Introduction to QFT
Bonus Assignment

Due on 24.02.16

1. The free real scalar field can be expanded as

$$\phi(x) = \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a(p)e^{ipx} + a^\dagger(p)e^{-ipx}] \,,$$

and the Klein-Gordon Hamiltonian is given by

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p : a^\dagger(p)a(p) + a(p)a^\dagger(p) : \ . \tag{1}$$

Show explicitly that $|k_1, k_2, ..., k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)...a^\dagger(k_n)|0\rangle$ is an eigenstate of H with energy $E_{k_1} + E_{k_2} + ... + E_{k_n}$.

2. Classical electrodynamics (without sources) is described by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \,, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

Derive the Maxwell equations by treating the components of A_μ as dynamical fields and write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

3. The proper way to introduce gauge fixing to quantize the Maxwell field is done by introducing a scalar auxiliary field B (Nakanishi-Lautrup field), modifying the Lagrangian to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2 \ . \tag{3}$$

Show that this is equivalent to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\alpha^{-1}}{2}(\partial^\mu A_\mu)^2 + \frac{\alpha^{-1}}{2}(\partial^\mu A_\mu + \alpha B)^2 \ . \tag{4}$$

Derive the the Euler-Lagrange equations for this system (now called quantum Maxwell equations) and show that this really reduces to the classical Lagrangian plus a gauge fixing term. Argue that $\alpha = 0$ (Landau gauge) is the quantum equivalent to Lorenz gauge.

4. Calculate the vacuum expectation values (i.e. express them in terms of propagators) and draw the corresponding diagrammatic representations for
   - (2%) $\langle 0|T\phi(w)\phi(x)\phi(y)\phi(z)|0\rangle$
   - (6%) $\langle 0|T\phi^2(x)\phi^2(y)|0\rangle$
   - (12%) $\langle 0|T\phi^4(x)\phi^4(y)|0\rangle$

5. Consider a field theory with three different scalar field A, B, C where we only have an interaction of the form $\mathcal{L}_{\text{int}} = gABC$. Look at the scattering process $AB \to AB$ and show that disconnected diagrams do not give any contributions to the scattering part of the S-Matrix element $\langle p'_A p'_B|S|p_A p_B\rangle$. Do this by follwing the steps:
   - Calculate the S-matrix element in 0th order of perturbation theory. Are the Feynman diagrams connected or disconnected? What's the physical meaning of this process?
   - What's the situation at order $\mathcal{O}(g)$?
   - Identify the leading order in perturbation theory where $AB \to AB$ scattering occurs.
   - Use the LSZ reduction formula to calculate $\langle p'_A p'_B|S|p_A p_B\rangle$ at leading order.
   - Is this generalizable to any order in perturbation theory and n-body scattering?

6. In addition to the canonical quantization, there is another way to formulate a quantized theory. This approach is called path integral formalism. Let us look at a simple matrix element for a scalar field theory:

$$\hat{\mathcal{L}} = \frac{1}{2}(\partial_\mu \hat{\phi})(\partial^\mu \hat{\phi}) - \frac{1}{2}m^2\hat{\phi}^2 - V(\hat{\phi})$$

- Show that the Hamiltonian of the system is given by

$$\hat{H} = \int d^3x \left(\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + V(\hat{\phi}) \right) .$$

Notice that we use $\hat{A}$ to explicitly show a quantity is an operator! Now we want to calculate the overlap of two states at different times, i.e. the matrix element

$$\langle \phi' | e^{-i\hat{H}t} | \phi \rangle .$$

For technical reasons we have to regularize our theory, i.e. formulate the theory on a finite grid with spacing a and we will take $a \to 0$ at the end. the Hamiltonian than reads

$$\hat{H} = \sum_{x \in \boldsymbol{R}^3} a^3 \left(\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\nabla_a\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + V(\hat{\phi}) \right) = \hat{H}_0(\hat{\Pi}) + \hat{U}(\hat{\phi}) .$$

Further we will introduce two complete sets of eigenstates with the following properties

$$\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle , \quad \mathbf{1} = \int \mathcal{D}\phi \, |\phi\rangle\langle\phi| , \quad \mathcal{D}\phi = \prod_{x \in \boldsymbol{R}^3} d\phi(x) ,$$

$$\hat{\Pi}(x)|\Pi\rangle = \Pi(x)|\Pi\rangle , \quad \mathbf{1} = \int \mathcal{D}\Pi \, |\Pi\rangle\langle\Pi| , \quad \mathcal{D}\Pi = \prod_{x \in \boldsymbol{R}^3} d\Pi(x) ,$$

$$\langle\phi'|\phi\rangle = \delta(\phi' - \phi) , \quad \langle\Pi'|\Pi\rangle = \delta(\Pi' - \Pi) , \quad \langle\phi|\Pi\rangle = \prod_{x \in \boldsymbol{R}^3} \frac{1}{\sqrt{2\pi}^3} \exp(ia^3\phi(x)\Pi(x)) .$$

- Begin by computing $\langle \phi' | e^{-i\hat{H}_0 t} | \phi \rangle$. Do this by introducing an appropriate set of complete eigenstates and solving a Gaussian functional integral (Evaluate this integral as it was a usual c-number integral). The result will be

$$\langle \phi' | e^{-i\hat{H}_0 t} | \phi \rangle = \prod_{x \in \boldsymbol{R}^3} \sqrt{\frac{a^3}{2\pi t}} \exp\left(-\frac{a^3}{2t}(\phi'(x) - \phi(x)) \right) . \tag{5}$$

- Now calculate $\langle \phi' | e^{-i\hat{H}t} | \phi \rangle$ by using the Trotter formula

$$\langle \phi' | e^{-i\hat{H}t} | \phi \rangle = \lim_{n_t \to \infty} \langle \phi' | \hat{W}_\epsilon | \phi \rangle , \quad \text{with} \quad \hat{W}_\epsilon = e^{-\epsilon\hat{U}/2} e^{-\epsilon\hat{H}_0} e^{-\epsilon\hat{U}/2} \quad \text{and} \quad \epsilon = t/n_t ,$$

  and inserting appropriate sets of eigenstates and using (5).

- Now take $a \to 0$ and show that $\langle \phi' | e^{-i\hat{H}t} | \phi \rangle = \int \mathcal{D}[\phi] e^{iS[\phi]}$, where $S[\phi]$ is the classical action without any operators,
  $\int \mathcal{D}[\phi] = \prod_{x \in \boldsymbol{R}^4} d\phi(x)$ and we have dropped an awkward normalization constant that does not play a role for practical calculations.