Einführung in das Standardmodell der Teilchenphysik

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Sheet 3

Hand-in: Wed 10th May 2017 (12am)

Problem 1: Magnetic moment of a spin-1/2 particle (4 Points)

In scalar quantum electrodynamics the Klein-Gordon equation is just

$$(D_\mu D^\mu + m^2)\varphi = 0$$

where the covariant derivative is $D_\mu = \partial_\mu + ieA_\mu$. Spin-$\frac{1}{2}$ particles obey the Dirac equation

$$(iD_\mu \gamma^\mu - m)\psi = 0\,.$$

In the non-relativistic limit, the Dirac equation produces the following Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + V(r) + \frac{e}{2m}\,\vec{B}.(\vec{L} + g\vec{S}) \tag{1}$$

where the factor g represents the relative strength of the intrinsic magnetic moment to the strength of the spin-orbit coupling and is predicted by the Dirac equation to be $g = 2$.

(a) (1 Point) Show that spin-$\frac{1}{2}$ particles satisfy also a modified Klein-Gordon equation with an additional term

$$(D_\mu D^\mu + m^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu})\psi = 0$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

(b) (2 Points) The additional term stands for extra effects of spin-$\frac{1}{2}$ over scalar particles namely for magnetic field interaction with the spin of the particle. Show that the extra term contains the term $\vec{B}.\vec{S}$.

Hint: Work in the Weyl representation and recall that $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k$.

(c) (1 Point) Interaction of a photon with a fermion in momentum space is given by

$$M^\mu = -ie\bar{u}(p_2)\gamma^\mu u(p_1)\,.$$

Show the Gordon identity for on-shell spinors

$$M^\mu = -ie\bar{u}(p_2)\gamma^\mu u(p_1) = -ie\bar{u}(p_2)\left[\frac{(p_1+p_2)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_2-p_1)_\nu}{2m}\right]u(p_1)$$

where the first part stands for the interaction of electric field with the point charge and the second part is the interaction of the magnetic field with the spin.

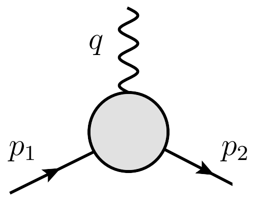
Problem 2: Anomalous magnetic moment of the muon (or electron) (16 Points)

Although the Dirac equation predicts for every fermion $g = 2$, experimental data point toward small deviations from this prediction. The experimental values for electron and muon are

$$g_e = 2.00231930436146 \qquad g_\mu = 2.00233184178$$

where the small differences are due to the different mass of the electron and the muon. The deviations from $g = 2$ are due to quantum corrections.

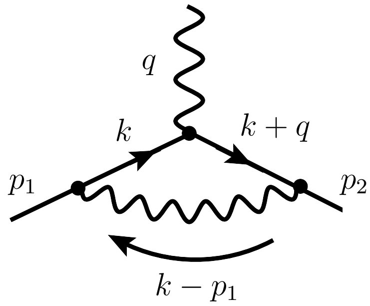
(a) (1 Point) Show that the general structure of an interaction of a photon with a fermion can be written in the following form



$$= -ie\bar{u}(p_2)\Big[F_1(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\Big]u(p_1)\,.$$

Hint: Consider all possible Lorentz structures admissible and apply momentum conservation, QED Ward identity, etc...

(b) (5 Points) Consider now the simplest quantum correction in QED, the diagram where an additional photon is exchanged between the incoming and the outgoing fermion.



The amplitude of such a diagram can be written as

$$\mathcal{M} = -e^3\epsilon_\mu(q)\bar{u}(p_2)\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(\not{q}+\not{k}+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{[(k-p_1)^2+i\epsilon][(q+k)^2-m^2+i\epsilon][k^2-m^2+i\epsilon]}u(p_1)\,.$$

Using the Feynman parameterization trick

$$\frac{1}{ABC} = 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{1}{[xA+yB+zC]^3}$$

with

$$A = k^2 - m^2 + i\epsilon \qquad B = (q+k)^2 - m^2 + i\epsilon \qquad C = (k-p_1)^2 + i\epsilon$$

and shifting the integration variable $k^\mu \mapsto k^\mu - yq^\mu + zp_1^\mu$ rewrite the amplitude as

$$\mathcal{M} = -2e^3\epsilon_\mu(q)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4k}{(2\pi)^4}\frac{N^\mu(k)}{(k^2-\Delta)^3}$$

where

$$\Delta = -xyq^2 + (1-z)^2m^2$$

and the numerator has the following structure

$$-\frac{1}{2}N^\mu = \bar{u}(p_2)\Big[(\not{k} - y\,\not{q} + z\,\not{p}_1)\gamma^\mu\,\not{q} + (\not{k} - y\,\not{q} + z\,\not{p}_1)\gamma^\mu(\not{k} - y\,\not{q} + z\,\not{p}_1)\Big]u(p_1)$$
$$+\bar{u}(p_2)\Big[m^2\gamma^\mu - 2m(2k^\mu - 2yq^\mu + 2zp_1^\mu + q^\mu)\Big]u(p_1)\,.$$

(c) (5 Points) Using $k^\mu k^\nu = \frac{1}{4} g^{\mu\nu} k^2$, the Gordon identity, the Ward identity, $x + y + z = 1$ and a lot of algebra (and e.g. the fact that terms linear in k lead to an integral which is zero for a symmetry reason), simplify the numerator N^μ to

$$-\frac{1}{2} N^\mu = \Big[-\frac{k^2}{2} + (1-x)(1-y)q^2 + (1 - 4z + z^2)m^2 \Big] \bar{u}(p_2)\gamma^\mu u(p_1) + imz(1-z)\bar{u}(p_2)\sigma^{\mu\nu} q_\nu u(p_1)$$
$$+ m(z-2)(x-y)\bar{u}(p_2) q^\nu u(p_1) \,.$$

(d) (1 Point) The first term in the numerator contributes to $F_1(q^2)$ and the last term vanishes due to the symmetry of the denominator $x \leftrightarrow y$. The term proportional to $\sigma^{\mu\nu} q_\nu$ is the only one that contributes to the anomalous magnetic moment. Show that its contribution is

$$F_2(q^2) = \frac{2m}{e}(4ie^3 m) \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \int \frac{d^4k}{(2\pi)^4} \frac{z(1-z)}{(k^2 - \Delta + i\epsilon)^3} \,.$$

(e) (4 Points) Applying an identity

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^3} = \frac{-i}{32\pi^2 \Delta}$$

simplify the $F_2(q^2)$ to

$$F_2(q^2) = \frac{\alpha}{\pi} m^2 \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{z(1-z)}{(1-z)^2 m^2 - xyq^2} \,.$$

Finally, evaluate $F_2(q^2 = 0)$ in order to obtain

$$F_2(0) = \frac{\alpha}{2\pi}$$

which leads to the final result

$$g = 2 + \frac{\alpha}{\pi} = 2.00232 \,.$$

We see that this result does not depend on the mass of the fermion and therefore it is simultaneously a correction to the magnetic moment of the electron and of the muon.