Introduction to QFT
Assignment 9

Will be discussed on 12.01.18

This assignment has to be handed in **not later than at noon 11.01.18**.

1. (30%) Adding ϕ^2 interactions to the Lagrangian

   We consider a free-field Lagrangian for a scalar field ϕ with mass m,

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \phi^2 \tag{1}$$

   We add to $\mathcal{L}_0$ a quadratic interaction given through $\mathcal{L}_i$ in terms of $g_2 \ll m^2$ as

$$\mathcal{L}_i = -\frac{g_2}{2}\phi^2 \tag{2}$$

   In order to compute S-matrix elements for a theory given by **Eq. 1**, we use the Feynman rules in position space given by

$$\begin{aligned} G\left(x-y\right) &= \mathrm{i}\int \frac{\mathrm{d}^4 k}{\left(2\pi\right)^4} \frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2 - m^2 + \mathrm{i}\epsilon} \\ \text{vertex factor} &= \left(-\mathrm{i}g_2\right)\int \mathrm{d}^4 z \end{aligned} \tag{3}$$

   At zeroth order in g_2 the amplitude for the field ϕ to propagate from x to y is simply given by the Feynman propagator in **Eq. 3**.

   (a) Draw the Feynman diagrams that describe the corrections to the amplitude of a particle propagating from point x to point y at orders $\left\{g_2, g_2^2, g_2^n\right\}$. Mark the points of interaction and label them as space-time points.

   (b) Evaluate the diagrams that you drew in (a) for $\mathcal{O}\left(g_2\right)$ and $\mathcal{O}\left(g_2^2\right)$ explicitly, meaning in full detail using **Eq. 3**.

   (c) Argue from the results obtained in (b) what the correction at $\mathcal{O}\left(g_2^n\right)$ would be.

   [*Hint*: Writing down the correct expression is not enough!]

   In this particular case all corrections up to infinite order can be resummed by interchanging summation and integration.

   (d) Find the fully summed propagator, e.g. $\sum_{n=0}^{\infty}\mathcal{O}\left(g_2^n\right)$. This propagator is the so-called exact propagator. What mass-shift would result in a propagator of the same form? What are the consequences of this?

2. (30%) Adding a linear term to the Lagrangian

   [*Hint*: do first **EX 1**, before attempting this exercise.]

   Again, we consider a free-field Lagrangian for a scalar field ϕ with mass m,

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \phi^2 \tag{4}$$

   This time, we add to $\mathcal{L}_0$ both a linear and a cubic self-interaction term given through $\mathcal{L}_i$ in terms of g_1 and g_3 such that $\frac{g_1 g_3}{m^4} \ll 1$ as

$$\mathcal{L}_i = -\frac{g_1}{2}\phi - \frac{g_3}{3!}\phi^3 \tag{5}$$

In order to compute S-matrix elements with the theory in **Eq. 4**, we use the Feynman rules in position space given by

$$\begin{aligned}
\Delta(x-y) &= \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\epsilon} \\
\text{linear vert. fact.} &= (-\mathrm{i}g_1)\int \mathrm{d}^4z \\
\text{cubic vert. fact.} &= (-\mathrm{i}g_3)\int \mathrm{d}^4z
\end{aligned} \tag{6}$$

At zeroth order in (g_1g_3) the amplitude for the field ϕ to propagate from x to y is again given by the Feynman propagator in **Eq. 6**.

(a) Draw the tree-level Feynman diagrams that describe the corrections to the amplitude of a particle propagating from point x to point y at orders $\mathcal{O}(g_1g_3)$, $\mathcal{O}\left((g_1g_3)^2\right)$ and $\mathcal{O}\left((g_1g_3)^3\right)$.

At every order one can argue by momentum conservation that the main contribution comes from a certain subclass of diagrams.

(b) Identify the diagrams in (a) that give the largest contribution and argue why the other diagrams are suppressed by at least one additional factor of $\propto \frac{1}{m^2}$

(c) From the set of diagrams identified in (b) compute in glorious detail the lowest order correction at $\mathcal{O}(g_1g_3)$. In analogy with previous exercise, generalise and argue the amplitude of this type of correction at $\mathcal{O}((g_1g_3)^n)$.

All corrections of the type computed in (c) can be resummed up to infinite order by interchanging summation and integration.

(d) Find the fully summed propagator, e.g. $\sum_{n=0}^{\infty}\mathcal{O}((g_1g_3)^n)$. Compute the effective mass of the field ϕ with the additional linear term.

There is more good news, not only is it possible to do the all-order resummation of set of diagrams identified in (b), it is actually possible to compute the full mass-correction that is caused by the presence of the linear term in the Lagrangian.

(e) Determine the potential $V(\phi)$ from $\mathcal{L}_0 + \mathcal{L}_1$ and find the value ϕ_{min} that minimises the potential.

(f) Transform the field $\phi \to \phi' + \phi_{\text{min}}$ in $V(\phi)$. Compute the linear, quadratic and cubic terms in ϕ'. Argue why the constant term of $V(\phi')$ is not relevant to us and compute the effective mass of the field ϕ'.

The effective mass in (f) and (d) are not equal, since the correction obtained through the field redefinition is the full contribution of the linear term and the result from (d) misses out on a lot of subdominant diagrams.

(g) Argue that the diagrams that have the largest contribution are actually those identified in (b) although the results in (f) and (d) are not (exactly) equal.

(h) **Christmas-bonus**: Guess the amplitude of the diagram drawn in **Fig. 1**!

3. (10%) Quantize the real scalar field according to Fermi statistics (that means replacing the commutators of fields and conjugate momenta by anti-commutators). Show that this procedure give rise to a theory that violates causality.

4. (30%) Kinks, domain walls and all that.

   We will work in (1 + 1) space time ($x^\mu = \{x^0, x^1\}$, $x_\mu x^\mu = x_0^2 - x_1^2$, etc.)

   Consider Lagrange density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi),$$

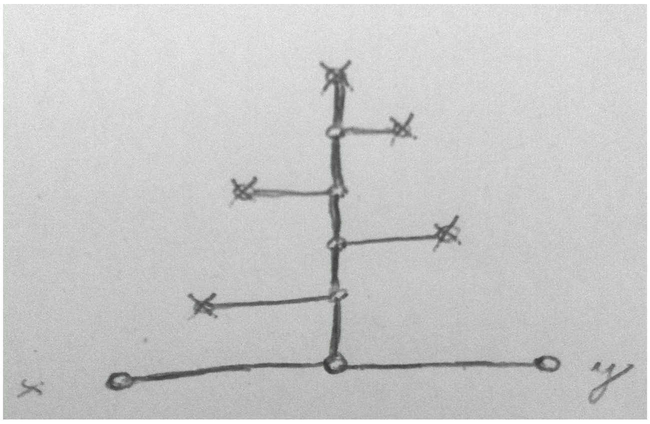

Figure 1: Feynman's christmas tree.

where ϕ is a real scalar field, $V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\mu^2}{4\lambda}$ and λ, μ are real numbers.

a) Show that $V(\phi)$ can be written as $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2$, where $v = \frac{\mu}{\sqrt{\lambda}}$.

b) Find the minima of the potential $V(\phi)$.

These minima are called vacuum expectation values of the field ϕ. In the following we will use boundary conditions

$$\phi(+\infty) = v,$$
$$\phi(-\infty) = -v.$$

c) Find the expression for a total energy of the field ϕ.

Now we will consider only time-independent fields.

d) Write down Euler-Lagrange equations for a time-independent field ϕ

e) Solve them using the boundary conditions from above.

[*Hint*: Recall the motion of a point-like particle in a potential $U(\phi) = -V(\phi)$.]

f) Draw your solutions. What are its asymptotic values?

g) Show that energy of your solution is proportional to $1/\lambda$. What does it mean? Is it possible to get the same result by perturbative methods?

h) What is the energy of this solution?

i) What will change if you use $\phi(+\infty) = -v$ and $\phi(\infty) = v$? Draw the solution for this case.

Two solutions you got are called kink and anti-kink respectively, they play a crucial role in the description of domain walls in ferromagnets and in some cosmological models
(see for example "Cosmological Consequences of the Spontaneous Breakdown of Discrete Symmetry" by Ya. B. Zeldovich, I. Yu. Kobzarev, L. B. Okun).

There is a deep connection between these objects and topology, namely one can show that there is no transformation to turn a kink into an anti-kink and wise versa. It implies that one can assign conserved charges to kink and anti-kink (so called topological charges).

j) Show that current $k^\mu = \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\phi$ is conserved for an arbitrary function ϕ
(here $\epsilon^{\mu\nu}$ is Levi-Civita symbol in 2D).

k) Find topological charges $Q_t = \int dx\, k^0$ for a kink and an anti-kink.