Einführung in das Standardmodell der Teilchenphysik

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Sheet 6

Hand-in: Wed 14th June 2017
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Problem 1: Useful $SU(N)$ identities

Prove the following identities involving color.

(a) (1 Point) The gluons transform in the adjoint representation of the $SU(N)$ where the generators are given by

$$\left(T_A^a\right)_{bc} = -if^{abc}\,.$$

Using the Jacobi identity

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0\,,$$

prove that the generators of the adjoint representation satisfy the standard commutation relation

$$\left[\,T_A^a, T_A^b\,\right] = if^{abc}\,T_A^c\,.$$

(b) (1 Point) Show that the infinitesimal transformation rule in case of a local $SU(N)$ symmetry for the gluon field A_μ^a is

$$A_\mu^a \mapsto A_\mu^a - f^{abc}\alpha^b A_\mu^c - \frac{1}{g_s}\,\partial_\mu \alpha^a\,,$$

which shows that A_μ^a indeed transforms in the adjoint representation of $SU(N)$.

(c) (2 Points) Use the decomposition of a general $N \times N$ matrix M in terms of the generators of the fundamental representation as

$$M_{ij} = \frac{1}{N}\,\mathrm{Tr}\,M\,\delta_{ij} + 2\,\mathrm{Tr}\big(M\,T_F^a\big)\,\big(T_F^a\big)_{ij}\,,$$

to prove that

$$if^{abc} = 2\,\mathrm{Tr}\big(T_F^c\,\left[\,T_F^a, T_F^b\,\right]\big)\,.$$

(d) (1 Point) Using the same relation as in part (c) prove the Fierz color identity

$$\delta_{ij}\delta_{kl} = \frac{1}{N}\,\delta_{ik}\delta_{lj} + 2\big(T_F^a\big)_{ik}\big(T_F^a\big)_{lj}\,.$$

(e) (2 Points) Using the relation

$$[T^a,[T^b,T^c]] = \{T^c,\{T^a,T^b\}\} - \{T^b,\{T^c,T^a\}\}$$

show that

$$f^{abc}f^{dec} = \frac{2}{N}(\delta^{ad}\delta^{be} - \delta^{ae}\delta^{bd}) + (d^{bce}d^{dac} - d^{bdc}d^{ace}).$$

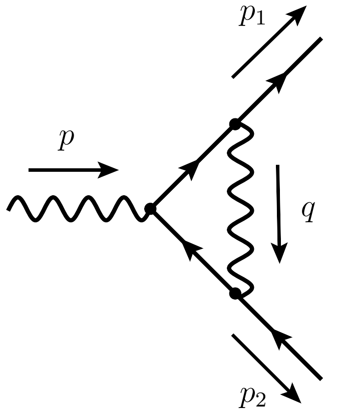
Show that in the familiar case of SU(2) this leads to

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Problem 2: Vertex correction in massless QED and QCD

Consider the only vertex correction diagram in the QED (and one of the most important vertex corrections in QCD).

(a) (4 Points) The amplitude in massless QED can be parametrized by a 4-vector Γ^μ and it can be written explicitly as



$$\mathcal{M} = -ie\,\mu^{\frac{4-D}{2}}\,\bar{u}(p_1)\,\Gamma^\mu\,v(p_2)$$
$$= -\left(e\,\mu^{\frac{4-D}{2}}\right)^3 \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\bar{u}(p_1)\,\gamma^\nu(\not{q}+\not{p}_1)\gamma^\mu(\not{q}-\not{p}_2)\gamma_\nu\,v(p_2)}{[q^2+i\varepsilon][(q+p_1)^2+i\varepsilon][(q-p_2)^2+i\varepsilon]}$$

Show that the amplitude can be written as

$$\bar{u}(p_1)\,\Gamma^\mu\,v(p_2) = -ie^2\mu^{4-D}\int_0^1 \mathrm{d}x \int_0^{1-x}\mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\bar{u}(p_1)\,N^\mu(q,p_1,p_2)\,v(p_2)}{((q+xp_1-yp_2)^2+sxy+i\varepsilon)^3}$$

where we have introduced the shorthand notation $s = 2p_1.p_2$ and $N^\mu = 2[(D-2)q^2+4q.p_1-4q.p_2-2s]\gamma^\mu - 4[(D-2)q^\mu+2p_1^\mu-2p_2^\mu]\,\not{q}$. In order to arrive at the expression above one has to use the Feynman parametrization trick for three propagators (Prove!)

$$\frac{1}{abc} = 2\int_0^1 \mathrm{d}x \int_0^{1-x}\mathrm{d}y\ \{a(1-x-y)+bx+cy\}^{-3}$$

(b) (2 Points) Perform a shift in the integration variable $q^\mu \to q^\mu - xp_1 + yp_2$ and use the fact that

$$\int \mathrm{d}^D q \frac{q^\mu q^\nu}{(q^2-A+i\varepsilon)^n} = \frac{1}{D}\,g^{\mu\nu}\int \mathrm{d}^D q\frac{q^2}{(q^2-A+i\varepsilon)^n},$$

to show that

$$\Gamma^\mu = -2ie^2\mu^{4-D}\gamma^\mu \int \mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\frac{(D-2)^2}{D}\,q^2 + s\,((2-D)xy+2x+2y-2)}{(q^2+s\,xy+i\varepsilon)^3}.$$

(c) (3 Points) Show that the first q^2-proportional part of the amplitude is

$$\Gamma_1^\mu = -2ie^2\mu^{4-D}\gamma^\mu\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\frac{(D-2)^2}{D}\,q^2}{(q^2+s\,xy+i\varepsilon)^3}$$
$$= \frac{e^2}{(4\pi)^2}\gamma^\mu\left(\frac{\mu^2}{-s}\right)^\varepsilon\left[\frac{c_\varepsilon}{\varepsilon}+1\right]$$

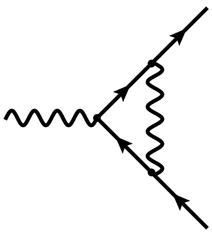
This ε-pole represents the ultraviolet divergence because the integral converges for $\varepsilon > 0$.

(d) (3 Points) Show that the remaining part of the amplitude is

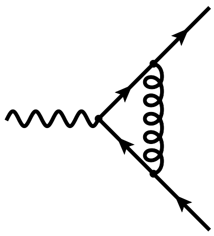
$$\begin{aligned}
\Gamma_2^\mu &= -2ie^2\mu^{4-D}\gamma^\mu \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{s\left((2-D)xy + 2x + 2y - 2\right)}{(q^2 + s\,xy + i\varepsilon)^3} \\
&= 2\frac{e^2}{(4\pi)^2}\gamma^\mu \left(\frac{\mu^2}{-s}\right)^\varepsilon c_\varepsilon \left[-\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} - \frac{54}{12} + \frac{\pi^2}{6}\right],
\end{aligned}$$

where this time the divergence doesn't come from the integral over the loop momentum but from the integrals over x and y. The divergence is of infrared origin as the integral would converge if $\varepsilon < 0$.

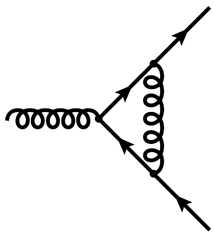
(e) (3 Points) If we would denote the amplitude of the QED vertex diagram as $\mathcal{M} = e^3\mathcal{M}_0$, what would the constants c_1 and c_2 be for the two possible QCD diagrams?



$\mathcal{M} = e^3\mathcal{M}_0$



$\mathcal{M} = c_1\mathcal{M}_0$



$\mathcal{M} = c_2\mathcal{M}_0$