Introduction to QFT
Assignment 2

Due on 11.11.16

1. (20%) Evaluate the commutators of the Dirac Hamiltonian, $H_D = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2$ with the following operators:
   a) $\boldsymbol{p} = -i\hbar\nabla$
   b) $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$
   c) $\boldsymbol{L}^2$
   d) $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} = \frac{i}{2}\boldsymbol{\gamma} \times \boldsymbol{\gamma}$ and $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$
   e) $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$
   f) $\boldsymbol{J}^2$
   g) $\boldsymbol{\Sigma} \cdot \frac{\boldsymbol{p}}{|p|}$

2. (30%) Show that

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma \cdot p + m \ ,$$

   and

$$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma \cdot p - m \ ,$$

3. (15%) The spin operator in the rest frame for a Dirac particle is defined by $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\Sigma}$. Prove that
   a) $\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$
   b) $\Sigma^i = -\gamma^5\gamma^0\gamma^i$,
   c) $S^2 = \frac{3}{4}$.

4. (15%) Without using any particular representation of the Dirac spinors, prove the following identities ($\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$):

$$2m\bar{u}(\boldsymbol{p}_1)\gamma_\mu u(\boldsymbol{p}_2) = \bar{u}(\boldsymbol{p}_1)[(p_1 + p_2)_\mu + i\sigma_{\mu\nu}(p_1 - p_2)^\nu]u(\boldsymbol{p}_2) \ ,$$

$$\bar{u}(\boldsymbol{p}_1)\sigma_{\mu\nu}(p_1 + p_2)^\nu u(\boldsymbol{p}_2) = i\bar{u}(\boldsymbol{p}_1)(p_1 - p_2)_\mu u(\boldsymbol{p}_2) \ .$$