# Einführung in das Standardmodell der Teilchenphysik

**Lectures:** Prof. M. Klasen & Prof. D. Frekers

**Exercises:** Dr. K. Kovařík, Dr. P. Scior, S. Schmiemann

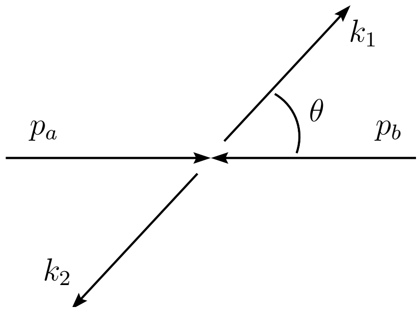
## Sheet 1

**Hand-in:** Wed 26th April 2017 (12am)

Postfach von S. Schmiemann oder P. Scior KP306

### Problem 1: Mandelstam variables

The center-of-mass system (CMS) is defined as the system where the sum of the incoming (and also the outgoing) 3-momenta vanishes. The great advantage of this reference frame is that the amount of energy delivered to the collision by the incoming particles in the CMS is exactly the amount that can directly be used to create new particles as the 3-momentum conservation doesn't require the final state particles to have any 3-momentum.




The Mandelstam variables are a way to describe the kinematics of a scattering process in an explicitly Lorentz invariant way. The Mandelstam variables for a $2 \to 2$ scattering process are defined in terms of the 4-momenta of the incoming and outgoing particles as

$$\begin{aligned} s &= (p_a + p_b)^2 = (k_1 + k_2)^2 \\ t &= (p_a - k_1)^2 = (p_b - k_2)^2 \\ u &= (p_a - k_2)^2 = (p_b - k_1)^2. \end{aligned}$$

(a) Show that the Mandelstam variables are connected through

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2.$$

which means that in the case of a $2 \to 2$ scattering process, there are only two independent Mandelstam variables.

(b) Consider now the process $e^- e^+ \to e^- e^+$ and verify that:

$$\begin{aligned} s &= \quad 4(k^2 + m^2), \\ t &= -2k^2(1 - \cos\theta), \\ u &= -2k^2(1 + \cos\theta), \end{aligned}$$

where $\theta$ is the center-of-mass scattering angle and $k = |p_a| = |p_b| = |k_1| = |k_2|$ is the common 3-momentum of incident and scattered electrons/positrons in the center-of-mass frame.

## Problem 2: Kinematics in the Center-of-Mass System

All calculations in collider physics result in predictions for particle decay widths $d\Gamma$ or cross-sections $d\sigma$. The cross-section is calculated using the relation

$$d\sigma = \frac{1}{F}\, |\mathcal{M}|^2 \, \mathrm{dPS}_n \,,$$

where $F$ is the flux factor, $\mathrm{dPS}_n$ stands for the phase-space integrals and $|\mathcal{M}|^2$ is the probability matrix element. The flux factor for incident particle beams along the $z$-axis is given by

$$F = 4E_aE_b|v_a^z - v_b^z| = 4|E_bp_a^z - E_ap_b^z| = 4\sqrt{(p_a\cdot p_b)^2 - m_a^2m_b^2}\,,$$

where $E_a, E_b, p_a$ and $p_b$ are the energies and momenta of the two incoming particles.

(a) Show that the flux can be rewritten using only Lorentz invariants as

$$F = 2\sqrt{\lambda(s, m_a^2, m_b^2)}\,.$$

where $\lambda(x, y, z)$ is the Källeén function defined as

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = (x - (\sqrt{y} - \sqrt{z})^2)(x - (\sqrt{y} - \sqrt{z})^2)\,.$$

(b) Show that a general $2 \to 2$ collision process can only take place if

$$s \geq (m_1 + m_2)^2\,,$$

where $m_1$ and $m_2$ are the masses of the final-state particles.

(c) Consider now the process $e^-e^+ \to e^-e^+$. Show that the process is physically allowed when $s \geq 4m^2$, $t \leq 0$ and $u \leq 0$. Moreover, determine the value of $t$ and $u$ for foreward and backward scattering.

(d) Show that in CMS, the energies and the 3-momenta of all particles in the $2 \to 2$ process can be expressed using just the masses and the Mandelstam variable $s$ as

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}\,, \qquad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}\,, \qquad |\vec{p_a}| = |\vec{p_b}| = \frac{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)}{2\sqrt{s}}\,,$$

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\,, \qquad E_2 = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}}\,, \qquad |\vec{k_1}| = |\vec{k_2}| = \frac{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{2\sqrt{s}}\,.$$

## Problem 3: Phase-space integration

The crucial part of the calculation of a cross-section $\sigma$ is the integration over the phase-space of the final-state particles. The Lorentz invariant phase-space integration element for two particles in the final state is given as

$$\mathrm{dPS}_2 = \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}\,(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_a - p_b)\,.$$

(a) Use the $\delta$-function and the previous results to rewrite the expression as

$$\mathrm{dPS}_2 = \frac{1}{16\pi^2}\frac{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{2s}\, d\cos\theta\, d\phi.$$

*Hint: In order to apply the $\delta$-function, transform the integration over the 3-momentum in the formula for* $\mathrm{dPS}_2$ *to an integration over 4-momenta*

$$\frac{\mathrm{d}^3 k_2}{2E_2} = \mathrm{d}^4 k_2\, \delta\big(k_2^2 - m_2^2\big) \ .$$

*and perform the integration over one 4-momentum. This should lead to*

$$\mathrm{dPS}_2 = \frac{1}{2E_1}\frac{d^3 k_1}{(2\pi)^3}\,(2\pi)\,\delta\big((p_a + p_b - k_1)^2 - m_2^2\big).$$

*Choose $|\vec{k}_1|$, $\theta$ and $\phi$ as the integration variables when integrating over the three components of the 3-momentum $\vec{k}_1$. The $z$-axis is conventionally chosen to be the direction of the 3-momentum $p_a$. Change integration variables and use energy $E_1$ instead of the momentum $|\vec{k}_1|$*

$$\int \mathrm{dPS}_2 = \int \frac{|\vec{k}_1|}{2}\frac{dE_1\, d\cos\theta\, d\phi}{(2\pi)^2}\,\delta((p_a + p_b - k_1)^2 - m_2^2).$$

(b) Rewrite the expression from (a) using Lorentz invariant variables by substituting the integration over the Mandelstam variable $t$ for the integration over $\cos\theta$

$$\int \mathrm{dPS}_2 = \int_{t_-}^{t_+} \frac{1}{16\pi^2}\frac{dt\, d\phi}{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)} \ .$$

Verify that the integration boundaries are

$$t_\pm = m_a^2 + m_1^2 - \frac{(s + m_a^2 - m_b^2)(s + m_1^2 - m_2^2)}{2s} \pm \frac{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{2s} \ .$$