Einführung in das Standardmodell der Teilchenphysik

Lectures: Prof. M. Klasen & Prof. D. Frekers
Exercises: Dr. K. Kovařík, Dr. P. Scior, S. Schmiemann

Sheet 4

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Postfach von S. Schmiemann oder P. Scior KP306

Problem 1: Renormalization

Let us discuss the renormalization of scalar ϕ^4 theory described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi\phi - \frac{\lambda}{4!}\phi^4 \ .$$

We do this in the most basic way, meaning that we regularize divergent integrals by introducing a UV-cutoff Λ and analyze the behavior of divergent integrals in terms of their asymptotic scaling with Λ.

(a) (2 Points) Determine the two divergent Feynman diagrams at 1-loop order.

(b) (2 Points) Use power counting, i.e. compare powers of momenta in the denominator and numerator (for more details, see e.g. Peskin-Schroeder p. 315-322) to argue that the diagrams diverge like Λ^2 and $\log\Lambda$ when $\Lambda \to \infty$.

(c) (3 Points) Show, in the limit of vanishing incoming and outgoing momentum, that the divergent part of the diagram contributing to the four point function is given by

$$\frac{i3\lambda^2}{32\pi^2}\log\frac{\Lambda^2}{m^2} \ .$$

Hint: To perform the 4-dimensional integral: switch to Euclidean space, i.e. continue analytically to $p_0 \to ip_{0E} \Rightarrow d^4p \to id^4p_E$ and $p^2 = p_\mu p^\mu \to -p_E^2 = -p_0p_0 - \vec{p}^2$. You do not need the $+i\epsilon$ term in the propagators. Use:

$$\int du \frac{u}{(u+m^2)^2} = \frac{m^2}{u+m^2} + \log|u+m^2| \ . \tag{1}$$

For large momentum transfer $q \gg m$, this diagram diverges like

$$\frac{3i\lambda^2}{32\pi^2}\log\frac{\Lambda^2}{q^2} \ .$$

The invariant matrix element for two body scattering in 1-loop order is the sum of the tree-level and the one-loop diagrams contributing to this process. It is given by

$$\mathcal{M} = -i\lambda + \frac{3i\lambda^2}{16\pi^2}\log\left(\frac{\Lambda}{q}\right) \ .$$

Renormalizing the theory now means that we have to choose our bare parameter λ in such a way that $\mathcal{M}$ is finite and independent of Λ, i.e. that the physics is unchanged when changing the

UV-cutoff.
Now let us assume we can measure $\mathcal{M}$ in an experiment (this can be done e.g. by scattering two pions of each other). In the experiment we will have some incoming momentum μ and we use the measured scattering amplitude to define a physical (or renormalized) coupling λ_R:

$$\mathcal{M} = -i\lambda_R = -i\lambda + \frac{3i\lambda^2}{16\pi^2}\log\left(\frac{\Lambda}{\mu}\right) .$$

We can also get the same relation between the renormalized coupling λ_R and the bare coupling λ by replacing all λ by λ_R in the Lagrangian and adding a term $-\frac{C}{4!}\phi^4$ to the Lagrangian that cancels the divergence from the four-point function. This is called adding a counter term.

(d) (4 Points) What is the value of C? Show that by adding the counter term you get the relation (to order $\mathcal{O}(\lambda^2)$)

$$\lambda_R = -\lambda + \frac{3\lambda^2}{16\pi^2}\log\left(\frac{\Lambda}{q}\right) , \tag{2}$$

where the bare coupling λ is now divergent.

(e) (5 Points) Show that the scattering amplitude can be written as

$$\mathcal{M} = -i\lambda_R + \frac{3i\lambda_R^2}{16\pi^2}\log\left(\frac{\mu}{q}\right) + \mathcal{O}(\lambda_R^3) .$$

By introducing the renormalized coupling λ_R we have made the scattering amplitude independent of the UV-cutoff and we can safely take $\Lambda \to \infty$. However, we have traded the UV-cutoff against a momentum scale μ where we have determined λ_R by experiment. This scale μ is of course arbitrary, i.e. we can measure the scattering amplitude at any momentum scale we want.
Again, we demand physics to be independent from our arbitrary choice of μ, i.e.

$$\mu\frac{\partial\mathcal{M}}{\partial\mu} = 0 ,$$

where we included an additional factor of μ to have a dimensionless derivative. The requirement results in the fact that $\lambda_R(\mu)$ will depend on the scale μ. This dependence is described by the so called β-function of the theory.

(f) (3 Points) Show that in our case we have

$$\beta(\lambda_R) = \mu\frac{\partial\lambda_R}{\partial\mu} = \frac{3\lambda_R^2}{16\pi^2} + \mathcal{O}(\lambda_R^3) .$$

(g) (3 Points) determine the running coupling $\lambda_R(\mu)$.