Introduction to QFT
Assignment 8

Will be discussed on 22.12.17

This assignment has to be handed in **not later than at noon 21.12.17**.

1. (30%) The Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A + \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_A^2\phi_A^2 - \frac{1}{2}m_b^2\phi_B^2 + \frac{\lambda}{4}\phi_A^2\phi_B^2 ,$$

describes two real fields that interact with each other. Calculate the lowest-order non-trivial matrix element $\langle k_B, p_B|S|k_A, p_A\rangle$ for the scattering process $AA \to BB$. Then calculate the matrix elements for the processes $AB \to AB$ and $AA \to AB$.

2. (30%) Consider two real scalar fields A, B with an interaction Lagrangian given by $\mathcal{L}_{\text{int}} = gA^2B$. Compute the invariant matrix element for the decay process $B \to AA$ at tree level.
Draw at least two different Feynman diagrams contributing to this process at one loop oder (order g^3) and at two loop oder (order g^5).

3. (40%) Wick's Theorem in an excursion into a non-relativistic QFT.

We consider a non-relativistic quantum field theory, inspired by low-energetic scattering in condensed matter systems, through a scalar field $\phi(\mathbf{r})$. This field has a continuous plane-wave decomposition given as,

$$\phi(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} a_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} . \quad (1)$$

The operator $a_{\mathbf{k}}$ satisfies the following canonical commutation relations,

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') . \quad (2)$$

(a) In what way does the expression in **Eq. 1** relate to the eigen-functions of the non-relativistic free-field hamiltonian H_0, given as

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 . \quad (3)$$

(b) Give expressions for the multi-particle states $|\mathbf{k}_1, \mathbf{k}_2\rangle$ and $|\mathbf{k}_3, \mathbf{k}_4\rangle$ in terms of the creation and annihilation operators and the vacuum. Then compute $\langle\mathbf{k}_3, \mathbf{k}_4|\mathbf{k}_1, \mathbf{k}_2\rangle$ by using the commutation relations in **Eq. 2** and give a diagrammatic representation of your result.

The density operator is given by the expression

$$\rho(\mathbf{r}) = \phi^\dagger(\mathbf{r})\phi(\mathbf{r}) \quad (4)$$

(c) Compute $\langle\mathbf{k}_3, \mathbf{k}_4|\rho(\mathbf{r})|\mathbf{k}_1, \mathbf{k}_2\rangle$ and explain in words what this quantity means. Try also to give a diagrammatic representation of the result, analogous to (b). What seems to happen with respect to momentum conservation?

We now add an interaction between different particles i and j, that can be expressed in terms of its Fourier components as,

$$V(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} V(\mathbf{q})\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \ . \tag{5}$$

(d) Compute, the following expression,

$$\langle \mathbf{k}_3, \mathbf{k}_4 | \int \mathrm{d}^3\mathbf{r}_i \int \mathrm{d}^3\mathbf{r}_j : \phi^\dagger(\mathbf{r}_i)\, \phi(\mathbf{r}_i) \left(\frac{V(\mathbf{r}_i - \mathbf{r}_j)}{2} \right) \phi^\dagger(\mathbf{r}_j)\, \phi(\mathbf{r}_j) : | \mathbf{k}_1, \mathbf{k}_2 \rangle \ . \tag{6}$$

Also give a diagrammatic representation of your result, what happens with respect to momentum conservation?

[*Hint*: here $: \ldots :$ means operators within are normal-ordered.]