Einführung in das Standardmodell der Teilchenphysik

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Sheet 5

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This problem set deals with the 1-loop renormalization of the electric charge in QED. This calculation is more involved than the calculation we did in scalar ϕ^4 theory, as there are more divergent diagrams and two of them lead to wave-function renormalization. However, we will make use of gauge invariance to show that we have to compute only one divergent diagram to determine the QED β-function.

Problem 1: Divergences in QED

(a) (2 Points) Use power counting to find all four divergent diagrams in QED to 1-loop order. *Hint: Diagrams with an odd number of external Photons always vanish due to charge conjugation symmetry ('Furry's theorem').*

(b) (1 Point) Show that the 1-loop photon four-point function diverges like

$$\mathcal{M} = c \log \Lambda (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \epsilon^\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma + \text{finite}$$

Problem 2: Ward identity

(a) (1 Point) The mode expansion of the photon field reads

$$A_\mu = \int \frac{d^4 p}{(2\pi)^4} \epsilon_\mu(p) e^{-ipx} \ .$$

Show that under a gauge transformation $A_\mu \to A'_\mu + \frac{1}{e}\partial_\mu \alpha$ the polarization vector transforms as

$$\epsilon(p) \to \epsilon'(p) = \epsilon(p) - \frac{i}{e} p_\mu \tilde{\alpha} \ .$$

(b) (2 Points) A matrix element of a process involving N external photons can be split up like

$$\mathcal{M} = \epsilon_{\mu_1} \ldots \epsilon_{\mu_N} \mathcal{M}^{\mu_1 \cdots \mu_N} \ .$$

Show that $\mathcal{M}^{\mu_1 \cdots \mu_N}$ is gauge invariant.

(c) (2 Points) Again use only gauge invariance to show that

$$p_{\mu_i} \mathcal{M}^{\mu_1 \cdots \mu_N} = 0 \ .$$

This result is called the 'Ward identity' and it is extremely useful to simplify a lot of calculations of QED diagrams.

(d) (1 Point) Apply the Ward identity to the result from (1b) show that the photon four point function has to be finite.

Thus, we have shown that there are only three divergent diagrams in QED to 1-loop order.

Problem 3: QED β-function

Suppose we have calculated all divergent diagrams in QED to 1-loop order. When we add counter terms to cancel all divergences the renormalized QED Lagrangian will look like

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi + e\bar{\psi}A\!\!\!/\psi \\ &-\frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} + i\delta_2\bar{\psi}\partial\!\!\!/\psi - \delta_m m\bar{\psi}\psi + \delta_e e\bar{\psi}A\!\!\!/\psi\ ,\end{aligned}$$

or in terms of the multiplicative renormalization constants

$$\mathcal{L} = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - Z_m m_R\bar{\psi}\psi + e_R\mu^{\epsilon/2}Z_e\bar{\psi}A\!\!\!/\psi\ ,$$

where we have inserted a regularization scale μ with $e = e_R\mu^{\epsilon/2}$, so that e_R will be dimensionless. This is necessary for dimensional regularization.

(a) (3 Points) Show, that gauge invariance of the Lagrangian implies that $Z_e = Z_2$ and that the connection of the renormalized coupling e_R and the bare coupling e_0 is given by

$$e_0 = e_R\mu^{\epsilon/2}Z_3^{-1/2}$$

We have now shown that from initially four (naively) diverging diagrams only one contributes to the renormalization of the electric charge. The contributing diagram is the one corresponding to the counter term canceling the divergence in the photon propagator. Let us now compute this divergence, i.e. the vacuum polarization.

(b) (6 Points) Retrace the derivation of the vacuum polarization done in the lectures using dimensional regularization , finding

$$i\Pi_{\mu\nu} = -\frac{ie_r^2}{2\pi^2}(g_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \log\frac{Q^2}{4\pi\mu^2} - \gamma - \frac{1}{2}\right)\ .$$

And show:

$$Z_3 = 1 - \frac{e_R^2}{6\pi^2\epsilon}\ .$$

Hint: Trace relations for the gamma matrices in n dimensions are: $\mathrm{Tr}\,1 = n$, $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = ng^{\mu\nu}$, $\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho = 0$, $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = n(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.

(c) (2 Points) Since the regularization scale μ is completely arbitrary, the bare couping e_0 is independent of μ. Use this to show that the β-function of QED to order $\mathcal{O}(e_R^3)$ is

$$\beta(e_R) = \mu\frac{\partial e_R}{\partial\mu} = \frac{e_R^3}{12\pi^2}\ .$$