B Appendix B Regularization

B.1 Integration parameters

To evaluate loop integrals in quantum field theory, it is often helpful to introduce Feynman or Schwinger parameters.

B.1.1 Feynman parameters

Feynman parameters are based on a number of easily verifiable mathematical identities. The simplest is

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2} = \int_0^1 dx\, dy\, \delta(x+y-1)\frac{1}{[xA+yB]^2}. \tag{B.1}$$

Other useful identities are

$$\frac{1}{AB^n} = \int_0^1 dx\, dy\, \delta(x+y-1)\frac{ny^{n-1}}{[xA+yB]^{n+1}}, \tag{B.2}$$

$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\frac{2}{[xA+yB+zC]^3}. \tag{B.3}$$

These are useful because they let us complete the square in the denominator. For example,

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k-p)^2} = \int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{[k^2+x((k-p)^2-k^2)]^2}$$
$$= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k-xp)^2-\Delta]^2}, \tag{B.4}$$

where $\Delta = -p^2x(1-x)$. Then we can shift $k \to k + xp$ leaving an integral that only depends on k^2.

B.1.2 Schwinger parameters

Another useful set of integration parameters are called Schwinger parameters. They are based on the following mathematical identities, which hold when $\mathrm{Im}(A) > 0$:

$$\frac{i}{A} = \int_0^\infty ds\, e^{isA}, \tag{B.5}$$

$$\left[\frac{i}{A}\right]^2 = \int_0^\infty s\, ds\, e^{isA}. \tag{B.6}$$

You can derive further identities by taking additional derivatives with respect to A. Also, Eq. (B.5) implies

$$\frac{1}{AB} = -\int_0^\infty ds \int_0^\infty dt\, e^{isA+itB} \tag{B.7}$$

when $\mathrm{Im}(A) > 0$ and $\mathrm{Im}(B) > 0$ (i.e. with Feynman propagators). These **Schwinger parameters** s and t have a nice physical interpretation: s and t are the proper times of the particles as they travel along their paths in the Feynman graph. This Schwinger proper-time interpretation is discussed in Chapter 32.

Note that writing $s+t=\tau$ and $x = \frac{t}{s+t}$, or $t = x\tau$ and $s = (1-x)\tau$, Eq.(B.7) becomes

$$\frac{1}{AB} = -\int_0^\infty \tau\, d\tau \int_0^1 dx\, e^{i\tau(A+(B-A)x)}$$
$$= \int_0^1 dx \frac{1}{[A+(B-A)x]^2}. \tag{B.8}$$

So the Feynman parameter x also has an interpretation, as the relative proper time $\frac{s}{s+t}$ of the two particles in the loop.

Other useful related identities are

$$\frac{1}{A^nB^m} = \frac{\Gamma(n+m)}{\Gamma(n)\,\Gamma(m)}\int_0^\infty ds \frac{s^{m-1}}{(A+Bs)^{n+m}}, \tag{B.9}$$

$$\frac{1}{AB} = \int_0^\infty ds \frac{1}{(A+Bs)^2}. \tag{B.10}$$

Schwinger parameters are used in Chapters 34 and 35.

B.2 Wick rotations

After introducing Feynman parameters and completing the square, one is often left with an integral over a loop momentum k^μ in Minkowski space. Once the $i\varepsilon$ factors are put in for Feynman propagators, 1-loop integrals often appear as

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^n}. \tag{B.11}$$

Assuming $\Delta > 0$ (you can check that Wick rotation still works for $\Delta < 0$ in Problem B.1), this integral has poles at $k_0 = \sqrt{\vec{k}^2+\Delta} - i\varepsilon$ and $k_0 = -\sqrt{\vec{k}^2+\Delta} + i\varepsilon$, as shown in Figure B.1. Since the poles are in the top-left and bottom-right quadrants of the k_0 complex plane, the integral over the figure-eight contour shown vanishes. Thus, the integrals over the real axis and the imaginary axis are equal and opposite. Therefore, we can substitute $k_0 \to ik_0$ so that $k^2 \to -k_0^2 - \vec{k}^2 = -k_E^2$, where $k_E^2 = k_0^2 + \vec{k}^2$ is the Euclidean momentum. This

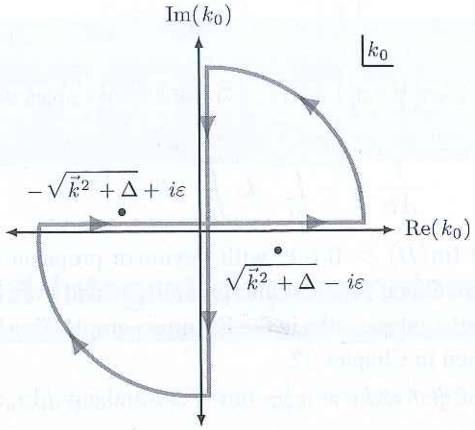

Fig. B.1 Wick rotations. Poles in integrations over Feynman propagators often have poles at at $k_0 = \pm\sqrt{\vec{k}^2+\Delta} \mp i\varepsilon$. Integrating over the real axis is then equivalent to integrating over the imaginary axis.

is known as a **Wick rotation**. After the Wick rotation, the $i\varepsilon$ will no longer play a role and we can just set $\varepsilon = 0$.

Once Wick-rotated, the integrals are evaluated in a straightforward way. We will need the formula for the surface area of the Euclidean 4-sphere: $\int d\Omega_4 = 2\pi^2$. Using this, we find

$$\int \frac{d^4k_E}{(2\pi)^4} f(k_E^2) = \frac{1}{16\pi^4}\int d\Omega_4 \int_0^\infty k_E^3\, dk_E\, f(k_E^2) = \frac{1}{8\pi^2}\int_0^\infty k_E^3\, dk_E\, f(k_E^2). \tag{B.12}$$

Then, for example, Eq. (B.11) with $n = 3$ is evaluated as

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^3} &= i\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(-k_E^2-\Delta)^3}\\ &= (-1)^3\frac{i}{8\pi^2}\int_0^\infty dk_E \frac{k_E^3}{(k_E^2+\Delta)^3}\\ &= \frac{-i}{32\pi^2\Delta}.\end{aligned} \tag{B.13}$$

Other useful formulas following from Wick rotations are

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2-\Delta+i\varepsilon)^4} = \frac{-i}{48\pi^2}\frac{1}{\Delta}, \tag{B.14}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^r} = i\frac{(-1)^r}{(4\pi)^2}\frac{1}{(r-1)(r-2)}\frac{1}{\Delta^{(r-2)}}, \quad r>2, \tag{B.15}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2-\Delta+i\varepsilon)^r} = i\frac{(-1)^{r-1}}{(4\pi)^2}\frac{2}{(r-1)(r-2)(r-3)}\frac{1}{\Delta^{(r-3)}}, \quad r>3, \tag{B.16}$$

and so on.

Keep in mind that the Wick rotation is just a trick for evaluating integrals. There is nothing physical about it. In addition, note that the Wick rotation can only be justified if there are no new poles that invalidate the contour rotation. This caveat is only relevant for 2-loop and higher integrals, which we will not encounter.

B.3 Dimensional regularization

The most important regularization scheme for modern applications is dimensional regularization ['t Hooft and Veltman, 1972]. The key observation is that an integral such as

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta+i\varepsilon)^2} \tag{B.17}$$

is divergent only if $d \geq 4$. If $d < 4$, then it will converge. If it is convergent we can Wick rotate, and the answer comes from analytically continuing all our formulas above to d dimensions.

B.3.1 Spinor algebra

In d dimensions, the metric is

$$g^{\mu\nu} = \mathrm{diag}(1,-1,-1,\cdots,-1), \tag{B.18}$$

which means that there is exactly one timelike dimension in even non-integer d. This metric satisfies

$$g^{\mu\nu}g_{\mu\nu} = d. \tag{B.19}$$

The Lorentz-invariant phase space is

$$d\Pi_{\mathrm{LIPS}} \equiv (2\pi)^d \prod_{\text{final states } j} \frac{d^{d-1}p_j}{(2\pi)^{d-1}}\frac{1}{2E_{p_j}}\delta^d(\Sigma p). \tag{B.20}$$

We can define spinor algebra to work the same way in $d = 4-\varepsilon$ dimensions as in $d = 4$. More precisely, we assume there are d four-dimensional γ-matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$. The identity matrix in spinor space satisfies $\mathrm{Tr}\mathbb{1}_{\alpha\beta} = 4$ as in four dimensions. In theories that involve γ_5 we also assume such a matrix exists satisfying

$$\{\gamma_5,\gamma_\mu\} = 0. \tag{B.21}$$

Theories with anomalies are the only places in which there can be subtleties with such a definition (see Chapter 30). An excellent discussion of spinors in various dimensions can be found in [Polchinski, 1998, Appendix B].

B.3.2 Scalar integrals

We will manipulate the expressions so that they are only functions of the magnitude of k. Then we will use

$$\int d^d k = \int d\Omega_d \int k^{d-1} dk, \tag{B.22}$$

where $d\Omega_d$ denotes the differential solid angle of the d-dimensional unit sphere. Explicitly,

$$d\Omega_d = \sin^{d-2}(\phi_{d-1}) \sin^{d-3}(\phi_{d-2}) \cdots \sin(\phi_2)\, d\phi_1 \cdots d\phi_{d-1}, \tag{B.23}$$

where ϕ_i is the angle to the i th axis, with $0 \le \phi_1 < 2\pi$ and $0 \le \phi_i < \pi$ for $i > 1$. For example, $d\Omega_2 = d\phi$. For $d = 3$, we normally write $\phi_1 = \phi$ and $\phi_2 = \theta$ giving

$$d\Omega_3 = d\cos\theta\, d\phi, \tag{B.24}$$

which is the usual volume element of a two-dimensional surface. Remember, d is the dimension of the solid volume, not the surface, which has dimension $d - 1$. The $(d-1)$-dimensional surface areas of a ball of radius 1 in integer dimensions are

$$\Omega_2 = \int d\Omega_2 = 2\pi \text{ (circle)}, \int d\Omega_3 = 4\pi \text{ (sphere)}, \int d\Omega_4 = 2\pi^2 \text{ (three-sphere)}, \ldots, \tag{B.25}$$

The equivalent volumes are

$$V_d = \Omega_d \int_0^R dr\, r^{d-1} = \Omega_d \frac{1}{d} R^d, \tag{B.26}$$

which are $V_2 = \pi R^2$, $V_3 = \frac{4}{3}\pi R^3$, $V_4 = \frac{1}{2}\pi^2 R^4$, etc.

For non-integer dimensions, the surface area formula can be derived using the same trick used for Gaussian integrals in Section 14.2.1:

$$(\sqrt{\pi})^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \int d\Omega_d \int_0^\infty dr\, r^{d-1} e^{-r^2} = \frac{1}{2}\Gamma\left(\frac{d}{2}\right) \int d\Omega_d, \tag{B.27}$$

so that

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \tag{B.28}$$

Alternatively, one can just integrate Eq. (B.23):

$$\begin{aligned} \Omega_d &= 2\pi \prod_{n=2}^{d-1} \left(\int_0^\pi d\phi_n \sin^{n-1}\phi_n\right) = 2\pi \prod_{n=2}^{d-1} \sqrt{\pi}\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\right) \\ &= 2\pi^{d/2} \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{2})} \cdots \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} = 2\pi^{d/2} \frac{\Gamma(1)}{\Gamma(\frac{d}{2})}. \end{aligned} \tag{B.29}$$

Using $\Gamma(1) = 1$, this reproduces Eq. (B.28).

In these expressions, $\Gamma(x)$ is the **Gamma function**, which is the analytic continuation of the factorial. For integer arguments, it evaluates to

$$\Gamma(1) = 1, \quad \Gamma(2) = 1, \quad \Gamma(3) = 2, \quad \Gamma(x) = (x-1)! \tag{B.30}$$

$\Gamma(z)$ has simple poles at 0 and all the negative integers. We will often need to expand $\Gamma(x)$ around the pole at $x = 0$:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) + \cdots, \tag{B.31}$$

where γ_E is the Euler–Mascheroni constant, $\gamma_E \approx 0.577$. Sometimes relations such as

$$\sin(\pi x) = \frac{\pi(1-x)}{\Gamma(x)\Gamma(2-x)}, \quad \cos(\pi x) = \left(\frac{1-2x}{2x}\right) \frac{\Gamma(1-x)\Gamma(1+x)}{\Gamma(2-2x)\Gamma(2x)}, \tag{B.32}$$

or the Euler β-function

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx (1-x)^{a-1} x^{b-1} \tag{B.33}$$

allow us to simplify expressions.

The integrals over Euclidean k_E are straightforward:

$$\int dk_E \frac{k_E^a}{(k_E^2 + \Delta)^b} = \Delta^{\frac{a+1}{2} - b} \frac{\Gamma(\frac{a+1}{2})\, \Gamma(b - \frac{a+1}{2})}{2\Gamma(b)}. \tag{B.34}$$

Equations (B.22), (B.28) and (B.34) can be combined into a general formula:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2a}}{(k^2 - \Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-\frac{d}{2}}} \frac{\Gamma(a + \frac{d}{2})\, \Gamma(b - a - \frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})}. \tag{B.35}$$

Special cases used in the text are

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} = \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right), \tag{B.36}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta + i\varepsilon)^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right), \tag{B.37}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta + i\varepsilon)^3} = \frac{d}{4} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right), \tag{B.38}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^3} = \frac{-i}{2(4\pi)^{d/2}} \frac{1}{\Delta^{3-\frac{d}{2}}} \Gamma\left(\frac{6-d}{2}\right). \tag{B.39}$$

This last integral is convergent in $d = 4$; however, the d-dimensional form is important for loops with IR divergences (see Chapter 20).

All dimensionally regulated versions of divergent integrals will have poles at $d = 4$. Therefore, we often expand $d = 4 - \varepsilon$ and drop terms of order ε. Another common convention is $d = 4 - 2\varepsilon$. If you are ever off by a factor of 2 in comparing to someone else's result, check the convention!

B.3.3 Field dimensions

Next, we should calculate the dimensions of all the fields and couplings in the Lagrangian. For the action to be dimensionless, the Lagrangian density should have mass dimension d. For example, in QED, the Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{B.40}$$

which implies the mass dimensions

$$[A_\nu] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}, \quad [m] = 1, \tag{B.41}$$

and also $[e] = \frac{4-d}{2}$. However, rather than have a non-integer dimensional coupling, it is conventional to take

$$e \to \mu^{\frac{4-d}{2}} e, \tag{B.42}$$

where μ is an arbitrary parameter of mass dimension 1. Then e remains dimensionless.

One usually only makes this change for the factors of e (or other gauge couplings) directly participating in a loop. If a loop graph is not one-particle irreducible, there may be other factors of e for which it is often simpler to leave four-dimensional. This is just a convention. If all factors of e are modified as in Eq. (B.42), the answer will still be correct, but may contain awkward logarithms of dimensionful scales when expanded around $d = 4$. These awkward logarithms drop out of physical quantities, of course, but they can be avoided at intermediate steps as well by only adding factors of μ to coupling constants participating in the loop.

The factors of μ coming from Eq. (B.42) modify loop integrals as

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^2}{(k^2-\Delta+i\varepsilon)^2} \to \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{e^2}{(k^2-\Delta+i\varepsilon)^2}. \tag{B.43}$$

Keep in mind that μ is *not* a large scale. It is *not* a UV cutoff. The dimensional regularization is removed when $d \to 4$, not when $\mu \to \infty$. Thus, μ is not like the Pauli–Villars mass M or a generic UV scale Λ. In fact, we will often use μ as a proxy for a physical infrared scale associated with a renormalization group point. Nevertheless, there are two unphysical parameters in dimensional regularization, ε and μ; both must drop out of physical predictions.

Including this factor of μ, the logarithmically divergent integral becomes

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^2}{(k^2-\Delta+i\varepsilon)^2} \to \mu^{4-d}\frac{ie^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}. \tag{B.44}$$

Now letting $d = 4 - \varepsilon$ we expand this around $\varepsilon = 0$ and get

$$\begin{aligned}\mu^{4-d}\frac{ie^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} &= \frac{ie^2}{16\pi^2}\left[\frac{2}{\varepsilon} + (-\gamma_E + \ln 4\pi + \ln\mu^2 - \ln\Delta) + \mathcal{O}(\varepsilon)\right] \\ &= \frac{ie^2}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln\frac{4\pi e^{-\gamma_E}\mu^2}{\Delta} + \mathcal{O}(\varepsilon)\right].\end{aligned} \tag{B.45}$$

The γ_E comes from the integral $\int \frac{d^dk}{k^4}$, the 4π comes from the phase space $\frac{1}{(2\pi)^d}$ and the μ comes from the μ^{4-d}. This combination, $4\pi e^{-\gamma_E}\mu^2$, shows up frequently, so we give it a symbol

$$\tilde{\mu}^2 \equiv 4\pi e^{-\gamma_E}\mu^2 \tag{B.46}$$

leading to

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^2}{(k^2-\Delta+i\varepsilon)^2} \to \frac{ie^2}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{\Delta} + \mathcal{O}(\varepsilon)\right]. \tag{B.47}$$

Sometimes we will omit the tilde and just write μ for $\tilde{\mu}$. Note that there is still a divergence in this expression as $\varepsilon \to 0$.

Dimensional regularization characterizes the degree to which integrals diverge at high energy through analytic properties of regulated results, rather than through powers of a cutoff scale. For example, the integral $\int \frac{d^4k}{(k^2-\Delta)^2}$ is logarithmically divergent. In d dimensions, the equivalent integral $\int \frac{d^dk}{(k^2-\Delta)^2} \sim \Gamma(\frac{4-d}{2})$ has a simple pole at $d = 4$, and no other poles for $d < 4$. A quadratically divergent integral, such as $\int \frac{d^4k}{k^2-\Delta}$, becomes $\int \frac{d^dk}{k^2-\Delta} \sim \Gamma(\frac{2-d}{2})$ in d dimensions. Expanding this result around $d = 4$ gives a $\frac{1}{\epsilon}$ pole as did the expansion of the logarithmically divergent integral. However, this does not mean that power divergences are absent with dimensional regularization. Rather they are hidden, as poles in integer $d < 4$. For example, the quadratic divergence translates to a pole in $\Gamma(\frac{2-d}{2})$ at $d = 2$. Thus, dimensional regularization translates the degree of divergence into the singularity structure of amplitudes in d dimensions.

Dimensional regularization can also be used to regulate IR-divergent integrals. For example, $\int d^dk \frac{1}{(k^2-m^2)k^4}$ is IR divergent for $d < 4$. We can evaluate this integral in $d = 4 - \varepsilon$ dimensions with $\varepsilon < 0$ instead of $\varepsilon > 0$. A nice feature of dimensional regularization as an IR regulator is that it can be used for both virtual graphs and phase space integrals.

Occasionally when using dimensional regularization we encounter an integral that is both UV and IR divergent; for example, the scaleless integral $\int \frac{d^dk}{k^4}$. This integral is not convergent for any d. Nevertheless, it is useful to be able to do such integrals. To progress, we can introduce an arbitrary scale Λ to divide the UV and IR regions of Euclidean momenta:

$$\begin{aligned}\int \frac{d^dk_E}{k_E^4} &= \Omega_d\int_0^\Lambda dk_E k_E^{d-5} + \Omega_d\int_\Lambda^\infty dk_E\, k_E^{d-5} \\ &= \Omega_d\left(\ln\Lambda - \frac{1}{\varepsilon_{\text{IR}}}\right) + \Omega_d\left(\frac{1}{\varepsilon_{\text{UV}}} - \ln\Lambda\right),\end{aligned} \tag{B.48}$$

where we have written $d = 4 - \varepsilon_{\text{IR}}$ for the first integral, assuming $\varepsilon_{\text{IR}} < 0$, and $d = 4 - \varepsilon_{\text{UV}}$ for the second integral, assuming $\varepsilon_{\text{UV}} > 0$. Rather than doing this split for every scaleless integral, since we know ε_{IR} and ε_{UV} must vanish from physical quantities, we often just set $\varepsilon_{\text{IR}} = \varepsilon_{\text{UV}} = \varepsilon$. When this is done, the integral is just 0. A simpler justification is that since there is no available quantity with non-zero mass dimension, scaleless integrals such as $\int \frac{d^dk}{k^4}$ must vanish in d dimensions.

Often we are interested in just the UV divergence of an integral, which can be extracted from a scaleless integral as

$$\left[\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4}\right]_{\text{UV-div}} = i\frac{\Omega_d}{(2\pi)^d}\frac{1}{\varepsilon_{\text{UV}}} = i\frac{2}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\frac{1}{\varepsilon_{\text{UV}}} = \frac{i}{8\pi^2}\frac{1}{\varepsilon_{\text{UV}}}. \tag{B.49}$$

This is a very useful shortcut to extracting the UV divergence.

B.3.4 k^μ integrals

We will often have integrals with factors of momenta, such as $k^\mu k^\nu$, in the numerator:

$$F^{\mu\nu}(\Delta) = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - \Delta)^n}. \tag{B.50}$$

These can be simplified using a trick. Since the integral is a tensor under Lorentz transformations but only depends on the scalar Δ, it must be proportional to the only tensor around, $g^{\mu\nu}$. Then, just by dimensional analysis, we must get the same thing as in an integral with $k^\mu k^\nu$ replaced by $ck^2 g^{\mu\nu}$ for some number c. Contracting with $g^{\mu\nu}$, we see that $c = \frac{1}{4}$ or more generally $c = \frac{1}{d}$. Therefore,

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{1}{d} g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta)^n}. \tag{B.51}$$

If there is just one factor of k^μ in the numerator, for example in

$$F(p^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k \cdot p}{(k^2 - p^2)^4}, \tag{B.52}$$

then the integrand is antisymmetric under $k \to -k$. Since we are integrating over all k, the integral must vanish. So we will only need to keep terms with even powers of k in the numerator.

B.4 Other regularization schemes

While dimensional regularization has a number of important advantages (it respects gauge invariance, it can regulate IR or UV divergences, no new fields are needed, etc.), it has the disadvantage of being unphysical. That is, one cannot think of analytical continuation into $4 - \varepsilon$ dimensions as representing some sort of short-distance deformation. A number of regulators that do have short-distance interpretations, such as the hard cutoff regulator or heat-kernel regulator, are discussed in Chapter 15 in the context of the Casimir effect. Those regulators are unfortunately not useful for general field theory calculations. Here we discuss two regulation schemes that do have widespread applicability, the derivative method and Pauli–Villars regularization, and briefly mention a few more.

B.4.1 Derivative method

A quick way to extract the UV divergence of an integral is by taking derivatives. Consider a logarithmically divergent integral, such as

$$\mathcal{I}(\Delta) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} = \infty. \tag{B.53}$$

If we take the derivative, the integral can be done:

$$\frac{d}{d\Delta}\mathcal{I}(\Delta) = \int \frac{d^4k}{(2\pi)^4} \frac{2}{(k^2 - \Delta + i\varepsilon)^3} = -\frac{i}{16\pi^2 \Delta}. \tag{B.54}$$

So,

$$\mathcal{I}(\Delta) = -\frac{i}{16\pi^2} \ln\frac{\Delta}{\Lambda^2}, \tag{B.55}$$

where Λ is an integration constant representing the UV cutoff and is formally infinite. Similarly, for a quadratically divergent integral, one could take the second derivative and then integrate twice to give

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - \Delta + i\varepsilon)^2} = 6 \int d\Delta \int d\Delta \left(\frac{-i}{48\pi^2}\frac{1}{\Delta}\right) = -\frac{i}{8\pi^2}\left(\Delta \ln\frac{\Delta}{\Lambda_1^2} + \Lambda_2^2\right) \tag{B.56}$$

for two integration constants Λ_1 and Λ_2.

The derivative method is not an ideal regulator. Since the cutoff Λ appears as a constant of integration, there is no way to relate Λ from one integral to Λ from another. In particular, cancellations that we expect due to constraints such as gauge invariance are not guaranteed to hold. Nevertheless, the derivative method is a quick way to check the coefficient of the logarithms appearing in any particular integral.

B.4.2 Pauli–Villars regularization

Pauli–Villars regularization requires that for each particle of mass m a new unphysical **ghost** particle of mass Λ be added with either the wrong statistics or the wrong-sign kinetic term. These new particles are designed to cancel exactly loop amplitudes with physical particles at asymptotically large loop momentum. For example, one can write down a Pauli–Villars Lagrangian for QED, which works at the 1-loop level, as

$$\mathcal{L}_{\text{PV}} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - e\widetilde{A\!\!\!/} - m)\psi + \frac{1}{4}\widetilde{F}_{\mu\nu}^2 - \frac{1}{2}\Lambda^2 \widetilde{A}_\mu^2 + \bar{\widetilde{\psi}}(i\partial\!\!\!/ - eA\!\!\!/ - e\widetilde{A\!\!\!/} - \Lambda)\widetilde{\psi}, \tag{B.57}$$

with $\widetilde{A}_\mu$ the ghost photon and $\widetilde{\psi}$ the ghost electron and $\widetilde{F}_{\mu\nu} = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu$. We assume that both the ghost photon and ghost electron have bosonic statistics; the ghost photon has a wrong-sign kinetic term.

For example, $\mathcal{L}_{\text{PV}}$ leads to a Feynman-gauge ghost-photon propagator of the form

$$\langle 0|T\{\widetilde{A}_\mu(x)\widetilde{A}_\nu(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{ig^{\mu\nu}}{p^2 - \Lambda^2 + i\varepsilon}. \tag{B.58}$$

Since this has the opposite sign from the photon propagator, it will cancel the photon's contribution, for example, to the electron self-energy loop for loop momenta $k^\mu \gg \Lambda$ (see Chapter 18). The sign of the residue of the propagator is normally dictated by unitarity – a particle whose propagator has the sign in Eq.(B.58) has negative norm, and would generate probabilities greater than 1. So, $\widetilde{A}_\mu$ cannot create or destroy physical on-shell particles. Thus, fields such as $\widetilde{A}_\mu$ are said to be associated with **Pauli–Villars ghosts**. The ghost electron propagator is the same as the regular electron propagator; however, ghost electron loops do not get a factor of -1 (since they are bosonic) and therefore cancel regular electron loops when $k^\mu \gg \Lambda$.

In more detail, an amplitude with Pauli–Villars regularization will sum over the real particle, with mass m, and the ghost particle, with fixed large mass $\Lambda \gg m$:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\varepsilon)^2} \to \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\varepsilon)^2} - \frac{1}{(k^2-\Lambda^2+i\varepsilon)^2}\right]. \tag{B.59}$$

For $k \gg \Lambda, m$ both terms in the new integrand scale as $\frac{1}{k^4}$ and so the integrand vanishes at least as $\frac{1}{k^6}$ making the integral convergent. We can now perform this integral by Wick rotation

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}&\left[\frac{1}{(k^2-m^2+i\varepsilon)^2} - \frac{1}{(k^2-\Lambda^2+i\varepsilon)^2}\right]\\ &= \frac{i}{8\pi^2}(-1)^2\int_0^\infty dk_E\left[\frac{k_E^3}{(k_E^2-m^2)^2} - \frac{k_E^3}{(k_E^2-\Lambda^2)^2}\right]\\ &= -\frac{i}{16\pi^2}\ln\frac{m^2}{\Lambda^2}\end{aligned} \tag{B.60}$$

so that

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\varepsilon)^2} \to \frac{i}{16\pi^2}\ln\frac{\Lambda^2}{m^2}. \tag{B.61}$$

Note that the coefficient of the logarithm is consistent with what we found using the derivative method, in Eq. (B.55) and with derivational regularization in Eq. (B.47).

When using Pauli–Villars regularization, the identity

$$\frac{1}{k^2-m^2} - \frac{1}{k^2-\Lambda^2} = \int_{m^2}^{\Lambda^2} \frac{-1}{(k^2-\Xi)^2}\,d\Xi \tag{B.62}$$

is often useful. It allows us to evaluate divergent integrals by squaring the propagator and adding an integration parameter Ξ. In fact, due to the identity

$$\int dm^2 \frac{d}{dm^2}\left[\frac{1}{k^2-m^2}\right] = \int dm^2 \frac{1}{(k^2-m^2)^2}, \tag{B.63}$$

Pauli–Villars can be viewed as a systematic implementation of the derivative method.

Pauli–Villars was historically important and serves a useful pedagogical function. Indeed, the introduction of Pauli–Villars ghosts is much more clearly a deformation in the UV, relevant at energy scales of order the Pauli–Villars mass or larger, than analytically continuing to $4-\varepsilon$ dimensions. However, in modern applications, Pauli–Villars is only occasionally useful. The problem is that complicated multi-loop diagrams necessitate many fictitious particles (one for each real particle will not do it; the Lagrangian $\mathcal{L}_{\text{PV}}$ only works at 1-loop). Thus, Pauli–Villars quickly becomes impractical. In addition, it is not useful in non-Abelian gauge theories, since a massive gauge boson breaks gauge invariance. (Pauli–Villars does work in an Abelian theory, at least at 1-loop, as long as the gauge boson couples to a conserved current.)

B.4.3 Other regulators

There are several other regulators that are sometimes used:

- Hard cutoff: $k_E < \Lambda$. This breaks Lorentz invariance, and usually every symmetry in the theory, but is perhaps the most intuitive regularization procedure.
- Point splitting. Divergences at $k \to \infty$ correspond to two fields approaching each other $x_1 \to x_2$. Point splitting puts a lower bound on this, $|x_1^\mu - x_2^\mu| > |\epsilon^\mu|$. This also breaks translation invariance and is impractical for gauge theories, but is useful in theories with composite operators.
- Lattice regularization. Although a lattice breaks both translation invariance and Lorentz invariance, it is possible to construct a lattice such that translation and Lorentz invariance are restored in the continuum limit (see Section 25.5).

Problems

B.1 Show that the Wick rotation still works if $\Delta < 0$.