Chapter 3
Korteweg-de Vries Equation

The Korteweg-de Vries (KdV) equation is the partial differential equation, derived by Korteweg and de Vries [14] to describe weakly nonlinear shallow water waves. The nondimensionalized version of the equation reads

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{3.1}$$

where $u = u(x, t)$. The factor of 6 is convenient for reasons of complete integrability, but can easily be scaled out if desired. Equation (3.1) was found to have *solitary wave solutions*, vindicating the observations of a solitary channel wave made by Russell [20].

3.1 Traveling wave solution

We are looking for a right traveling wave solution of the form [24]

$$u(\xi) := u(x - ct),$$

such as $u \to 0$, $u_\xi \to 0$ and $u_{\xi\xi} \to 0$ as $\xi \to \pm\infty$. Substitution into Eq. (3.1) leads to the ODE

$$u_{\xi\xi\xi} + 6uu_\xi - cu_\xi = 0.$$

An integration with respect to ξ yields

$$u_{\xi\xi} = -3u^2 + cu + c_1,$$

where c_1 is a constant of integration. Since $u \to 0$, $u_\xi \to 0$ and $u_{\xi\xi} \to 0$ as $\xi \to \pm\infty$, $c_1 = 0$. A second integration yields

$$\frac{1}{2}u_\xi^2 = -u^3 + \frac{1}{2}cu^2 + c_2,$$

where $c_2 = \text{const} = 0$. That is, the last equation can be written as

$$d\xi = \frac{du}{u\sqrt{c-2u}},$$

which can be integrated, yielding

$$u(\xi) = \frac{c}{2}\,\text{sech}^2\left(\frac{1}{2}\sqrt{c}\,(\xi - \xi_0)\right),$$

where ξ_0 is an arbitrary constant. In (x,t) coordinates the traveling wave solution reads

$$u(x,t) = \frac{c}{2}\,\text{sech}^2\left(\frac{1}{2}\sqrt{c}\,(x - x_0 - ct)\right). \tag{3.2}$$

Equation (3.2) describes the localized traveling wave solution with a negative amplitude (see Fig. 3.1 (a)), which is called *a soliton*. The term soliton was first introduced by Zabusky and Kruskal [28], who studied Eq. (3.1) with periodic boundary conditions numerically. They found [28, 24, 15] that initial condition of the form $u(x, 0) = \cos(2\pi x/L)$, $x \in [0, L]$ broke up into a train of solitary waves with successively large amplitude. Moreover the solitons seems to be almost unaffected in shape by passing through each other (though this could cause a change in their position). An example of two-soliton solution is shown on Fig. 3.1 (b).
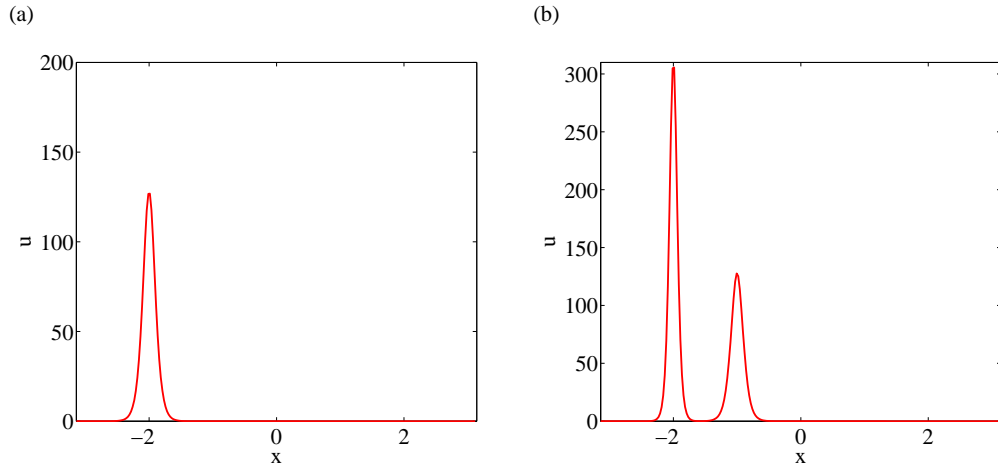
(a)   (b)

Fig. 3.1 Solitary solutions of KdV equation (3.1). (a)A single-soliton solution (3.2) for $c = 5$, calculated for $t = 0$ and $x_0 = -2$. (b) Two-soliton solution.

3.2 Numerical treatment

Consider the KdV Eq. (3.1) on the interval $x \in [-\pi, \pi]$ with the initial condition in form of the superposition of two solitons with velocities c_1 and c_2

$$u(x,0) = \frac{c_1^2}{2}\operatorname{sech}^2\left(\frac{c_1(x+2)}{2}\right) + \frac{c_2^2}{2}\operatorname{sech}^2\left(\frac{c_2(x+1)}{2}\right),$$

and periodic boundary conditions [25]. Notice that Eq. (3.1) is stiff. The stiffness results from the term u_{xxx} and appears as rapid linear oscillation of the heigh-wavenumber modes. Our goal is to constract the numerical solution of Eq. (3.1) using the method of integratig factors (see Appendix D). To this aim we rewrite the KdV equation (3.1) as

$$u_t + 3\left(u^2\right)_x + u_{xxx} = 0,$$

with Fourier transform

$$\hat{u}_t + 3ik\mathscr{F}[u^2] - ik^3\hat{u} = 0.$$

Now we multiply by *the integrating factor* e^{-ik^3t} and obtain

$$e^{-ik^3t}\hat{u}_t + 3ike^{-ik^3t}\mathscr{F}[u^2] - ik^3e^{-ik^3t}\hat{u} = 0.$$

Defining

$$\hat{U} = e^{-ik^3t}\hat{u}$$

the last relation is equivalent to

$$\hat{U}_t + ik^3\hat{U} + 3ike^{-ik^3t}\mathscr{F}[u^2] - ik^3\hat{U} = 0,$$

i.e.,

$$\hat{U}_t + 3ike^{-ik^3t}\mathscr{F}[u^2] = 0.$$

That is, in Fourier space we can rewrite the KdV equation (3.1) as

$$\hat{U}_t + 3ike^{-ik^3t}\mathscr{F}\left[\left(\mathscr{F}^{-1}(e^{ik^3t}\hat{U})\right)^2\right] = 0. \qquad (3.3)$$

Now, the time-intergation of the resulting equation can be done by, e.g., the standart RK4 method. The result of the calculation is shown on Fig. The simulation uses 256 grid points and is de-aliased using the usual $2/3$ rule. One can see, the solitons pass through each other as expected with only a change in phase.

Fig. 3.2 Numerical solution of the KdV equation (3.1) on the interval $x \in [-\pi, \pi]$ within IFM scheme, combined with the standart RK method. The initial condition is a superposition of two solitons with velocities $c_1^2 = 25$ and $c_2^2 = 16$. The solitons pass through each other as expected with only a change in phase