Introduction to QFT
Assignment 1

Due on 02.11.17

This assignment has to be handed in **not later than at noon 02.11.17**.

1. (45%) Dirac Hamiltonian ($\hbar = 1, c = 1$) is written as

$$H_D = \sum_{i=1}^{3} \hat{\alpha}_i \hat{p}_i + \hat{\beta} m.$$

   (a) Show that matrices α_i and β are hermitian.

   (b) Show that they are traceless.

   (c) Show that the dimensionality of α_i and β has to be even.
   [*Hint*: what are the eigenvalues of α_i and β?]

   In more often used notation Dirac Hamiltonian can be written as

$$H_D = \sum_{i=1}^{3} \gamma^0 \left(\gamma^i \hat{p}_i + m \right).$$

   (d) Show that transformation of all four matrices can be written in a form $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$, where $\mu = 0, 1, 2, 3$.

   (e) Using the relation between α_i's and β (derived in the lecture) show that the γ-matrices obey $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}$.

   The fifth matrix γ^5 is given by product $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Using these definitions show that next identities hold (**do not use any particular representation**):

   (f) $\{\gamma^\mu, \gamma^\nu\} = 0$, for $\mu \neq \nu$

   (g) $\left\{\gamma^\mu, \gamma^5\right\} = 0$

   (h) $\left(\gamma^5\right)^2 = 1$, (by 1 we mean $\mathbb{1}$)

   (i) $\not{p}^2 = p^2$, where $\not{p} \equiv \gamma_\mu p^\mu$,

   (j) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon^{\mu\nu\rho\lambda}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda$ where $\epsilon^{\mu\nu\rho\lambda}$ is Levi-Civita tensor defined such that $\epsilon^{\mu\nu\rho\lambda} = +1$ for μ, ν, ρ, λ an even permutation of 0, 1, 2, 3, $\epsilon^{\mu\nu\rho\lambda} = -1$ for an odd permutation of 0, 1, 2, 3 and $\epsilon^{\mu\nu\rho\lambda} = 0$ otherwise.

   (k) $\gamma_\mu \gamma^\mu = 4$

   (l) $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

(m) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

2. (30 %) There are infinitely many different choices to represent the Dirac matrices γ^μ. Two of the most often used representations are

   Dirac representation: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$

   Weyl representation: $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$

   (a) Calculate the explicit form of γ^5 in both representations.

   (b) Check explicitly in both Weyl and Dirac representation that $\psi_R(x) = P_R\psi(x)$ and $\psi_L(x) = P_L\psi(x)$ are eigenstates of the chirality operator γ^5.

   We have now introduced the concept of chirality ("handedness") and we call ψ_R the right-handed and ψ_L the left-handed component of a Dirac fermion. The Weyl representation is particularly useful for the concept of chirality.

   (c) Show that $\left[H_D, \gamma^5\right] \neq 0$.

3. (25%)

   As practical application of the KG-equation one can study a special type of atoms, these are atoms wherein one or more electrons were replaced by negatively charged scalar particles: σ^-.

   The charged sigma-field $\phi(\mathbf{r}, t)$ is a complex scalar field obeys the Klein-Gordon equation given in Eq. (1). Here, $A_0(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the electromagnetic scalar and vector potentials.

$$\left((\partial_t - \mathrm{i}qA_0(\mathbf{r}, t))^2 - (\nabla - \mathrm{i}q\mathbf{A}(\mathbf{r}, t))^2 + m_\sigma^2\right)\phi(\mathbf{r}, t) = 0 \qquad (1)$$

   [*Hint*: look up your QM lecture notes.]

   (a) Find the scalar potential $A_0(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ that the pion is affected by in the vicinity of a stationary atom nucleus with proton number Z. Express these function in the Gaussian system of units (in contrast to either the SI or Heaviside-Lorentz formulations) and with natural units, e.g. $\epsilon_0 = \hbar = c = k_B = 1$.

   (b) Apply the found expressions in (a) to Eq. 1 and use separation of variables to solve the time-dependent part of the resulting equation. What is the equation that the spatial part of $\phi(\mathbf{r}, t)$ must obey?

   (c) Expand the '∇' operator and solve the angular part of the equation that is found in terms of special functions. What is the equation that the radial part of $\phi(\mathbf{r}, t)$ must obey?

   (d) Determine the energy spectrum of the atom from the radial equation found in (c) in terms of the given variables in Eq. 1 and quantum numbers

n, l and m familiar from the hydrogen atom.

(e) **EXTRA**(not graded) Expand the solution of (d) in Ze^2, the effective fine-structure constant, up to 4^{th} order (no worries the odd orders are zero anyway).