3.2.3 Running coupling (26.6)

Mittwoch, 31. Mai 2017
08:33

Renormalised Lagrangian for quark-gluon interaction:

$$\mathcal{L} = \mu^{\frac{4-d}{2}} g_R Z_1 A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j = \mu^{\frac{4-d}{2}} g_R \frac{Z_1}{Z_2\sqrt{Z_3}} A^{(0)}_\mu \bar{\psi}^{(0)}_i \gamma^\mu T^a_{ij} \psi^{(0)}_j$$

Bare coupling:

$$g_0 = g_R \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-d}{2}}$$

Must be μ-independent, since there was no μ in the unrenormalised Lagrangian, i.e.:

$$0 = \mu \frac{d}{d\mu} g_0 = \mu \frac{d}{d\mu}\left[g_R \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-d}{2}} \right] = g_R \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{\varepsilon}{2}} \left[\frac{\varepsilon}{2} + \frac{\mu}{g_R} \frac{d}{d\mu} g_R + \frac{\mu}{Z_1/(Z_2\sqrt{Z_3})} \frac{d}{d\mu} \frac{Z_1}{Z_2\sqrt{Z_3}} \right]$$

Expanding perturbatively and counting the δ_i as $O(g_R^2)$, this gives

$$\boxed{\beta(g_R) := \mu \frac{d}{d\mu} g_R} = g_R \left[\left(-\frac{\varepsilon}{2}\right) - \mu \frac{d}{d\mu}\left(\delta_1 - \delta_2 - \frac{\delta_3}{2} \right) \right] + \ldots$$

Solving this iteratively with $\mu \frac{d\delta_i[g_R(\mu)]}{d\mu} = \mu \frac{\partial \delta_i}{\partial g_R} \frac{\partial g_R}{\partial \mu} = \frac{d\delta_i}{dg_R} g_R \left(-\frac{\varepsilon}{2}\right) + \ldots,$

we have:

$$\beta(g_R) = -\frac{\varepsilon}{2} g_R + \frac{\varepsilon}{2} g_R^2 \frac{\partial}{\partial g_R}\left(\delta_1 - \delta_2 - \frac{\delta_3}{2} \right)$$

where one usually sets $\varepsilon \to 0$.

Using the one-loop counterterms above, we find:

$$\boxed{\beta(g_R) = -\frac{g_R^3}{16\pi^2}\left[\frac{11}{3} C_A - \frac{4}{3} n_f T_f \right] =: -\frac{g_R^3}{16\pi^2} \beta_0}$$

Note that the ξ- (gauge-) dependence has canceled.

Renormalised Lagrangian for the triple-gluon interaction:

$$\boxed{\beta(g_R) = -\frac{\varepsilon}{2} g_R + \frac{\varepsilon}{2} g_R^2 \frac{\partial}{\partial g_R}\left(\delta_{A^3} - \frac{3}{2}\delta_3 \right)}$$

leads to the same answer, as it must due to gauge invariance.

Strong "fine structure constant":

$$\mu \frac{d}{d\mu} \alpha_s = \mu \frac{d}{d\mu}\left(\frac{g_s^2}{4\pi} \right) = \frac{g_s}{2\pi} \beta(\alpha_s) = -\frac{\alpha_s^2}{2\pi} \beta_0$$

Since $n_f = 6 < 17$, $\beta_0 = 11 - \frac{2n_f}{3} > 0$, and therefore α_s decreases with μ $\Rightarrow$ "asymptotic freedom" of QCD.

Solution:

$$\boxed{\alpha_s = \frac{2\pi}{\beta_0} \frac{1}{\ln \frac{\mu}{\Lambda_{QCD}}}}$$

Landau pole at Λ_{QCD} introduced through dimensional transmutation as a boundary condition set by the renormalization condition at a particular scale.

Measuring α_s fixes Λ_{QCD}

Measuring α_s fixes Λ_{QCD}.

Higher orders:

$$\beta(\alpha_s) = -\varepsilon\alpha_s - 2\alpha_s\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \left(\frac{\alpha_s}{4\pi}\right)^3\beta_2 + \left(\frac{\alpha_s}{4\pi}\right)^4\beta_3 + O(\alpha_s^5)\right]$$

The QCD β-function is currently known to fourth order [van Ritbergen et al., 1997].

Charge universality:

In QED, $Z_1 = Z_2$ exactly in the on-shell scheme due to gauge invariance.

Thus, Z_e related to Z_3.

Similar condition in QCD:

$$\frac{Z_1}{Z_2} = \frac{Z_{1c}}{Z_{\tilde{1}3}} = \frac{Z_{A^3}}{Z_3} = \frac{\sqrt{Z_{A^4}}}{\sqrt{Z_3}}$$

Using the one-loop QCD renormalization constants, we have

$$\delta_1 - \delta_2 = \delta_{1c} - \delta_{3c} = \delta_{A^3} - \delta_3 = \frac{1}{2}(\delta_{A^4} - \delta_3) = -\frac{1}{\varepsilon}\frac{g_s^2}{32\pi^2} C_A (\xi + 3)$$

i.e. the condition is indeed satisfied.

$\boxed{\text{3.2.4 Running mass (23.2)}}$

Bare mass $m_0 = Z_m m_R$ should be μ-independent, since there was no μ in unren. $\mathcal{L}$, i.e.

$$0 = \mu\frac{d}{d\mu}m_0 - \mu\frac{d}{d\mu}[Z_m m_R] = Z_m m_R\left[\frac{\mu}{m_R}\frac{dm_R}{d\mu} + \frac{\mu}{Z_m}\frac{dZ_m}{d\mu}\right]$$

"Anomalous mass dimension" (name from Callan-Symanzik equation, see below)

$$\gamma_m := \frac{\mu}{m_R}\frac{dm_R}{d\mu} = -\frac{\mu}{Z_m}\frac{dZ_m}{d\mu} = -\frac{1}{Z_m}\frac{dZ_m}{dg_s}\mu\frac{dg_s}{d\mu} = -\frac{1}{1+\delta_m}\left(\frac{2}{g_s}\delta_m\right)\left(-\frac{\varepsilon}{2}g_s\right) = \delta_m\,\varepsilon$$

$$\Rightarrow \boxed{\gamma_m = -\frac{3g_s^2}{8\pi^2}C_F}$$

$\boxed{\text{3.2.5 Renormalization group equation (23.4)}}$

Bare Green's function:

$$G^{(0)}_{n,m} = \langle 0|T[A^{(0)}_{\mu_1}\cdots A^{(0)}_{\mu_n}\varphi^{(0)}_1\cdots\varphi^{(0)}_m]|0\rangle$$

Must be μ-independent. Related to the renormalized Green's function by

$$G^{(0)}_{n,m} = Z_3^{n/2}Z_2^{m/2}G_{n,m}$$

Renormalized Green's function:

$$G_{n,m}(p,\mu,g_R,m_R) = \langle 0|T[A_{\mu_1}\cdots A_{\mu_n}\varphi_1\cdots\varphi_m]|0\rangle$$

Finite, but depends on μ explicitly and implicitly through renormalized parameters.

Renormalized group equation:

$$0 = \mu\frac{d}{d\mu}G^0_{n,m} = Z_3^{n/2}Z_2^{m/2}\left(\mu\frac{\partial}{\partial\mu} + \frac{n}{2}\frac{\mu}{Z_3}\frac{dZ_3}{d\mu} + \frac{m}{2}\frac{\mu}{Z_2}\frac{dZ_2}{d\mu} + \mu\frac{dg_R}{d\mu}\frac{\partial}{\partial g_R} + \mu\frac{\partial m_R}{\partial\mu}\frac{\partial}{\partial m_R}\right)$$

$$\times G_{n,m}$$

Defining:

$$\gamma_3 = \frac{\mu}{Z_3}\frac{dZ_3}{d\mu} \quad , \quad \gamma_2 = \frac{\mu}{Z_2}\frac{dZ_2}{d\mu}$$

leads to the Callan-Symanzik equation

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} + \frac{n}{2}\gamma_3 + \frac{m}{2}\gamma_2 + \beta\frac{\partial}{\partial g_R} + \gamma_m m_R \frac{\partial}{\partial m_R}\right) G_{n,m} = 0}$$

3.2.6 Renormalizability (21.2)

In natural units, action S has mass dimension 0 (like Planck action $\hbar$).

$\Rightarrow$ Lagrangian density, which must be integrated over 4 space-time dim's, has dimension 4.

$\Rightarrow$ Dimensions of fields: $[\phi] = 1$, $[A_\mu] = 1$, $[\psi] = \frac{3}{2}$, since $[\partial_\mu] = 1$.

Index of divergence:

For a general interaction Lagrangian:

$$\mathcal{L}_i = g_i (\partial_\mu)^{d_i} (\phi_i)^{b_i} (\psi)^{f_i}$$

the index of divergence is defined as

$$\boxed{\delta_i := d_i + b_i + \frac{3}{2} f_i - 4}$$

Superficial degree of divergence:

$$\boxed{D := 4L - 2B_I - F_I + \sum_i n_i d_i}$$

for a Feynman diagram with L loops, each with $\int d^4k$, B_I internal bosons (each with k^{-2}), F_I internal fermions (each with k^{-1}), and n_i vertices of type d_i.

Number of loops is

$$L = B_I + F_I - \sum_i n_i + 1$$

(check for $2\to 2$ at tree-level and one-loop!)

Each external line connects to only one vertex, but internal line with two, so

$$\sum_i n_i b_i = B + 2B_I \quad , \quad \sum_i n_i f_i = F + 2F_I$$

where B and F are number of external boson and fermion lines.

Therefore:

$$\boxed{D = 4 - B - \frac{3}{2}F + \sum_i n_i \delta_i}$$

Renormalizability:

Case $\delta_i < 0$: Super-renormalizable theory. D decreases as number of vertices increases, so only lowest order diagrams are divergent.
Example: Scalar field theory with $\mathcal{L}_i = \lambda\phi^3$, where $\delta_i = 3-4 = -1$

Case $\delta_i = 0$: Renormalizable theory. Coupling is dimensionless.
Logarithmical divergences in all orders. Example: SM.

Case $\delta_i > 0$: Non-renormalizable theory.

Example: Fermi's 4-fermion interaction.