Introduction to QFT
Assignment 10

Will be discussed on 19.01.18

This assignment has to be handed in **not later than at noon 18.01.18**.

1. (50%) Electrons in magnetic field (QM)

   Consider a Dirac equation for the electron interacting with electro-magnetic field A_μ

$$(i\gamma^\mu D_\mu - m)\,\psi = 0, \tag{1}$$

   where $D_\mu = \partial_\mu - ieA_\mu$. Do not use any particular representation unless stated explicitly.

   a) Show that $i\sigma^{\mu\nu}D_\mu D_\nu = \frac{e}{2}\,\sigma^{\mu\nu}F_{\mu\nu}$, where $\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

   b) Acting on the Dirac equation by $(i\gamma^\mu D_\mu + m)$ show that

$$\left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\psi = 0. \tag{2}$$

   Since the Dirac theory is gauge invariant we are free to chose the form of of the field A_μ. So we can choose $A_0 = A_3 = 0$ and $A_1 = -\frac{1}{2}Bx_2$ and $A_2 = \frac{1}{2}Bx_1$, where x_1, x_2 are components of a 3-vector $\mathbf{x}$. Moreover we will consider only weak fields.

   c) Show that for the filed A_μ with components given above

$$(D_i)^2 = \Box + e\mathbf{B}\cdot[\mathbf{x}\times\mathbf{p}] + O\left(A_i^2\right), \tag{3}$$

   where $i = 1, 2, 3$, $\mathbf{B} = (0, 0, B)$ and $\mathbf{p}$ is the usual momentum operator from QM.

   Now we will use the Dirac basis. In this basis

$$\sigma^{ij} \quad = \epsilon^{ijk}\begin{bmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{bmatrix}, \tag{4}$$

   where σ^k are usual Pauli matrices. Also one can show that for the slow electrons among the four components Dirac wave function $\psi = \begin{bmatrix}\phi\\ \chi\end{bmatrix}$ (written in terms of two two-component spinors ϕ and χ) only first two components survive and thus $\psi \approx \begin{bmatrix}\phi\\ 0\end{bmatrix}$.

   d) Show that the operator $e/2\,\sigma^{\mu\nu}F_{\mu\nu}$ acting on ϕ gives

$$2e\,\mathbf{B}\cdot\mathbf{S}, \tag{5}$$

   where $\mathbf{S} = \sigma/2$ (usual spin operator from QM).

   e) Using approximation from above show that in the Dirac basis one can write Eq.2 as

$$\left[\partial_t^2 - \Box - e\mathbf{B}\cdot(\mathbf{L} + 2\mathbf{S}) + m^2\right]\phi = 0 \tag{6}$$

   Now we assume that the function ϕ can be written as a product of a fast oscillating exponent e^{-imt} and a slowly oscillating function Ψ (which means that we can drop terms like $\partial_t^2\Psi$).

   f) Substituting $\phi = e^{-imt}\Psi$ show that Eq.6 can be written as

$$-i\frac{\partial}{\partial_t}\Psi = \frac{1}{2m}\,\triangle\,\Psi + \frac{1}{2m}e\mathbf{B}\cdot(\mathbf{L} + 2\mathbf{S})\,\Psi. \tag{7}$$

   What is the meaning of the second term on the right hand side?

2. (50%) Angular momentum of the Dirac field (classical field theory)

Consider a transformation of a Dirac field ψ

$$\psi \to \left(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi(x + \delta x), \tag{8}$$

where $x + \delta x = x^\mu + \omega^\mu_\nu x^\nu$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and all transformation understood to be infinitesimally small $|\omega| \ll 1$.

a) Show that under the transformation in Eq.8

$$\psi \to \psi - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\psi + \omega^\lambda_\beta x^\beta \partial_\lambda \psi. \tag{9}$$

Now we want to find corresponding the Noether currents. It will be more convenient to consider the terms in Eq.9 separately.

b) Find the Noether current corresponding to the last term on the right hand side of the Eq.9. Express the answer in terms of the Energy-Momentum tensor

$$\mathcal{T}^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\nu \psi - \mathcal{L}\delta^\mu_\nu \tag{10}$$

[*Hint*: Recall the way you derived the expression for the Energy-Momentum tensor.]

c) Now do the same for the second term on the right hand side of Eq.9

If you have done everything correctly you should get a tensor of the third rank which will be anti-symmetric with respect to the interchange of one pair of its indices ($\mathcal{M}^{\mu\alpha\beta} = -\mathcal{M}^{\mu\beta\alpha}$).

d) Set $\mu = 0$ so $\mathcal{M}^{\alpha\beta} \equiv \mathcal{M}^{0\alpha\beta}$. Substituting the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ and using Dirac representation of gamma matrices find M^{12}. What is the physical meaning of the other components of $\mathcal{M}^{\alpha\beta}$?

[*Hint*: Use Eq.4.]