Introduction to QFT
Assignment 3

Will be discussed on 17.11.17

This assignment has to be handed in **not later than at noon 16.11.17**.

1. (45%) A scalar field ϕ transforms under a Lorentz transformation as:

$$S(\Lambda)^{-1}\,\phi(x)\,S(\Lambda) = \phi\left(\Lambda^{-1}x\right) \ . \tag{1}$$

(a) Demonstrate $[\phi(x), \mathcal{J}^{\mu\nu}] = \mathcal{L}^{\mu\nu}\phi$ using the infinitesimal expansion of $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}$ in Eq. 1 with $\mathcal{J}^{\mu\nu}$ as the generators of the Lorentz algebra and $\mathcal{L}^{\mu\nu}$ defined through: $\mathcal{L}^{\mu\nu} = \mathrm{i}\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$.

(b) Use (a) to demonstrate now that $[[\phi(x), \mathcal{J}^{\mu\nu}], \mathcal{J}^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x)$.

(c) Use the results from (a) and (b) to show that the following relation holds:

$$[\phi(x), [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]] = \left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(x) \ . \tag{2}$$

[*Hint*: Do not forget that the generators of the Lorentz group $\mathcal{J}^{\mu\nu}$ form a *Lie algebra*.]

(d) Rewrite the right-hand side of Eq. 2 such that it is at most linear in $\mathcal{L}^{\mu\nu}$.

(e) The result of (d) can be used to verify the following expression:

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = \frac{1}{\mathrm{i}}\left(g^{\mu\rho}\mathcal{J}^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma) \ , \tag{3}$$

up to terms that commute with ϕ and its derivatives.

(f) EXTRA(not graded) How does one call the possible term that could commute with ϕ and its derivatives? Does it exist for the Lorentz algebra?

2. (15%) Dirac Bilinears and Lorentz group.

Consider a set of matrices: $\{1, \gamma^{\mu}, \gamma^{5}, \gamma^{\mu}\gamma^{5}, \sigma^{\mu\nu}\}$,
where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$.

(a) How many linearly independent matrices do we have?

The Lorentz group acts on the Dirac spinors in the following way $\psi'(x') = S(\Lambda)\psi(x)$, where $S(\Lambda) = e^{-\frac{i}{4}\epsilon_{\mu\nu}\sigma^{\mu\nu}}$ and $\epsilon_{\mu\nu}$ is an antisymmetric tensor which contains parameters of the transformation.

Consider the following bilinear forms: $\bar{\psi}\psi$, $i\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$, $\bar{\psi}\sigma^{\mu\nu}\psi$.

(b) You have seen in the lectures that under the action of the Lorentz group $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$ transform as scalars. Demonstrate now that the bilinear forms $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi$ transform as vectors and that $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms as a tensor.

3. (45%) Dirac Bilinears and Discrete Symmetries.

In this exercise we consider the Dirac spinors as operators. Recall from your lectures on QM that under a unitary transformation U an operator $\hat{O}$ transforms as

$$\hat{O}' = U\,\hat{O}\,U^{-1}.$$

The parity operator $\hat{P}$ is defined for Dirac spinors as $\psi(t, -\mathbf{x}) = \hat{P}\psi = \gamma^0\psi(t, \mathbf{x})$.

(a) Show that $\mathcal{P}\psi_L = \psi_R$ and $\mathcal{P}\psi_R = \psi_L$

(b) In your lectures you have seen how operators $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$ transform under the operator $\hat{P}$. Find the transformations of the other bilinear forms.

The particle-antiparticle conjugation operation $\hat{C}$ is defined as $\psi^c = \hat{C}\psi = \mathcal{C}\bar{\psi}^T$, where $\mathcal{C}$ is a certain representation dependent matrix that obeys $\mathcal{C}^\dagger\gamma^\mu\mathcal{C} = (-\gamma^\mu)^*$.

(c) Without specifying a representation show that $\psi^c(x)$ transforms under the Lorentz group in the same way as ψ

In Weyl or Dirac representation $\mathcal{C} = \gamma^2\gamma^0$. **Until the end of this exercise you have to use either Dirac or Weyl basis**.

(d) Show that $\psi^c = \gamma_2\psi^*$

(e) Show that $\hat{C}\psi_L = \psi_R^c$ and $\hat{C}\psi_R = \psi_L^c$

(f) Study how bilinear forms $\bar{\psi}\psi$, $i\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\sigma^{\mu\nu}\psi$ transform under the particle-antiparticle conjugation operation $\hat{C}$.

The time reversal operator is defined (in the Dirac or Weyl basis) as $\hat{T} = \gamma_1\gamma_3 K$ where K is an operator that conjugates all complex numbers standing to the right of it, in other words $\hat{T}$ is an anti-unitary operator (e. g. $\hat{T}\alpha\gamma^\mu\psi = \alpha^*(\gamma^\mu)^*\psi$).

(g) Study how bilinear forms $\bar{\psi}\psi$, $i\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\sigma^{\mu\nu}\psi$ transform under time reversal operation $\hat{T}$.

Hint: In both Dirac and Weyl representations the only complex matrix is $\gamma^2 = -\left(\gamma^2\right)^*$.

(h) EXTRA(not graded) Make sure that $\psi'(t') = \hat{T}\psi(t)$ satisfies the Dirac equation with $t' = -t$.