
Problem Sheet 6: **To hand in until: 10.07.2017**

Problem 1: Stability of stripes and squares in the Swift-Hohenberg equation

Consider a two-dimensional Swift-Hohenberg equation

$$\partial_t \phi = \varepsilon\, \phi - (\Delta + 1)^2 \phi - \phi^3\,, \tag{1}$$

$\phi = \phi(\mathbf{r},\, t)$, $\mathbf{r} = (x,\, y)$ about the onset of an instability of the homogeneous state $\phi_0 = 0$ at $\varepsilon = 0$ with $q = q_c = 1$. In order to investigate the competition between stripes and squares a general ansatz which incorporates both patterns can be performed:

$$\phi(\mathbf{r},\, t) = \sum_{j=1}^{4} A_j(t)\, e^{\mathrm{i}\mathbf{q}_j \cdot \mathbf{r}}\,, \tag{2}$$

where A_j, $j = 1 \ldots 4$ are complex amplitudes with $A_{j+2} = A_j^*$ and $\mathbf{q}_1 = -\mathbf{q}_3$, $\mathbf{q}_2 = -\mathbf{q}_4$, $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$.

a) Substitute Eq. (2) into Eq. (1).

b) Project the resulting equation on the subspace, which is spanned by the plane waves $e^{\mathrm{i}\mathbf{q}_j \cdot \mathbf{r}}$. Show that the resulting *amplitude equations* read

$$\dot{A}_1 = \varepsilon\, A_1 - A_1 \left(3|A_1|^2 + 6|A_2|^2\right)\,, \tag{3}$$

$$\dot{A}_2 = \varepsilon\, A_2 - A_2 \left(3|A_2|^2 + 6|A_1|^2\right)\,. \tag{4}$$

c) Let $A_0 = (A_{01}, A_{02})^T \in \mathbb{R}^2$ be the steady state solution of the system (3)-(4), describing stripes or squares. To analyze their stability, consider a perturbed solution

$$A_j(t) = A_{0j} + a_j e^{\lambda t}\,, \qquad j = 1,\, 2\,. \tag{5}$$

Show that the linearisation in a_j results in the following linear eigenvalue problem

$$\lambda\, \mathbf{a} = J\, \mathbf{a}\,, \tag{6}$$

where $\mathbf{a} = (a_1\,, a_2)^T$ and

$$\mathbf{J} = \begin{pmatrix} \varepsilon - (9A_{01}^2 + 6A_{02}^2) & -12A_{01}A_{02} \\ -12A_{01}A_{02} & \varepsilon - (6A_{01}^2 + 9A_{02}^2) \end{pmatrix}\,. \tag{7}$$

d) Show that the stripes steady state solution of the system (3)-(4) is given by

$$A_{01} = \pm\sqrt{\frac{\varepsilon}{3}}\,, \qquad A_{02} = 0\,. \tag{8}$$

Using the trace-determinant criterion, show now that stripes exist and are stable for $\varepsilon > 0$.

e) Show that the squares steady state solution (3)-(4) is given by

$$A_{01} = A_{02} = \pm\frac{\sqrt{\varepsilon}}{3}\,. \tag{9}$$

Again, using the trace-determinant criterion, demonstrate that squares exist for $\varepsilon > 0$, but are always unstable.

f) Consider now a generalized Swift-Hohenberg equation

$$\partial_t \phi = \varepsilon\, \phi - (\Delta + 1)^2 \phi - b\phi^3 - c\phi\Delta^2(\phi^2)\,, \tag{10}$$

where b and c have arbitrary signs. Show that in this case the ansatz (2) leads to the following set of amplitude equations

$$\dot{A}_1 = \varepsilon\, A_1 - A_1\left(\alpha|A_1|^2 + \beta|A_2|^2\right)\,, \tag{11}$$

$$\dot{A}_2 = \varepsilon\, A_2 - A_2\left(\alpha|A_2|^2 + \beta|A_1|^2\right)\,, \tag{12}$$

where $\alpha = 3b + 16c$, $\beta = 6b + 16c$.

g) Investigate now the stability of stripes and squares for (10). Show that the stripe steady state solution

$$A_{01} = \pm\sqrt{\frac{\varepsilon}{\alpha}}\,, \quad A_{02} = 0 \tag{13}$$

exists for $\varepsilon > 0$, $b > -\dfrac{16}{3}c$ and is stable for all positive values of ε. Further, consider the squares steady state solution

$$A_{01} = A_{02} = \pm\sqrt{\frac{\varepsilon}{\alpha + \beta}} \tag{14}$$

and demonstrate that it exists for $b > -\dfrac{32\,c}{9}$ and is stable for $b < 0$.