
Problem Sheet 2: **To hand in until: 15.05.2017**

Problem 1: Scaling of nonlinear PDEs

Consider a one-dimensional Kuramoto-Sivashinsky equation of the form

$$\partial_t\psi = a_1\,\partial_x^2\psi + a_2\,\partial_x^4\psi + a_3\,(\partial_x\psi)^2\ , \tag{1}$$

where a_1, a_2, a_3 are real coefficients. Show that using the rescaling of dependent and independent variables, $x' = x/L$, $t' = t/T$ and $\phi(x', t') = \psi/\psi_0$ with L, T and ψ_0 to be determined, Eq. (1) can be written in the parameterless form

$$\partial_{t'}\phi = -\partial_{x'}^2\phi - \partial_{x'}^4\phi + (\partial_{x'}\phi)^2\ .$$

Thereby write the transformation equations explicitly.

Problem 2: Cole–Hopf transformation

Consider the viscid Burgers' equation

$$\partial_t u + u\,\partial_x u = \nu\partial_x^2 u\,, \tag{2}$$

where $u = u(x,\,t)$ and ν is the viscosity coefficient. In 1950 Hopf and in 1951 Cole independently introduced the method to solve the viscid Burgers' equation that has come to be known as the Cole-Hopf transformation.

a) Show that with the change of variable $u = \partial_x\psi$, Eq. (2) can be written as

$$\partial_t\psi + \frac{1}{2}\,(\partial_x\psi)^2 = \nu\partial_x^2\psi\,. \tag{3}$$

b) Show now that the ansatz $\psi = -2\nu\log\varphi$ transforms Eq. (3) into the heat equation

$$\partial_t\varphi = \nu\partial_x^2\varphi\,. \tag{4}$$

c) Solve the heat equation with the initial condition $\varphi(x, 0) = \exp\left(-\frac{1}{2\nu}\int\limits_0^x u(x', 0)\,dx'\right)$.

d) Reconstruct now the solution of the viscid Burgers' equation (2).

Problem 3: Korteweg-de Vries Equation

The Korteweg-de Vries Equation describes weakly nonlinear shallow water waves. The non-dimensional version of the equation reads

$$\partial_t u + 6\,u\,u_x + \partial_x^3 u = 0\,, \tag{5}$$

where $u = u(x,\,t)$.

a) Show that substitution of the traveling wave ansatz $u(\xi) = u(x - c\,t)$ with $u \to 0$, $u_\xi \to 0$ and $u_{\xi\xi} \to 0$ as $\xi \to \pm\infty$ into Eq. (5) leads to the equation

$$u_{\xi\xi\xi} + 6\,u\,u_\xi - c\,u_\xi = 0\,.$$

b) Show now that after two integrations with respect to ξ the last equation reduces to

$$d\xi = \frac{du}{u\sqrt{c-2\,u}}\,.$$

c) Integrate this equation using, i.e., `Maple` or `Mathematica` and verify that the solution of Eq. (5) is given by

$$u(x,t) = \frac{c}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}(x-x_0-ct)\right)\,, \tag{6}$$

where x_0 is an arbitrary constant. Equation (6) describes the localized traveling wave solution with a positive amplitude, which is called a soliton.

d) Draw the soliton solution (6) and discuss its properties.