2.3 Renormalization

Mittwoch, 10. Mai 2017
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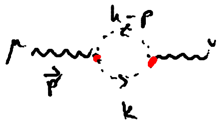
2.3.1 Vacuum polarization (16)

Scalar QED



$$= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i(2k^\mu - p^\mu)}{(k-p)^2 - m^2 + i\varepsilon} \cdot \frac{i(2k^\nu - p^\nu)}{k^2 - m^2 + i\varepsilon}$$

$$= 2ie^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

Embedded diagrams → keep photon off-shell, Lorentz indices uncontracted.

Sum of two diagrams:

$$i\Pi_2^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + \cancel{2p^\mu k^\nu} + \cancel{2p^\nu k^\mu} - \cancel{p^\mu p^\nu} + 2g^{\mu\nu}[(p-k)^2 - m^2]}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]}$$

Lorentz invariance:

$$\Pi_2^{\mu\nu} = \Delta_1(p^2, m^2)\, p^2 g^{\mu\nu} + \Delta_2(p^2, m^2)\, p^\mu p^\nu$$

Δ_2 contributes to gauge-dependent part $(1-\xi)\, p^\mu p^\nu$ in propagator → must drop out, need only

$$\Pi_2^{\mu\nu} = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2g^{\mu\nu}[(p-k)^2 - m^2]}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]}$$

Feynman parameters:

$$\boxed{\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[A + (B-A)x]^2}}$$

Using $A = (p-k)^2 - m^2 + i\varepsilon$, $B = k^2 - m^2 + i\varepsilon$

$$A + (B-A)x = [k - p(1-x)]^2 + p^2 x(1-x) - m^2 + i\varepsilon$$

Loop integral:

$$\Pi_2^{\mu\nu} = ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{-4k^\mu k^\nu + 2g^{\mu\nu}[(p-k)^2 - m^2]}{[(k - p(1-x))^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2}$$

Shift loop momentum $k^\mu \to k^\mu + p^\mu(1-x)$, measure remains unchanged

$$\Pi_2^{\mu\nu} = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{-2k^\mu k^\nu + g^{\mu\nu}[k^2 + x^2 p^2 - m^2]}{[k^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2}$$

We have dropped terms in p^μ and $p^\mu p^\nu$ (see above) and terms linear in k^μ, since we integrate over k.

Dimensional regularization:

$$\boxed{\int \frac{d^4k}{(2\pi)^4} \Rightarrow \int \frac{d^dk}{(2\pi)^d}} \quad \text{with} \quad \boxed{d = 4 - \varepsilon}$$

Regularizes both UV and IR for logarithmically divergent integrals.

Using $k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}$ (read Schwartz, Sec. B.3.4)

$$\Pi_2^{\mu\nu} = 2ie^2 \mu^{4-d} g^{\mu\nu} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(1-\frac{2}{d})k^2 + x^2p^2 - m^2}{[k^2 + \underbrace{p^2x(1-x) - m^2}_{-\Delta} + i\epsilon]^2}$$

Using the formulas from App. B:

- $$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{[k^2 - \Delta + i\epsilon]^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right)$$
- $$\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - \Delta + i\epsilon]^2} = \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right)$$

Insert in $\Pi_2^{\mu\nu}$:

$$\Pi_2^{\mu\nu} = -2 \frac{e^2}{(4\pi)^{d/2}} g^{\mu\nu} \mu^{4-d} \int_0^1 dx$$

$$\times \left[\left(1-\frac{d}{2}\right) \Gamma\left(1-\frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} + \left(x^2p^2 - m^2\right) \Gamma\left(2-\frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right]$$

Using $\Gamma(2-\frac{d}{2}) = (1-\frac{d}{2})\Gamma(1-\frac{d}{2})$ this simplifies to

$$\Pi_2^{\mu\nu} = -2 \frac{e^2}{(4\pi)^{d/2}} p^2 g^{\mu\nu} \Gamma\left(2-\frac{d}{2}\right) \mu^{4-d} \int_0^1 dx \; x(2x-1) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

For completeness, we also give the result with $p^\mu p^\nu$ terms:

$$\Pi_2^{\mu\nu} = \frac{-2e^2}{(4\pi)^{d/2}} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \Gamma\left(2-\frac{d}{2}\right) \mu^{4-d} \int_0^1 dx \; x(2x-1) \left(\frac{1}{m^2 - p^2x(1-x)}\right)^{2-\frac{d}{2}}$$

$\Rightarrow$ Satisfies Ward Identity: $p_\mu \Pi_2^{\mu\nu} = 0$. ✓

Expanding in $d = 4-\varepsilon$:

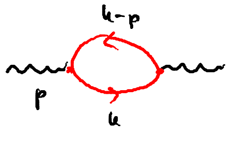
$$\Pi_2^{\mu\nu} = -\frac{e^2}{8\pi^2} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \int_0^1 dx \; x(2x-1) \left[\frac{2}{\varepsilon} + \ln\left(\frac{4\pi e^{-\gamma_E} \mu^2}{m^2 - p^2 x(1-x)}\right) + O(\varepsilon) \right].$$

If $Q^2 = -p^2 > 0$ and $m \ll Q$, integration over x leads to

$$\boxed{\Pi_2^{\mu\nu} = -\frac{e^2}{48\pi^2} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \left(\frac{2}{\varepsilon} + \ln \frac{\tilde{\mu}^2}{Q^2} + \frac{8}{3} \right)}$$

where $\boxed{\tilde{\mu}^2 := 4\pi e^{-\gamma_E} \mu^2}$ as in modified Minimal Subtraction scheme ($\overline{MS}$).

QED



$$= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{i}{k^2 - m^2} \mathrm{Tr}\left[\gamma^\mu (\not{k} - \not{p} + m) \gamma^\nu (\not{k} + m)\right]$$

Trace theorems for Dirac matrices, integrals as above, lead to:

$$\Pi_2^{\mu\nu} = -\frac{e^2}{2\pi^2} p^2 g^{\mu\nu} \int_0^1 dx \; x(1-x) \left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right) + O(\varepsilon) \right]$$

So we find for large $Q^2 = -p^2 \gg m^2$:

$$\boxed{\Pi_2^{\mu\nu} = \frac{-e^2}{12\pi^2} p^2 g^{\mu\nu} \left(\frac{2}{\varepsilon} + \ln \frac{\tilde{\mu}^2}{Q^2} + \frac{5}{3} + O(\varepsilon) \right]}$$

Full result (with $p^\mu p^\nu$ piece):

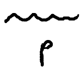
Full result (with $p^\mu p^\nu$ piece):

$$\boxed{\Pi_2^{\mu\nu} = \frac{-8e^2}{(4\pi)^{d/2}} \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \Gamma\left(2-\frac{d}{2}\right) \mu^{4-d} \times \int_0^1 dx\, x(1-x) \left(\frac{1}{m^2 - p^2 x(1-x)}\right)^{2-\frac{d}{2}}}$$

Satisfies Ward identity. (✓)

Renormalization:

Corrected photon propagator:

$iG^{\mu\nu} =$  $+$  $=$

$$= -i\frac{g^{\mu\nu}}{p^2} + \frac{-i}{p}\, i\Pi_2^{\mu\nu} \frac{-i}{p^2} + p^\mu p^\nu \text{ terms} =$$

$$= -i\frac{[1-e^2\Pi_2(p^2)]\, g^{\mu\nu}}{p^2} + p^\mu p^\nu \text{ terms}$$

Corrected Coulomb potential (in Fourier space):

$$V(p) = e^2 \frac{1-e^2\Pi_2(p^2)}{p^2}$$

Definition of renormalized charge, defined at scale p_0:

$$\boxed{\begin{aligned} e_R^2 &:= p_0^2 V(p_0^2) = e^2 - e^4 \Pi_2(p_0^2) + \ldots \quad \Leftrightarrow \\ e^2 &= e_R^2 + e_R^4 \Pi_2(p_0^2) + \ldots \end{aligned}}$$

Since $\Pi_2(p_0^2)$ is infinite, so is e, but only e_R is observable.

Potential at a different scale p:

$$p^2 V(p^2) = e^2 - e^4\Pi_2(p^2) + \ldots = e_R^2 - e_R^4\left[\Pi_2(p^2) - \Pi_2(p_0^2)\right] + \ldots$$

Taking $p_0 \to 0$ corresponds to $r \to \infty$, i.e. macroscopically defined charge. Since

$$\Pi_2(p^2) - \Pi_2(p_0^2) = -\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x) \ln\left[1 - \frac{p^2}{m^2} x(1-x)\right]$$

Small distance limit $\left(r = \frac{1}{p} \ll \frac{1}{m}\right)$:

$$\boxed{V(p^2) \simeq \frac{e_R^2}{p^2}\left\{1 + \frac{e_R^2}{12\pi^2} \ln\frac{-p^2}{m^2} + O(e_R^4)\right\}}$$

Letting $m \to 0$ leads to a mass singularity, but potential difference is independent of m.