right)^4 + \Omega_m^{(0)} \left(\frac{a_0}{a} \right)^3 + \Omega_\Lambda^{(0)} \right]. \quad (3.1)$$

Once the evolution of the scale factor and the Hubble parameter $H(a)$ as a function of the scale factor is known, it can be used to determine time $t(a)$ as a function of the scale factor via:

$$t(a) = \int_0^a \frac{da'}{\dot{a}'} = \int_0^a \frac{da}{aH(a)}$$

In [43] it is shown that the scale factor can also be expressed in terms of the entropic effective number of degrees of freedom g_s :

$$a \propto (g_s^{\frac{1}{3}} T)^{-1} \implies a(T) = \left(\frac{g_s^{(0)}}{g_s} \right)^{\frac{1}{3}} \frac{T_0}{T}.$$

It is also possible to express the total energy density in relativistic particles ρ_r in terms of the energetic effective number of degrees of freedom g_ρ :

$$\rho_r = \frac{T^4}{30} \pi^2 g_\rho.$$

This offers the opportunity to use the DOF's tabulated data computed in [45] to compute the scale factor $a(t)$ as a function of time. As discussed in [10], equation (3.1) is used to compute the Hubble parameter $H(a)$, where $\rho(a)$ is determined using the DOFs, $H_0 = 67.39 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m^{(0)} = 0.3142$ and $\Omega_\Lambda^{(0)} = 0.6858$ [46]. To obtain continuous functions for the numerical computation, the tabulated relations $g_s(T)$ and $g_\rho(T)$, together with the numerically determined scale factor $a(t)$, are interpolated.

As explained in [10], the time dependence of the Hubble parameter is used to compute the VOS parameters as a function of time. As we have seen in subsection 2.3.3, if the universe expands as a power law, $a(t) \propto t^\beta$, where $\beta \in (0, 1)$ is constant, the VOS equations (2.18) and (2.19) have a scaling solution of the form $L = \xi_\beta t$ with a constant RMS velocity $v = v_\beta$. For a spatially flat universe ($\Omega = 1 \Leftrightarrow \kappa = 0$), this solution is given by [31]:

$$\xi_\beta^2 = \frac{k(v_\beta)(k(v_\beta) + \tilde{c})}{4\beta(1 - \beta)} \quad v_\beta^2 = \frac{k(v_\beta)(1 - \beta)}{\beta(k(v_\beta) + \tilde{c})},$$

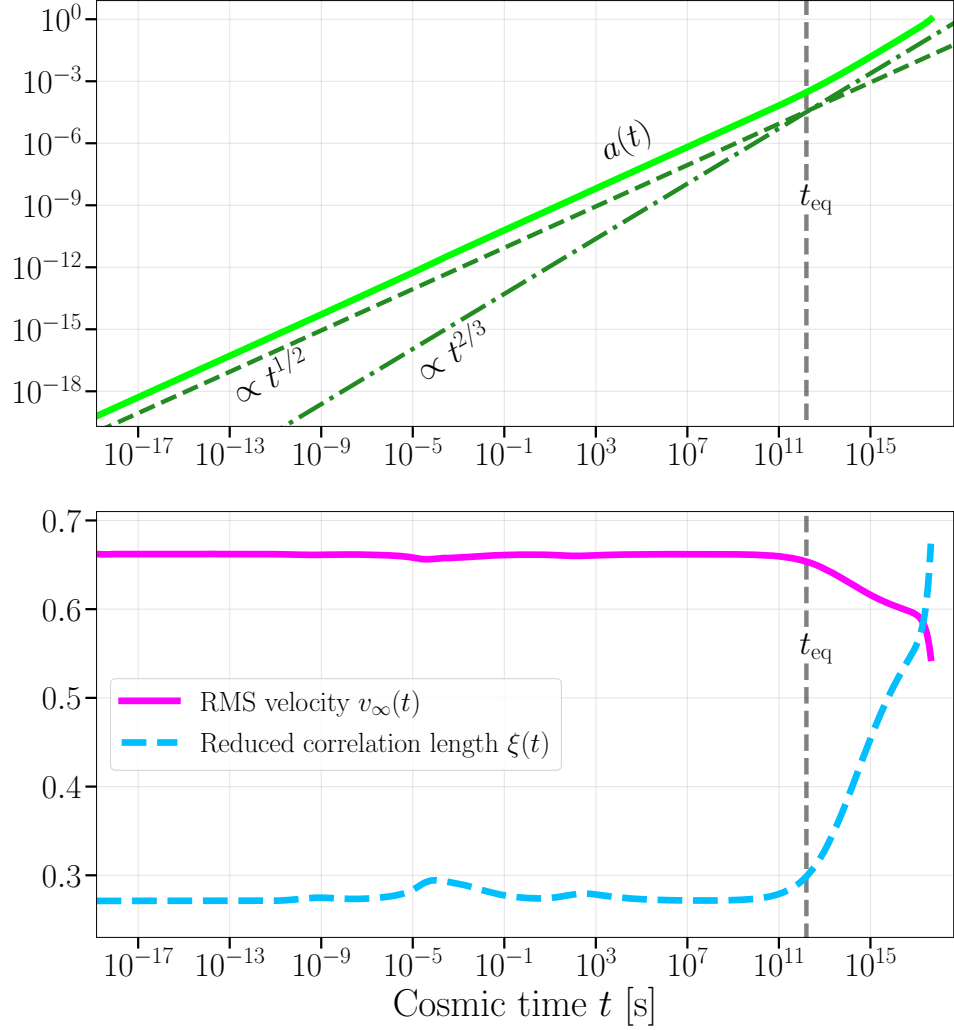


Figure 3.1: The upper panel shows the time evolution of the scale factor $a(t)$ in a lime-coloured solid line. In addition, lines are drawn proportional to $t^{1/2}$ (dashed) and to $t^{2/3}$ (dash-dot). The lower panel shows the time evolution of the root-mean-square velocity $v_\infty(t)$ in a magenta-coloured solid line and the reduced correlation length $\xi(t)$ in a deepskyblue-coloured dashed line. In both panels, the time of radiation-matter equality t_{eq} , the time when $\rho_r(t) = \rho_m(t)$, is drawn. This figure is inspired by Figure 1 from [10].

where for the momentum parameter k a fit to an ultra-high resolution network simulation that determined a transition between the non-relativistic regime and the relativistic regime with the correct asymptotic limits is used [31]:

$$k(v) = \frac{2\sqrt{2}}{\pi}(1-v^2)(1+2\sqrt{2}v^3)\frac{1-8v^6}{1+8v^6},$$

The solution for the radiation-dominated era $\beta = 1/2$ is used as an initial condition to solve the VOS differential equations (2.19) and (2.18). In figure 3.1, the time evolution of the scale factor $a(t)$ and the VOS parameters $\xi(t)$ and $v(t)$ are shown.

Instead of the complete sum in equations (2.28) and (2.29), it is sufficient to compute the contribution of the fundamental mode:

$${}^{(1)}\Omega_{\text{GW}}^{II,\{I\}}(f) = \frac{16\pi}{3f} \left(\frac{G\mu}{H_0} \right)^2 \Gamma \int_{t_{\text{ini}}}^{t_0} dt' \left(\frac{a(t')}{a(t_0)} \right)^5 n\left(\frac{2ka(t')}{fa(t_0)}, t'\right) \{\mathbf{a}\}.$$

The complete spectrum is then given as:

$$\Omega_{\text{GW}}^{II,\{I\}}(f) = \frac{1}{\zeta_{n_{\text{max}}}(q)} \sum_{k=1}^{n_{\text{max}}} \frac{1}{k^q} {}^{(k)}\Omega_{\text{GW}}^{II,\{I\}}(f),$$

where $\zeta_{n_{\text{max}}}(q) = \sum_{k=1}^{n_{\text{max}}} k^{-q}$ and ${}^{(k)}\Omega_{\text{GW}}^{II,\{I\}}(f) = {}^{(1)}\Omega_{\text{GW}}^{II,\{I\}}(f/k)$ [10]. In this thesis, $n_{\text{max}} = 10^5$ is used.

The values of Newton's constant of gravitation $G = 6.70861 \times 10^{-39} \hbar c (\text{GeV}/c^2)^{-2}$ and for the reduced Planck constant $\hbar = 6.582119514 \times 10^{-22} \text{ MeV s}$ are taken from [47]. The value of the Planck time $t_{\text{Pl}} = 5.391247 \times 10^{-44} \text{ s}$ is taken from [48]. Furthermore, we assume $g = 1$ for the computations.

3.2 LISA: Laser Interferometer Space Antenna

Over the last few decades, several projects with the aim of detecting gravitational waves to unveil the Gravitational Universe have been founded. After winning the Nobel Prize in 2017 [49] for the first-ever direct observation of gravitational waves in 2015 [50], the Laser Interferometer Gravitational-Wave Observatory (LIGO) [51] is most likely the most famous project. Other projects are, for example, the collaboration partners of LIGO, VIRGO [52], and the Japanese project Kamioka Gravitational Wave Detector (KAGRA) [53]. These are all collaborations that work with ground-based detectors. Due to various fundamental and technical noises, as described, for example, in [54], the detectors have a constrained frequency band $[\sim \text{Hz}, \sim \text{kHz}]$ to which they are sensitive.

However, it is also possible to use the pulsars in the Milky Way to detect gravitational

waves. Pulsars are rapidly rotating neutron stars that emit beams of electromagnetic radiation along an axis, which does not have to align with their rotational axis. Whenever this beam intersects our line of sight, we can measure this beam; the pulsar acts similarly to a lighthouse. For certain pulsars, it is possible to predict their averaged pulse shape up to years. Since gravitational waves are perturbations of the spacetime metric, they influence the measured pulse. By observing several pulsars and measuring these deviations from predicted pulse arrival times, one can detect gravitational waves [55]. This technique is named Pulsar Timing Array (PTA) and is utilised by collaborations such as the North American Nanohertz Observatory for Gravitational Waves (NANOGrav) [56]. However, PTA analyses are constrained to gravitational waves with nanohertz frequency and are used to study the stochastic gravitational-wave background [57].

LISA is a planned project to measure gravitational waves with millihertz frequency and so fills the gap between PTA analysis and terrestrial gravitational wave detectors. As we will see, cosmic strings are predicted to have a strong signal in LISA's sensitive frequency range.

3.2.1 Introduction to LISA

Before we have a look at the Signal-to-Noise ratio of LISA, we should briefly summarise what LISA is. This subsection is based on [58, 59].

Just as LIGO, KARGA and VIRGO, LISA is based on optical interferometry, but instead of a ground-based detector, it is positioned in an orbit far from Earth. This way, we get rid of the noise that comes from ground motion. Moreover, space provides an environment containing less thermal, magnetic and gravitational fluctuation compared to Earth. Detectors that measure gravitational waves with millihertz frequencies require a certain scale, for which the scale of a stable environment, as in space, is necessary. LISA consists of three identical spacecraft that form an approximately equilateral triangle with a side length of 2.5×10^9 m. The constellation follows an Earth-trailing heliocentric orbit approximately 20° behind Earth. Each spacecraft is equipped with two lasers with a wavelength of 1064 nm. 30 cm telescopes are used for transmitting and receiving the lasers between the spacecrafts [60]. Inside each spacecraft are two test masses, free-falling 46 mm cubes of an Au-Pt alloy, serving as geodesic reference objects. A “drag-free control” is used to ensure that these test

masses remain floating in free-fall. The lasers are utilised to measure the variations in distance between the test masses. As an example, let us consider a gravitational wave with a plus polarisation (2.4) travelling along the z -axis [16]:

$$h_{ab}^{\text{TT}}(t) = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}_{ab} \cos[\omega(t)],$$

where $ab \in \{1, 2\}$ are indices in the transverse (x, y) plane. Thus, the metric is given by:

$$ds^2 = c^2 dt^2 - (1 - h \cos[\omega(t)]) dx^2 - (1 + h \cos[\omega(t)]) dy^2 - dz^2.$$

Consider two objects that are separated by a distance L along the y axis. The physical distance between them is given by:

$$\rho^2 = L^2(1 + h \cos[\omega(t)]).$$

So two free-falling objects change their physical distance by $\Delta L = \rho - L$, Therefore, the phase of a laser shifts by:

$$\Delta\phi \approx \frac{2\pi}{\lambda} \frac{L}{2} h \cos[\omega(t)],$$

where λ is the wavelength of the laser. The order of gravitational wave amplitude h of GW150914, the merger of two black holes, is approximately 10^{-21} [16]. It is not possible to apply a Michelson-Morley interferometer, because the beams lose too much intensity to diffraction over the large 2.5×10^9 m distances. Instead, two lasers are used for each link, such that at each end of an arm the phase of a transmitted laser and the phase of a received signal is measured. The received signal contains the delayed phase of the received laser plus the gravitational wave. Therefore, the signals are independent, and the noise of the lasers does not cancel out. However, by combining the results of each spacecraft, it is possible to reduce the laser noise and extract the signal of the gravitational wave.

The triangular formation offers the chance to measure the polarisation of the gravitational waves. It is planned that LISA will launch around the year 2034 and will operate in science mode for at least four years, up to 10 years. A sketch of LISA orbiting around the Sun and its operating principle is shown in figure 3.2.

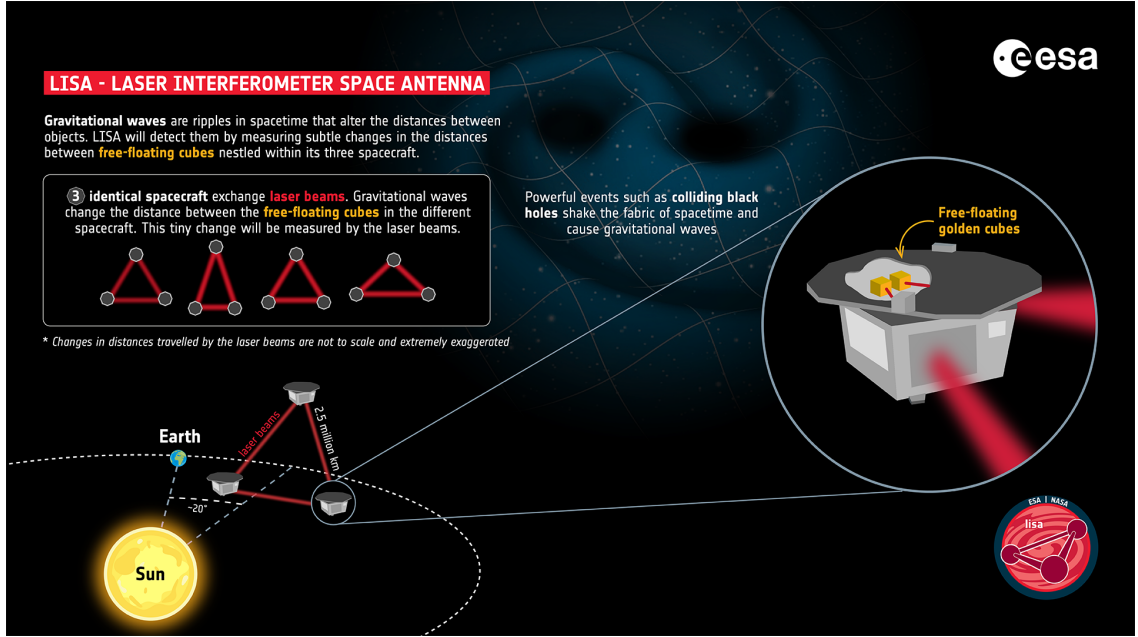


Figure 3.2: A sketch of the experimental setup and operating principle of LISA. ESA/ATG Medialab, ©ESA (ESA Standard Licence), [61].

3.2.2 The Signal-to-Noise ratio

As already mentioned, we use LISA's SNR to compare the signal of type-I cosmic strings to the signal of type-II cosmic strings. To be more precise, we study the difference between type-I and type-II signals and use this difference as the signal of interest, as explained in the next chapter. This subsection shows how the SNR is calculated and how it is utilised for the comparison. This subsection is based on [62, 41].

While the normalised energy density $h^2\Omega_{\text{GW}}$ is a useful expression for the gravitational wave, for the derivation of the SNR it is easier to work with the spectral density $S(f)$. For a stochastic gravitational wave background, as assumed for cosmic strings, the signal is expected to be isotropic, stationary and unpolarized, so that the following relation is satisfied:

$$h^2\Omega_{\text{GW}} = \frac{2\pi^2}{3} \left(\frac{h}{H_0} \right)^2 f^3 S(f). \quad (3.2)$$

The measured signal $d(t)$ at the detector is given as:

$$d(t) = s(t) + n(t), \quad (3.3)$$

where $s(t)$ is the signal of gravitational waves and $n(t)$ is the noise of the detector. The SNR of an auto-correlation statistic S is defined in terms of the difference in the expectation values of S under the assumption of a gravitational wave signal and no gravitational wave signal and the positive square root of the variance σ^2 as:

$$\varrho = \frac{\mu}{(\sigma^2)^{1/2}} = \frac{\langle S \rangle_{\mu \neq 0} - \langle S \rangle_{\mu = 0}}{(\langle S^2 \rangle - \langle S \rangle^2)^{1/2}},$$

where $\langle S \rangle_{\mu=0}$ is the detector's signal without assuming a gravitational wave signal, and $\langle S \rangle_{\mu \neq 0}$ is the detector's signal assuming a gravitational wave signal. The auto-correlation statistic S of the detector's signal (3.3) in terms of the Fourier transforms is given as:

$$S = \int_{-\infty}^{\infty} df \tilde{d}(f) \tilde{d}^*(f) \tilde{Q}(f) \quad (3.4)$$

where $\tilde{Q}(f)$ is a real filter function, and we took the limit of large T_{obs} . For μ we get:

$$\mu = \frac{T_{\text{obs}}}{2} \int_{-\infty}^{\infty} df \mathcal{R}(f) S_{\text{signal}}(f) \tilde{Q}(f), \quad (3.5)$$

where $\mathcal{R}(f)$ is the signal respond function of the detector and $S_{\text{signal}}(f)$ is defined via: $\langle \tilde{s}(f) \tilde{s}^*(f') \rangle = \frac{1}{2} \delta(f - f') \mathcal{R}(f) S_{\text{signal}}(f)$. The variance σ is given as:

$$\sigma^2 = \frac{T_{\text{obs}}}{2} \int_{-\infty}^{\infty} df (\mathcal{R}(f) [S_{\text{noise}}(f) + S_{\text{signal}}(f)])^2 \tilde{Q}^2(f), \quad (3.6)$$

where $S_{\text{noise}}(f)$ is defined via: $\langle \tilde{n}(f) \tilde{n}^*(f') \rangle = \frac{1}{2} \delta(f - f') \mathcal{R}(f) S_{\text{noise}}(f)$. With an optimal filter function that maximise the SNR $\tilde{Q}(f) = c \mathcal{R}^{-1} S_{\text{signal}} / (S_{\text{noise}}(f) + S_{\text{signal}}(f))^2$, where c is an arbitrary normalization constant, we obtain:

$$\rho = \left[\frac{T_{\text{obs}}}{2} \int_{-\infty}^{\infty} df \left(\frac{S_{\text{signal}}(f)}{S_{\text{noise}}(f) + S_{\text{signal}}(f)} \right)^2 \right]^{1/2} \quad (3.7)$$

$$= \left[T_{\text{obs}} \int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{h^2 \Omega_{\text{signal}}(f)}{h^2 \Omega_{\text{noise}}(f) + h^2 \Omega_{\text{signal}}(f)} \right)^2 \right]^{1/2}. \quad (3.8)$$

For this equation, it is assumed that the noise of the detector goes to infinity for frequencies outside the LISA frequency band.

Before [62] was published, it was conventional to always work with a weak gravitational wave background signal, where $h^2\Omega_{\text{signal}} \ll h^2\Omega_{\text{noise}}$, independent of the actual signal of interest. In the derivation of this weak signal limit SNR $\rho_{\text{weak signal}}$, the variance of just the noise is considered without the assumption of a gravitational wave signal $\sigma_{\mu=0}$. One sees that equation (3.8) becomes:

$$\rho_{\text{weak signal}} = \left[T_{\text{obs}} \int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{h^2\Omega_{\text{signal}}(f)}{h^2\Omega_{\text{noise}}(f)} \right)^2 \right]^{1/2}. \quad (3.9)$$

However, this SNR should only be used if the condition is satisfied; otherwise, the SNR of arbitrary signals (3.8) is the correct one. While for the SNR for weak signals no limit exists, the correct formula (3.8) is limited by:

$$\rho_{\text{max}} = \sqrt{T_{\text{obs}}(f_{\text{max}} - f_{\text{min}})} \approx 11,235.$$

In this thesis, we apply both SNR independent of the gravitational wave signal and discuss the results of it.

More details of the presented derivation and calculation are given in [62, 41] or in the Appendix B.

3.2.3 LISA's Noise

Last but not least, we need to compute LISA's noise $h^2\Omega_{\text{noise}}$. This subsection is based on [62].

The noise of LISA consists of two parts: an intrinsic instrumental strain noise S_{noise} and confusion noise S_{gcn} caused by galactic double white dwarf binaries:

$$S_{\text{noise}}(f) = S_{\text{inst}}(f) + S_{\text{gcn}}(f).$$

S_{inst} can be expressed in terms of the detector's response function $\mathcal{R}$ and the detector noise PSD D_{inst} :

$$S_{\text{inst}} = \frac{D_{\text{inst}}}{\mathcal{R}}.$$

Originally shown in [63], D_{inst} consists of two stationary contributions:

$$D_{\text{inst}}(f) = \frac{1}{L^2} D_{\text{oms}}(f) + 2 \left[1 + \cos^2 \left(\frac{f}{f_\star} \right) \right] \frac{D_{\text{acc}}(f)}{(2\pi f)^4 L^2}, \quad (3.10)$$

where $L = 2.5 \text{ Gm}$ is the length of an arm and $f_\star = 19.09 \text{ mHz}$ is LISA's transfer frequency [63]. The first contribution, the optical metrology noise, is given by:

$$D_{\text{oms}}(f) = (1.5 \times 10^{-11} \text{ m})^2 \left[1 + \left(\frac{2 \text{ mHz}}{f} \right)^4 \right] \text{ Hz}^{-1},$$

and the second contribution, the single test mass acceleration noise, is given by:

$$D_{\text{acc}}(f) = (3 \times 10^{-15} \text{ m s}^{-2})^2 \left[1 + \left(\frac{0.4 \text{ mHz}}{f} \right)^2 \right] \left[1 + \left(\frac{f}{8 \text{ mHz}} \right)^4 \right] \text{ Hz}^{-1}.$$

The formula for the detector's response function $\mathcal{R}$ is taken from [62] and shown in Appendix B. The galactic confusion noise has been studied in [60], and they found analytical fits to the confusion noise of the functional form¹:

$$S_{\text{gcn}}(f) = A \left(\frac{1 \text{ mHz}}{f} \right)^{7/3} e^{-(f/f_{\text{ref}})^\alpha - \beta f \sin(\kappa f)} [1 + \tanh(\gamma(f_{\text{knee}} - f))].$$

We use for the overall amplitude $A = 9 \times 10^{-38} \text{ Hz}^{-1}$ and for the reference frequency $f_{\text{ref}} = 1 \text{ Hz}$. Moreover, we assume an observation time of $T_{\text{obs}} = 4 \text{ years}$, so that all other parameters are given in table 3.1. For LISA's frequency band, the interval

Table 3.1: Parameters for the galactic confusion noise S_{gcn} , for an observation time $T_{\text{obs}} = 4 \text{ years}$. The values are taken from [60] (respectively [62]).

α	$\beta [\text{mHz}^{-1}]$	$\kappa [\text{mHz}^{-1}]$	$\gamma [\text{mHz}^{-1}]$	$f_{\text{knee}} [\text{mHz}^{-1}]$
0.138	-0.221	0.521	1.680	1.13

$[f_{\text{min}} = 10^{-5}, f_{\text{max}} = 10^0]$ is chosen. The characteristic noise $\sqrt{f S(f)}$ and the noise energy density $h^2 \Omega_{\text{noise}}$ are presented in figure 3.3. Then, the dimensionless energy density of the noise $h^2 \Omega_{\text{noise}}$ is given using equation (3.2):

$$h^2 \Omega_{\text{noise}} = \frac{2\pi^2}{3} \left(\frac{h}{H_0} \right)^2 f^3 S_{\text{noise}}(f). \quad (3.11)$$

¹As also presented in [62], in this thesis the second sign in the e function is altered compared to the formula (3) from [60].

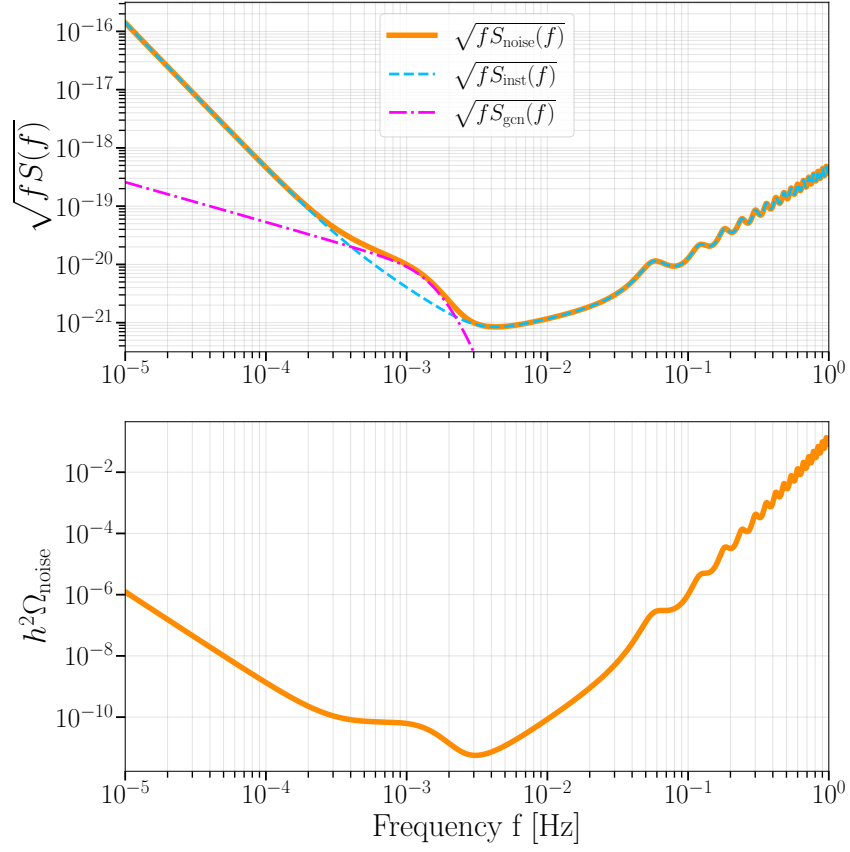


Figure 3.3: The upper panel shows the dimensionless characteristic strain $\sqrt{fS(f)}$ of LISA combined noise S_{noise} (orange solid line), the instrumental noise S_{inst} (blue dotted line), and the noise of the galactic binaries S_{gcn} (magenta dot-dash line). The lower panel presents the dimensionless energy density of the noise $h^2\Omega_{\text{noise}}$.

Results

After planting all the necessary ingredients, we can begin harvesting our results by comparing the gravitational-wave spectra of type-I and type-II cosmic strings. The first section presents a comparison for which we utilise the SNR for a weak signal (3.9). In the second section, we use the SNR for arbitrary signals (3.8) and discuss the differences. Moreover, it should be mentioned that the parameter spaces of type-I and type-II spectra are being studied with a finite resolution. For type-I, the logarithm of the VEV $\log_{10}(\nu/\text{GeV})$ lies inside the interval $[9.5; 14.5]$ and the resolution is $[0.05]$, and the logarithm of the scalar boson's mass $\log_{10}(m_\phi/\text{GeV})$ lies in the interval $[1; 8]$ with a resolution of 0.1. In this parameter space exists a region for which no SNR can be determined due to a cutoff of the spectrum. In the following, this region is always marked in grey in all heatmaps. For type-II, the logarithm of the dimensionless string tension $G\mu$ is inside the interval $[-30; -6]$ with a resolution of 0.01.

4.1 Type-I versus Type-II Signals

The question we want to answer is: Can we distinguish between gravitational wave background (GWB) signals from type-I and type-II cosmic strings measured by LISA? For this reason, we focus on type-I and type-II spectrum pairs, where both spectra are similar from a LISA perspective. To determine such a pair, we use the SNR formula (3.9), where we set $\Omega_{\text{signal}} = |\Omega_{\text{GW}}^{(I)} - \Omega_{\text{GW}}^{(II)}|$:

$$\rho_0 := \left[T_{\text{obs}} \int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{h^2 \Omega_{\text{GW}}^{(I)}(f; \nu, m_\phi) - h^2 \Omega_{\text{GW}}^{(II)}(f; G\mu)}{h^2 \Omega_{\text{noise}}(f)} \right)^2 \right]^{1/2}.$$

$\Omega_{\text{GW}}^{(I)}$ is the GWB signal from type-I cosmic strings and $\Omega_{\text{GW}}^{(II)}$ from type-II cosmic strings. From now on, we call this quantity ρ_0 and the SNR calculated with formula (3.9)

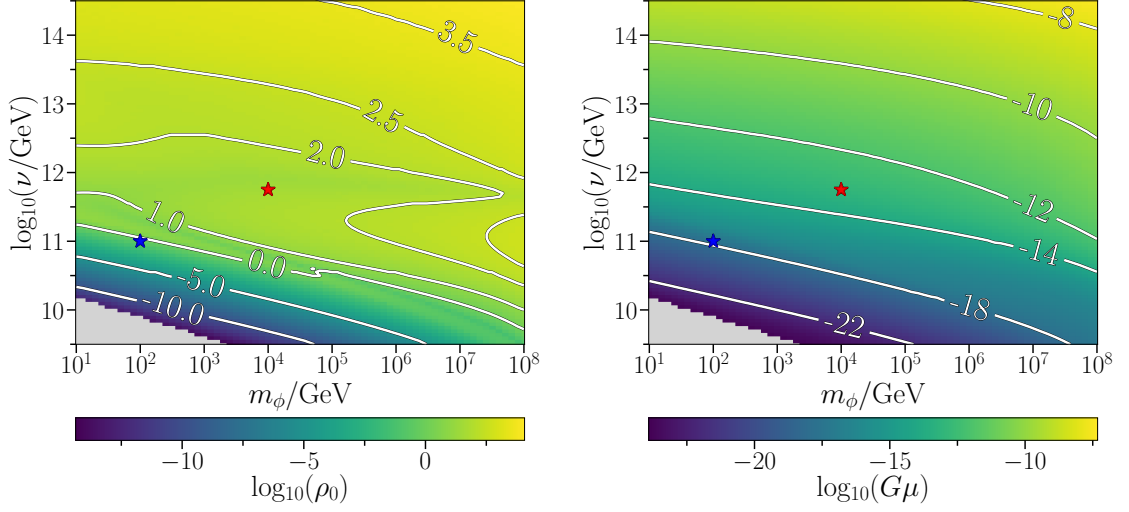


Figure 4.1: The minimised SNR of the difference between GWB signals from type-I and type-II cosmic strings computed using the formula valid for weak signal strengths (3.9). The right panel shows the corresponding values of the type-II parameter $G\mu$.

ρ' . As we have seen in subsection 2.3.4, the spectrum from type-I cosmic strings depends on the VEV ν and the scalar boson's mass m_ϕ , while the spectrum from type-II cosmic strings only depends on the string tension $G\mu$. Thus, we pick one type-I spectrum, so we choose one combination of ν and m_ϕ , and by minimising ρ_0 over $G\mu$ we find the type-II spectrum that is the closest to the chosen type-I spectrum from a LISA perspective.

In figure 4.1, one can see the results for the complete parameter space of interest. For a first idea of how to interpret these values, the SNR for GWB signals just from type-I and just from type-II cosmic strings are demonstrated in Appendix C.1 in figure C.1 and C.2. Obviously, the SNR ρ_0 can only be less than or equal to the sum of the SNR ρ' of the corresponding type-I and type-II spectra. Therefore, the lower part of the right panel in figure 4.1, corresponds to a region in parameter space where the SNR ρ' for type-I and type-II is already low. For the rest of the parameter space, the GWB signals from type-I and type-II cosmic strings differ because the SNR ρ_0 is not close to zero. However, for higher values of ν and m_ϕ , the difference between ρ_0 and the SNR ρ' of the pure spectra increases. One can, for instance, compare the contour line of 1 in figure 4.1 with the same line in figure C.1 and see that in figure 4.1 it is

slightly higher. Moreover, in figure 4.1 there is no ρ_0 greater than 10^5 . However, let us discuss two examples in more detail in the following two subsections.

4.1.1 Example: red star

The first example is marked with a red star in figure 4.1 and in most of the other plots presented in this thesis. For that example, $\nu = 10^{11.75}$ GeV and $m_\phi = 10^4$ GeV were chosen, and the determined type-II partner has a string tension of $G_\mu = 10^{-13.19}$. The SNR of the difference is $\rho_0 \approx 35.83$, while the pure SNR for these type-I strings is $\rho' \approx 1753$ and for type-II it is $\rho' \approx 1511$. As also presented in figure C.1 and C.2, this is the very edge of where the formula for weak signal (3.9) can be used as a rough approximation.

The two spectra are shown in figure 4.4 together with the sensitivity of LISA. One recognises that these two spectra are almost indistinguishable within LISA's frequency range. That is why the SNR ρ_0 is almost two orders of magnitude lower than ρ' . However, we can also study the spectra's slope. One sees that both spectra can be described by a combination of three power laws $\propto f^p$. A visualisation of this description is shown in Appendix C.1 in figure C.3. Therefore, we take the derivative of $\log_{10}(h^2\Omega_{\text{GW}})$ with respect to $\log_{10}(f/\text{Hz})$ in order to determine p :

$$\frac{d \log_{10}(h^2\Omega_{\text{GW}})}{d \log_{10}(f)} \propto \frac{d \log_{10}(f^p)}{d \log_{10}(f)} = p.$$

We demonstrate the results in what we call the phase space, which is shown in figure 4.3. One notices that even in the phase space both spectra look very similar; they have at least a similar shape. However, if we take the frequencies into account, we are able to distinguish clearly between both of them. Because we see that only for the frequency $f = 10^{-2}$ they are similar, this means the points for this frequency are close in the phase space, but besides this specific point, they are completely different. This means that for no other frequency are they at the same position in phase space.

So to answer our question: yes, we can distinguish between the GWB signals from these type-I cosmic strings and GWB signals from type-II cosmic strings. First, the SNR ρ_0 is significantly greater than one and second, they do look different in phase space.

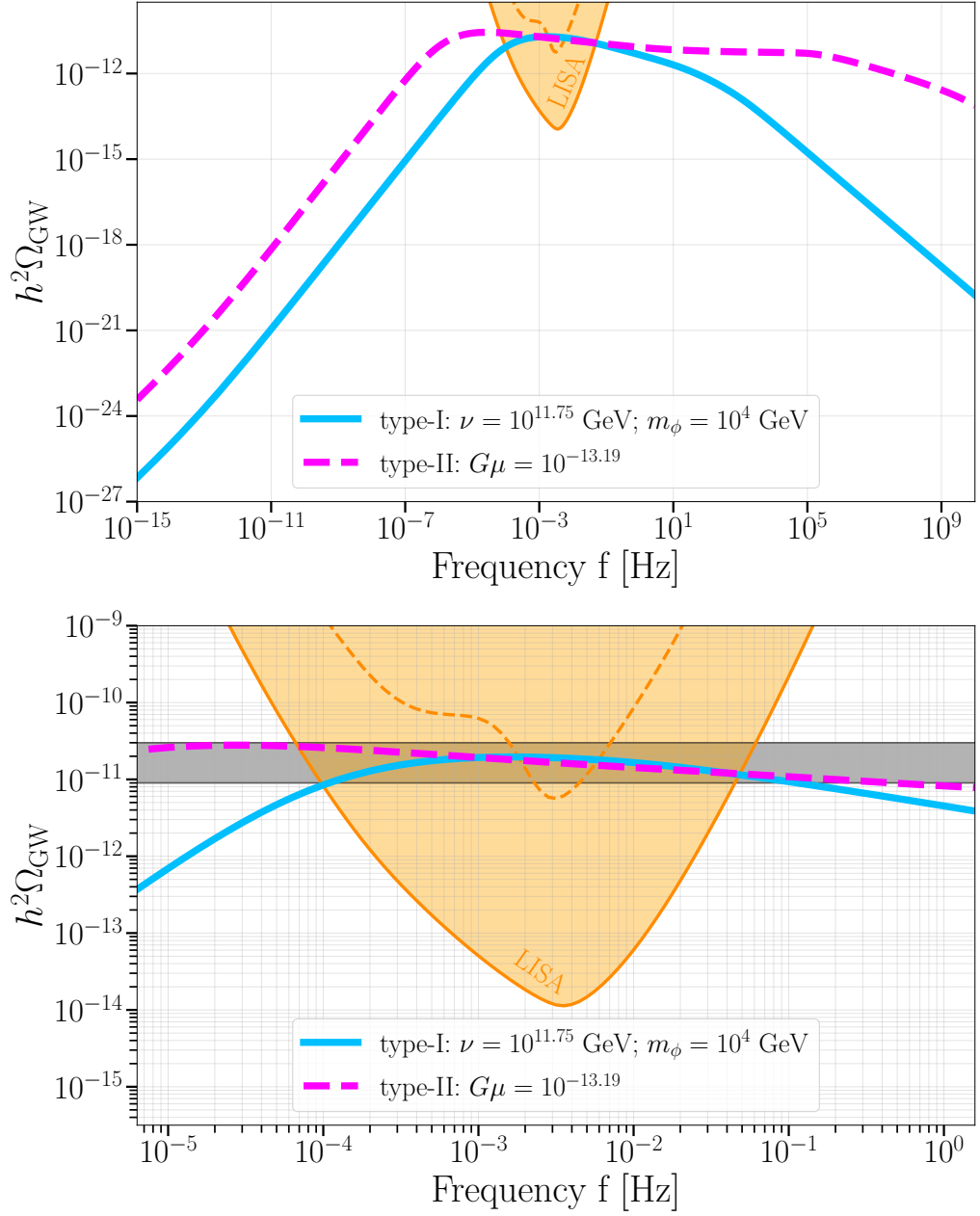


Figure 4.2: Gravitational wave spectra from type-I cosmic strings with $\nu = 10^{11.75}$ GeV (blue solid line) and $m_\phi = 10^4$ GeV and from type-II cosmic strings with $G\mu = 10^{-13.19}$ (magenta dashed line). The orange solid curve shows LISA's sensitivity using the Power-Law Integrated sensitivity curve [64]. The dashed orange curve shows LISA's noise density $h^2 \Omega_{\text{noise}}$ (3.11) or Appendix C.3. The lower panel is a zoomed-in version of the upper panel. The black coloured region demonstrates the range of $h^2 \Omega_{\text{GW}}$ LISA is sensitive to in this particular example and is also shown in figure 4.3.

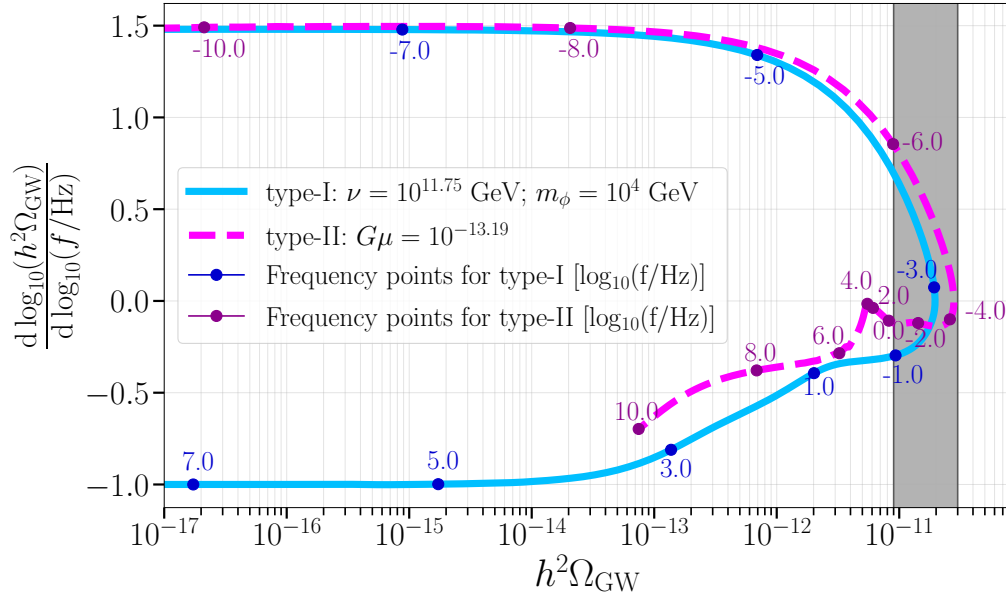


Figure 4.3: The phase space of the cosmic strings for the first example, where type-I cosmic strings have $\nu = 10^{11.75}$ GeV and $m_\phi = 10^4$ GeV (blue solid line) and type-II cosmic strings have $G\mu = 10^{-13.19}$ (magenta dashed line). The black coloured region demonstrates the range of $h^2\Omega_{\text{GW}}$ LISA is sensitive to in this particular example and is also shown in figure 4.4. The dark blue and purple points indicate the frequency of the spectrum for the corresponding amplitude $h^2\Omega_{\text{GW}}$.

4.1.2 Example: blue star

Let us consider another example, which is marked with a blue star in figure 4.1 and in most of the other plots presented in this thesis. This time we choose the type-I string parameters $\nu = 10^{11}$ GeV and $m_\phi = 10^2$ GeV. Therefore, the type-II parameter is, in this case, $G\mu = 10^{-13.19}$. For this pair, we have $\rho_0 = 0.31$. The GWB signal from these type-I cosmic strings has an SNR of $\rho' \approx 0.59$, and for type-II it is $\rho' \approx 0.53$. These low SNR's ρ' indicate that it will be very difficult to measure the GWB signal from these type-I or type-II strings, but it works as an example to demonstrate the following points.

We see that in this case the SNR's ρ_0 and ρ' are much lower, as well as the difference between them. However, we also see that both spectra differ more within the frequency range of LISA. Thus, the low SNR ρ_0 is not caused by greater similarity between the two spectra but by a smaller SNR ρ' of the GWB signal from type-I and type-II strings. Nevertheless, we also have a look at the phase space, which is shown

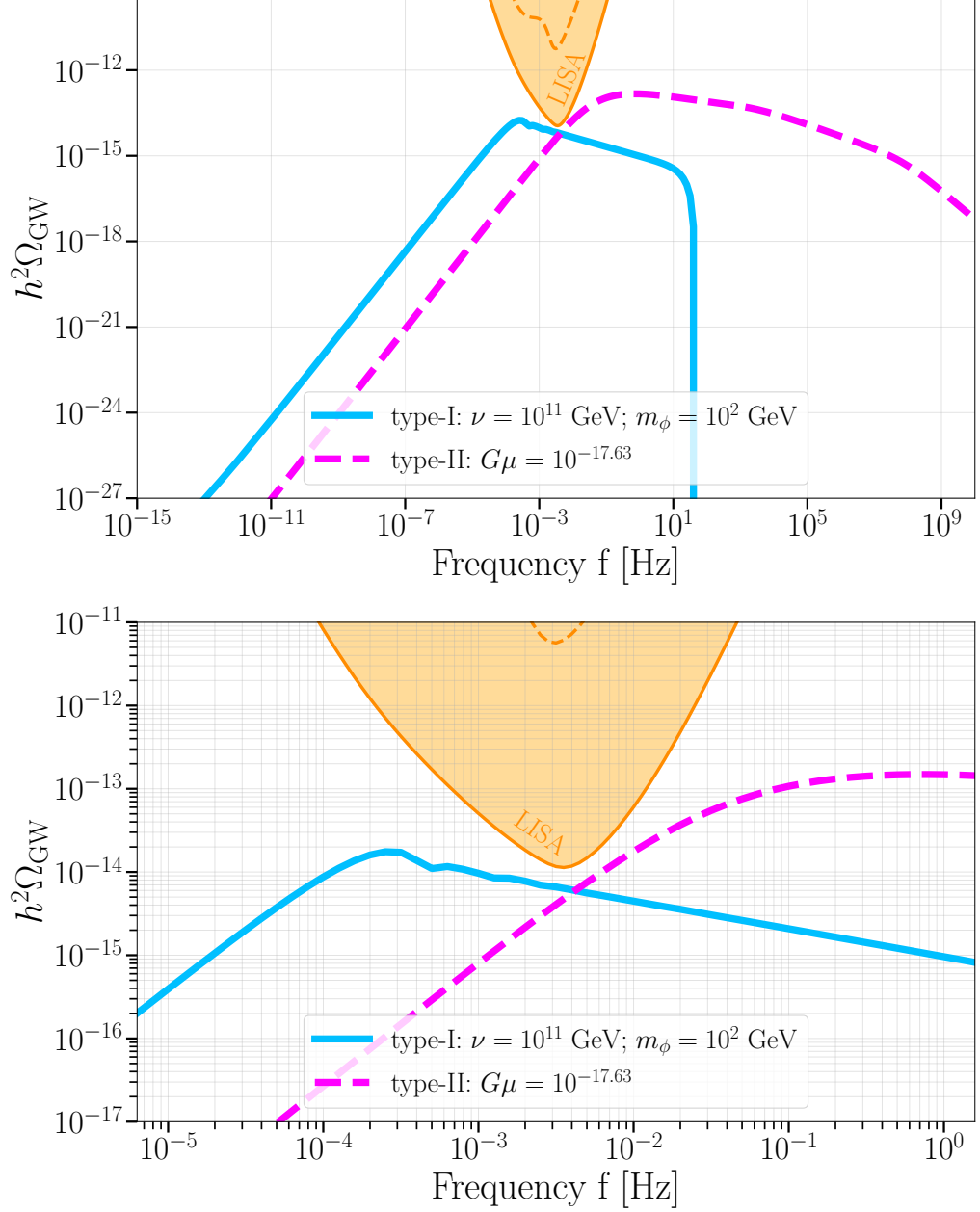


Figure 4.4: Gravitational wave spectra from type-I cosmic strings with $\nu = 10^{11}$ GeV (blue solid line) and $m_\phi = 10^2$ GeV and from type-II cosmic strings with $G\mu = 10^{-17.63}$ (magenta dashed line). The orange solid curve shows LISA's sensitivity using the Power-Law Integrated sensitivity curve [64] or Appendix C.3. The dashed orange curve shows LISA's noise density $h^2\Omega_{\text{noise}}$ (3.11). The lower panel is a zoomed-in version of the upper panel.

in figure 4.5. This time, we can use the combination of two power laws to describe the spectra, which is illustrated in Appendix C.1 in figure C.4. And we notice that in this case the shapes do not look that similar. Moreover, there is no frequency for which the two spectra are at the same position in phase space.

Therefore, in case of detection of these signals, we are also able to distinguish between the two spectra. This time, the SNR ρ_0 may be small, but the shapes of the spectra are different, as well as their behaviour in phase space.

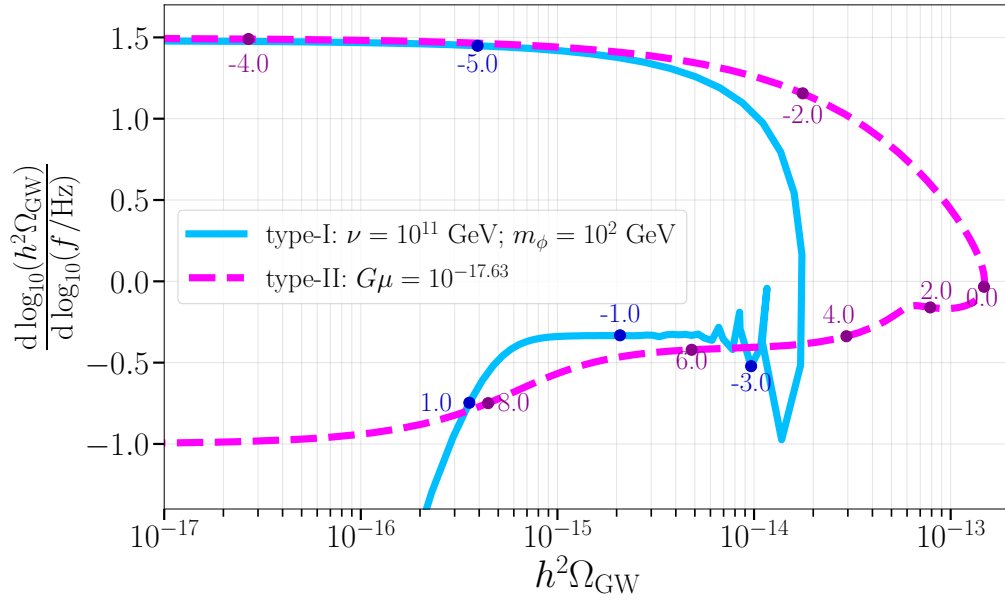


Figure 4.5: The phase space of the cosmic strings for the second example, where type-I cosmic strings have $\nu = 10^{11}$ GeV and $m_\phi = 10^2$ GeV (blue solid line) and type-II cosmic strings have $G\mu = 10^{-17.63}$ (magenta dashed line). The dark blue and purple points indicate the frequency of the spectrum for the corresponding amplitude $h^2 \Omega_{\text{GW}}$.

4.2 The New SNR Formula

Let us now discuss the impact of changing the SNR formula valid for weak signal strength (3.9) to the SNR formula valid for arbitrary signal strengths (3.8). For that, we have to think about how we change ρ_0 . We have two options: either we expect a GWB signal from type-I cosmic strings and are interested in how similar the GWB signal from type-II cosmic strings is, then we use:

$$\rho_1 := \left[T_{\text{obs}} \int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{h^2 \Omega_{\text{GW}}^{(I)}(f; \nu, m_\phi) - h^2 \Omega_{\text{GW}}^{(II)}(f; G\mu)}{h^2 \Omega_{\text{noise}}(f) + h^2 \Omega_{\text{GW}}^{(I)}(f; \nu, m_\phi)} \right)^2 \right]^{1/2}, \quad (4.1)$$

or the other way around, and then we use:

$$\rho_2 := \left[T_{\text{obs}} \int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{h^2 \Omega_{\text{GW}}^{(I)}(f; \nu, m_\phi) - h^2 \Omega_{\text{GW}}^{(II)}(f; G\mu)}{h^2 \Omega_{\text{noise}}(f) + h^2 \Omega_{\text{GW}}^{(II)}(f; G\mu)} \right)^2 \right]^{1/2}. \quad (4.2)$$

The results for the first case are shown in figure 4.6 and for the second case in 4.7.

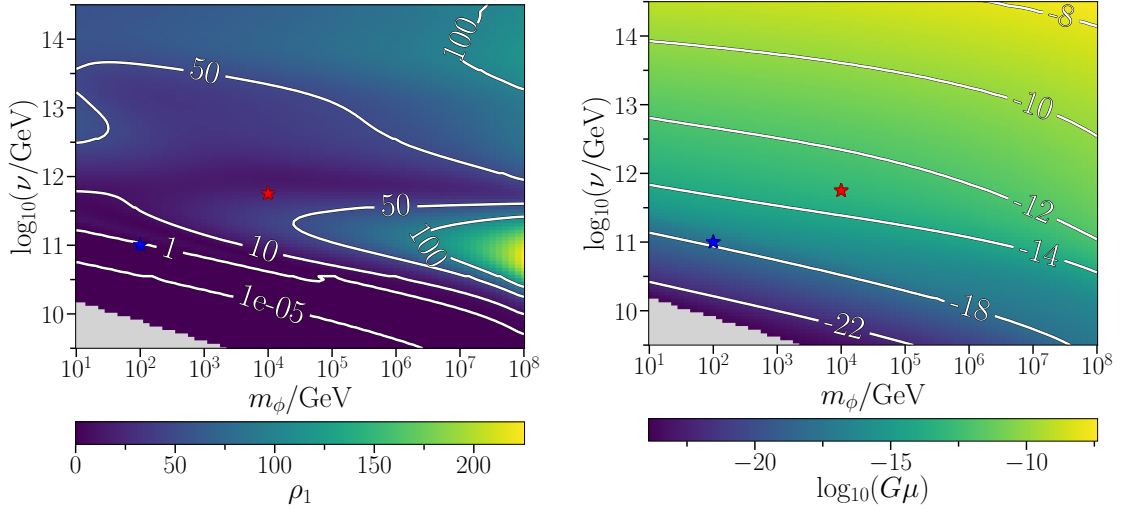


Figure 4.6: The minimised SNR of the difference between GWB signals from type-I and type-II cosmic strings computed using formula (4.1). The right panel shows the corresponding values of the type-II parameter $G\mu$.

As expected, the SNR for strong signals changes, so that in two cases ρ_1 and ρ_2 are lower compared to ρ_0 for values significantly larger than 1. For weaker signals, all three methods match. One has to keep in mind that the SNR ρ_0 in figure 4.1 is given

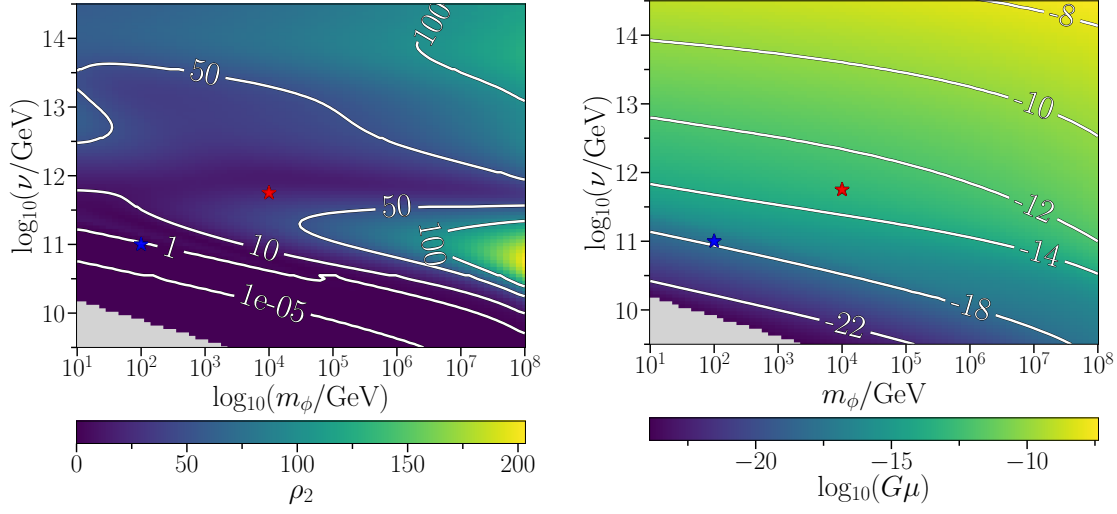


Figure 4.7: The minimised SNR of the difference between GWB signals from type-I and type-II cosmic strings computed using formula (4.2). The right panel shows the corresponding values of the type-II parameter $G\mu$.

on a logarithmic scale. For illustration, the contour lines for $\rho_{0,1,2} = 1$ and $\rho_{0,1,2} = 10$ are given in the figures 4.1, 4.6 and 4.7.

Furthermore, we see that the maximum of ρ_1 in figure 4.6 and the maximum of ρ_2 in figure 4.7 are at the same position, but the maximum of ρ_0 in figure 4.1 is at a totally other position. Nevertheless, the corresponding $G\mu$ values are almost identical in all three cases. The biggest difference between the $\log_{10}(G\mu)$ values yields 0.22. We can again have a look at our examples and compare the values of ρ_0 , ρ_1 and ρ_2 given in table 4.1. We see that for the first example, where the SNR's are larger, it makes a difference which formula one uses for the computations. In the second example, where we have weak signals, one can use both differences. This is exactly what we expect.

Table 4.1: The SNR values of the difference between GWB signals from type-I and type-II cosmic strings computed using the three different formulas.

	ρ_0	ρ_1	ρ_2
red star	35.83	19.61	19.52
blue star	0.31	0.31	0.31

Conclusion

In this thesis, we have studied the gravitational wave spectra from type-I and type-II cosmic strings. We focused on the region in the type-I parameter space, which is spanned by the vacuum expectation value ν_ϕ and the mass of the scalar boson m_ϕ , where $\log_{10}(\nu_\phi/\text{GeV}) \in [9.5; 14.5]$ and $\log_{10}(m_\phi/\text{GeV}) \in [1; 8]$. For all gravitational wave spectra that are generated by type-I cosmic strings within this parameter space region, we have determined the parameter of type-II cosmic strings $G\mu$, that yields the gravitational wave spectrum closest to the type-I spectrum from the perspective of LISA.

For this study, we have utilised the Signal-to-Noise Ratio (SNR) for LISA, where we used the difference between the gravitational wave spectrum from type-I and type-II cosmic strings as the signal, and minimised over the type-II parameter $G\mu$. For the formula for the calculation of the SNR, we have applied the weak-signal limit, what has been done typically in the literature, and the formula valid for arbitrary signal strengths. For the latter one, one has to decide whether one expects a signal from type-I cosmic strings and is interested in how well a signal from type-II cosmic strings can imitate it or vice versa. The values of the type-II parameter $G\mu$ for all three cases have been shown. It turns out that between these three cases, the maximal deviation of the values of $\log_{10}(G\mu)$ is 0.22.

Furthermore, it was shown that for detectable signals, the gravitational wave spectra from type-I and type-II cosmic strings with an SNR of order unity or higher, within this parameter space region, are distinguishable, since the SNR of their difference is of order unity or higher as well. However, it was confirmed that for strong signals, the weak-limit formula for the SNR is not sufficient, and one has to use the formula for arbitrary signal strengths in this analysis.

In addition to the SNR analysis, another way of distinguishing between the type-I and type-II spectra has been presented. For two type-I and type-II spectra pairs, the spectra have been shown in the phase space spanned by the logarithmic derivative

of the gravitational wave spectrum and its amplitude. Thereby, it was shown that although the two spectra might be similar in the frequency domain as well as in the phase space, it is possible to differentiate between them, because they can be at different places in the phase space for the same frequency.

It would be interesting to extend this analysis to other detectors—such as LIGO, VIRGO and KAGRA—in future projects, so that a general distinction between spectra from type-I and type-II cosmic strings can be ensured, independent of the detector used. Moreover, one could extend this analysis to further effects of cosmic strings. As, for instance, shown in [65], there exists the possibility of cosmic strings that carry a current, which influences the gravitational wave spectrum [66]. The comparison presented in this thesis could be extended to conducting strings.

Acknowledgments

I would like to thank my supervisor, Doa Hashemi Asl, for supervising this project, answering my questions, and discussing my results. I would also like to thank Kai Schmitz for supervising and for the opportunity to write my bachelor's thesis in his wonderful research group.

Calculations (or parts of them) for this publication were performed on the HPC cluster PALMA II of the University of Münster, subsidised by the DFG (INST 211/667-1).

The Moments of Gravitational Waves



This Appendix shall outline the steps in deriving the quadrupole formula 2.1, proceeding from the approximate Fourier transform of the energy-momentum tensor to the quadrupole expression and is based on [14].

Applying the approximated e function (2.6) to the Fourier transform of the energy-momentum tensor (2.5) gives us an approximation of the energy-momentum tensor:

$$T_{kl} \left(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{n}}{c}, \vec{x}' \right) \approx T_{kl} \left(t - \frac{r}{c}, \vec{x}' \right) + \frac{x'^i n^i}{c} \partial_t T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_t^2 T_{kl}$$

We introduce the moments:

$$\begin{aligned} S^{ij}(t) &= \int d^3x T^{ij}(t, \vec{x}), & S^{ij,k}(t) &= \int d^3x T^{ij}(t, \vec{x}) x^k, \\ S^{ij,kl}(t) &= \int d^3x T^{ij}(t, \vec{x}) x^k x^l, \end{aligned}$$

so that the metric perturbation is expressed in terms of the moments:

$$h_{ij}^{\text{TT}}(t, \vec{x}) = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \left[S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \dots \right]_{\text{ret}}.$$

Similarly, we define the moments of T^{00}/c^2 :

$$\begin{aligned} M &= \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}), & M^i &= \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i, \\ M^{ij} &= \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j, \end{aligned}$$

and of the moment density $(1/c)T^{0i}$:

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, \vec{x}), \quad P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, \vec{x}) x^j, \\ P^{i,jk} = \frac{1}{c} \int d^3x T^{0i}(t, \vec{x}) x^j x^k.$$

In flat spacetime $\partial_\mu T^{\mu\nu} = 0$ is satisfied, so that $\partial_0 T^{00} = \partial_i T^{i0} = \partial_i T^{0i}$. If we consider a volume V which contains the whole source, such that $T^{\mu\nu}$ vanishes at the boundary ∂V , we get the following relations:

$$\dot{M}^{ij} = P^{i,j} + P^{j,i} \quad \text{and} \quad \dot{P}^{i,j} = S^{i,j} \implies S^{ij} = \frac{1}{2} \ddot{M}_{ij}.$$

So the metric perturbation takes on the following form:

$$\implies \left[h_{ij}^{\text{TT}}(t, \vec{x}) \right]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\hat{n}) \ddot{M}^{kl}(t - r/c).$$

We split the moments as:

$$M^{kl} = \left(M^{kl} - \frac{1}{3} \delta^{kl} M_{ii} \right) + \frac{1}{3} \delta^{kl} M_{ii},$$

And define a new moment, which will be the quadrupole moment:

$$Q^{ij} := M^{ij} - \frac{1}{3} \delta^{ij} M_{kk} = \int d^3r \rho(t, \vec{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right),$$

as we can see in a comparison to the quadrupole moment of electrodynamics:

$$\int d^3r \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right) \rho_e(\vec{r})$$

We plug the quadrupole moment into the equation of the metric perturbation and obtain the equation we wanted:

$$\left[h_{ij}^{\text{TT}}(t, \vec{x}) \right]_{\text{quad}} = \frac{1}{r} \frac{2G}{c^4} \ddot{Q}_{ij}^{\text{TT}}(t - r/c).$$

LISA's Singal-to-Noise Ratio

B

This appendix is an addition to subsection 3.2.2 and is based on [62, 41]. For an even more detailed explanation, it is recommended to go to the references.

B.1 The spectral density

A stochastic gravitational wave at a given point $\mathbf{x} = 0$ is expressed in the TT-gauge via:

$$h_{ab}(t) = \sum_{A=+, \times} \int_{-\infty}^{+\infty} df \int d\hat{\Omega} \tilde{h}_A(f, \hat{\Omega}) e^{-2\pi i f t} e_{ab}^A(\hat{\Omega}), \quad (\text{B.1})$$

where $\hat{\Omega}$ is a unit vector pointing in the wave's propagation direction. For the polarisation, the equation $e_{ab}^A(\hat{\Omega}) e^{A', ab}(\hat{\Omega}) = 2\delta^{AA'}$ holds. Then the spectral density S is defined as:

$$\langle \tilde{h}_A^*(f, \hat{\Omega}) \tilde{h}_{A'}(f', \hat{\Omega}') \rangle = \delta(f - f') \frac{1}{4\pi} \delta^2(\hat{\Omega}, \hat{\Omega}') \delta_{AA'} \frac{1}{2} S(f), \quad (\text{B.2})$$

where $1/4\pi$ is a normalisation factor and $1/2$ convention, and compensates negative frequencies. Instead, one could also work with a double-sided spectral density. Using both equations (B.1) and (B.2) gives us:

$$\langle h_{ab}(t) h^{ab}(t) \rangle = 4 \int_{f=0}^{f=\infty} d(\log f) f S(f). \quad (\text{B.3})$$

The gravitational energy density can be obtained by spatial averaging over several wavelengths $\rho_{\text{GW}} = \frac{1}{32\pi G} \langle \dot{h}_{\text{ab}} \dot{h}^{\text{ab}} \rangle$. Thus, we get by using equations (B.1), (B.2) and (B.3):

$$\begin{aligned} \rho_{\text{GW}} &= \frac{4}{32\pi G} \int_{f=0}^{f=\infty} d(\log f) f (2\pi f)^2 S(f) \\ \Rightarrow \Omega_{\text{GW}} &= \frac{1}{\rho_c} \frac{d\rho}{d \log f} = \frac{4\pi^2}{3H_0} f^3 S(f), \end{aligned}$$

which is used for equation (3.2) for a single-sided spectral density.

B.2 The auto-correlation statistic

The relation between the measured signal $s(t)$ of a gravitational wave experiment and the gravitational strain $h_{\text{ab}}(t)$ is given by the detector function D^{ab} :

$$s(t) = D^{\text{ab}} h_{\text{ab}}(t).$$

By inserting equation (B.1), we obtain:

$$s(t) = \sum_{A=+, \times} \int_{-\infty}^{+\infty} df \int d\hat{\Omega} \tilde{h}_A(f, \hat{\Omega}) e^{-2\pi i f t} \underbrace{D^{\text{ab}} e_{\text{ab}}^A(\hat{\Omega})}_{F_A(\hat{\Omega})}, \quad (\text{B.4})$$

where we introduce the detector pattern function $F_A(\hat{\Omega})$. The Fourier transform is:

$$\tilde{s}(f) = \sum_{A=+, \times} \int d\hat{\Omega} \tilde{h}_A(f, \hat{\Omega}) F_A(\hat{\Omega}).$$

The auto-correlation statistic of an output $d(t)$ is given as:

$$S = \int_{-T_{\text{obs}}/2}^{T_{\text{obs}}/2} dt \int_{-T_{\text{obs}}/2}^{T_{\text{obs}}/2} dt' d(t) d(t') Q(t - t')$$

where T_{obs} is the observation time and $Q(t - t')$ the real filter function. By inserting the Fourier transforms of $d(t) = s(t) + n(t)$ and $Q(t - t')$ and assuming $T \rightarrow \infty$, we get S in terms of the Fourier transforms:

$$S = \int_{-\infty}^{+\infty} df \tilde{d}^*(f) \tilde{d}(f) \tilde{Q}(f) \\ \Rightarrow \langle S \rangle_{\mu \neq 0} = \int_{-\infty}^{+\infty} df \langle \tilde{d}^*(f) \tilde{d}(f) \rangle \tilde{Q}(f).$$

We assume no correlation between noise n and signal s , so that $\langle sn \rangle = 0$. Moreover, we have the following equation for stochastic and Gaussian signals:

$$\langle \tilde{s}(f) \tilde{s}^*(f') \rangle = \frac{1}{2} \delta(f - f') \mathcal{R}(f) S_{\text{signal}}(f) \\ \langle \tilde{n}(f) \tilde{n}^*(f') \rangle = \frac{1}{2} \delta(f - f') D_{\text{noise}}(f),$$

where for the first equation we use equation (B.4) and define [67]:

$$\mathcal{R}(f) := \int \frac{d\hat{\Omega}}{4\pi} \left[\sum F^{A,*}(\hat{\Omega}) F^A(\hat{\Omega}) \right],$$

and for the second equation we have $D_{\text{noise}}(f) = \mathcal{R}(f) S_{\text{noise}}(f)$ the detector noise PSD. We use $\delta(0) = \int_{-T_{\text{obs}}/2}^{T_{\text{obs}}/2} dt = T_{\text{obs}}$ to get:

$$\langle S \rangle_{\mu \neq 0} = \frac{T_{\text{obs}}}{2} \int_{-\infty}^{+\infty} df \mathcal{R}(f) (S_{\text{noise}}(f) + S_{\text{signal}}(f)) \tilde{Q}(f).$$

Under the assumption that we have perfect knowledge of the detector, we have a similar expression of the expectation value under the assumption of no gravitational wave signal:

$$\langle S \rangle_{\mu=0} = \frac{T_{\text{obs}}}{2} \int_{-\infty}^{+\infty} df \mathcal{R}(f) S_{\text{noise}}(f) \tilde{Q}(f),$$

so that μ is given as:

$$\mu = \frac{T_{\text{obs}}}{2} \int_{-\infty}^{+\infty} df \mathcal{R}(f) S_{\text{signal}}(f) \tilde{Q}(f),$$

as it is already shown in equation (3.5). For σ , one has to calculate $\langle S^2 \rangle - \langle S \rangle^2$, which is demonstrated in [62].

B.3 The response function of the detector

The formula is taken from [62], where it is stated that it is taken from [68].

$$\begin{aligned}
 u^2 \mathcal{R}(u, \gamma) = & s_{2u} \left[s_{\gamma/2}^2 \left(\frac{1}{u} + \frac{2}{u^3} \right) + c_{\gamma/2}^2 (2 \operatorname{Si}(2u) - \operatorname{Si}(2u_+) - \operatorname{Si}(2u_-)) \right] \\
 & + c_{2u} \left[s_{\gamma/2}^2 \left(\frac{1}{6} - \frac{2}{u^2} \right) + c_{\gamma/2}^2 (2 \operatorname{Ci}(2u) - \operatorname{Ci}(2u_+) - \operatorname{Ci}(2u_-) + \ln c_{\gamma/2}^2) \right] \\
 & - \frac{s_{u_+ - u_-}}{32u s_{\gamma/2}^3} \left(21 - 28c_\gamma + 7c_{2\gamma} + \frac{3 - c_\gamma}{u^2} \right) + \frac{c_{u_+ - u_-}}{8u^2 s_{\gamma/2}^2} (1 + s_{\gamma/2}^2) \\
 & - 2s_{\gamma/2}^2 (\operatorname{Ci}(2u) - \operatorname{Ci}(u_+ - u_-) + \ln s_{\gamma/2}) + \frac{3 - c_\gamma}{12} - \frac{1 - c_\gamma}{u^2},
 \end{aligned}$$

where the following notations are used:

$$s_x = \sin(x), \quad c_x = \cos(x), \quad u_\pm = u \pm u \sin\left(\frac{\gamma}{2}\right),$$

and

$$\operatorname{Si}(x) = \int_0^x dt \frac{\sin(t)}{t}, \quad \operatorname{Ci}(x) = - \int_x^\infty dt \frac{\cos(t)}{t}. \quad (\text{B.5})$$

Additional Plots



In this appendix, the plots that were mentioned in the text for illustrative purposes are presented.

C.1 SNR for GWB signals from type-I and type-II cosmic strings

This section shows the pure SNR for GWB signals from type-I [C.1](#) and type-II cosmic strings [C.2](#), using both formulas: the one for arbitrary signals (3.8) and the one only for weak signals (3.9), for an observation time of $T_{\text{obs}} = 4$ years. For the SNR for GWB signals from type-II cosmic strings, a similar figure was already shown in [62] for

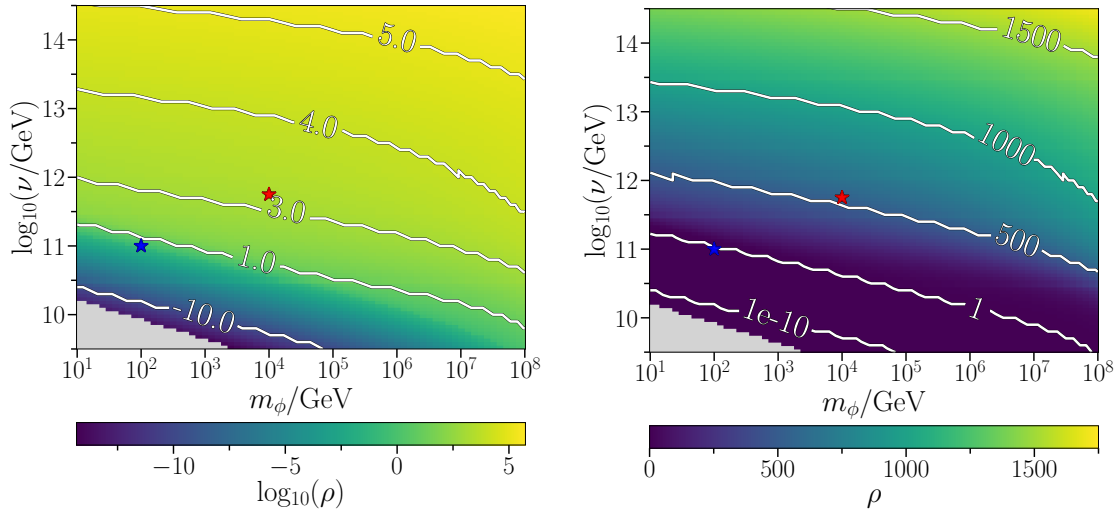


Figure C.1: The SNR for GWB signals from type-I cosmic strings. The right panel shows the results using the formula for weak signal strength (3.9); the right panel shows the results using the formula for arbitrary signal strength. While for the left panel the SNR ρ is given in a logarithmic scale, in the right panel it is given in a linear scale.

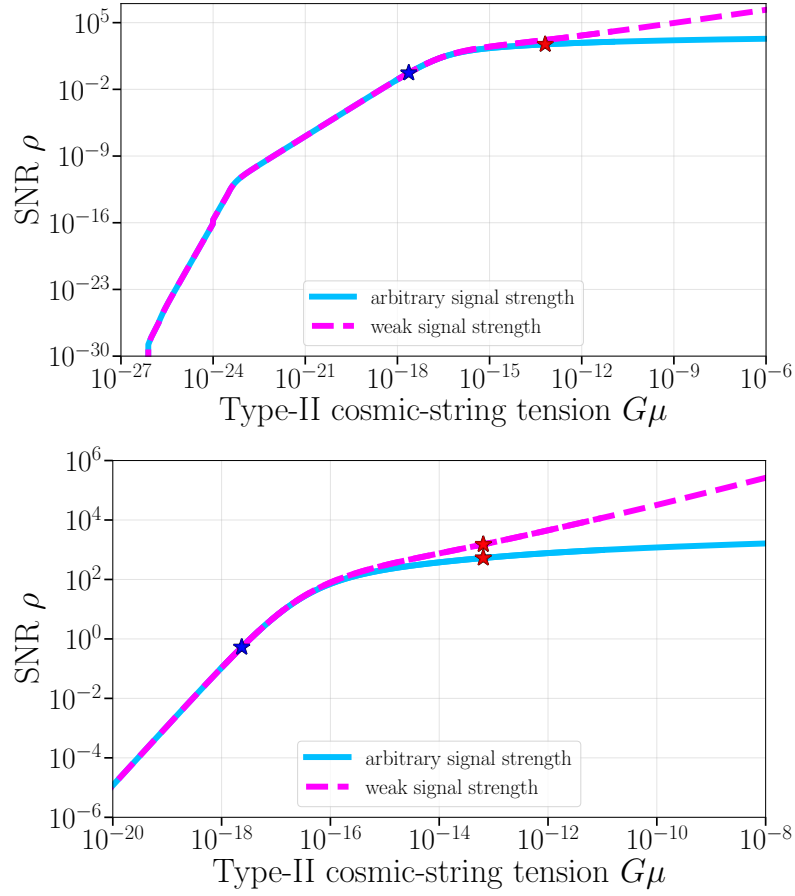


Figure C.2: The SNR for GWB signals from type-II cosmic strings. The blue solid line shows the results using the formula for weak signal strength (3.9); the magenta coloured dashed line shows the results using the formula for arbitrary signal strength. The lower panel shows a zoomed-in version of the upper one. For an observation time of $T_{\text{obs}} = 4$ years, a similar figure is shown in [62].

$T_{\text{obs}} = 1$ year. One can see that for high SNR values the results of the two formulas deviate from each other, while for low SNR values they are equal.

C.2 Power-law approximations of the spectra

This section shows the spectra demonstrated in this thesis with the power-law approximations. One can see that both spectra are roughly described by combinations of power law. The power laws are shown in green coloured dash-dot lines. Figure C.3 presents the spectra of the first example for the GWB signal from type-I cosmic

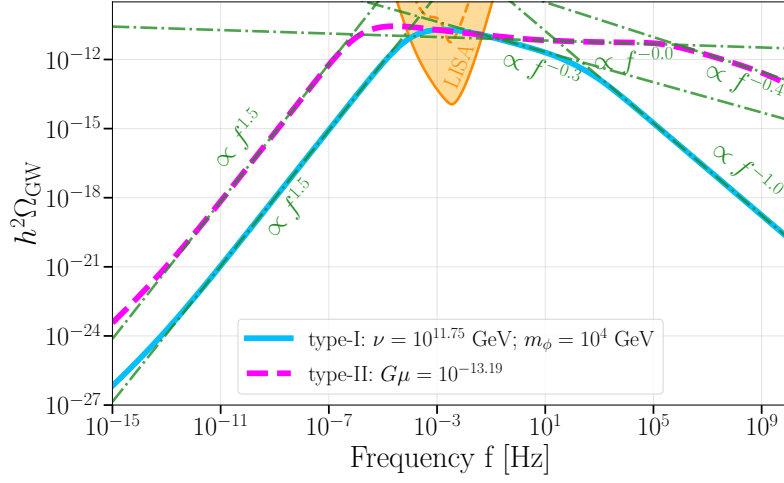


Figure C.3: The spectrum from type-I cosmic strings with $\nu = 10^{11.75}$ GeV and $m_\phi = 10^4$ GeV (blue solid line) and the spectrum from type-II cosmic strings with $G\mu = 10^{-13.19}$. The green dash-dotted lines show the power-law approximation of the two spectra.

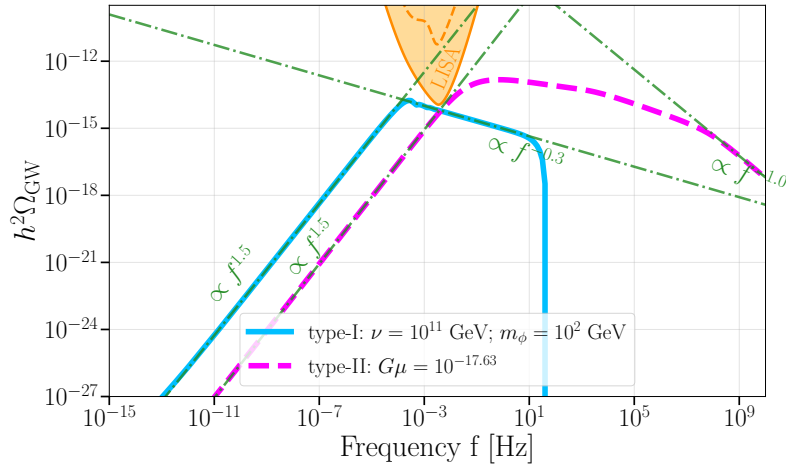


Figure C.4: The spectrum from type-I cosmic strings with $\nu = 10^{11}$ GeV and $m_\phi = 10^2$ GeV (blue solid line) and the spectrum from type-II cosmic strings with $G\mu = 10^{-17.63}$. The green dash-dotted lines show the power-law approximation of the two spectra.

strings with $\nu = 10^{11.75}$ GeV and $m_\phi = 10^4$ GeV, and from type-II cosmic strings with $G\mu = 10^{-13.19}$. Figure C.3 presents the spectra of the second example for the GWB signal from type-I cosmic strings with $\nu = 10^{11}$ GeV and $m_\phi = 10^2$ GeV, and from type-II cosmic strings with $G\mu = 10^{-17.63}$.

C.3 Power-law integrated sensitivity curves

This section presents the idea of the Power-law integrated sensitivity curves that are shown in the figures of the spectra. It was developed in [64], where it is explained in more detail.

The idea is that every GW signal that follows a power law and is below this sensitivity curve has an SNR less than 1. Therefore, we consider all kinds of power-law signals $\Omega_\beta(f/f_{\text{ref}})^\beta$, where Ω_β is a constant amplitude for each β , f_{ref} is an arbitrary reference frequency, and β is the power-law index. To study different power-law signals, we have to choose different β values over a large range, for example, $\beta \in [-20, 20]$. We use the SNR formula (3.9), to normalise Ω_β , such that the SNR $\rho = 1$:

$$\Omega_\beta = \frac{\rho}{\sqrt{T_{\text{obs}}}} \left[\int_{f_{\text{min}}}^{f_{\text{max}}} df \left(\frac{(f/f_{\text{ref}})^\beta}{\Omega_{\text{signal}}(f)} \right)^2 \right]^{-1/2}.$$

Then we take the maximum of these power-law signals for each frequency:

$$\Omega_{\text{PLISC}}(f) = \max_{\beta} \left[\Omega_\beta \left(\frac{f}{f_{\text{ref}}} \right)^\beta \right].$$

The idea is presented in figure C.5.

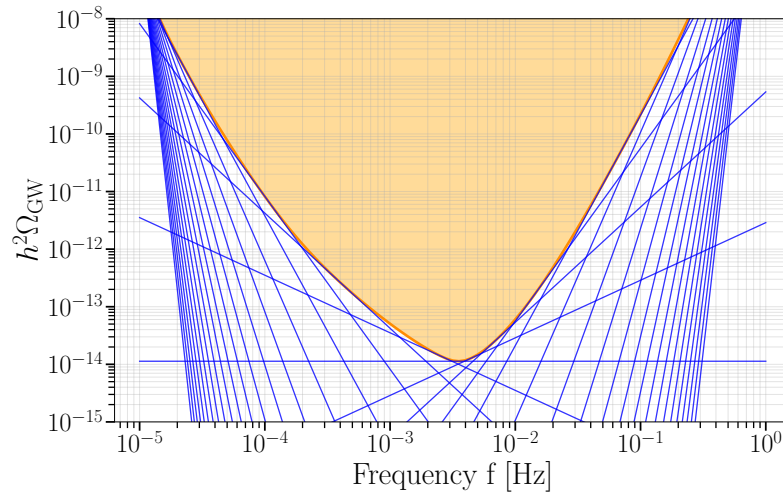


Figure C.5: The power-law integrated sensitivity curve in orange and the normalised power-laws in blue.

Bibliography

- [1] M. Thomson. *Modern particle physics*. eng. Cambridge: Cambridge university press, 2013.
- [2] R. Jeannerot, J. Rocher, and M. Sakellariadou. “How generic is cosmic string formation in SUSY GUTs”. In: *Physical Review D* 68.10 (Nov. 2003), p. 103514. DOI: [10.1103/PhysRevD.68.103514](https://doi.org/10.1103/PhysRevD.68.103514). URL: <http://arxiv.org/abs/hep-ph/0308134> (visited on 08/17/2026).
- [3] B. J. Holzer. “Introduction to Particle Accelerators and their Limitations”. In: *CERN Yellow Reports* (Feb. 2016), 29 Pages. DOI: [10.5170/CERN-2016-001.29](https://doi.org/10.5170/CERN-2016-001.29). URL: <http://arxiv.org/abs/1705.09601> (visited on 08/17/2026).
- [4] B. Povh, K. Rith, C. Scholz, F. Zetsche, and W. Rodejohann. *Teilchen und Kerne: Eine Einführung in die physikalischen Konzepte*. ger. 9. Aufl. 2014. Springer-Lehrbuch. Berlin, Heidelberg: Springer Spektrum, 2014. DOI: [10.1007/978-3-642-37822-5](https://doi.org/10.1007/978-3-642-37822-5).
- [5] K. Schmitz. “Modern Cosmology, an Amuse-Gueule”. In: 2022, pp. 37–70. DOI: [10.1007/978-3-031-05625-3_3](https://doi.org/10.1007/978-3-031-05625-3_3). URL: <http://arxiv.org/abs/2203.04757> (visited on 08/17/2026).
- [6] A. Einstein. “Die Grundlage der allgemeinen Relativitätstheorie”. German. In: *Annalen der Physik* 354.7 (1916), pp. 769–822. DOI: [10.1002/andp.19163540702](https://doi.org/10.1002/andp.19163540702).
- [7] A. Einstein. “Über Gravitationswellen”. In: *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (Jan. 1918), pp. 154–167. URL: <https://ui.adsabs.harvard.edu/abs/1918SPAW.....154E> (visited on 08/06/2026).

- [8] N. Christensen. “Stochastic Gravitational Wave Backgrounds”. en. In: *Reports on Progress in Physics* 82.1 (Jan. 2019), p. 016903. DOI: [10.1088/1361-6633/aae6b5](https://doi.org/10.1088/1361-6633/aae6b5). URL: <http://arxiv.org/abs/1811.08797> (visited on 08/14/2026).
- [9] T. W. B. Kibble. “Topology of cosmic domains and strings”. en. In: *Journal of Physics A: Mathematical and General* 9.8 (Aug. 1976), pp. 1387–1398. DOI: [10.1088/0305-4470/9/8/029](https://doi.org/10.1088/0305-4470/9/8/029). URL: <https://iopscience.iop.org/article/10.1088/0305-4470/9/8/029> (visited on 08/18/2026).
- [10] K. Schmitz and T. Schroeder. “Gravitational waves from cosmic strings for pedestrians”. In: *Journal of Cosmology and Astroparticle Physics* 2026.01 (Jan. 2026), p. 025. DOI: [10.1088/1475-7516/2026/01/025](https://doi.org/10.1088/1475-7516/2026/01/025). URL: <http://arxiv.org/abs/2412.20907> (visited on 04/16/2026).
- [11] A. Einstein. “Näherungsweise Integration der Feldgleichungen der Gravitation”. In: *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (Jan. 1916), pp. 688–696. URL: <https://ui.adsabs.harvard.edu/abs/1916SPAW.....688E> (visited on 08/06/2026).
- [12] J. Pinochet. “General relativity in a nutshell I”. In: *Physica Scripta* 98.12 (Dec. 2023), p. 126103. DOI: [10.1088/1402-4896/ad0c34](https://doi.org/10.1088/1402-4896/ad0c34). URL: <http://arxiv.org/abs/2401.08612> (visited on 08/03/2026).
- [13] J. Pinochet. “General relativity in a nutshell II”. en. In: *Physica Scripta* 98.12 (Dec. 2023), p. 126104. DOI: [10.1088/1402-4896/ad0c15](https://doi.org/10.1088/1402-4896/ad0c15). URL: <http://arxiv.org/abs/2401.12219> (visited on 08/03/2026).
- [14] M. Maggiore. *Gravitational waves: volume 1: theory and experiments*. eng. Oxford: Oxford University Press, 2007. DOI: [10.1093/acprof:oso/9780198570745.001.0001](https://doi.org/10.1093/acprof:oso/9780198570745.001.0001).
- [15] J. Pinochet. *Classical Tests of General Relativity Part I: Looking to the Past to Understand the Present*. en. Aug. 2020. DOI: [10.48550/arXiv.2008.11177](https://doi.org/10.48550/arXiv.2008.11177). URL: <http://arxiv.org/abs/2008.11177> (visited on 08/03/2026).
- [16] T. Fließbach. *Allgemeine Relativitätstheorie*. ger. 7. Aufl. 2016. Berlin, Heidelberg: Springer Berlin Heidelberg, 2016. DOI: [10.1007/978-3-662-53106-8](https://doi.org/10.1007/978-3-662-53106-8).
- [17] A. Vilenkin. “Cosmic Defects”. en. In: (Oct. 1996), p. 005. URL: <https://arxiv.org/abs/astro-ph/9610125> (visited on 07/30/2026).

- [18] J. Schwichtenberg. *Physics from Symmetry*. eng. 2nd ed. 2018. Undergraduate Lecture Notes in Physics. Cham: Springer, 2018. DOI: [10.1007/978-3-319-66631-0](https://doi.org/10.1007/978-3-319-66631-0).
- [19] A. Vilenkin and E. P. S. Shellard. *Cosmic strings and other topological defects*. eng. Reprint. Cambridge monographs on mathematical physics. Cambridge [u.a.]: Cambridge Univ. Press, 2001.
- [20] A. Abokhalil. “The Higgs Mechanism and Higgs Boson: Unveiling the Symmetry of the Universe”. en. In: (June 2023). URL: <https://arxiv.org/abs/2306.01019> (visited on 07/30/2026).
- [21] A. Gangui. *Topological Defects in Cosmology*. Oct. 2001. DOI: [10.48550/arXiv.astro-ph/0110285](https://doi.org/10.48550/arXiv.astro-ph/0110285). URL: <http://arxiv.org/abs/astro-ph/0110285> (visited on 04/16/2026).
- [22] B. C. Hall. *An Elementary Introduction to Groups and Representations*. en. May 2000. DOI: [10.48550/arXiv.math-ph/0005032](https://doi.org/10.48550/arXiv.math-ph/0005032). URL: <http://arxiv.org/abs/math-ph/0005032> (visited on 07/27/2026).
- [23] J. Goldstone, A. Salam, and S. Weinberg. “Broken Symmetries”. en. In: *Physical Review* 127.3 (Aug. 1962), pp. 965–970. DOI: [10.1103/PhysRev.127.965](https://doi.org/10.1103/PhysRev.127.965). URL: <https://link.aps.org/doi/10.1103/PhysRev.127.965> (visited on 07/28/2026).
- [24] R. Gross and A. Marx. *Festkörperphysik*. ger. 2., aktualisierte Aufl. De Gruyter Studium. Berlin: De Gruyter Oldenbourg, 2014. DOI: [10.1524/9783110358704](https://doi.org/10.1524/9783110358704).
- [25] D. Croon and D. J. Weir. “Gravitational Waves from Phase Transitions”. en. In: *Contemporary Physics* 65.2 (Apr. 2024), pp. 75–96. DOI: [10.1080/00107514.2024.2423496](https://doi.org/10.1080/00107514.2024.2423496). URL: <http://arxiv.org/abs/2410.21509> (visited on 07/29/2026).
- [26] B. Hartmann and J. Kunz, eds. *Gravity, Cosmology, and Astrophysics: A Journey of Exploration and Discovery with Female Pioneers*. en. Vol. 1022. Lecture Notes in Physics. Cham: Springer Nature Switzerland, 2023. DOI: [10.1007/978-3-031-42096-2](https://doi.org/10.1007/978-3-031-42096-2). URL: <https://link.springer.com/10.1007/978-3-031-42096-2> (visited on 04/16/2026).
- [27] Smoot Group Cosmology. *The Birth of the Universe*. URL: https://aether.lbl.gov/eunhwa_webpage_2/birth.html (visited on 07/30/2026).

- [28] S. F. Saipuddin, A. Hashim, and N. E. Suhaimi. “Type I and Type II Superconductivity”. en. In: *Superconducting Materials*. Ed. by Y. Slimani and E. Hannachi. Singapore: Springer Nature Singapore, 2022, pp. 123–146. DOI: [10.1007/978-981-19-1211-5_5](https://doi.org/10.1007/978-981-19-1211-5_5). URL: https://link.springer.com/10.1007/978-981-19-1211-5_5 (visited on 07/23/2026).
- [29] K. Saikawa. “A review of gravitational waves from cosmic domain walls”. In: *Universe* 3.2 (May 2017), p. 40. DOI: [10.3390/universe3020040](https://doi.org/10.3390/universe3020040). URL: <http://arxiv.org/abs/1703.02576> (visited on 08/11/2026).
- [30] C. J. A. P. Martins and E. P. S. Shellard. “Quantitative string evolution”. en. In: *Physical Review D* 54.4 (Aug. 1996), pp. 2535–2556. DOI: [10.1103/PhysRevD.54.2535](https://doi.org/10.1103/PhysRevD.54.2535). URL: <https://link.aps.org/doi/10.1103/PhysRevD.54.2535> (visited on 07/31/2026).
- [31] C. J. A. P. Martins and E. P. S. Shellard. “Extending the velocity-dependent one-scale string evolution model”. en. In: *Physical Review D* 65.4 (Jan. 2002), p. 043514. DOI: [10.1103/PhysRevD.65.043514](https://doi.org/10.1103/PhysRevD.65.043514). URL: <https://link.aps.org/doi/10.1103/PhysRevD.65.043514> (visited on 07/31/2026).
- [32] R. A. Matzner. “Interaction of U(1) cosmic strings: Numerical intercommutation”. In: *Computers in Physics* 2 (Sept. 1988), pp. 51–64. DOI: [10.1063/1.168306](https://doi.org/10.1063/1.168306). URL: <https://ui.adsabs.harvard.edu/abs/1988ComPh...2...51M> (visited on 08/01/2026).
- [33] C. J. A. P. Martins, J. N. Moore, and E. P. S. Shellard. “Unified model for vortex-string network evolution”. In: *Physical Review Letters* 92.25 (June 2004), p. 251601. DOI: [10.1103/PhysRevLett.92.251601](https://doi.org/10.1103/PhysRevLett.92.251601). URL: <http://arxiv.org/abs/hep-ph/0310255> (visited on 08/01/2026).
- [34] J. J. Blanco-Pillado and K. D. Olum. “Stochastic gravitational wave background from smoothed cosmic string loops”. In: *Physical Review D* 96.10 (Nov. 2017), p. 104046. DOI: [10.1103/PhysRevD.96.104046](https://doi.org/10.1103/PhysRevD.96.104046). URL: <http://arxiv.org/abs/1709.02693> (visited on 08/01/2026).
- [35] Chung-Ang University Particle Physics Theory Group. *CAU Lecture: Gravitational Waves from the Early Universe: Part III – Kai Schmitz*. June 4, 2021. URL: <https://www.youtube.com/watch?v=xFI8PBsAqY> (visited on 08/01/2026).

- [36] P. Auclair, J. J. Blanco-Pillado, D. G. Figueroa, A. C. Jenkins, M. Lewicki, M. Sakellariadou, S. Sanidas, L. Sousa, D. A. Steer, J. M. Wachter, and S. Kuroyanagi. “Probing the gravitational wave background from cosmic strings with LISA”. In: *Journal of Cosmology and Astroparticle Physics* 2020.04 (Apr. 2020), pp. 034–034. DOI: [10.1088/1475-7516/2020/04/034](https://doi.org/10.1088/1475-7516/2020/04/034). URL: <http://arxiv.org/abs/1909.00819> (visited on 08/02/2026).
- [37] J. J. Blanco-Pillado, K. D. Olum, and B. Shlaer. “The number of cosmic string loops”. In: *Physical Review D* 89.2 (Jan. 2014), p. 023512. DOI: [10.1103/PhysRevD.89.023512](https://doi.org/10.1103/PhysRevD.89.023512). URL: <http://arxiv.org/abs/1309.6637> (visited on 08/02/2026).
- [38] K. J. Bae, A. Datta, R. Maji, and W.-I. Park. *Cogenesis of baryon and lepton number asymmetries matching the EMPRESS Data*. Sept. 2025. DOI: [10.48550/arXiv.2509.13098](https://doi.org/10.48550/arXiv.2509.13098). URL: <http://arxiv.org/abs/2509.13098> (visited on 04/16/2026).
- [39] A. Afzal. “Gravitational Waves from $\mathbb{S}^1$ Strings in a Neutrino Mass Model”. en. In: *Physics of the Dark Universe* 51 (Feb. 2026), p. 102204. DOI: [10.1016/j.dark.2025.102204](https://doi.org/10.1016/j.dark.2025.102204). URL: <http://arxiv.org/abs/2509.11107> (visited on 08/03/2026).
- [40] Y. Cui, S. P. Martin, D. E. Morrissey, and J. D. Wells. “Cosmic Strings from Supersymmetric Flat Directions”. In: *Physical Review D* 77.4 (Feb. 2008), p. 043528. DOI: [10.1103/PhysRevD.77.043528](https://doi.org/10.1103/PhysRevD.77.043528). URL: <http://arxiv.org/abs/0709.0950> (visited on 04/16/2026).
- [41] M. Maggiore. “Gravitational Wave Experiments and Early Universe Cosmology”. In: *Physics Reports* 331.6 (July 2000), pp. 283–367. DOI: [10.1016/S0370-1573\(99\)00102-7](https://doi.org/10.1016/S0370-1573(99)00102-7). URL: <http://arxiv.org/abs/gr-qc/9909001> (visited on 08/09/2026).
- [42] *LISA Consortium*. en-US. URL: <https://www.lisamission.org> (visited on 08/06/2026).
- [43] M. Kunz. *Cosmolgy II*. English. Tech. rep. University of Geneva, June 2005. URL: https://fiteoweb.unige.ch/~kunzm/lectures/cosmo_II_2005.pdf (visited on 08/05/2026).

- [44] S. Weinberg. *Gravitation and cosmology: principles and applications of the general theory of relativity*. eng. New York Chichester Brisbane [etc.]: J. Wiley & sons, 1972.
- [45] K. Saikawa and S. Shirai. “Precise WIMP Dark Matter Abundance and Standard Model Thermodynamics”. en. In: *Journal of Cosmology and Astroparticle Physics* 2020.08 (Aug. 2020), pp. 011–011. doi: [10.1088/1475-7516/2020/08/011](https://doi.org/10.1088/1475-7516/2020/08/011). URL: <http://arxiv.org/abs/2005.03544> (visited on 08/05/2026).
- [46] P. Collaboration et al. “Planck 2018 results. VI. Cosmological parameters”. en. In: *Astronomy & Astrophysics* 641 (Sept. 2020), A6. doi: [10.1051/0004-6361/201833910](https://doi.org/10.1051/0004-6361/201833910). URL: <http://arxiv.org/abs/1807.06209> (visited on 08/05/2026).
- [47] *Review of Particle Physics* | *Phys. Rev. D*. URL: <https://journals.aps.org/prd/abstract/10.1103/PhysRevD.98.030001> (visited on 08/04/2026).
- [48] *National Institute of Standards and Technology*. URL: <https://www.nist.gov/> (visited on 08/04/2026).
- [49] *Nobel Prize in Physics 2017*. en-US. URL: <https://www.nobelprize.org/prizes/physics/2017/summary/> (visited on 08/07/2026).
- [50] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. en. In: *Physical Review Letters* 116.6 (Feb. 2016), p. 061102. doi: [10.1103/PhysRevLett.116.061102](https://doi.org/10.1103/PhysRevLett.116.061102). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.116.061102> (visited on 08/07/2026).
- [51] *LIGO Lab | Caltech | MIT*. URL: <https://www.ligo.caltech.edu/> (visited on 08/07/2026).
- [52] *Die Geheimnisse des Universums erforschen: Gravitationswellen-Detektoren starten ihren nächsten Beobachtungslauf – Virgo*. en-GB. URL: <https://www.virgo-gw.eu/die-geheimnisse-des-universums-erforschen-gravitationswellen-detektoren-starten-ihren-nachsten-beobachtungslauf/> (visited on 08/07/2026).
- [53] *KAGRA 大型低温重力波望遠鏡 – KAGRA 大型低温重力波望遠鏡のホームページです。*. ja. URL: <https://gwcenter.icrr.u-tokyo.ac.jp/> (visited on 08/07/2026).

- [54] A. Buikema et al. “Sensitivity and performance of the Advanced LIGO detectors in the third observing run”. en. In: *Physical Review D* 102.6 (Sept. 2020), p. 062003. DOI: [10.1103/PhysRevD.102.062003](https://doi.org/10.1103/PhysRevD.102.062003). URL: <https://link.aps.org/doi/10.1103/PhysRevD.102.062003> (visited on 08/07/2026).
- [55] S. R. Taylor. *The Nanohertz Gravitational Wave Astronomer*. en. May 2021. DOI: [10.48550/arXiv.2105.13270](https://doi.org/10.48550/arXiv.2105.13270). URL: <http://arxiv.org/abs/2105.13270> (visited on 08/07/2026).
- [56] *NANOGrav*. en. May 2026. URL: <https://nanograv.org/> (visited on 08/07/2026).
- [57] G. Agazie et al. “The NANOGrav 15-year Data Set: Evidence for a Gravitational-Wave Background”. en. In: *The Astrophysical Journal Letters* 951.1 (July 2023), p. L8. DOI: [10.3847/2041-8213/acdac6](https://doi.org/10.3847/2041-8213/acdac6). URL: <http://arxiv.org/abs/2306.16213> (visited on 08/07/2026).
- [58] J. Baker et al. *The Laser Interferometer Space Antenna: Unveiling the Millihertz Gravitational Wave Sky*. en. July 2019. DOI: [10.48550/arXiv.1907.06482](https://doi.org/10.48550/arXiv.1907.06482). URL: <http://arxiv.org/abs/1907.06482> (visited on 08/07/2026).
- [59] M. Colpi et al. *LISA Definition Study Report*. Feb. 2024. DOI: [10.48550/arXiv.2402.07571](https://doi.org/10.48550/arXiv.2402.07571). URL: <http://arxiv.org/abs/2402.07571> (visited on 08/07/2026).
- [60] N. Cornish and T. Robson. “Galactic binary science with the new LISA design”. en. In: *Journal of Physics: Conference Series* 840 (May 2017), p. 012024. DOI: [10.1088/1742-6596/840/1/012024](https://doi.org/10.1088/1742-6596/840/1/012024). URL: <http://arxiv.org/abs/1703.09858> (visited on 08/10/2026).
- [61] *Construction of ESA’s ambitious LISA mission begins*. en. URL: https://www.esa.int/Science_Exploration/Space_Science/LISA/Construction_of_ESA_s_ambitious_LISA_mission_begins (visited on 08/07/2026).
- [62] K. Schmitz and J. D. Romano. *Signal-to-Noise Ratio Contours for LISA*. July 2026. DOI: [10.48550/arXiv.2607.08445](https://doi.org/10.48550/arXiv.2607.08445). URL: <http://arxiv.org/abs/2607.08445> (visited on 08/10/2026).
- [63] T. Robson, N. Cornish, and C. Liu. “The construction and use of LISA sensitivity curves”. In: *Classical and Quantum Gravity* 36.10 (May 2019), p. 105011. DOI: [10.1088/1361-6382/ab1101](https://doi.org/10.1088/1361-6382/ab1101). URL: <http://arxiv.org/abs/1803.01944> (visited on 08/10/2026).

- [64] E. Thrane and J. D. Romano. “Sensitivity curves for searches for gravitational-wave backgrounds”. en. In: *Physical Review D* 88.12 (Dec. 2013), p. 124032. DOI: [10.1103/PhysRevD.88.124032](https://doi.org/10.1103/PhysRevD.88.124032). URL: <http://arxiv.org/abs/1310.5300> (visited on 08/13/2026).
- [65] M. Lilley, F. D. Marco, J. Martin, and P. Peter. “Nonabelian Bosonic Currents in Cosmic Strings”. In: *Physical Review D* 82.2 (July 2010), p. 023510. DOI: [10.1103/PhysRevD.82.023510](https://doi.org/10.1103/PhysRevD.82.023510). URL: <http://arxiv.org/abs/1003.4601> (visited on 08/21/2026).
- [66] I. Y. Rybak and L. Sousa. “Emission of gravitational waves by superconducting cosmic strings”. In: *Journal of Cosmology and Astroparticle Physics* 2022.11 (Nov. 2022), p. 024. DOI: [10.1088/1475-7516/2022/11/024](https://doi.org/10.1088/1475-7516/2022/11/024). URL: <http://arxiv.org/abs/2209.01068> (visited on 08/21/2026).
- [67] N. J. Cornish. “Detecting a stochastic gravitational wave background with the Laser Interferometer Space Antenna”. In: *Physical Review D* 65.2 (Dec. 2001), p. 022004. DOI: [10.1103/PhysRevD.65.022004](https://doi.org/10.1103/PhysRevD.65.022004). URL: <http://arxiv.org/abs/gr-qc/0106058> (visited on 08/10/2026).
- [68] C. Zhang, Q. Gao, Y. Gong, B. Wang, A. J. Weinstein, and C. Zhang. “Full analytical formulas for frequency response of space-based gravitational wave detectors”. In: *Physical Review D* 101.12 (June 2020), p. 124027. DOI: [10.1103/PhysRevD.101.124027](https://doi.org/10.1103/PhysRevD.101.124027). URL: <http://arxiv.org/abs/2003.01441> (visited on 08/10/2026).