2.1.9 Spinors (10.3, 11.2, 13.1)

Free Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi := \bar{\psi}(i\not{\partial} - m)\psi \tag{2.33}$$

Describes free fermions of mass m. Adjoint bispinor is $\bar{\psi} = \psi^\dagger \gamma^0$.

Equation of motion: Dirac equation

$$(i\not{\partial} - m)\psi = 0 \tag{2.34}$$

Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.35}$$

Therefore $(\gamma^0)^2 = 1$ with eigenvalues ± 1, $(\gamma^i)^2 = -1$ with eigenvalues $\pm i \to \gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$.

Commutator:

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{2.36}$$

Build bilinear quantities with simple Lorentz transformations from $\bar{\Psi}$ and Ψ. $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^5\Psi$, $\bar{\Psi}\gamma^\mu\Psi$, $\bar{\Psi}\gamma^\mu\gamma^5\Psi$, $\bar{\Psi}\sigma^{\mu\nu}\Psi$ transform as scalar, pseudoscalar, vector, axial vector, and (anti-symmetric) tensor.

Dirac-Pauli representation:

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^i = \beta\boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{2.37}$$

Also need later

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^{5\dagger}, \tag{2.38}$$

again in the Dirac-Pauli representation, satisfying $(\gamma^5)^2 = 1$ and $\{\gamma^5, \gamma^\mu\} = 0$.

Weyl (chiral) representation:

$$\alpha^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \ \beta = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{2.39}$$

which implies that

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.40}$$

Commonly used in "small mass" or "high-energy" limit, since (large) momentum term in Dirac equation proportional to $\boldsymbol{\alpha}$ is then (block) diagonal.

Spin projectors:

$$P_R \;=\; \frac{1+\gamma_5}{2} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_L \;=\; \frac{1-\gamma_5}{2} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.41}$$

Projection operators satisfy $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$.

Quantized spin-1/2 fields:

$$\psi(x) \;=\; \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} [a_s(p) u_s(p) e^{-ipx} + b_s^\dagger(p) v_s(p) e^{ipx}], \tag{2.42}$$

$$\bar{\psi}(x) \;=\; \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} [a_s^\dagger(p) \bar{u}_s(p) e^{ipx} + b_s(p) \bar{v}_s(p) e^{-ipx}] \tag{2.43}$$

Equal-time anticommutator: $\{a_s(p), a^\dagger_{s'}(p')\} = \{b_s(p), b^\dagger_{s'}(p')\} = (2\pi)^3 \delta(p-p')\delta_{ss'}$.

Spinor basis:

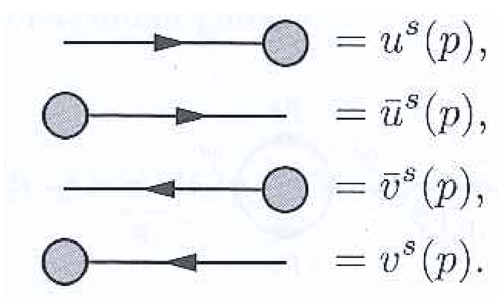
$$u_s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi_s \\ \sqrt{p.\bar{\sigma}}\xi_s \end{pmatrix} \quad , \quad v_s(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta_s \\ -\sqrt{p.\bar{\sigma}}\eta_s \end{pmatrix} \tag{2.44}$$

with $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, $\xi_1 = \eta_1 = (1,0)$ and $\xi_2 = \eta_2 = (0,1)$. They satisfy

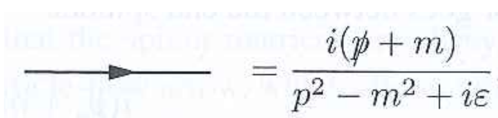
$$\bar{u}u = 2m \qquad , \quad \bar{v}v = -2m \tag{2.45}$$

$$\sum_s u_s(p)\bar{u}_s(p) = (\not{p} + m) \quad , \quad \sum_s v_s(p)\bar{v}_s(p) = (\not{p} - m) \tag{2.46}$$

Feynman rules for external particles:



Feynman rule for propagator:



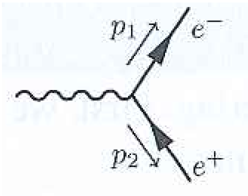
2.1.10 Feynman rule(s) for QED (13.1)

Expanded Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \frac{1}{4}F_{\mu\nu}^2, \tag{2.47}$$

where we used $D_\mu = \partial_\mu + ieA_\mu$. Landau gauge ($\xi = 0$) enforces $\partial_\mu A^\mu = 0$.

Interaction vertex:



$$= -ie\gamma^\mu.$$

No factor of momentum $\rightarrow$ same for any combination of incoming or outgoing fields.

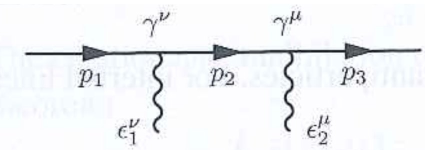
Index μ contracted with polarization vector ε_μ (external photon) or propagator $\Pi_{\mu\nu}$ (internal).

Dirac matrix $\gamma^\mu_{\alpha\beta}$ sandwiched between spinors, e.g. $\bar{u}_\alpha \gamma^\mu_{\alpha\beta} u_\beta$.

Feynman diagrams:

- Dirac matrices picked up against fermion flow direction
- Averages/sums over initial/final-state spins

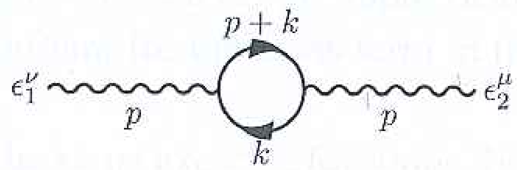
Tree-level example:



$$= (-ie)^2 \bar{u}(p_3)\gamma^\mu \frac{i(\not{p}_2 + m)}{p_2^2 - m^2 + i\varepsilon}\gamma^\nu u(p_1)\epsilon^2_\mu(q_2)\,\epsilon^1_\nu(q_1)$$

One-loop example:

$$i\mathcal{M} =$$



$$= e^2 \epsilon^{2\star}_\mu \epsilon^1_\nu \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\varepsilon}\gamma^\nu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}\right]$$

Integrate over internal loop momentum, multiply with (-1) for Fermi-Dirac statistics.

2.1.11 Dirac algebra (13.2) [SKIP]

Cyclic property of the trace:

$$\mathrm{Tr}[AB...C] = \mathrm{Tr}[B...CA] \tag{2.48}$$

Traces:

$$\begin{aligned}
\mathrm{Tr}[\not a_1 \dots \not a_n] &= \mathrm{Tr}[\gamma_5 \not a_1 \dots \not a_n \gamma_5] = (-1)^n \mathrm{Tr}[\not a_1 \dots \not a_n \gamma_5 \gamma_5] = (-1)^n \mathrm{Tr}[\not a_1 \dots \not a_n], && (2.49)\\
\mathrm{Tr}[\not a_1 \dots \not a_{2n+1}] &= 0, && (2.50)\\
\mathrm{Tr}[\not a \not b] &= \mathrm{Tr}[\not b \not a] = \frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] a_\mu b_\nu = \mathrm{Tr}[1] g^{\mu\nu} a_\mu b_\nu = 4a.b, && (2.51)\\
\mathrm{Tr}[\not a_1 \dots \not a_n] &= 2a_1.a_2 \mathrm{Tr}[\not a_3 \dots \not a_n] - \mathrm{Tr}[\not a_2 \not a_1 \not a_3 \dots \not a_n] = \dots \\
&= a_1.a_2 \mathrm{Tr}[\not a_3 \dots \not a_n] - a_1.a_3 \mathrm{Tr}[\not a_2 \not a_4 \dots \not a_n] + \dots \\
&+ a_1.a_n \mathrm{Tr}[\not a_2 \dots \not a_{n-1}], && (2.52)\\
\mathrm{Tr}[\not a_1 \not a_2 \not a_3 \not a_4] &= a_1.a_2 \mathrm{Tr}[\not a_3 \not a_4] - a_3.a_1 \mathrm{Tr}[\not a_2 \not a_4] + a_1.a_4 \mathrm{Tr}[\not a_2 \not a_3] \\
&= 4a_1.a_2\, a_3.a_4 - 4a_1.a_3\, a_2.a_4 + 4a_1.a_4\, a_2.a_3, && (2.53)\\
\mathrm{Tr}\,\gamma^5 &= \mathrm{Tr}[\gamma^5\gamma^0\gamma^0] = -\mathrm{Tr}[\gamma^0\gamma^5\gamma^0] = -\mathrm{Tr}\,\gamma^5 = 0, && (2.54)\\
\mathrm{Tr}[\gamma^5 \not a \not b] &= +\mathrm{Tr}[\gamma^5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1}\gamma_\lambda] a^\mu b^\nu = \mathrm{Tr}[\gamma_\lambda\gamma^5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1}] a^\mu b^\nu \\
&= -\mathrm{Tr}[\gamma^5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\lambda^{-1}] a^\mu b^\nu = 0, && (2.55)\\
\mathrm{Tr}[\gamma^5 \not a \not b \not c \not d] &= -4i\varepsilon^{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta, && (2.56)\\
\mathrm{Tr}[\not a_1 \not a_2 \dots \not a_{2n}] &= \mathrm{Tr}[C \not a_1 C^{-1} C \not a_2 C^{-1} \dots C \not a_{2n} C^{-1}] \\
&= (-1)^{2n} \mathrm{Tr}[\not a_1^T \not a_2^T \dots \not a_{2n}^T] = \mathrm{Tr}[\not a_{2n} \dots \not a_1]^T = \mathrm{Tr}[\not a_{2n} \dots \not a_1], && (2.57)
\end{aligned}$$

using $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ with charge conjugation matrix $C = -i\gamma^2\beta$.

Contracted products:

$$\begin{aligned}
\gamma_\mu\gamma^\mu &= \frac{1}{2}(\gamma_\mu\gamma^\mu + \gamma^\mu\gamma_\mu) = g^\mu_{\ \mu} = 4, && (2.58)\\
\gamma_\mu \not a \gamma^\mu &= \gamma_\mu\gamma^\nu a_\nu \gamma^\mu = a_\nu \gamma_\mu (2g^{\mu\nu} - \gamma^\mu\gamma^\nu) = 2\not a - 4\not a = -2\not a, && (2.59)\\
\gamma_\mu \not a \not b \gamma^\mu &= \gamma_\mu \not a \gamma^\lambda b_\lambda \gamma^\mu = \gamma_\mu \not a 2g^{\lambda\mu} b_\lambda - \gamma_\mu \not a \gamma^\mu \not b = 2\not b \not a + 2\not a \not b = 4a.b, && (2.60)\\
\gamma_\mu \not a \not b \not c \gamma^\mu &= \gamma_\mu \not a \not b 2g^{\nu\mu} c_\nu - \gamma_\mu \not a \not b \gamma^\mu \not c = 2\not c \not a \not b - 4a.b \not c \\
&= 4\not c\, a.b - 2\not c \not b \not a - 4a.b \not c = -2\not c \not b \not a, && (2.61)\\
\gamma_\mu \not a \not b \not c \not d \gamma^\mu &= \gamma_\mu \not a \not b \not c 2g^{\mu\nu} d_\nu - \gamma_\mu \not a \not b \not c \gamma^\mu \not d = 2\not d \not a \not b \not c + 2\not c \not b \not a \not d. && (2.62)
\end{aligned}$$

Cf. also App. A (PDF on the web).