# Introduction to QFT
# Assignment 5

Will be discussed on 01.12.17

This assignment has to be handed in **not later than at noon 30.11.17**.

1. (36%) By Noether's theorem the Lagrangian of a scalar-field theory has four conserved charges related to the $\mu = 0$ components of the stress-energy tensor given in Eq. 1.

$$T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi - \mathcal{L}\delta^{\mu}_{\nu} \tag{1}$$

   (a) Define the canonical momentum conjugate $\pi\left(\mathbf{x}\right)$ to the scalar field $\phi\left(\mathbf{x}\right)$ and use it to express the four conserved charges that follow from Eq. 1 in terms of both $\pi\left(\mathbf{x}\right)$ and $\phi\left(\mathbf{x}\right)$. What is the physical interpretation of these four conserved charges?

   (b) The fields $\pi\left(\mathbf{x}\right)$ and $\phi\left(\mathbf{y}\right)$ are promoted to operators. Give the equal-time canonical commutation relations between and among $\pi\left(\mathbf{x}\right)$ and $\phi\left(\mathbf{y}\right)$.

   The fields $\pi\left(\mathbf{x}\right)$ and $\phi\left(\mathbf{y}\right)$ can be decomposed in terms of the creation operator $a^\dagger_{\mathbf{p}}$ and annihilation operator $a_{\mathbf{p}}$ as follows:

$$\phi\left(\mathbf{x}\right) = \int \frac{d^3\mathbf{p}}{\left(2\pi\right)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$

$$\pi\left(\mathbf{x}\right) = -\mathrm{i} \int \frac{d^3\mathbf{p}}{\left(2\pi\right)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$

   (c) Compute the inverse expressions for $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ in terms of $\phi$ and $\pi$.

   (d) Compute the commutation relations between and among $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}'}$ such that the canonical commutation relation in (b) hold.

   (e) Express the conserved charges from (a) in terms of (momentum) integrals over $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}'}$ in normal ordering. Give a physical interpretation to result that you find.

2. (36%) The complex Klein-Gordon fields $\phi(\mathbf{x})$ and $\phi^\dagger(\mathbf{x})$ can be expressed in terms of two independent real Klein-Gordon fields $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ as

$$\phi(\mathbf{x}) = \frac{\phi_1(\mathbf{x}) + \mathrm{i}\phi_2(\mathbf{x})}{\sqrt{2}}\ , \quad \phi^\dagger(\mathbf{x}) = \frac{\phi_1(\mathbf{x}) - \mathrm{i}\phi_2(\mathbf{x})}{\sqrt{2}}\ .$$

(a) Use the equal-time commutation relations between fields and their conjugate momenta for the real fields to find the respective relations for the complex fields.

(b) The Fourier transform of the complex Klein-Gordon field can be written as

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-\mathrm{i}\mathbf{p}\mathbf{x}} + b^\dagger_{\mathbf{p}} e^{\mathrm{i}\mathbf{p}\mathbf{x}}).$$

Write the creation and annihilation operators of the complex fields in terms of those of the real fields $a^{(1)}_{\mathbf{p}}$, ${a^{(1)}}^\dagger_{\mathbf{p}}$ and $a^{(2)}_{\mathbf{p}}$, ${a^{(2)}}^\dagger_{\mathbf{p}}$. Use the commutation relations between creation and annihilation operators for the real fields to find the respective relations for the complex fields.

[*Hint*: Operators of different scalar fields commute.]

(c) What are the number operators of the field $\phi$? Write the energy and momentum operators of the complex Klein-Gordon field in terms of the number operators.

(d) Write the charge operator of the complex Klein-Gordon field in terms of the number operators. What is the physical meaning of different terms in the expression you obtained?

3. (28%) Let

$$f(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \tilde{f}(p) e^{-\mathrm{i}\mathbf{p}\mathbf{x}} ,$$

be a classical field that solves the Klein-Gordon equation. Define the operators

$$a = C \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \tilde{f}^*(\mathbf{p}) a(\mathbf{p}) , \quad a^\dagger = C \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \tilde{f}(\mathbf{p}) a^\dagger(\mathbf{p}) ,$$

where $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are the annihilation and creation operators for the Klein-Gordon field, and $C$ is a normalization constant.

(a) Calculate the following commutators

$$[a(\mathbf{p}), a^\dagger] , \quad [a(\mathbf{p}), a] .$$

(b) Prove that

$$[a(\mathbf{p}), (a^\dagger)^n] = n \frac{C\tilde{f}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} (a^\dagger)^{n-1} .$$

(c) Show the the state $|z\rangle = e^{-|z|^2/2} e^{z a^\dagger} |0\rangle$ is an eigenstate of the operator $a(\mathbf{p})$.