# Chapter 6
# Korteweg-de Vries Equation

The Korteweg-de Vries (KdV) equation is the partial differential equation, derived by Korteweg and de Vries [13] to describe weakly nonlinear shallow water waves. The nondimensionalized version of the equation reads

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}, \tag{6.1}$$

where $u = u(x,t)$. The factor of 6 is convenient for reasons of complete integrability, but can easily be scaled out if desired. Equation (6.1) was found to have *solitary wave solutions*, vindicating the observations of a solitary channel wave made by Russell [23].

## 6.1 Traveling wave solution

In the same way as in Sec. 5.1 we look for a right traveling wave solution of the form [27]

$$u(\xi) := u(x - ct),$$

such as $u \to 0$, $u_\xi \to 0$ and $u_{\xi\xi} \to 0$ as $\xi \to \pm\infty$. Substitution into Eq. (6.1) leads to the ODE

$$u_{\xi\xi\xi} - 6uu_\xi - cu_\xi = 0.$$

An integration with respect to $\xi$ yields

$$u_{\xi\xi} = 3u^2 + cu + c_1,$$

where $c_1$ is a constant of integration. Since $u \to 0$, $u_\xi \to 0$ and $u_{\xi\xi} \to 0$ as $\xi \to \pm\infty$, $c_1 = 0$. A second integration yields

$$\frac{1}{2}u_\xi^2 = u^3 + \frac{1}{2}cu^2 + c_2,$$

where $c_2 = \text{const} = 0$. That is, the last equation can be written as

$$d\xi = \frac{du}{u\sqrt{2u+c}},$$

which can be integrated, yielding

$$u(\xi) = -\frac{c}{2}\,\text{sech}^2\left(\frac{1}{2}\sqrt{c}\,(\xi-\xi_0)\right),$$

where $\xi_0$ is an arbitrary constant. In $(x,t)$ coordinates the traveling wave solution reads

$$u(x,t) = -\frac{c}{2}\,\text{sech}^2\left(\frac{1}{2}\sqrt{c}\,(x-x_0-ct)\right). \tag{6.2}$$

Equation (6.2) describes the localized traveling wave solution with a negative amplitude (see Fig. 6.1 (a)), which is called *a soliton*. The term soliton was first introduced by Zabusky and Kruskal [30], who studied Eq. (6.1) with periodic boundary conditions numerically. They found [30, 27, 14] that initial condition of the form $u(x,0) = \cos(2\pi x/L)$, $x \in [0,L]$ broke up into a train of solitary waves with successively large amplitude. Moreover the solitons seems to be almost unaffected in shape by passing through each other (though this could cause a change in their position). An example of two-soliton solution is shown on Fig. 6.1 (b).

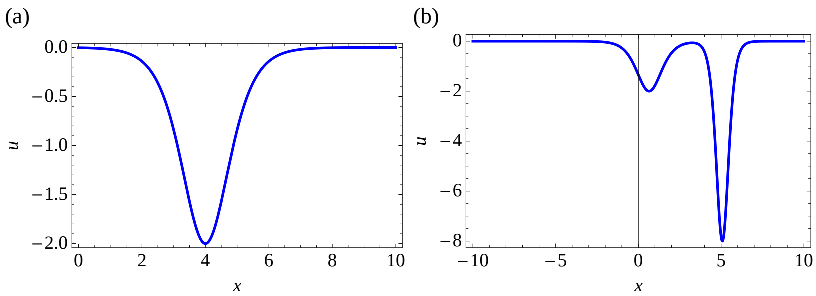


**Fig. 6.1** Solitary solutions of KdV equation (6.1). (a)A single-soliton solution (6.2) for $c = 2$, calculated for $t = 1$. (b) A two-soliton solution, calculated at $t = 0.3$.

## 6.2 Numerical treatment

Consider the KdV Eq. (6.1) on the interval $x \in [-L, L]$ with initial condition

$$u(x,0) = f(x) := -N(N+1)\,\text{sech}^2(x),$$

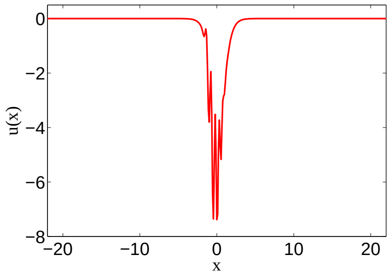


**Fig. 6.2** Numerical solution of the KdV Eq. (6.1) on the interval $t \in [-22, 22]$, using the explicit schema (6.3). Space and time discretization steps are $\triangle x = 0.11$, $\triangle t = 5e-4$, respectively. After several intergation steps numerical instability can be observed.

where $N$ is an amount of solitons and periodic boundary conditions [12]. The first idea is to apply central difference to spatial derivatives on the right hand side and forward difference to the time derivative on the left, as in contrast to the wave equation (4.1) or the sine-Gordon equation (5.1) only the information about initial position of $u$ is known and the artificial point $u_i^{-1}$ can not be calculated. That is, the simple explicit schema reads:

$$\frac{u_i^{j+1} - u_i^j}{\triangle t} = 3\,u_i^j\,\frac{u_{i+1}^j - u_{i-1}^j}{\triangle x} - \frac{u_{i+2}^j - 2\,u_{i+1}^j + 2\,u_{i-1}^j - u_{i-2}^j}{2\,\triangle x^3}\,,$$

or, with $h = \triangle t / \triangle x$

$$u_i^{j+1} = u_i^j + 3\,h\,u_i^j\left(u_{i+1}^j - u_{i-1}^j\right) - \frac{h}{2\,\triangle x^2}\left(u_{i+2}^j - 2\,u_{i+1}^j + 2\,u_{i-1}^j - u_{i-2}^j\right). \qquad (6.3)$$

Since Eq. (6.1) is nonlinear, the direct verification of the stability of the scheme (6.3) with the help of von Neumann analysis (see Sec. 1.3). However, one can examine the stability of the *liner equation*

$$u_t = -u_{xxx}\,. \qquad (6.4)$$

Using the usual ansatz (1.21) the following criterium for Eq. (6.4) can be obtained [12]

$$\triangle t \le \frac{1}{m}\,\triangle x^3\,, \qquad (6.5)$$

where

$$m = \max|\sin(2\,k\,\triangle x) - 2\,\sin(k\,\triangle x)| = \frac{3\sqrt{3}}{2} \simeq 2.6\,.$$

That is, the linear equation (6.4) is conditionally stable, what is not surprising for explicit schemata. However, if we apply the schema (6.3), one can see that after several intergation steps a numerical instability occurs (see Fig. 6.2). That is, the schema (6.3) is unstable and has to be modified.

The first idea is to modify the relation for the time derivative on the right hand side. As was mentioned above, the direct usage of the central difference formula is impossible due to initial condition. On the other hand, the artificial point $u_i^{-1}$ is essential only on the first time step. Hence, on the first time step ($j = 0$) the schema (6.3) can be used, whereas for $j = 1, \dots, T$ the central difference formula is applied:

$$\frac{\partial u}{\partial t} \rightarrow \frac{u_i^{j+1} - u_i^{j-1}}{2 \triangle t} .$$

In addition, we replace $u_i^j$ on the right hand side by the average, namely

$$u_i^j \rightarrow \frac{1}{3}\left(u_{i-1}^j + u_i^j + u_{i+1}^j\right) .$$

That is, the final modified schema reads

$$u_i^{j+1} = u_i^{j-1} + 2h\left(u_{i-1}^j + u_i^j + u_{i+1}^j\right)\left(u_{i+1}^j - u_{i-1}^j\right) - \frac{h}{\triangle x^2}\left(u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j\right) . \tag{6.6}$$

Let us apply the modified schema (6.6) to Eq. (6.1) for the case of two-soliton solution. That is, we solve Eq. (6.1 on the interval $x \in [-L, L]$ according to

| | |
|---|---|
| `Space interval` | $L = 10$ |
| `Space discretization step` | $\triangle x = 0.18$ |
| `Time discretization step` | $\triangle t = 2e-3$ |
| `Amount of time steps` | $T = 1e+5$ |

We start with initial condition

$$u(x,0) = f(x) := -6\,\text{sech}^2(x) ,$$

and apply periodic boundary condition. Notice that in the presented case the linear stability condition (6.5) is fulfilled. The result of calculation is presented on Fig. 6.3. The localized initial condition decomposes into two solitons with different depths and velocities, moving in the same direction. In addition, the solitons collide at some time moment, and the deeper soliton overtank the smaller one.

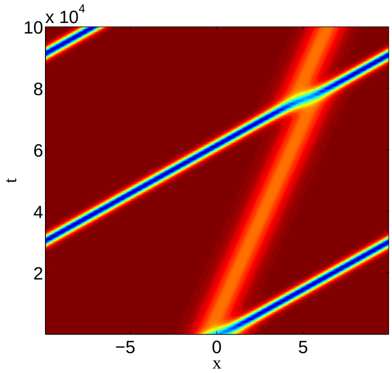


**Fig. 6.3** Space-time representation of the numerical two-soliton solution of the KdV Eq. (6.1) on the interval $t \in [-10, 10]$, using the modified schema (6.6). Space and time discretization steps are $\triangle x = 0.18$, $\triangle t = 2e-5$, respectively.