Appendix D
Exponential Time Differencing und Integrating Factor Methods

We are interested in the solution of nonlinear evolution equation of the form

$$u_t = \mathscr{L}u + \mathscr{N}(u), \tag{D.1}$$

$u = u(x,t)$, $x \in \Omega \subset \mathbb{R}^n$, $n = 1,2,3$ and $t \in [0, T]$. We also supply the initial conditions

$$u(x,0) = u_0(x) \quad \text{in}\,\Omega$$

and boundary conditions, i.e., periodic. The operators $\mathscr{L}$ and $\mathscr{N}$ denote linear and nonlinear parts, respectively.
Here we concentrate on time-discretization schemes with *exact* treatment of the *linear* part for solving the system of ODE's in question. That is, if the nonlinear term of the equation is zero, then the scheme reduces to the evalution of the exponential function of the operator representing the linear term. This approach is profitable if the corresponding system of ODE's for the mode amplitudes is *stiff*. Notice, that in general, in spectral and pseudo-spectral simulations, the *linear terms* are responsible for the stiffness of the set of ODE's for the mode's amplitudes. Indeed, if n is the order of the highest spatial derivative, the time scale, corresponding to the k'th mode, scales as $\mathscr{O}(k^{-n})$ for large k, that is, the highest modes evolve on short time scales [6].

D.1 Exponential Time Differencing Methods (ETD)

In order to simplify the notation we replace the linear operator $\mathscr{L}$ by a scalar q, i.e.,

$$u_t = qu + \mathscr{N}(u, t). \tag{D.2}$$

First we multiply (D.2) by *the integrating factor* e^{-qt} and integrate the equation over a single time step from $t = t_n$ to $t = t_n + h$ and obtain the *exact* relation

$$\big(u(t_{n+1})e^{-qh} - u(t_n)\big)e^{-qt_n} = \int_{t_n}^{t_n+h} e^{-qt}\mathscr{N}(u, t)dt \tag{D.3}$$

or, equivalently,

$$u(t_{n+1}) = u(t_n)e^{qh} + e^{qh}\int_0^h e^{-q\tau}\mathscr{N}(u(x, t_n+\tau), t_n+\tau)d\tau. \tag{D.4}$$

The difference between different ETD methods consist in difference approximations to the integral in the equation above.

The simplest approximation to the integral in (D.4) is that $\mathscr{N}$ is constant between $t = t_n$ and $t = t_{n+1} := t_n + h$, i.e.,

$$\mathscr{N} = \mathscr{N}_n + \mathscr{O}(h),$$

where $u_n = u(t_n)$ and $\mathscr{N}_n = \mathscr{N}(u_n, t_n)$. Then Eq. (D.4) becomes the scheme **ETD1**, given by

$$u_{n+1} = u_n e^{qh} + \mathscr{N}_n \frac{e^{qh} - 1}{q}. \tag{D.5}$$

Now let us consider the higher-order approximation of the form

$$\mathscr{N} = \mathscr{N}_n + \tau \frac{\mathscr{N}_n - \mathscr{N}_{n-1}}{h} + \mathscr{O}(h^2).$$

Then one obtains the so-called scheme **ETD2**

$$u_{n+1} = u_n e^{qh} + \mathscr{N}_n \frac{(1 + hq)e^{qh} - 1 - 2hq}{hq^2} + \mathscr{N}_{n-1} \frac{-e^{qh} + 1 + hq}{hq^2}. \tag{D.6}$$

Notice that ETD schemes of arbitary order can also be derived [6]. Other possibility is to use ETD schemes, combined with Runge-Kutta methods [6] or so-called Integrating Factor Methods, briefly discussed below.

D.2 Integrating Factor Methods (IFM)

The method of Integrating Factors (IFM) is also based on the idea that the problem in question can be transformed so that the linear part of the system is solved exactly. Integrating factor methods are usually obtained by rewriting (D.2) as [3, 6]

$$\frac{d}{dt}\left(ue^{-qt}\right) = e^{-qt}\mathscr{N}(u). \tag{D.7}$$

and then applying a time-stepping scheme to this equation.

For example, the forward Euler approximation reduces to

$$u_{n+1} = e^{qh}\Big(u_n + h\mathscr{N}(u_n)\Big) \tag{D.8}$$

In the same manner IFM can be embedded into different Runge-Kutta schemata. We mention only RK2 (Heun-Method):

$$u_{n+1} = u_n e^{qh} + \frac{h}{2}\Big(\mathscr{N}_n e^{qh} + \mathscr{N}((u_n + h\mathscr{N}_n)e^{qh}, t + h)\Big) \tag{D.9}$$

Other IFM schemata are discussed in, e.g., [3, 6] in more details.