Einführung in das Standardmodell der Teilchenphysik

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Sheet 10

Hand-in: Wed 12th Juli 2017 (12am)
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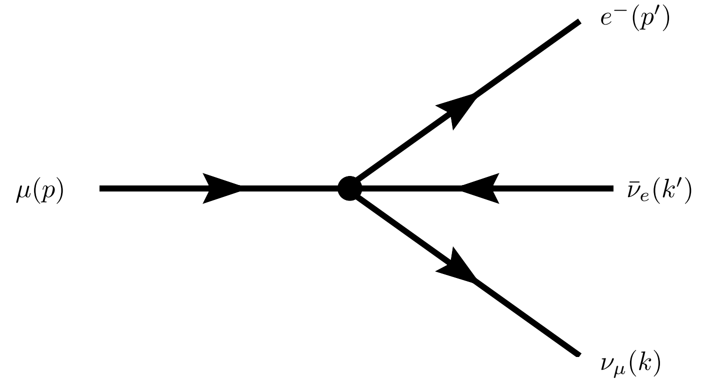
Problem 1:Decay of the Muon in the Fermi theory [20 Points]

Before the rise of the Standard Model and before the discovery of the intermediate $W^{\pm}$ and Z^0 bosons, the weak phenomena were described (rather successfully) by Enrico Fermi using an effective theory where the interactions included 4 fermions. Due to the 4-fermion vertex the theory was not renormalizable but yielded surprisingly consistent results at tree level. The lagrangian relevant for the muon decay[1] includes only an interaction between left-handed components of the fermions which reflects the known violation of parity by the weak interactions.



$$\begin{aligned}\mathcal{L} &= \frac{G_F}{\sqrt{2}} \left(\overline{\nu}_\mu \gamma^\alpha (1-\gamma^5)\mu\right) \left(\overline{e}\,\gamma_\alpha (1-\gamma^5)\nu_e\right) \\ &= \frac{G_F}{\sqrt{2}} \left(\overline{e}\,\gamma^\alpha (1-\gamma^5)\mu\right) \left(\overline{\nu}_\mu\,\gamma_\alpha (1-\gamma^5)\nu_e\right) .\end{aligned}$$

$\overline{\nu}_\mu$, ν_e, μ and $\overline{e}$ indicates the corresponding spinors of the particles. The first line is the so-called charge exchange form of the lagrangian and the following line is the charge retention form which is obtained by the Fierz identity. The process we are going to calculate is a 3-body decay of muon into electron, muon neutrino and electron anti-neutrino, see figure above.

(a) (1 Points) Write down the matrix element $\mathcal{M}$ for the muon decay.

(b) (2 Points) Square the amplitudes summing over the polarizations

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{G_F^2}{2} \operatorname{Tr}\left[(\not{p} + m_\mu)\gamma^\alpha (1-\gamma^5)\not{k}\,\gamma^\beta (1-\gamma^5)\right] \times \operatorname{Tr}\left[\not{k}'\gamma_\alpha (1-\gamma^5)(\not{p}' + m_e)\gamma_\beta (1-\gamma^5)\right] .$$

[1] We will not give the part of the Fermi lagrangian for quarks nor attempt a full treatment of the Fermi theory but we refer for a nice introduction to the book "Quark and Leptons" by F. Halzen and A.D. Martin.

(c) (2 Points) Evaluate the traces:

$$\begin{aligned}\mathrm{Tr}&\left[(\not{p}+m_\mu)\gamma^\alpha(1-\gamma^5)\not{k}\,\gamma^\beta(1-\gamma^5)\right]=\\&=8\left(p^\alpha k^\beta+p^\beta k^\alpha-(p.k)g^{\alpha\beta}+i\varepsilon^{\mu\alpha\nu\beta}p_\mu k_\nu\right)\,,\\\mathrm{Tr}&\left[\not{k}'\gamma_\alpha(1-\gamma^5)(\not{p}'+m_e)\gamma_\beta(1-\gamma^5)\right]=\\&=8\left(k'^\alpha p'^\beta+k'^\beta p'^\alpha-(p'.k')g^{\alpha\beta}+i\varepsilon^{\mu\alpha\nu\beta}k'_\mu p'_\nu\right)\,.\end{aligned}$$

(d) (2 Points) Multiply the traces using the following relation for the ε tensors

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma}=-2\left(\delta^\alpha_\rho\delta^\beta_\sigma-\delta^\alpha_\sigma\delta^\beta_\rho\right)$$

and verify that the final expression for the squared matrix element is

$$\sum_{\text{spins}}|\mathcal{M}|^2=32\,G_F^2\left[4(p.k')(p'.k)\right]\,.$$

(e) (2 Points) Before coming to the phase-space integral, express everything in the reference frame of the decaying muon where the scalar products are

$$\begin{aligned}p.k' &= m_\mu\,\omega',\\ p'.k &= \tfrac{1}{2}m_\mu^2-\tfrac{1}{2}m_e^2-m_\mu\,\omega',\end{aligned}$$

where ω and ω' are the energies of the muon neutrino and electron anti-neutrino respectively.

(f) (3 Points) This squared matrix element should be inserted into the decay width formula

$$\Gamma=\frac{1}{2m_\mu}\int\left(\prod_f\frac{\mathrm{d}^3k_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}|^2(2\pi)^4\delta^{(4)}\left(p_A-\sum k_f\right)\,.$$

Use the hint to derive

$$\Gamma=\frac{G_F^2}{2m_\mu\pi^3}\int\mathrm{d}\omega'\,\mathrm{d}E'_p\,m_\mu\,\omega'\left(m_\mu^2-m_e^2-2m_\mu\,\omega'\right)\,,$$

with the integration boundaries

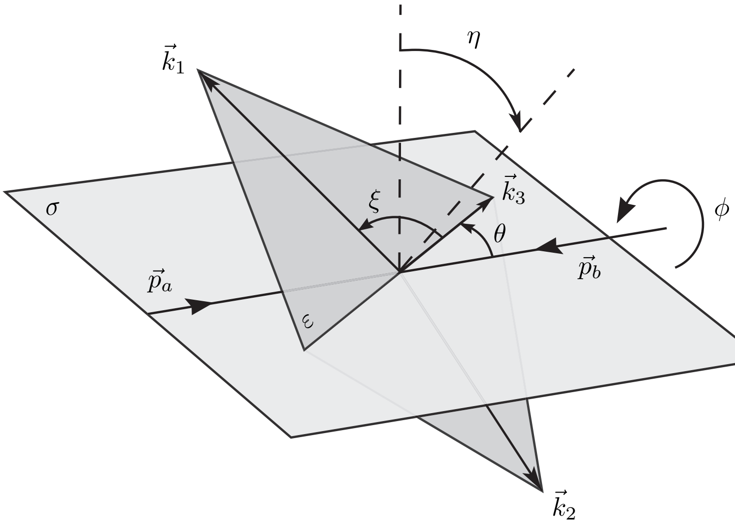
$$\begin{aligned}\omega' &\in \left(\tfrac{1}{2}\left(m_\mu-E'_p-\sqrt{E'^2_p-m_e^2}\right),\tfrac{1}{2}\left(m_\mu-E'_p+\sqrt{E'^2_p-m_e^2}\right)\right),\\ E'_p &\in \left(m_e\,,\frac{m_\mu}{2}+\frac{m_e^2}{2m_\mu}\right).\end{aligned}$$

HINT: The phase-space element is

$$\left(\prod_f\frac{\mathrm{d}^3k_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\left(p_A-\sum k_f\right)=\frac{1}{8}\frac{1}{(2\pi)^5}\,\mathrm{d}k_1^0\,\mathrm{d}\eta\,\mathrm{d}k_3^0\,\mathrm{d}\cos\theta\,\mathrm{d}\phi\,,$$

where k_1, k_2 and k_3 are the final state 4-momenta and the angles η, θ and ϕ are all defined in the figure below. The difference between the $1\to3$ decay and $2\to3$ process is that the decay matrix element in the rest frame of the decaying particle does not depend on any of the angles η, θ and ϕ but only on the energies k_1^0, k_3^0 and the relative angle between the 3-momenta ξ which is fixed by 4-momentum conservation to

$$\cos\xi=\frac{(\sqrt{s}-k_1^0-k_3^0)^2+m_1^2+m_3^2-m_2^2-(k_1^0)^2-(k_3^0)^2}{2\sqrt{(k_1^0)^2-m_1^2}\,\sqrt{(k_3^0)^2-m_3^2}}\,.$$



On top of that the 4-momentum conservation determines the integration limits on the energies (which in this case are the only non-trivial integration parameters).

$$
\begin{aligned}
(k_3^0)^{\min} &= m_3\,, \qquad\qquad (k_3^0)^{\max} = \frac{\sqrt{s}}{2} - \frac{(m_1+m_2)^2 - m_3^2}{2\sqrt{s}}\,,\\
(k_1^0)^{\max}_{\min} &= \frac{1}{2\tau}\left[\sigma\,(\tau + m_+ m_-) \pm |\vec{k}_3|\,\sqrt{(\tau - m_+^2)(\tau - m_-^2)}\right]\,,
\end{aligned}
$$

with

$$
\sigma = \sqrt{s} - k_3^0\,, \qquad \tau = \sigma^2 - |\vec{k}_3|^2\,, \qquad m_\pm = m_1 \pm m_2\,.
$$

For the decay we use $s = m_\mu^2$ and the 4-momenta and the masses are set as follows

$$
\begin{aligned}
k_1 &= k', & m_1 &= 0,\\
k_2 &= k, & m_2 &= 0,\\
k_3 &= p', & m_3 &= m_e.
\end{aligned}
$$

(g) (3 Points) Perform the integration over ω', define $r = m_e/m_\mu$

$$
\Gamma = \frac{G_F^2}{2 m_\mu \pi^3} \int \mathrm{d}E_p' \,\frac{m_\mu^2}{6} \sqrt{E_p'^2 - m_\mu^2 r^2}\,\left(-4E_p'^2 - 2m_\mu^2 r^2 + 3E_p' m_\mu (1+r^2)\right)\,.
$$

(h) (3 Points) Use the following integrals

$$
\begin{aligned}
\int \sqrt{x^2 - a^2}\mathrm{d}x &= \frac{1}{2}\,x\sqrt{x^2-a^2} - \frac{1}{2}\,a^2 \log\left(x + \sqrt{x^2-a^2}\right)\,,\\
\int x\,\sqrt{x^2 - a^2}\mathrm{d}x &= \frac{1}{3}\left(x^2 - a^2\right)^{3/2}\,,\\
\int x^2\,\sqrt{x^2 - a^2}\mathrm{d}x &= \sqrt{x^2-a^2}\left(\frac{x^3}{4} - \frac{a^2 x}{8}\right) - \frac{1}{8}\,a^4 \log\left(x + \sqrt{x^2-a^2}\right)\,,
\end{aligned}
$$

to perform the integration over the energy of the electron anti-neutrino. Set $t = r^2$ to derive:

$$
\frac{1}{\tau_\mu} = \Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - 8t - 12t^2 \log t + 8t^3 - t^4\right)\,.
$$

(i) (2 Points) Using this formula, convert the lifetime of the muon

$$\tau_\mu = 2.197019 \pm 0.000021\,\mu s,$$

into a value for the Fermi coupling constant G_F. The error above is only the experimental one. There is a theoretical error coming from not including higher-order corrections into the conversion formula. This error turns out to be significantly bigger than the experimental one. This is the motivation for calculating loop corrections to the muon decay.