

BACHELOR'S THESIS

C^0 -INEXTENDIBILITY OF
THE SCHWARZSCHILD SPACETIME

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Abstract

We start by giving a physical motivation for proving the C^0 -inextendibility of a spacetime. As one of the first results for showing C^0 -inextendibility, we investigate the maximal analytically extended Schwarzschild spacetime and give necessary theoretical backgrounds on Lorentzian geometry, Einstein's (vacuum) equations and the Schwarzschild solution including a non-zero cosmological constant. After quoting required results from [1], [2], [3] and [4] and sketching the proof of C^0 -inextendibility for the Schwarzschild spacetime we begin by assuming C^0 -extendibility and separate into two cases: *leaving the Kruskal manifold through the Schwarzschild exterior* and *through the interior*. In both cases, a contradiction is deduced. For the latter, the so-called spacelike diameter is shown to be bounded by assumption but infinite by the geometry of the Schwarzschild black hole. Finally, we try to adapt the proof to the case of a non-vanishing cosmological constant for the de Sitter-Schwarzschild spacetime.

1 Introduction

This work aims at elucidating the proof of C^0 -inextendibility of the maximal analytically extended Schwarzschild spacetime. The Schwarzschild solution plays an important role in general relativity as it is the simplest black hole solution to the Einstein vacuum equations. Because the upcoming proof is the initial introduction of displayed methods the first step in understanding and solving the problem is investigating the most simplified model. The problem addressed in general terms is the notion of a suitably regular Lorentzian manifold as it is used in the strong cosmic censorship conjecture:

For generic asymptotically flat initial data for the vacuum Einstein equations $\text{Ric}(g) = 0$, the maximal globally hyperbolic development is inextendible as a *suitably regular Lorentzian manifold*. [1, p. 320]

The cosmic censorship hypotheses were conceived by Penrose in e. g. [5] and are further illustrated in [6, ch. 1 & 2] by Hawking and Penrose. After prescribing initial data to a physical system we would classically expect to have determined the evolution of the system. This evolution in the words of general relativity is given by the maximal globally hyperbolic development which is the maximal unique development of initial data. If this development was further extendible but not determined by initial data, we could not predict any physics on the extended parts. In the classical but dynamical theory of general relativity, such a situation should be avoided which is the statement of the strong cosmic censorship conjecture above. There-

fore, a sensible definition of *suitably regular* should comply with the most general physical assumptions which here is assumed to be continuity. The regularity of a Lorentzian manifold that describes the spacetime we live in is characterised by the regularity of its corresponding metric. A well-known result for the Schwarzschild spacetime is its C^2 -inextendibility which states that there is no larger spacetime containing the Schwarzschild spacetime as a proper subset and having a twice continuously differentiable metric. This follows from a curvature blow-up in the centre of the Schwarzschild black hole. In this thesis we show that the maximal analytically extended Schwarzschild solution (the maximal C^2 -regular extension) is in fact C^0 -inextendible, i. e. we cannot find an extension of the spacetime even if for a merely continuous metric. This is formally stated as Theorem I in subsection 2.3. The difficulty of proving this result lies in the breakdown of the notion of curvature, geodesics and the Einstein equations themselves for a non-differentiable metric. The minimal regularity required to make sense of these in a weak formulation is $C^{1,1}$ (differentiable with Lipschitz continuous first derivatives) or $W^{1,2}$ (first weak derivatives in L^2). The works this thesis is built upon are [1] and [2] by Jan Sbierski. The parts of proof presented here are limited in their extent but detailed and thorough in explanation. They closely follow the line of argument in [2] and quote several results of [2], [3] and [4] without proving them.

2 Theoretical Foundations

We start by giving a brief introduction into Lorentzian geometry while the reader is assumed familiar with differential geometry.

2.1 Basics of Lorentzian geometry

Here basic definitions, notions and terminology of Lorentzian geometry as a subcategory of semi-Riemannian geometry are listed.

Definition 2.1 (Lorentzian manifolds & spacetimes). A Lorentzian manifold (M, g) is defined as a smooth n -dimensional manifold M together with a Lorentzian metric g (a semi-Riemannian metric with signature $(-, +, \dots, +)$). A Lorentzian manifold paired with a time orientation (see below) is called a spacetime.

Such manifolds are meant to represent the physical spacetime we live in. This general formulation with manifolds as opposed to e. g. \mathbb{R}^3 allows for curved space and time where the distance is a local property. That means that the metric is non-constant and depends on where in the manifold you look at. The semi-Riemannian metric being positive definite only up to a

subspace of dimension $n - 1$ characterises a time coordinate which forms a causal structure corresponding to the finiteness of the speed of light which here is put to 1. In a spacetime a point $p \in M$ describes a physical event at a specific place, the last $n - 1$ coordinates, and time, the zeroth coordinate.

Definition 2.2 (causal structure). Given a Lorentzian manifold (M, g) we call a vector $0 \neq v \in T_p M$ in the tangent space of a point $p \in M$

1. **timelike** if $g(v, v) < 0$,
2. **null** if $g(v, v) = 0$ and
3. **spacelike** if $g(v, v) > 0$.

The zero vector is by convention spacelike. The term **causal** describes timelike as well as null vectors.

At every point p of a Lorentzian manifold (M, g) these notions divide the tangent space $T_p M$ into a double time cone consisting of all timelike vectors separated by its boundary, the double light cone of all null vectors, from the set of spacelike vectors (cf. Fig. 1). The double time and light cones together form a double causal cone.

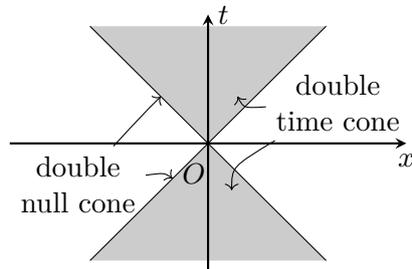


Figure 1: The double time and null cone in the tangent space at a point of a Lorentzian manifold where the vector ∂_t is timelike and ∂_x spacelike.

Definition 2.3 (time orientation). We call a Lorentzian manifold (M, g) time oriented if there exists a function choosing one cone of the timelike double cone at each $p \in M$ such that there is a smooth vector field V in a neighbourhood U of p in which V lies everywhere in the chosen time cone.

Having such a smooth function translates into patching the local choice of a time cone consistently together on the entire manifold. If a Lorentzian manifold is time oriented a future and past can be defined by the chosen time cone. As a consequence we are enabled to talk about events in physical space and time. The time orientation, however, on choosing one or the other

time cone is an arbitrary choice, hence, can be chosen as pleased. Moreover, for two timelike vectors v, w to point in the same time cone $g(v, w) < 0$ must hold.

Definition 2.4. The above terms describing causality are generalised to the following objects in differential geometry of a Lorentzian manifold (M, g) :

- Smooth vector fields of M are called timelike/null/spacelike if they are so at every point $p \in M$.
- Here, causal/timelike/null/spacelike curves are defined as piecewise smooth curves through M with causal/timelike/null/spacelike tangent vector at every point of the curve. Additionally, the tangent vector before and after a discontinuity must point in the same time/light cone.
- Subsets of the tangent space $T_p(M)$ of M and for a $p \in M$ are timelike/null/spacelike if every inlying non-zero vector is.
- A hypersurface N of M is called timelike/null/spacelike if the tangent space $T_p(N) \subset T_p(M)$ is timelike/null/spacelike for every $p \in N$. This is equivalent to the induced metric \bar{g} on N being negative definite/degenerate/positive definite.

To the terms *timelike*, *null* and *causal* one may add future/past directed in respect to the chosen time orientation.

Timelike vectors or timelike curves describe directions or paths of massive observers through spacetime which are restricted to move slower than light. Null curves describe the path of massless particles, e. g. photons, moving at the speed of light. Therefore, only if two events are connected by a causal curve there can be a causal relation between them.

Definition 2.5 (Lorentzian length). Let (M, g) be a Lorentzian manifold. The length of causal curves $\gamma : I \rightarrow M$ with $I \subset \mathbb{R}$ an interval is given by the Lorentzian length functional

$$L(\gamma) := \int_I \sqrt{-g(\dot{\gamma}(s), \dot{\gamma}(s))} ds. \quad (2.1)$$

For spacelike curves, it is adjusted to

$$L(\gamma) := \int_I \sqrt{g(\dot{\gamma}(s), \dot{\gamma}(s))} ds. \quad (2.2)$$

The meaning of the Lorentzian length for causal curves is the **proper time**, i. e. the time experienced by an on-the-curve moving observer. Geodesics,

the paths of free-falling observers, in Riemannian geometry minimise their length. This holds true for spacelike geodesics. Timelike geodesics, however, maximise their Lorentzian length, i. e. the proper time experienced by an observer following the geodesic.

Definition 2.6 (timelike future (past)). Let (M, g) be a spacetime. The timelike future (past) in M of a point $p \in M$ is defined by the set

$$I^\pm(p, M) = \{q \in M \mid \exists \text{ future (past) directed timelike curve from } p \text{ to } q\}. \quad (2.3)$$

These sets describe the events that may influence or be influenced by an event p excluding communication at the speed of light. In other words, the points lying in the timelike future $I^+(p, M)$ of p can be reached starting at p by a massive observer, and respectively, p can be reached by massive observers starting at some point in the timelike past $I^-(p, M)$ of p . The next two definitions rely on the causal connection of events.

Definition 2.7 (Cauchy hypersurface). A Cauchy hypersurface is a spacelike hypersurface Σ of a Lorentzian manifold (M, g) with which every inextendible causal curve of M intersects once and only once.

Definition 2.8 (global hyperbolicity). A spacetime is called globally hyperbolic if and only if it admits a Cauchy hypersurface Σ .

The existence of such a hypersurface implies that every spacetime event can be causally related to an event on Σ . The notion of globally hyperbolic plays an important role in the concept of determinism. Given initial data on a Cauchy hypersurface Σ the evolution into a so-called globally hyperbolic development, i. e. a globally hyperbolic Lorentzian manifold agreeing with the initial data on Σ and satisfying the *equation of motion* for a spacetime (see below), exists and is uniquely determined by the initial data. From a classical point of view, the maximal globally hyperbolic development is expected to be inextendible in a suitable sense. According to this the future development or respectively the time evolution of known physical data would be predictable. This formalises the motivation given in the introduction and appears in the strong cosmic censorship conjecture.

Definition 2.9 (extendibility). Let (M, g) be an n -dimensional Lorentzian manifold. If there exists an n -dimensional Lorentzian manifold (\tilde{M}, \tilde{g}) with M isometrically embedded as a proper subset, time orientation preserving and \tilde{g} being C^k -regular we call (M, g) **C^k -extendible** for a $k \in \mathbb{N}_0$. If there does not exist such (\tilde{M}, \tilde{g}) we call (M, g) **C^k -inextendible**. We call M extended to \tilde{M} and \tilde{M} the extension of M .

If (M, g) is isometrically embedded in (\tilde{M}, \tilde{g}) then the metrics agree. More precisely via the pushforward, $g_p(v, w) = \tilde{g}_{\iota(p)}(d\iota_p v, d\iota_p w)$ with the corresponding isometry $\iota : M \hookrightarrow \tilde{M}$ holds for all $p \in M$ and $v, w \in T_p M$. In the

following however, we will not distinguish between the embedded $\iota(M) \subsetneq \tilde{M}$ and the manifold M itself and consider it a subset of \tilde{M} . This is solely for simplicity in notation. We thus say $\tilde{g}|_M = g$. As mentioned before, one can also look at extendibility properties on Sobolev spaces.

The most important background is now given. In addition, the following two notions will appear later on.

Definition 2.10 (achronal). If A is a subset of a spacetime M in which $q \notin I^+(p, M)$ holds for every $p, q \in A$ it is said to be achronal.

Note as an example that a Cauchy hypersurface is achronal.

Definition 2.11 (future (past) boundary). Let (M, g) be a spacetime which can be C^0 -extended to an \tilde{M} . In this larger manifold \tilde{M} the future (past) boundary $\partial^\pm(M)$ of M is defined by the set of all points $p \in \tilde{M}$ for which a timelike curve $\tilde{\gamma} : [-1, 0] \rightarrow \tilde{M}$ exists that lies entirely in M except its endpoint $\tilde{\gamma}(0) \in \tilde{M} \setminus M$. Additionally $\tilde{\gamma}|_{[-1, 0)}$ is future (past) directed in M .

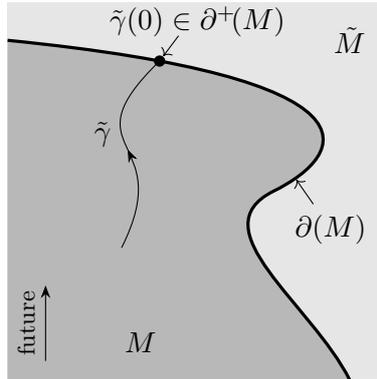


Figure 2: A future directed timelike curve $\tilde{\gamma}$ leaving the manifold M via its endpoint lying on the future boundary of M thus lying in the extension \tilde{M} without M .

Remember that we consider M as a proper subset of \tilde{M} . Therefore, when we talk about the (future or past) boundary of the manifold M it is always meant as the boundary of a subset of its extension. As an easy corollary one notes that by defining a sequence of points on the respective timelike curve towards the boundary and the continuity of the curve we instantly see that $(\partial^+(M) \cup \partial^-(M)) \subseteq \partial(M)$. The situation is illustrated in Figure 2.

2.1.1 Conventions

- The physical constants c and G for the speed of light and gravitational constant are normalised to 1.
- Connectedness and smoothness (at least C^2) of all appearing manifolds is tacitly assumed. The assumption of connectedness seems physically sensible and dropping it would result in having to proof every statement on each disconnected patch. Remark 2.2 of [7, pp. 8f.] discusses the smoothness assumption.
- The term *neighbourhood* is considered to refer to an open neighbourhood.
- The zeroth coordinate x_0 is considered the time coordinate, all others as the spatial coordinates \underline{x} .
- The Einstein convention for summing over repeated indices is used.
- Greek letters as indices run from 0 to $d = n-1$ where n is the dimension of the manifold and Latin ones from 1 to d traversing the spatial coordinates. Analysing the extendibility behaviour of spacetimes in higher dimensions of $d \geq 3$ may prove helpful in further understanding constructs such as black branes from string theory which are solutions to the Einstein equations with additional spatial dimensions [8].
- Covectors can be decomposed in the basis $\{dx^i\}$ whereas vectors of the tangent space are written in the basis of the partial derivatives $\{\partial_i\}$.
- When talking about physical interpretations we usually assume the dimension of the spacetime to be $3 + 1$.
- We use the notation \tilde{a} for a curve or subset of the extension whereas writing a corresponds to the restriction on the original manifold.

2.2 Einstein's equations

The Einstein equations (or Einstein's equations or Einstein field equations) are a set of partial differential equations in the sense of differential geometry that represent the equation of motion for spacetime itself. Let (M, g) be a n -dimensional spacetime. If the metric fulfils these differential equations it describes a physical spacetime. They govern the change of curvature and distance across the manifold. We postulate the **Einstein equations** as follows:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (2.4)$$

Here, $R_{\mu\nu}$ is the Ricci curvature tensor, which becomes R , the Ricci scalar, after contraction. $T_{\mu\nu}$ is the energy-momentum tensor describing the density and flow of energy, equivalent to matter, and momentum in the given spacetime. A motivation for these equations (note that μ, ν run from 0 to n) by Einstein is found in [9, pp. 79ff., ch. 4]. While there are multiple approaches and approximations to define the energy-momentum tensor which include describing perfect fluids, scalar fields satisfying the Klein-Gordon equations or describing electromagnetic fields (cf. e. g. [10, pp. 238ff., ch. 6] or for short [11, p. 3938]) we concentrate on what Albert Einstein called the ‘fine marble’ [12, p. 370], i. e. the left hand side of (2.4).

2.2.1 Vacuum Einstein equations

The vacuum Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \quad (2.5)$$

This formulation of the Einstein equations involves purely mathematically derived descriptions of a spacetime whereas the ‘low grade wood’, i. e. the energy-momentum tensor according to Einstein [12, p. 370], requires physical modelling on a particular system. Putting $T_{\mu\nu}$ to zero leaves us with a description of spacetime in vacuum without any matter. The vacuum equations of this form for dimensions $n > 2$ can be rewritten to a much simpler equation via the calculation

$$R \stackrel{\text{by def.}}{\stackrel{(2.5)}{\downarrow}} g^{\mu\nu} R_{\mu\nu} \stackrel{(2.5)}{\downarrow} \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = \frac{1}{2}\text{tr}(\text{Id}_n)R = \frac{n}{2}R. \quad (2.6)$$

Therefore if (2.5) holds $R = 0$ and consequently $R_{ij} = 0$. The latter is equivalent to (2.5). Thus, we call a Lorentzian manifold satisfying the vacuum Einstein equations **Ricci-flat**.

The trivial solution to the vacuum equations is the flat space with no curvature, i. e. $R_{\mu\nu}$ vanishes identically. It is called the **Minkowski space**. The **Minkowski metric** $m_{\mu\nu}$ is constant and in Cartesian coordinates given by the diagonal matrix $m_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$.

2.2.2 Cosmological constant

One may write the Einstein equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (2.7)$$

The newly appearing $\Lambda \in \mathbb{R}$ is the so-called cosmological constant. For the Einstein equations, it may act like an integration constant but also can give

rise to physical interpretations. Its effects are as of the energy-momentum tensor which is why it is sometimes moved to the right of the equation and interpreted as the intrinsic energy density of the vacuum. There are close connections to quantum field theory and dark energy which could explain the accelerated expansion of the universe. For further information on this see [10, ch. 4.5, 8.4 & 8.7].

Reevaluating (2.6) results in

$$R = \frac{2n}{n-2}\Lambda \quad \text{and} \quad R_{\mu\nu} = \frac{2}{n-2}\Lambda g_{\mu\nu}, \quad (2.8)$$

the vacuum equations with a cosmological constant.

2.3 Schwarzschild solution

This work takes a look at the simplest black hole solution to the vacuum Einstein equations, the $d + 1$ -dimensional Schwarzschild solution, which is up to isometry the unique spherically symmetric, asymptotically flat $d + 1$ -dimensional spacetime around a non-rotating and non-charged mass where $d \geq 3$. A black hole solution admits an area or a submanifold from where no information or observers including light can escape. This area is called a **black hole** and has a boundary that is called the **event horizon**. The Schwarzschild black hole in spatial coordinates is a sphere in d dimensions. The sphere itself is of dimension $d - 1$. The submanifold outside of the black hole and event horizon is called the **Schwarzschild exterior** whereas the black hole as submanifold inside the event horizon is called the **Schwarzschild interior**. The metric as a solution to (2.5) can be written in spherical coordinates as

$$g = - \left(1 - \frac{2m}{r^{d-2}}\right) dt^2 + \left(1 - \frac{2m}{r^{d-2}}\right)^{-1} dr^2 + r^2 \mathring{\gamma}_{d-1} \quad (2.9)$$

on the exterior

$$M_{\text{ext}} := \mathbb{R}_t \times \left((2m)^{\frac{1}{d-2}}, \infty\right)_r \times \mathbb{S}^{d-1} \quad (2.10)$$

as well as on the interior

$$M_{\text{int}} := \mathbb{R} \times \left(0, (2m)^{\frac{1}{d-2}}\right) \times \mathbb{S}^{d-1}. \quad (2.11)$$

Since this solution is spherically symmetric we use the standard metric $\mathring{\gamma}_{d-1}$ on the $d - 1$ -dimensional unit sphere \mathbb{S}^{d-1} . For $d = 3$ $\mathring{\gamma}_2$ just becomes $d\theta^2 + \sin^2\theta d\phi^2$. The time orientation in the exterior is prescribed by the vector field ∂_t being future directed everywhere in M_{ext} . In M_{int} a smooth choice is to put $-\partial_r$ future directed.

Using so called **ingoing Eddington-Finkelstein coordinates** one can find coordinates that isometrically extent M_{ext} and M_{int} to a Lorentzian manifold

$$M_{EF} = \mathbb{R}_v \times (0, \infty)_r \times \mathbb{S}^{d-1} \quad (2.12)$$

with
$$g_{EF} = - \left(1 - \frac{2m}{r^{d-2}} \right) dv^2 + 2dvdr + r^2 \gamma_{d-1}.$$

These coordinates are now completely regular at the event horizon $r = 2m$ and the time orientation, prescribing $-\partial_r$ to be future directed, preserves the time orientation of M_{int} and M_{ext} .

2.3.1 Kruskal manifold

The maximal analytic Schwarzschild solution, however, is called the **Kruskal manifold**. In the carried out parts of proof of the main statement below only the explicit form of the Schwarzschild interior will be needed. Details on the Kruskal manifold are given in e. g. [10, pp. 222ff., ch. 5.7]. A conformal diagram, the Penrose diagram, of the Kruskal manifold is shown in Figure 3. This type of illustration preserves the causal structure of the spacetime. For more information on Penrose diagrams see [10, pp. 471ff., app. H] or [13, pp. 46ff., ch. 3.2.1].

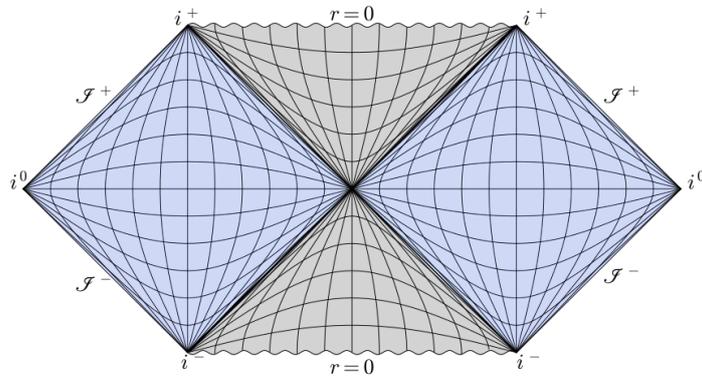


Figure 3: Penrose diagram of the maximal analytically extended Schwarzschild spacetime, the Kruskal spacetime. The blue shaded area on the right is our Schwarzschild exterior whereas the upper grey shaded area is the interior with the singularity at $r = 0$. Lines connecting i^+ with i^+ or i^+ with i^- are of constant r . [14]

The Kruskal manifold is a globally hyperbolic C^2 -extension of the above manifolds but itself is C^2 -inextendible. This follows from a curvature blow-

up which is characterised by a singular scalar curvature, the Kretschmann scalar, for $r \rightarrow 0$. Curvature is quantised by tensors, therefore independent of coordinates. These tensorial curvature quantities rely on the second derivative of the metric which precisely is the reason that the same argument does not hold for C^0 -inextendibility.

Theorem I. *The maximal analytic Schwarzschild spacetime (M_{Kruskal}, g) in $d + 1$ dimensions as a solution to the vacuum Einstein equations is C^0 -inextendible for any $d \geq 3$.*

This is the main statement of the thesis and will be proved in section 4.

2.3.2 Global hyperbolicity of the Schwarzschild interior

In the proof of Theorem I we look at Cauchy hypersurfaces of the Schwarzschild interior. For this M_{int} must be globally hyperbolic. We claim

$$\Sigma = \{p \in M_{\text{int}} \mid r(p) = c\} \quad (2.13)$$

is a Cauchy hypersurface of M_{int} for any constant $c \in (0, (2m)^{1/(d-2)})$. From (2.9) or (2.12) calculate the induced metric on Σ

$$\bar{g} = - \left(1 - \frac{2m}{c^{d-2}} \right) dv^2 + c^2 \hat{\gamma}_{d-1}. \quad (2.14)$$

It holds

$$c^{d-2} < 2m \quad \Rightarrow \quad - \left(1 - \frac{2m}{c^{d-2}} \right) > 0 \quad \Rightarrow \quad \bar{g} \text{ is positive definite.}$$

So, Σ is indeed a spacelike hypersurface. For it to be Cauchy every inextendible causal curve needs to intersect Σ exactly once. This is seen directly in Figure 3 by placing a causal future cone in the upper grey shaded area, i. e. the Schwarzschild interior. Every inextendible causal curve starting in the Schwarzschild interior eventually falls in the singularity at $r = 0$ and hence passes every line of constant r in Figure 3. More precisely, every causal curve starting in the Schwarzschild interior will fall into the singularity in finite affine parameter time. To see this one first calculates

$$g(\dot{\gamma}, -\partial_r) = - \left(1 - \frac{2m}{r^{d-2}} \right)^{-1} \dot{r} \quad (2.15)$$

for a curve $\gamma(s) = (t, r, \omega)$ in M_{int} and ω the projection on the $(d - 1)$ -sphere. When choosing γ future directed timelike (2.15) must be less than 0 because $-\partial_r$ is also future directed timelike. In the interior r^{d-2} is smaller than $2m$ thus $\dot{r} < 0$. For future directed timelike geodesic we can find a stronger result. Entering the vector fields of symmetry, so called Killing

fields, ∂_t and ∂_ϕ into the Euler-Lagrange equation with Lagrangian $\frac{1}{2}g(\dot{\gamma}, \dot{\gamma})$ two conserved quantities are revealed:

$$E := \left(1 - \frac{2m}{r^{d-2}}\right) \dot{t} \quad \text{and} \quad L := r^2 \prod_{i=1}^{d-2} \sin^2 \theta_i \dot{\phi} \quad (2.16)$$

where E is the energy resulting from time-translation symmetry and L the angular momentum from rotational or spherical symmetry. The θ_i are the first $d-2$ coordinates on the $(d-1)$ -sphere and ϕ is the last. Together with (2.15) for a causal geodesic γ that is future directed one can find that $\dot{r} \leq -|E| < 0$ [13, p. 44f.]. This means every future inextendible causal geodesic reaches $r = 0$ in finite parameter time. Causal geodesics are length, or proper time for timelike geodesics, maximising. It follows that every inextendible future directed (i. e. $\dot{r} < 0$) causal curve must also reach $r = 0$ in finite parameter time. For past directed curves the argument is reversed. Finally, inextendible causal curves starting in the Schwarzschild interior intersect all hypersurfaces of constant $r \in (0, (2m)^{\frac{1}{d-2}})$ or in other words, Σ is a Cauchy hypersurface. Such hypersurfaces are used later in subsection 4.2.3.

2.3.3 Schwarzschild with cosmological constant

From (2.8) one can derive the simplest spherically symmetric solution similarly to how it is done with $\Lambda = 0$. This solution combines the (*anti*-)de Sitter solution which is the not necessarily spherically symmetric simplest solution to the vacuum equations (2.8) with positive (negative) cosmological constant with the Schwarzschild solution (2.9) from above. It is known as the **(anti-)de Sitter-Schwarzschild solution**. Putting the dimension $n = 1 + 3$ results in

$$g_\Lambda := - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.17)$$

In higher dimension, we use the effective parameter Λ' for the cosmological constant Λ and write

$$g_{\Lambda,n} := - \left(1 - \frac{2m}{r^{n-3}} - \Lambda' r^2\right) dt^2 + \left(1 - \frac{2m}{r^{n-3}} - \Lambda' r^2\right)^{-1} dr^2 + r^2 \hat{\gamma}_{n-2} \quad (2.18)$$

where Λ' may still depend on the dimension n [15, p. 645] but is proportional to Λ [16, p. 3]. The term

$$1 - \frac{2m}{r^{n-3}} - \Lambda' r^2 \quad (2.19)$$

will diverge towards negative infinity for $r \rightarrow 0$ and towards positive / negative infinity if Λ' is negative / positive for $r \rightarrow \infty$. Looking for extrema

we set its derivative in r to 0:

$$(n-3)\frac{2m}{r^{n-2}} - 2\Lambda'r = 0. \quad (2.20)$$

This has a (unique) solution only for $\Lambda' > 0$. Therefore, if $\Lambda' < 0$ then (2.19) is monotonely increasing and has one root, i. e. has one horizon. Assuming $\Lambda' > 0$ and plugging (2.20) into (2.19) we find that the maximum only lies above 0 if

$$\Lambda' < \frac{n-3}{n-2} \frac{1}{(2(n-2)m)^{\frac{2}{n-3}}} \quad (2.21)$$

holds. This is $(32m)^{-1}$ for $n = 4$. It follows that the spacetime has two horizons for small but positive Λ' where $g_{\Lambda,n}$ is singular. As for (2.9) of Schwarzschild one can find an extension whose metric is regular at the two horizons. The de Sitter-Schwarzschild is the simplest spacetime admitting a black hole event horizon as well as a cosmological event horizon. The first originates from the Schwarzschild horizon and the second from the de Sitter horizon. [17, p. 1] If Λ' is now greater then the right hand side of (2.21) there is no horizon and (2.19) is smaller zero for every $r > 0$. The time orientation is chosen analogously to Schwarzschild with zero cosmological constant, for (2.19) greater zero ∂_t and if below zero $-\partial_r$. Keeping in mind when and in what cases (2.19) has positive or negative sign lets us repeat any calculations and observations of subsection 2.3.2 in analogue. In addition, the black hole of Schwarzschild, i. e. the curvature singularity and geodesic incompleteness for $r \rightarrow 0$, is generally preserved for a non-zero cosmological constant. [17] It appears as if Theorem I should also hold for a non-zero cosmological constant. This is further investigated in section 5.

2.4 Toolkit for the proof of Theorem I

This section provides most of what is needed for the proof of Theorem I. The following auxiliary results are not proved here and stem from works [3], [4], [1] and [2]. A main tool for completing the proof of Theorem I in subsection 4.2 is the *spacelike diameter* which will be introduced below in subsection 2.4.2. subsection 4.2.2, a crucial step using the spacelike diameter, requires the two propositions of subsection 2.4.3.

2.4.1 Two fundamental ingredients

To prove Theorem I we need the following two theorems. They set up the main idea of the proof as sketched in section 3.

Theorem II. *Let (M, g) be a globally hyperbolic spacetime with a C^0 -extension to (\tilde{M}, \tilde{g}) . There exists a future directed timelike curve $\tilde{\gamma} : [-1, 0] \rightarrow$*

\tilde{M} with $\tilde{\gamma}(0) \in \partial(M)$ and $\tau := \tilde{\gamma}|_{[-1,0)} : [-1,0) \rightarrow M$ being a future directed and future inextendible timelike geodesic in M .

This theorem replaces the weaker formulation (Lemma 2.17) of the original proof in [1, p. 333] which asserted the existence of such a curve without necessarily being a geodesic. This simplifies subsection 4.1 of the proof together with the next Theorem. Theorem II is proved as Theorem 3.3 in [3, pp. 7ff.]. There it requires the future boundary to be non-empty but this is directly given by the existence of a timelike curve leaving M from Lemma 2.17 in [1, p. 333]. This timelike curve terminates at a point on $\partial(M) \subseteq \tilde{M}$. By choosing the time orientation accordingly and Definition 2.11 this point lies in the future boundary of M .

Theorem III. *Let $\gamma : [-1,0) \rightarrow M_{\text{Kruskal}}$ be a future directed and future inextendible timelike geodesic (not necessarily affinely parametrised). Then γ either ends in timelike infinity and is future complete or ends at the singularity, meaning $(r \circ \gamma)(s) \rightarrow 0$ for $s \rightarrow 0$.*

This theorem is a rephrasing of Proposition 36 in chapter 13 of [4, pp. 396ff.] and is one of few results in this thesis exclusive to the maximal analytic Schwarzschild spacetime. The proof includes a case separation for the endpoint of an incomplete curve to lie in the interior, exterior or horizon of the Kruskal manifold and uses the properties of Kruskal coordinates to predict the behaviour of the curve similarly to what is done in subsection 2.3.2. Theorem III combines with Theorem II and introduces two separate cases, the curve being future complete and terminating in timelike infinity and the curve *falling* into the singularity at $r = 0$. The first case we describe by saying that the curve is leaving M_{Kruskal} through the exterior whereas for the latter we say it leaves through the interior. To apply these two theorems we first need to assume there is a C^0 -extension to M_{Kruskal} contrary to the statement of Theorem I. The proof then will be deduced by contradiction in section 4. Subsection 4.1 and 4.2 present the mentioned case separation.

2.4.2 Adjustable cones, spacelike diameter and openness of the timelike future

For the proof we must go to locally near-Minkowski coordinates in which we can analyse the geometry by looking at the manifold as a Euclidean space $R \subset \mathbb{R}^{d+1}$ where we write $\langle \cdot, \cdot \rangle_{d+1}$ for the respective Euclidean inner product and $|\cdot|_{d+1}$ for the associated norm. In this case, our coordinates are given by $x_\mu e_\mu$ for $x_\mu \in \mathbb{R}$ and $\{e_\mu\}$ being the standard basis.

Definition 2.12. We define a forward (backward) cone in Euclidean \mathbb{R}^{d+1} space in respect to the cosine of the angle $\vartheta \in (0, \pi/2)$ spanning from the

zeroth coordinate axis to the cone by

$$C_{\cos \vartheta}^{\pm} := \left\{ x \in \mathbb{R}^{d+1} \mid \pm \frac{\langle x, e_0 \rangle_{d+1}}{|x|_{d+1}} > \cos \vartheta \right\}. \quad (2.22)$$

Note that the backward cone is just inverted along the zeroth coordinate axis (cf. Fig. 4).

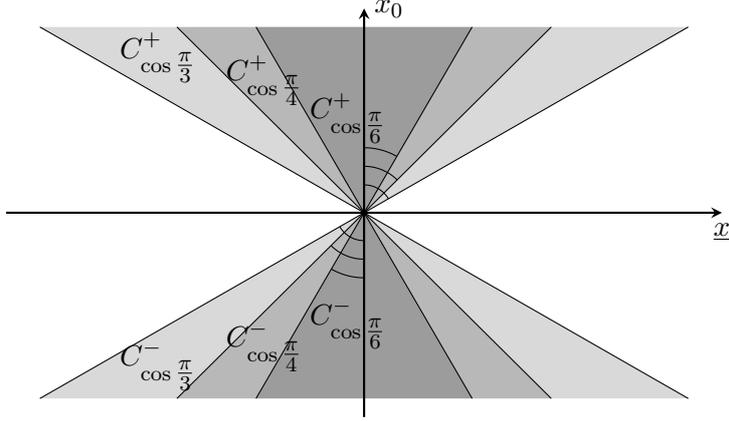


Figure 4: Forward and backward cones for different angles from the x_0 -axis.

In Minkowski space, the timelike future and past of a point x are precisely given by $x + C_{\cos \pi/4}^{\pm}$. The idea is to be able to narrow or widen the timelike Minkowski future $C_{\cos \pi/4}^{\pm}$ such that in a chart where the metric is only near Minkowski, it is an open subset of the actual future of a point (for narrowing) or contains the actual future as an open subset (for widening).

The now-defined quantity will later deliver a crucial part of the proof of Theorem I. It will be used in subsection 4.2 and work in a sense analogously to the future (in-)completeness appearing in subsection 4.1.

Definition 2.13 (spacelike diameter). Let (M, g) be a connected globally hyperbolic Lorentzian manifold. We write $\underline{\Sigma}(M) := \{\Sigma \subset M \mid \Sigma \text{ is Cauchy hypersurface of } M\}$ as well as $\Upsilon(\Sigma, p, q) := \{\gamma : [0, 1] \rightarrow \Sigma \mid \gamma \text{ is piecewise smooth, } \gamma(0) = p, \gamma(1) = q\}$ for a spacelike hypersurface Σ and $p, q \in \Sigma$. We then define the spacelike diameter of a subset $A \subseteq M$ as follows:

$$\begin{aligned} \text{diam}_s(A) &:= \sup_{\Sigma \in \underline{\Sigma}(M)} \text{diam}(\Sigma \cap A) \\ &= \sup_{\substack{\Sigma \in \underline{\Sigma}(M) \\ p, q \in (\Sigma \cap A)}} \left(\inf_{\gamma \in \Upsilon(\Sigma \cap A, p, q)} L(\gamma) \right) \end{aligned} \quad (2.23)$$

where L is the Lorentzian length functional for spacelike curves (cf. Def. 2.5).

The term *spacelike diameter* suggests we somehow measure the extent of spacelike hypersurfaces of A . In fact, by (2.23) we see that we take the supremum of the diameter of specific spacelike hypersurfaces, subsets of Cauchy hypersurfaces. Remember that for Riemannian manifolds geodesics are characteristic for minimising the distance between two points. Since a spacelike hypersurface is a Riemannian manifold the infimum in (2.23) is an alternative to writing ‘length of a geodesic connecting p and q on Σ ’. So, what we actually do is find the length of the longest geodesic on all Cauchy hypersurfaces intersected with A which just is a more refined idea of *extent of spacelike hypersurfaces of A* .

Before coming to the proof of Theorem I a final Proposition is given. It is proved as Proposition 2.6 in [1, p. 329].

Proposition 2.14. *For a Lorentzian manifold (M, g) where g is a continuous metric the timelike future (past) $I^\pm(p, M)$ of any point $p \in M$ is open in M .*

Note that this statement must specifically hold for Lorentzian manifolds with a mere C^0 -regular metric. The statement is only used in subsection 4.2.3 showing an unbounded length of certain spacelike curves that lie in the timelike future of some fixed point in the Schwarzschild interior.

2.4.3 Auxiliary results for proving finiteness of the spacelike diameter in the Schwarzschild interior

The curve γ as well as $\tilde{\gamma}$ in the next two Propositions will eventually be the curves received from Theorem II and Theorem III. The following Propositions will be needed in subsection 4.2.2. They allow us to uniformly bound the spacelike diameter of $I^+(\tilde{\gamma}(-\mu), M_{\text{int}})$ in M_{int} .

Proposition 2.15. *Let (M, g) be a globally hyperbolic Lorentzian manifold that is C^0 -extended to a manifold (\tilde{M}, \tilde{g}) . Now assume a future directed timelike curve $\tilde{\gamma} : [-1, 0] \rightarrow \tilde{M}$ with $\tilde{\gamma}(0) \in \partial(M)$ and $\gamma = \tilde{\gamma}|_{[-1, 0]}$, $\text{Im}(\gamma) \subseteq M$ together with a chart $\tilde{\varphi} : \tilde{U} \rightarrow \tilde{\varphi}(\tilde{U}) \subseteq \mathbb{R}^{d+1}$, $\tilde{\gamma}(0) \in \tilde{U}$ fulfilling*

$$(\tilde{\varphi} \circ \tilde{\gamma})(s) = (s, 0, \dots, 0) \quad \text{for } s \in (-\epsilon, 0] \quad \text{and} \quad |\tilde{g}_{\mu\nu} - m_{\mu\nu}| < \delta \quad (2.24)$$

is given for an $\epsilon > 0$ and a $\delta > 0$ such that the cones $C_{\cos \pi/6}^\pm$ are contained in the time cones of \tilde{g} in these coordinates. Then, there exist $\epsilon_0 \in [\epsilon, 0)$ and $\epsilon_1 > 0$ satisfying $(-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \subseteq \tilde{\varphi}(\tilde{U})$ such that there exists a Lipschitz continuous function $f : (-\epsilon_1, \epsilon_1)^d \rightarrow (-\epsilon_0, \epsilon_0)$ with following properties:

- a) $\{(x_0, \underline{x}) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \mid x_0 < f(\underline{x})\} \subseteq \tilde{\varphi}(M \cap \tilde{U})$,
 - b) $\{(x_0, \underline{x}) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \mid x_0 = f(\underline{x})\} \subseteq \tilde{\varphi}(\partial^+(M) \cap \tilde{U})$.
- (2.25)

Additionally, $\{(x_0, \underline{x}) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \mid x_0 = f(\underline{x})\}$, i. e. the graph of f , is achronal in $(-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$.

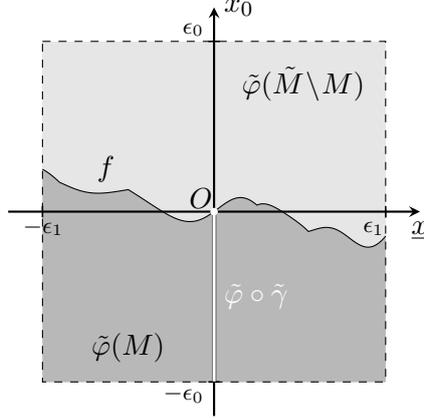


Figure 5: The setting of Proposition 2.15 in coordinates and inclusion relations (2.25) illustrated.

The graph of f is the future boundary of M in coordinates. The area underneath represents a subset of M near the boundary, around $\tilde{\gamma}$ and in coordinates. The graph of f and the area above lie exclusively in the extension \tilde{M} , not in M . One can imagine this proposition to draw a (Lipschitz) continuous line (projected on the two-dimensional plane for time and spatial coordinates) in near-Minkowski coordinates between the manifold and its extension (cf. Fig. 5). It is a structural result holding for any C^0 -extension of a globally hyperbolic Lorentzian manifold and is thus not particularly acquainted with the Schwarzschild spacetime. For a proof see Proposition 1 of [2, pp. 3ff.]. An interesting aspect of this proposition is that $f(\underline{x}) < \epsilon_0$ holds for all $\underline{x} \in (-\epsilon_1, \epsilon_1)^d$. This explicitly shows that one cannot think of the extension onto $r = 0$ in the case of the Schwarzschild interior of adding a single point to the manifold but rather adding an open neighbourhood. It resembles the difference of viewing $r = 0$ as a hypersurface rather than as a point.

Proposition 2.16. *Let (M_{int}, g) be the $(d + 1)$ -dimensional Schwarzschild interior and let $\tilde{\varphi}$ and $\tilde{\gamma}$ be as in the conditions of Proposition 2.15 and ϵ_0, ϵ_1 as well as f then be given by Proposition 2.15. There exists a $\mu > 0$ such that*

- a) $\tilde{\varphi}\left(I^+(\tilde{\gamma}(-\mu), M_{\text{int}})\right) \subseteq \{(x_0, \underline{x}) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \mid x_0 < f(\underline{x})\}$,
 - b) $(-\epsilon_0, -\frac{49}{50}\epsilon_0) \times (-\epsilon_1, \epsilon_1)^d \subseteq I^-((\tilde{\varphi} \circ \tilde{\gamma})(-\mu), (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d)$.
- (2.26)

Achieving these may seem rather trivial when looking at Figure 6, choosing a small enough μ such that $\tilde{\gamma}(-\mu)$ is very close to the graph of f , putting ϵ_1 smaller ϵ_0 and anticipating that we can choose the timelike future and past of $\tilde{\gamma}(-\mu)$ to be arbitrarily close to the Minkowski cones $C_{\cos \pi/4}^\pm$ starting at that point (cf. Fig. 6). However, this proposition is proved as Step 1 of chapter 4 in [2] spanning over pages 8 to 13, uses properties of the chart and curve and is rather technical. Additionally, future one-connectedness of the Schwarzschild interior is needed. A definition and a proof for the Schwarzschild interior is given in [1], Definition 2.12 (pp. 331f.) and Proposition 6.5 (pp. 362ff.).

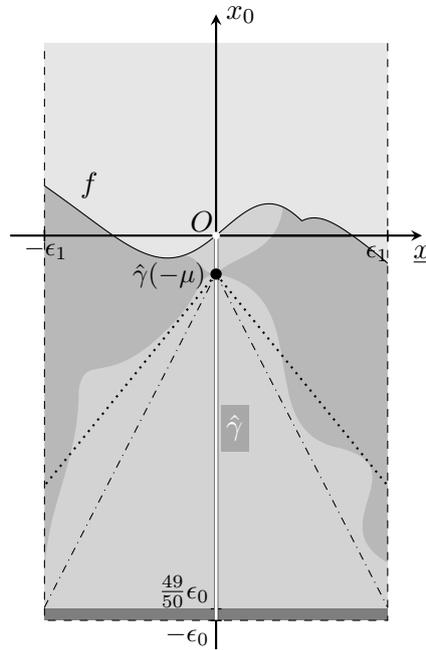


Figure 6: The two inclusion relations (2.26) of Proposition 2.16 illustrated in coordinates. $\hat{\gamma}$ is short for the curve $\tilde{\varphi} \circ \tilde{\gamma}$. The light-shaded areas under the graph of f resemble the timelike future and past of $\hat{\gamma}(-\mu)$. The coordinates are chosen such that the past is near the Minkowski cone $C_{\cos \pi/4}^-$ (dotted line) and includes the narrowed cone $C_{\cos \pi/6}^-$ (dash-dotted line).

Proposition 2.16 connects the abstract manifold to local near-Minkowski coordinates, here in $(-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$. In these, the required results will be proved.

How Proposition 2.15 and 2.16 will be helpful is explicated in more detail in subsection 3.2.1 of the sketch of the proof of Theorem I.

3 Sketch of the Proof

The following structure of subsections is revisited in the actual proof. First, the existence of a C^0 -extension \tilde{M} of M_{Kruskal} is assumed. Applying Theorem II and subsequently Theorem III leaves us with a timelike curve $\tilde{\tau}$ leaving M_{Kruskal} and two cases to be sorted out. In both, a contradiction is deduced. Remember, τ is the future directed and future inextendible geodesic $\tilde{\tau}|_{M_{\text{Kruskal}}}$ on M_{Kruskal} .

3.1 Timelike geodesic leaving through the exterior

The aim is to find a reparametrisation and a chart containing the geodesic up to the point $\tilde{\tau}(0)$ where it leaves the manifold. The chart is a priori of the extension \tilde{M} . In this chart we want to estimate the length of τ up to $\tilde{\tau}(0)$ in M_{Kruskal} . Note that the Lorentzian length functional is independent of parametrisation. Since our geodesic τ is according to Theorem II future inextendible and timelike in M_{Kruskal} one can expect its potential future completeness, meaning its infinite length, to come from the curve running in future timelike direction. Therefore, we want to control the zeroth coordinate in our chart. This is achieved by first constructing the chart centred at the asymptotic endpoint $\tilde{\tau}(0)$ of our geodesic τ which lies on the boundary of M_{Kruskal} , i. e. not in M_{Kruskal} . Then we start from local Minkowski coordinates at $\tilde{\tau}(0)$ and parametrise $\tilde{\tau}$ such that the zeroth coordinate, the time coordinate, in near-Minkowski coordinates of the chart is linear in the curve parameter. By this its impact on the length functional (cf. Def. 2.5) in the tangent vector of $\tilde{\tau}$ is eliminated. We now deal with the spatial coordinates of the curve. The curve is timelike, thus in Minkowski space, the spatial or spacelike extent of the curve is bounded in the time cone of the curve's starting point. We want to use the same argument for our near-Minkowski coordinates which we prescribe to be so close to Minkowski that the timelike future of the starting point is entirely contained in a slightly widened forward cone (similarly to the timelike past in Figure 6 but inverted). From this, we can bound the norm of the on spatial coordinates projected tangent vectors. In this chart, the choice of parametrisation and the locally bounded metric components that are near Minkowski are sufficient to calculate a finite length of τ up to $\tilde{\tau}(0)$ in the constructed chart now restricted to M_{Kruskal} . This construction of course only works in a small neighbourhood of $\tilde{\tau}(0)$, however, we know that the rest of the curve then lies on a compact interval completely in M_{Kruskal} and hence also has finite length. The finite length of the future inextendible curve τ implies future incompleteness, so the first case of Theorem III is dismissed.

3.2 Timelike curve leaving through the interior

This part requires more effort than the previous. Two additional propositions (Prop. 2.15 & 2.16) are needed that are not proved in this thesis.

Theorem III left us with the option of our curve $\tilde{\tau}$ leaving through the interior, i. e. through the singularity at $r = 0$. We now assume to only be in the Schwarzschild interior. At first, a chart on a small neighbourhood around $\tilde{\tau}$ is constructed that maps to an open cuboid in \mathbb{R}^{d+1} in which $\tilde{\tau}$ is a straight line and Proposition 2.15 draws an achronal graph f representing the boundary of the Schwarzschild interior as a subset of its extension (cf. Fig. 5). As a note, we will now say τ instead of $\tilde{\tau}$ when acting in M_{int} or its image on the cuboid. The reason for this construction is to proof the finiteness of the spacelike diameter in these near-Minkowski coordinates. This will much ease coming up with a description of curves whose length can be uniformly bounded.

In a sense, the procedure is analogue to the curve leaving through the exterior where we know the curve should be future complete due to the geometry but by assumption can bound its length. This is basically arguing with infiniteness and finiteness of the timelike diameter. Sbierski for this proof found one can use the analogue of the spacelike diameter.

3.2.1 Finite spacelike diameter

The spacelike diameter takes the supremum over all Cauchy hypersurfaces (cf. Def. 2.13). Therefore, we deduce a bound for the length of curves on an arbitrary Cauchy hypersurface Σ . However, we do not look at the spacelike diameter of the entire M_{int} but rather want to find a subset of M_{int} . Consequently, we only look at Cauchy hypersurfaces Σ having a non-empty intersection with this subset. The subset should fulfil two conditions.

- a) The subset should be contained entirely in the neighbourhood for our near-Minkowski coordinates and thus lie under the graph of f in the cuboid.

This is to ensure that we can deal with any curve lying in the intersection of Σ with this subset in our local coordinates.

As a second property, we would like to find a complete and smooth description of Σ intersected with the subset in our near-Minkowski neighbourhood in the cuboid. This enables us to calculate any lengths of curves in Σ because we need to restrict our near-Minkowski metric to Σ without any of the induced metric components diverging. Otherwise, we could not bound any lengths (cf. Def. 2.5). To achieve this, the idea is to draw straight lines $\sigma_{\underline{x}}$ starting at the bottom of Figure 7 and going up along constant $\underline{x} \in \mathbb{R}^d$ which

allows us to characterise Σ by the intersection with each of these straight lines in coordinates. The chart is chosen such that the *upwards* direction in Figure 7 is timelike.

- b) For the intersection to actually occur for every in coordinates straight line $\sigma_{\underline{x}}$ the subset must be such that the straight lines start in the past of any Σ .

Since the $\sigma_{\underline{x}}$ would intersect the graph of f in our cuboid they are future inextendible in M_{int} and by the defining properties of a Cauchy hypersurface this is enough to guarantee the existence and uniqueness of an intersection such.

These two conditions we lay on the required subset are precisely the results of Proposition 2.16, (2.26a) and (2.26b), the looked after subset being the timelike future of some point on τ in M_{int} . This point $\tau(-\mu)$ is given by a $\mu > 0$.

As described, by b) we find a function h via the $\sigma_{\underline{x}}$ parametrising the intersection of Σ with the near-Minkowski cuboid (cf. Fig. 8). Smoothness of h follows from finding a smooth function u whose zero level set is locally Σ and which itself can be done by the smoothness of Σ as a submanifold in coordinates. Σ in our cuboid is spacelike and the metric components are bounded because they are near Minkowski. From this, we conclude that the spatial derivatives of h in coordinates are also finite. This bound only depends on how close the metric is chosen to Minkowski. If we now take arbitrary p and q on Σ intersected with the timelike future of $\tau(-\mu)$ and by a) can go to coordinates of our coordinates in which the graph of h is flat (cf. Fig. 10) we can calculate the length to be bounded because p and q in coordinates lie in the cuboid of finite dimensions and the induced metric components on Σ in coordinates are bounded. The latter holds because of the bounded near-Minkowski metric and the finite derivative of h .

This means we find on Σ a curve connection p and q of bounded length where the bound only depends on how near the metric is to Minkowski in the cuboid, the spatial dimensions of the cuboid and the dimension d (cf. (4.22)), therefore is uniform for any p, q and Σ . Remember the spacelike diameter (cf. Def. 2.13) takes the infimum of all possible curves connecting points p and q and subsequently the supremum over p, q and Σ .

Henceforth, the spacelike diameter of the timelike future of $\tau(-\mu)$ is uniformly bounded.

3.2.2 Infinite spacelike diameter

The existence of a C^0 extension is not needed. Reminding again of Definition 2.13, we now need to find some sequence of p and q or of Cauchy hypersur-

faces Σ for which the shortest curve connecting p and q in the intersection of Σ with the timelike future of $\tau(-\mu)$ in M_{int} is bounded. The μ originates from Proposition 2.16 as explained in the previous subsection. Because by this the spacelike diameter of the timelike future of $\tau(-\mu)$ must be finite contrary to the result of above.

We start at some point in the timelike future of $\tau(-\mu)$. This can be chosen to be any point $\tau(s_0)$ for $s_0 \in (-\mu, 0)$ because τ is future directed timelike. Proposition 2.14 states that the set is open. Accordingly, we find a neighbourhood of $\tau(s_0)$ still in the timelike future. In coordinates of (2.9) the neighbourhood is propagated in $-\partial_r$ direction. Since this is set to be future directed in the Schwarzschild interior we remain in the timelike future of $\tau(s_0)$. We extract a closed interval of the t coordinate lying in the timelike future for all $r > 0$ below an initial r_0 and some constant projection ω_0 on the \mathbb{S}^{d-1} unit sphere. Knowing that the hypersurfaces $\Sigma_n := \{r = n^{-1}\}$ are Cauchy hypersurfaces of M_{int} for every fixed $n \in \mathbb{N}_{\geq r_0^{-1}}$ (cf. subsec. 2.3.2) we find the geodesic or shortest curve between the endpoints of the extracted t interval with $r = n^{-1}$ and ω_0 is just the linear or straight line in the t coordinate with constant $r = n^{-1}$ and ω_0 (cf. Fig. 11). Calculating the length of these curves on Σ_n with its induced metric results in an expression growing unbounded with $n \rightarrow \infty$. The supremum over all of these $\Sigma_{n \geq r_0^{-1}}$ and p_n, q_n must be smaller or equal to the spacelike diameter (cf. Def. 2.13) and at the same time is infinite. This concludes the proof.

4 Proof of C^0 -inextendibility

For a general idea of the individual steps see the corresponding subsections of section 3.

Proof of Theorem I. The proof is by contradiction. We assume there exists a C^0 -extension $\iota : M_{\text{Kruskal}} \hookrightarrow \tilde{M}$ of the maximal analytic Schwarzschild spacetime (M_{Kruskal}, g) to a spacetime (\tilde{M}, \tilde{g}) .

Theorem II gives a future directed timelike curve $\tilde{\tau} : [-1, 0] \rightarrow \tilde{M}$ and the restriction $\tau : [-1, 0) \rightarrow M_{\text{Kruskal}}$ to M_{Kruskal} which is a future directed timelike and future inextendible geodesic in M_{Kruskal} . Subsequently, Theorem III gives two options: either τ leaves M_{Kruskal} through the exterior being future complete or through the interior at the singularity.

4.1 τ leaving through the exterior

We now discuss the first option and want to find coordinates in which the length of τ up to the boundary $\partial(M_{\text{Kruskal}})$ can easily be calculated to be

finite. This would give its future incompleteness. The result is independent of the Kruskal manifold and uses no exclusive properties of it.

4.1.1 Reparametrising τ and constructing a chart around it

The tangent vector to $\tilde{\tau}$ is everywhere timelike. Without loss of generality, i. e. after possible linear reparametrisation,

$$\tilde{g}(\dot{\tilde{\tau}}(0), \dot{\tilde{\tau}}(0)) = -1 \quad (4.1)$$

shall hold. By widening the timelike future cone of Minkowski space $C_{\cos \pi/4}^+$ to $C_{\cos \pi/3}^+ = C_{1/2}^+$ and assuming $\tilde{g}_{\mu\nu} - m_{\mu\nu}$ is bounded by a sufficiently small $\delta \in (0, 1)$ in a chart $\tilde{\varphi} : \tilde{U} \rightarrow \tilde{\varphi}(\tilde{U})$ we can ensure that $\frac{d}{ds}(\tilde{\varphi} \circ \tilde{\tau})(s) \in C_{1/2}^+$ for every $s \in [-1, 0]$ such that $\tilde{\tau}(s) \in \tilde{U}$. Such a chart is found by centring it at $\tilde{\tau}(0)$ and applying the Gram-Schmidt orthonormalisation procedure with \tilde{g} to the tangent space of $\tilde{\tau}(0)$ with fixed and normalised $\frac{\partial}{\partial x^0}$. The metric \tilde{g} now equals the Minkowski metric $m_{\mu\nu}$ at $\tilde{\tau}(0)$ in coordinates. If necessary, we shrink \tilde{U} such that

$$|\tilde{g}_{\mu\nu} - m_{\mu\nu}| < \delta \quad (4.2)$$

holds in these coordinates by the continuity of \tilde{g} . We now write $(\tilde{\varphi} \circ \tilde{\tau})(s) = (\tilde{\tau}_0(s), \tilde{\underline{\tau}}(s)) \in \mathbb{R} \times \mathbb{R}^d$. Since the chart is centered around $\tilde{\tau}(0)$ we have $\tilde{\tau}_0(0) = 0$. Further, because the spatial coordinate components of g represent a positive definite Riemannian metric, for $\frac{d}{ds}(\tilde{\varphi} \circ \tilde{\tau})(s)$ to be timelike the zeroth component $\tilde{\tau}_0(s)$ must be non-zero for all s . Via the inverse function theorem we can find an $s_0 \in (-1, 0)$ such that $\tilde{\tau}(s) \in U$, $\tilde{\tau}_0^{-1}(s)$ exists and $(\tilde{\tau} \circ \tilde{\tau}_0^{-1})(s) \in U$ for all $s \in (s_0, 0)$ where we now put $U = \tilde{U} \cap M_{\text{Kruskal}}$. Reparametrising $\tilde{\tau}$ and restricting to M_{Kruskal} gives the (not necessarily affinely parametrised) future timelike geodesic $\hat{\tau} : (s_0, 0) \rightarrow \mathbb{R}^{d+1}$ which is a reparametrisation of τ in coordinates as follows:

$$\hat{\tau} = \left(\tilde{\varphi} \circ \tilde{\tau} \circ \tilde{\tau}_0^{-1} \right) \Big|_{(s_0, 0)} = \left(\tilde{\varphi}|_U \circ \tau \circ \tilde{\tau}_0^{-1} \Big|_{(s_0, 0)} \right) \quad (4.3)$$

$$\text{or respectively} \quad \hat{\tau}(s) = \left(s, \tilde{\underline{\tau}}(\tilde{\tau}_0^{-1}(s)) \right) = (s, \hat{\underline{\tau}}(s)) \quad \forall s \in (s_0, 0).$$

$(\tau \circ \tau_0^{-1})(s)$ is still timelike and lies in $U \subseteq \tilde{U}$. Hence, its tangent vectors $\dot{\hat{\tau}}(s)$ which are rescaled tangent vectors of $(\tilde{\varphi}|_U \circ \tau)$ in U remain in $C_{1/2}^+$ for all $s \in (s_0, 0)$ from which follows

$$\frac{1}{2} < \frac{\langle \dot{\hat{\tau}}(s), e_0 \rangle_{d+1}}{|\dot{\hat{\tau}}(s)|_{d+1}} = \frac{1}{\sqrt{1 + |\hat{\underline{\tau}}(s)|_d}}. \quad (4.4)$$

Rearranging gives $|\hat{\underline{\tau}}(s)|_d < 3$ for all $s \in (s_0, 0)$.

4.1.2 Future incompleteness of τ

The length of $\tau|_{[-1, \tilde{\tau}_0^{-1}(s_0)]}$ is finite since it is defined on a compact interval entirely in M_{Kruskal} . The remaining length which is independent of the parametrisation of τ is given by

$$\begin{aligned} \int_{s_0}^0 \sqrt{-g(\dot{\hat{\tau}}(s), \dot{\hat{\tau}}(s))} ds &= \int_{s_0}^0 \sqrt{-\left(g_{00} + 2 \sum_{i=1}^d g_{0i} \dot{\hat{\tau}}_i(s) + \sum_{i,j=1}^d g_{ij} \dot{\hat{\tau}}_i(s) \dot{\hat{\tau}}_j(s)\right)} ds \\ &< \int_{s_0}^0 \sqrt{1 + \delta + 2 \cdot d \cdot \delta \cdot 3 + 0} ds \\ &= |s_0| \sqrt{1 + \delta(6d + 1)} < \infty. \end{aligned} \tag{4.5}$$

Here, (4.2) is used for the isometric embedded g as well as the bound for $|\dot{\hat{\tau}}(s)|_d$ from above. It follows that τ has a finite length and is therefore not future complete. Thus τ cannot leave through the exterior.

4.2 τ leaving through the interior

We first remind the reader of the auxiliary results in subsection 2.4.3 which enable us to handle the local geometry as we transition from $M_{\text{Kruskal}} \subset \tilde{M}$ to $\partial^+(M_{\text{Kruskal}}) \subseteq \tilde{M} \setminus M_{\text{Kruskal}}$. The spacelike diameter (cf. Def. 2.13) is a crucial quantity of this section. It will be taken in regard to the timelike future of a point on τ short of the boundary $\partial(M)$. This point needs to be chosen carefully which is the task and purpose of Proposition 2.16.

To also derive a contradiction in the second case of Theorem III we look at the spacelike diameter and show that by the assumption of the existence of a C^0 -extension this quantity grows beyond bounds, however simultaneously must be bounded according to the geometry of the Schwarzschild interior. This will be the desired contradiction.

4.2.1 Constructing a chart around τ

As before we work with the future directed timelike curve $\tilde{\tau} : [-1, 0] \rightarrow \tilde{M}$ whose restriction $\tau = \tilde{\tau}|_{[-1, 0)}$ on M_{Kruskal} is future inextendible. By Theorem III and the previous subsection we now know such a curve cannot be future complete in M_{Kruskal} and must therefore leave M_{Kruskal} through the singularity, i. e. $(r \circ \tau)(s) \rightarrow 0$ as $s \rightarrow 0$. Consequently and without loss of generality, we assume τ lies in $M_{\text{int}} \subset M_{\text{Kruskal}}$.

First, the tangent vector to $\tilde{\tau}$ at $\tilde{\tau}(0)$ is normalised and mapped to $\frac{\partial}{\partial x_0}$. An orthonormal basis of the tangent space with this future timelike direction

fixated is mapped to the standard basis in coordinates. At $\tilde{\tau}(0)$ the metric now is the Minkowski metric $m_{\mu\nu}$. Continuing to a neighbourhood and centring the chart at $\tilde{\tau}(0)$ gives us a map we call $\tilde{\psi}$. The chart is constructed such that the components of $\frac{d}{ds}(\tilde{\psi} \circ \tilde{\tau})(s)$ are $\dot{\tilde{\tau}}_0(0) = 1$ and $\dot{\tilde{\tau}}(0) = (0, \dots, 0)$ which lets us find a neighbourhood of $\tilde{\tau}(0)$ and an $\epsilon > 0$ where $\tilde{\tau}_0^{-1}(s)$ exists and is smooth for $s \in (-\epsilon, 0]$ via the inverse function theorem and subsequently

$$F(x_0, \underline{x}) := \left(\tilde{\tau}^{-1}(x_0), \underline{x} - \tilde{\tau}(\tilde{\tau}_0^{-1}(x_0)) \right) \in \mathbb{R} \times \mathbb{R}^d \quad (4.6)$$

is a diffeomorphism onto its image where \underline{x} are the coordinates x_1, \dots, x_d . We now use the coordinate map $\tilde{\varphi} := (F \circ \tilde{\psi})$. The differential of F at 0 is

$$dF_0 = \left(\begin{array}{c|c} \frac{d}{dx_0}(\tilde{\tau}_0^{-1}(x_0)) & 0 \\ \hline -\dot{\tilde{\tau}} \circ \tilde{\tau}_0^{-1} \cdot \frac{d}{dx_0}(\tilde{\tau}_0^{-1}(x_0)) & \text{Id}_d \end{array} \right)_0 = \text{Id}_{1+d}. \quad (4.7)$$

We see $\tilde{g}_{F(0)}(dF_0 v, dF_0 w) = \tilde{g}_0(v, w)$ for all $v, w \in T_0 \tilde{M}$ and hence the time orientation is preserved and \tilde{g}_0 is still the Minkowski metric. Now choose a neighbourhood \tilde{U} small enough such that \tilde{g} remains close to $m_{\mu\nu}$ by continuity of \tilde{g} . The time cones of Minkowski $C_{\cos \pi/4}^\pm$ contain $C_{\cos \pi/6}^\pm$. Therefore, we find a $\delta \in (0, 1)$ such that the time cones of \tilde{g} also include $C_{\cos \pi/6}^\pm$ (these are the time cones of $-\frac{1}{3}dx_0^2 + dx_1^2 + \dots + dx_d^2$) as long as $|\tilde{g}_{\mu\nu} - m_{\mu\nu}| < \delta$ holds. A small δ ensures that the causal orientation of \tilde{g} is not altered. By shrinking the neighbourhood \tilde{U} of $\tilde{\tau}(0)$, if necessary even more, the following are now fulfilled:

- a) $\tilde{\tau} : [-1, 0] \rightarrow \tilde{M}$ with $\tilde{\gamma}(0) \in \partial^+(M)$ and $\text{Im}(\tilde{\gamma}|_{[-1, 0)}) \subseteq M$,
- b) $(\tilde{\varphi} \circ \tilde{\tau})(s) = (s, 0, \dots, 0)$ for $s \in (-\epsilon, 0]$,
- c) $\tilde{g}_{\mu\nu}(0) = m_{\mu\nu}$,
- d) $|\tilde{g}_{\mu\nu}(x) - m_{\mu\nu}| < \delta$ for $x \in \tilde{\varphi}(\tilde{U})$ and
- e) $-\frac{1}{3}dx_0^2 + dx_1^2 + \dots + dx_d^2 \prec \tilde{g}(x)$ for $x \in \tilde{\varphi}(\tilde{U})$,

where the last property is tantamount to the time cones of the left metric being subsets of the time cones of \tilde{g} . Listed properties are essential for enabling us to use Proposition 2.15. With the conditions of (4.8) Proposition 2.15 produces an $f : (-\epsilon_1, \epsilon_1)^d \rightarrow (-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 \in (\epsilon, 0)$, $\epsilon_1 > 0$ which tries to *draw a line* in between M_{int} and \tilde{M} in $\tilde{\varphi}(\tilde{U})$ (see again (2.25b)). We can restrict this construction to (M_{int}, g) on $U := M_{\text{int}} \cap \tilde{\varphi}^{-1}((-\epsilon_0, \epsilon) \times (-\epsilon_1, \epsilon_1)^d)$ with $\varphi : U \rightarrow \varphi(U)$, $\varphi = \tilde{\varphi}|_U$. Properties b), d) and e) of (4.8) then still hold for the future directed timelike curve τ excluding the zero in b) and restricting to $\varphi(U)$ in d) and e). Furthermore, Proposition 2.16 finds a μ such that the inclusion relations (2.26) for the timelike future and past of $\tau(-\mu)$ hold.

4.2.2 Finiteness of the spacelike diameter

The aim is to show that the spacelike diameter of the timelike future of $\tau(-\mu)$ in M_{int} is finite. This means we want to find a finite constant only depending on δ and the dimension d which uniformly bounds the diameter of $I^+(\tau(-\mu), M_{\text{int}}) \cap \Sigma$ for any Cauchy hypersurface Σ of M_{int} .

Given now any Cauchy hypersurface Σ of M_{int} whose intersection $\Sigma \cap I^+(\tau(-\mu), M_{\text{int}})$ is non-empty. Consequently, $\tau(-\mu)$ lies in the timelike past of Σ in M_{int} , then also $I^-(\tau(-\mu), M_{\text{int}}) \subseteq I^-(\Sigma, M_{\text{int}})$ holds. Due to (2.26a) the points of intersection can also be found in our coordinate chart $\varphi(U)$. The situation is shown in Figure 7.

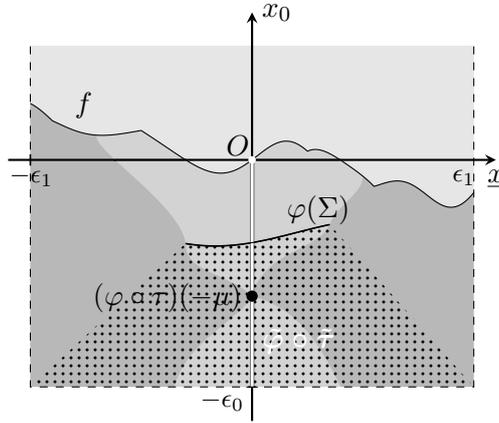


Figure 7: Timelike future and past of $(\varphi \circ \tau)(-\mu)$ under the graph of f are lightened. Given a Cauchy hypersurface Σ intersecting with the timelike future of $\tau(-\mu)$ in the coordinates of Proposition 2.15 the past of $(\varphi \circ \tau)(-\mu)$ lies in the past of $\varphi(\Sigma \cap U)$ (dotted area).

Inclusion relation (2.26b) of Proposition 2.16 stays true but for illustrative reasons Figure 7 does not depict it (cf. Fig. 6). To find a bound on the length of curves connecting arbitrary two points in $\Sigma \cap I^+(\tau(-\mu), M_{\text{int}})$ we first want to find a usable description of $\varphi(\Sigma)$. Now, look at straight lines in x_0 from $-\epsilon_0$ up to the graph of f along fixed $\underline{x} \in (-\epsilon_1, \epsilon_1)$ in $\varphi(U)$ (cf. Fig. 8). The corresponding future inextendible curves in M_{int} are given by

$$\sigma_{\underline{x}} : (-\epsilon_0, f(\underline{x})) \rightarrow M_{\text{int}}, \quad \sigma_{\underline{x}}(s) = \tilde{\varphi}^{-1}(s, \underline{x}). \quad (4.9)$$

By the construction of the coordinate chart, these are future directed timelike curves. The curves start in

$$\varphi^{-1}\left(\left(-\epsilon_0, -\frac{49}{50}\epsilon_0\right) \times (-\epsilon_1, \epsilon_1)^d\right) \subset U, \quad (4.10)$$

meaning they lie only in this set and its timelike future in M_{int} . (2.26b) of Proposition 2.16 tells us that they actually start in (cf. Fig. 6)

$$\varphi^{-1}\left(I^-\left((\varphi \circ \tau)(-\mu), (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d\right)\right) \subseteq I^-(\tau(-\mu), M_{\text{int}}) \subseteq I^-(\Sigma, M_{\text{int}}). \quad (4.11)$$

By this and their future inextendibility, they must intersect Σ . Since Σ is a Cauchy hypersurface $\sigma_{\underline{x}}$ intersects Σ for each $\underline{x} \in (-\epsilon_1, \epsilon_1)^d$ exactly once. Through this, define $h(\underline{x}) : (-\epsilon_1, \epsilon_1)^d \rightarrow (-\epsilon_0, \epsilon_0)$ with $h(\underline{x}) \in (-\epsilon_0, f(\underline{x}))$ giving the x_0 -coordinate of the intersection of $\sigma_{\underline{x}}$ with Σ .

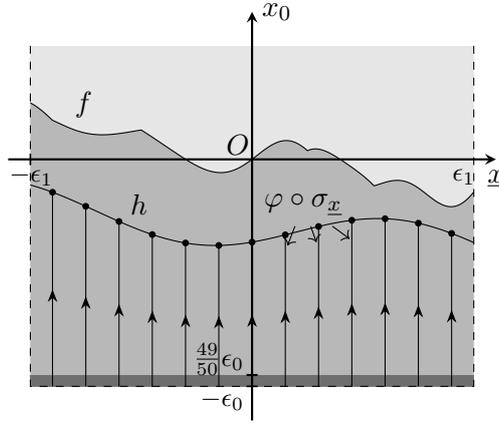


Figure 8: Finding a parametrisation h of $\varphi(\Sigma \cap U)$ via the straight lines $\varphi \circ \sigma_{\underline{x}}$ starting in $(-\epsilon_0, -\frac{49}{50}\epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$.

If we can now show that h is smooth describing $\varphi(\Sigma \cap U)$ we could deduce a bound of the components of the induced metric on Σ . This then directly leads to a uniform bound on the length of curves in $\varphi(\Sigma \cap U)$.

To see that h indeed is smooth, first note that $N = \varphi(\Sigma \cap U)$ is a smooth submanifold of $(-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$ of dimension d because Σ is a hypersurface of M_{int} . Also, $\{(h(\underline{x}), \underline{x}) \mid \underline{x} \in (-\epsilon_1, \epsilon_1)^d\} \subseteq N$. For an arbitrary $\underline{x}_0 \in (-\epsilon_1, \epsilon_1)^d$, let V be a neighbourhood in N with chart $\psi : \mathbb{R}^{d+1} \supset N \supseteq V \rightarrow W \subset \mathbb{R}^d$ centered around $(h(\underline{x}_0), \underline{x}_0)$ (cf. Fig. 9). Because ψ is a submanifold map from \mathbb{R}^{d+1} to \mathbb{R}^d it is a diffeomorphism. The differential of its inverse at 0 $d\psi_0^{-1}$, i. e. the Jacobian of ψ^{-1} at 0, has rank d because $d\psi_0^{-1}$ is a bijection from the tangent space $T_0\mathbb{R}^d$ onto its image and thus is an injection from the tangent space $T_0\mathbb{R}^d$ to $T_{(h(\underline{x}_0), \underline{x}_0)}\mathbb{R}^{d+1}$. Without loss of generality, assume the last d rows of $d\psi_0^{-1}$ are linearly independent. From this, we construct a smooth scalar function of which some neighbourhood of $(h(\underline{x}_0), \underline{x}_0)$ is a level set. Such a function will allow us to use the implicit

function theorem. Define

$$\begin{aligned} \chi : \mathbb{R}^{d+1} \supset \mathbb{R} \times W &\rightarrow V \subseteq N \subset \mathbb{R}^{d+1}, \\ (x_0, \underline{x}) &\mapsto (x_0 + \psi_0^{-1}(\underline{x}), \psi_1^{-1}(\underline{x}), \dots, \psi_d^{-1}(\underline{x})) \end{aligned} \quad (4.12)$$

where $\psi_i^{-1}(\underline{x})$ is the i -th component of $\psi^{-1}(\underline{x}) \in \mathbb{R}^{d+1}$ for $i \in \{0, \dots, d\}$ and any $(x_0, \underline{x}) \in V$. The differential of χ at 0 is now

$$d\chi_0 = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & \overline{d\psi}_0 \end{array} \right) \quad (4.13)$$

with $\overline{d\psi}_0$ being the invertible square matrix that is given by $d\psi_0$ without its first row. It follows that $d\chi_0$ is invertible and hence χ is a diffeomorphism on some neighbourhood of 0 by the inverse function theorem. Note that $\chi(0) = (h(\underline{x}_0), \underline{x}_0)$. Using the projection $\pi : \mathbb{R} \times W \rightarrow \mathbb{R}$ on the first coordinate we see that the map

$$\begin{aligned} u : (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d &\supseteq U_0 \rightarrow \mathbb{R}, \\ (x_0, \underline{x}) &\mapsto (\pi \circ \chi^{-1})(x_0, \underline{x}) \end{aligned} \quad (4.14)$$

is smooth and has level set $u^{-1}(0) = \varphi(\Sigma \cap U) \cap U_0$ where $U_0 \subseteq (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$ is the domain on which χ^{-1} exists. This also is still a neighbourhood of $(h(\underline{x}_0), \underline{x}_0)$. Figure 9 displays the setting.

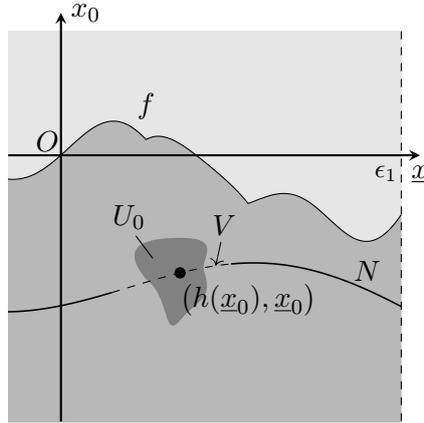


Figure 9: The point $(h(\underline{x}_0), \underline{x}_0)$ of arbitrary \underline{x}_0 on N with neighbourhoods $V \subset N$ and $U_0 \subset \varphi(U)$.

We have $u(h(\underline{x}_0), \underline{x}_0) = 0$ and by construction the partial derivative $\frac{\partial}{\partial x_0} u$ evaluated at $(h(\underline{x}_0), \underline{x}_0)$ is non-zero. If it was not the timelike vector $\frac{\partial}{\partial x_0}$

would be tangent to the spacelike hypersurface N at $(h(\underline{x}_0), \underline{x}_0)$. This condition fulfilled, we can apply the implicit function theorem which tells us that for a neighbourhood $W_0 \subseteq W \subseteq (-\epsilon_1, \epsilon_1)^d$ of \underline{x}_0 there exists a smooth function $w : W_0 \rightarrow \mathbb{R}$ whose graph is contained in the level set $u^{-1}(0)$, i. e. $u(w(\underline{x}), \underline{x}) = 0$ for all $\underline{x} \in W_0$. Therefore, the points $(w(\underline{x}), \underline{x})$ lie on $\varphi(\Sigma \cap U)$ which implies that $h|_{W_0} = w$ is smooth around \underline{x}_0 . Since $\underline{x}_0 \in (-\epsilon_1, \epsilon_1)^d$ was arbitrary we have smoothness of h everywhere.

Remark 4.1. The smoothness of h basically just follows from the smoothness of our manifold and Σ as a smooth submanifold. Local existence of a smooth function whose level set identifies with a smooth submanifold is a known result but for completeness here explicated. Its proof is inspired by [18].

Define $\omega : (-\epsilon_1, \epsilon_1)^d \rightarrow N \subset (-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$ through $\omega(\underline{x}) = (h(\underline{x}), \underline{x})$. This function is smooth, invertible and its inverse ω^{-1} represents a coordinate chart of the submanifold N . The by g induced metric \bar{g} on N is then calculated in components and in these coordinates to be

$$\bar{g}_{ij} = \left(\overset{\text{pullback}}{\omega^* g} \right)_{ij} = g_{\mu\nu} \frac{\partial \omega^\mu}{\partial x_i} \frac{\partial \omega^\nu}{\partial x_j} = g_{00} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} + g_{0j} \frac{\partial h}{\partial x_i} + g_{i0} \frac{\partial h}{\partial x_j} + g_{ij}. \quad (4.15)$$

These are to be bound uniformly. For this, it now becomes clear that smoothness of h is needed. We constructed our chart such that $g_{\mu\nu}$ already is bounded independently of Σ (cf. (4.8d)). We now look at the (spatial) derivatives of h .

The tangent vectors $\frac{\partial}{\partial x_i} \omega(\underline{x})$ to the Cauchy hypersurface Σ are spacelike. Hence,

$$\frac{\partial}{\partial x_i} \omega(\underline{x}) = \frac{\partial h(\underline{x})}{\partial x_i} \partial_0 + \partial_i \quad (4.16)$$

satisfies the inequality

$$0 \leq g \left(\frac{\partial h(\underline{x})}{\partial x_i} \partial_0 + \partial_i, \frac{\partial h(\underline{x})}{\partial x_i} \partial_0 + \partial_i \right) = \left(\frac{\partial h(\underline{x})}{\partial x_i} \right)^2 g_{00} + 2 \left(\frac{\partial h(\underline{x})}{\partial x_i} \right) g_{0i} + g_{ii} \quad (4.17)$$

for all $\underline{x} \in (-\epsilon_1, \epsilon_1)^d$ and $i \in \{1, \dots, d\}$. For each $i = 1, \dots, d$ the roots of the corresponding parabola on the right in $\frac{\partial h(\underline{x})}{\partial x_i}$ as variable are

$$\left(\frac{\partial h(\underline{x})}{\partial x_i} \right)_{\pm} = \frac{-g_{0i} \mp \sqrt{g_{0i}^2 - g_{ii}g_{00}}}{g_{00}}. \quad (4.18)$$

Since the coefficient of the leading monomial $g_{00} < -1 + \delta < 0$ is negative $\frac{\partial h(\underline{x})}{\partial x_i}$ must lie in the closed interval spanned by the roots dependent on \underline{x} in

(4.18) for the inequality of (4.17) to be fulfilled. With (4.8d) we can bound the derivative independently of h and \underline{x} as follows:

$$\left| \frac{\partial h(\underline{x})}{\partial x_i} \right| \leq \max \left| \left(\frac{\partial h(\underline{x})}{\partial x_i} \right)_{\pm} \right| < \frac{\delta}{1-\delta} + \sqrt{\frac{\delta^2}{(1-\delta)^2} + \frac{1+\delta}{1-\delta}} = \frac{1+\delta}{1-\delta} > 0 \quad (4.19)$$

for all $i \in \{1, \dots, d\}$. Coming back to (4.15) we deduce

$$|\bar{g}_{ij}| < (1+\delta) \left(\frac{1+\delta}{1-\delta} \right)^2 + 2\delta \frac{1+\delta}{1-\delta} + 1 + \delta = 2 \left(\frac{1+\delta}{1-\delta} \right)^2 \quad (4.20)$$

holds for all $i, j \in \{1, \dots, d\}$. Clearly, this bound only depends on δ and not on Σ .

Finally, we investigate the length of curves connecting any two points $p, q \in I^+(\tau(-\mu), M_{\text{int}}) \cap \Sigma$. Inclusion relation (2.26a) of Proposition 2.16 together with the construction of f in Proposition 2.15 to satisfy (2.25a) implies that p and q lie in $U \cap \Sigma$. As a consequence, there exist $\underline{x}, \underline{y} \in (-\epsilon_1, \epsilon_1)^d$ with $\omega(\underline{x}) = \varphi(p)$ and $\omega(\underline{y}) = \varphi(q)$. In $(-\epsilon_1, \epsilon_1)^d$, treated as coordinates of N , we connect p and q or respectively $\underline{x} = \omega^{-1}(\varphi(p))$ and $\underline{y} = \omega^{-1}(\varphi(q))$ by a straight line $\sigma : [0, 1] \rightarrow (-\epsilon_1, \epsilon_1)^d$, $\sigma(s) = \underline{x} + s(\underline{y} - \underline{x})$. The curves in their respective ambient manifold are depicted in Figure 10.

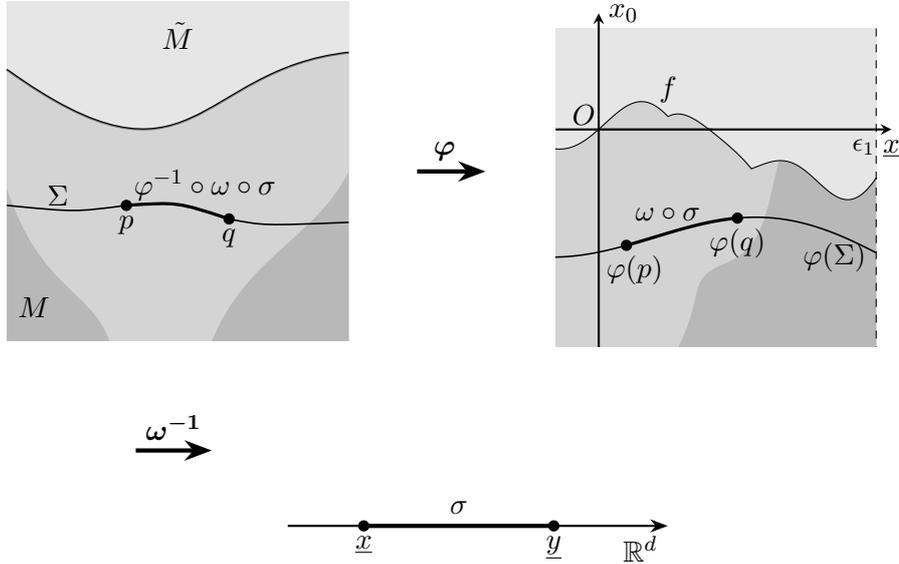


Figure 10: Illustrated transition from a curve in the original manifold M to coordinates in $(-\epsilon_0, \epsilon_0) \times (-\epsilon_1, \epsilon_1)^d$ to coordinates of N in $(-\epsilon_1, \epsilon_1)^d \subset \mathbb{R}^d$ where it becomes a straight line.

The length of the shortest curve connecting p and q in Σ is given by the length of the shortest curve in N connecting $\varphi(p)$ and $\varphi(q)$ which is at

maximum the length of σ connecting \underline{x} and \underline{y} in $(-\epsilon_1, \epsilon_1)^d$. That we compute to be

$$\begin{aligned}
L(\sigma) &= \int_0^1 \sqrt{\bar{g}(\dot{\sigma}(s), \dot{\sigma}(s))} ds \\
&= \int_0^1 \sqrt{\sum_{i,j=1}^d (\underline{y}_i - \underline{x}_i) \bar{g}_{ij}(\sigma(s)) (\underline{y}_j - \underline{x}_j)} ds \\
&< \int_0^1 \sqrt{\sum_{i,j=1}^d 2\epsilon_1 \cdot 2 \left(\frac{1+\delta}{1-\delta}\right)^2 \cdot 2\epsilon_1} ds \\
&= 2\sqrt{2} d \epsilon_1 \frac{\delta+1}{\delta-1}.
\end{aligned} \tag{4.21}$$

This bound is independent of p, q or $\underline{x}, \underline{y}$ as well as Σ . For the spacelike diameter of the timelike future of $\tau(-\mu)$ in M_{int} it follows

$$\text{diam}_s \left(I^+(\tau(-\mu), M_{\text{int}}) \right) \leq L(\varphi^{-1} \circ \omega \circ \sigma) = L(\sigma) < 2\sqrt{2} d \epsilon_1 \frac{\delta+1}{\delta-1} < \infty. \tag{4.22}$$

This is the desired result. The following subsection will show that the same spacelike diameter is infinite.

4.2.3 Infiniteness of the spacelike diameter

The result of this subsection that the spacelike diameter of M_{int} is infinite holds without any assumptions of this proof and is an intrinsic property of the Schwarzschild interior. However, using the previously constructed timelike curve τ offers a simple line of argument.

As in the previous part, we look at the spacelike diameter of $I^+(\tau(-\mu), M_{\text{int}})$. We write $\tau(-\frac{\mu}{2}) = (t_0, r_0, \omega_0) \in \mathbb{R} \times (0, (2m)^{\frac{1}{d-2}}) \times \mathbb{S}^{d-1} = M_{\text{int}}$ in standard coordinates where ω_0 is the projection of $\tau(-\frac{\mu}{2})$ onto the unit sphere \mathbb{S}^{d-1} . τ is future directed timelike. The point $\tau(-\frac{\mu}{2})$ obviously lies in the timelike future of $\tau(-\mu)$ which is open by Proposition 2.14. Hence, there exists an $\lambda > 0$ such that every point of the set $[t_0, t_0 + \lambda] \times r_0 \times \omega_0$ can still be connected to $\tau(-\mu)$ via timelike curves. To these, concatenate curves in $-\partial_r$ direction and because $-\partial_r$ is set future directed timelike in M_{int} remain in $I^+(\tau(-\mu), M_{\text{int}})$. Therefore,

$$[t_0, t_0 + \lambda] \times (0, r_0] \times \omega_0 \subseteq I^+(\tau(-\mu), M_{\text{int}}). \tag{4.23}$$

We now claim that the diameter of the Cauchy hypersurfaces

$$\Sigma_n := \left\{ p \in M_{\text{int}} \mid r(p) = n^{-1} \right\} \quad \text{for all } n \in \mathbb{N}_{\geq n_0} \quad (4.24)$$

with $n_0 \in \mathbb{N}$, $n_0 \geq r_0^{-1}$ grows beyond bound for $n \rightarrow \infty$. The induced metric on Σ_n is calculated to be

$$\bar{g}_n = - \left(1 - 2mn^{d-2} \right) dt^2 + n^{-2} \hat{\gamma}_{d-1} \quad (4.25)$$

where $\hat{\gamma}_{d-1}$ is the standard metric on the d -dimensional sphere. Looking now at the pairs of points

$$p_n := \left(t_0, \frac{1}{n}, \omega_0 \right) \quad \text{and} \quad q_n := \left(t_0 + \lambda, \frac{1}{n}, \omega_0 \right). \quad (4.26)$$

The shortest curve connecting p_n with q_n in Σ_n has of course constant $r = n^{-1}$ and constant projection on the $(d-1)$ -sphere. This follows directly from the positive definiteness of the round metric. Every perturbation to ω_0 in (4.28) would lead to increasing the length. Remember, the shortest curve connecting these two points is the (up to parametrisation) unique geodesic of the spacelike submanifold Σ_n . For \bar{g}_n the geodesic equation for the t -coordinate can be easily computed to be just $\frac{d^2}{ds^2} t = 0$. It instantly follows that $\gamma_n : [0, \lambda] \rightarrow \Sigma_n$,

$$\gamma_n(s) = \left(t_0 + s, \frac{1}{n}, \omega_0 \right) \quad (4.27)$$

is the wanted curve. The points p_n , q_n and the curves γ_n all lie in (4.23) and thus in $I^+(\tau(-\mu), M_{\text{int}})$.

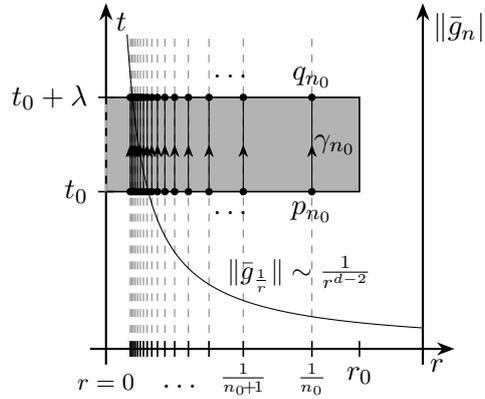


Figure 11: The shaded area belongs to the left t -axis and is the set from (4.23) which intersects all Cauchy hypersurfaces $\Sigma_n \geq n_0$ and contains our γ_n , p_n and q_n . $\|\bar{g}_n\|$, the maximal component of the matrix \bar{g}_n , is shown in dependence of r .

Imagine a from right to left, i. e. $-\partial_r$ future direction, expanding future cone containing the shaded area in Figure 11. The graph shows the divergence

of the metric components of \bar{g}_n for Σ_n approaching $r = 0$. This corresponds to the curvature blow-up at $r = 0$. For all $n \geq n_0$ the γ_n have length

$$\begin{aligned} L(\gamma_n) &= \int_0^\lambda \sqrt{\bar{g}_n(\dot{\gamma}_n(s), \dot{\gamma}_n(s))} ds \\ &= \int_0^\lambda \sqrt{2mn^{d-2} - 1} ds \\ &= \lambda \sqrt{2mn^{d-2} - 1}. \end{aligned} \tag{4.28}$$

For $d \geq 3$ which is indeed assumed for the Schwarzschild solution and writing $A = I^+(\tau(-\mu), M_{\text{int}})$ it follows

$$\begin{aligned} \text{diam}_s(A) &= \sup_{\substack{\Sigma \in \Sigma(M) \\ p, q \in (\Sigma \cap A)}} \left(\inf_{\gamma \in \Upsilon(\Sigma \cap A, p, q)} L(\gamma) \right) \\ &\geq \sup_{\Sigma_n \geq n_0} \left(\inf_{\gamma \in \Upsilon(\Sigma_n \cap A, p_n, q_n)} L(\gamma) \right) \\ &= \sup_{n \geq n_0} L(\gamma_n) \\ &= \sup_{n \geq n_0} \lambda \sqrt{2mn^{d-2} - 1} \\ &= \infty. \end{aligned} \tag{4.29}$$

The spacelike diameter of $I^+(\tau(-\mu), M_{\text{int}})$ therefore is infinite.

Together with the result of the previous subsection, this concludes the proof of C^0 -inextendibility by contradiction.

□_{Theorem I}

5 Proof with non-zero cosmological constant

Introducing the cosmological constant into the Schwarzschild solution we review if the proof of C^0 -inextendibility still holds. For a reminder revisit subsection 2.2.2 and 2.3.3.

Anti-de Sitter spacetimes are not globally hyperbolic [19, p. 38]. Therefore, we look at the case of positive Λ' . Assuming global hyperbolicity and the existence of curvature blow-up at $r = 0$ as for Schwarzschild for the de Sitter-Schwarzschild spacetime which may depend on parameters m and Λ' try to execute the proof. For contradiction, we also assume a C^0 -extension

exists. First, Theorem II holds as it is a general result for globally hyperbolic C^0 -extensions. Next, the proof of Theorem III in [4, pp. 396ff.] can relatively easily be adjusted remembering the methods and results from subsection 2.3.2 and 2.3.3. Excluding the case of the curve $\tilde{\tau}$ of Theorem II being future complete and not leaving through the curvature singularity is done exactly the same as for the Schwarzschild exterior in subsection 4.1. So the given curve must leave through $r = 0$. Showing boundedness for the spacelike diameter in a neighbourhood of $\tilde{\tau}$ leaving the manifold requires the propositions of subsection 2.4.3. Proposition 2.15 again is independent of the Kruskal manifold whereas Proposition 2.16 requires future one-connectedness in a neighbourhood of the singularity. If this is given one checks that adding the cosmological constant on page 1 of [2] does not change the argumentation. Then, the same bound for the spacelike diameter can be constructed in the coordinates of Proposition 2.15. Finally, we need the geometry near the singularity to behave similar in the Schwarzschild interior to get unboundedness. If close enough to $r = 0$ the Σ_n of subsection 4.2.3 are spacelike and Cauchy is explicated in subsection 2.3.3. The on Σ_n induced metric now reads

$$\bar{g}_n = - \left(1 - 2mn^{d-2} - \Lambda'n^{-2} \right) dt^2 + n^{-2} \dot{\gamma}_{d-1}. \quad (5.1)$$

The curves of (4.27) for a $\lambda > 0$ in the same line of argument lie in the future of $\tau(-\mu)$ of Proposition 2.16. The length of these is now

$$L(\gamma_n) = \lambda \sqrt{2mn^{d-2} + \Lambda'n^{-2} - 1}. \quad (5.2)$$

As before (cf. (4.29)) the spacelike diameter of the future of $\tau(-\mu)$ in the de Sitter-Schwarzschild spacetime is

$$\sup_{n \geq n_0} \lambda \sqrt{2mn^{d-2} + \Lambda'n^{-2} - 1} = \infty \quad (5.3)$$

where the new term does not change the behaviour towards $r \rightarrow 0$ or $n \rightarrow \infty$.

We conclude, under the given circumstances explained above the proof of Theorem I is easily adjusted to the Schwarzschild spacetime with positive cosmological constant.

6 Conclusion and outlook

C^0 -inextendibility of the maximal analytically extended Schwarzschild spacetime shows that the conclusion of the strong cosmic censorship conjecture as read in section 1 is valid for the strongest regularity desirable. However, the Schwarzschild interior is not exactly considered generic as its singularity structure changes under perturbations of initial data. Still, the methods used

and shown in the proof are expected to be useful when investigating other solutions to the Einstein (vacuum) equations and their lowest regularity extensions. As mentioned in the introduction this is why the Schwarzschild solution although its interior not exactly being considered physical plays an important role in general relativity. In fact, only Theorem III, Proposition 2.16 and subsubsection 4.2.3 take reference to the Kruskal manifold or the Schwarzschild interior.

The existence of a geodesic leaving a C^0 -extended manifold and entering the extension as well as finding a nice chart in the extension around the point where the geodesic leaves the original manifold with a Lipschitz continuous line drawing the boundary hold for any C^0 -extensions by Theorem II and Proposition 2.15 (thus also for higher regularity extensions). Firstly, one imagines Theorem III and its proof in [4, pp. 396ff.] to be relatively easily adjusted to other maximal analytically extended black hole spacetimes. Secondly, the proof of Proposition 2.16 in [2, pp. 8ff.] itself requires the future one-connectedness of the Schwarzschild interior. Finally, if in addition the spacelike diameter in the geometry around the point of leaving the manifold for a spacetime is checked to be infinite, e. g. as done for Schwarzschild in subsubsection 4.2.3 or de Sitter-Schwarzschild in section 5, the proof then can be directly transferred to this spacetime. As shown, one example of such a spacetime is the de Sitter-Schwarzschild spacetime. The proof is swiftly adapted for showing C^0 -inextendibility in this case. According to [1, p. 325], other spacetimes for which this proof is adjustable to include the *de Sitter spacetime*, *Nariai* and *Plebański-Hacyan*. This kind of proof establishes the spacelike diameter as a useful tool to explore extendibility for certain spacetimes.

Except for a particular expanding universe solution, the Kasner solution with some negative p_i , the techniques of the proof as displayed in this thesis are not sufficient and one needs to develop other methods. This is due to a finite spacelike diameter near the singularities. [1, pp. 325f.] The author of [1] and [2], Jan Sbierski, recently proved C^0 -inextendibility for a certain class of FRLW-models [20]. For this, Proposition 2.15 is again used which shows the usefulness of the methods introduced for Theorem I. Another important result or tool in this context is Theorem 1 of [3, p. 937] which states that globally hyperbolic spacetimes are C^0 -inextendible if they are timelike geodesically complete. In addition to regularity properties, singularities can be considered weaker and stronger than others. The Schwarzschild singularity e. g. is considered *strong* singularity. However, these notions do not relate directly to the regularity class of the metric as one might expect [1, p. 322].

Until quantum gravity takes over one needs to stick to the assumption of classical predictability and therefore continue investigating the statement of the strong cosmic censorship and the lowest regularity class for which inex-

tendibility holds. Despite not finding room for the left out parts of the proof, i. e. the proof of Proposition 2.15 and of Proposition 2.16, in this thesis, one should read and analyse these as this offers a qualitative understanding of dealing with local geometry of singularities. For verifying the result of C^0 -inextendibility of the de Sitter-Schwarzschild spacetime a more thorough investigation of the details, especially of the future one-connectedness near the singularity, is still needed.

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