Nützliche Relationen für Differentialoperatoren

Skalare Funktionen: $\phi(\vec{r}),\ \psi(\vec{r})$ Vektorfunktionen: $\vec{a}(\vec{r}),\ \vec{b}(\vec{r})$ konstanter Vektor: $\vec{c}$

Spezialfälle:

$$\text{div}\,\vec{r} = 3 \quad , \quad \text{rot}\,\vec{r} = 0 \quad , \quad \text{rot}\left(\vec{c}\times\vec{r}\right) = 2\vec{c} \quad , \quad (\vec{a}\cdot\nabla)\vec{r} = \vec{a} \quad , \quad \text{grad}\,\phi\left(r\right) = \frac{\partial\phi}{\partial r}\frac{\vec{r}}{r}$$

Produktregeln:

$$\text{grad}\,(\phi\psi) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi$$

$$\text{grad}\,(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times\text{rot}\,\vec{b} + \vec{b}\times\text{rot}\,\vec{a}$$

$$\text{div}\,(\phi\,\vec{a}) = \phi\,\text{div}\,\vec{a} + \vec{a}\,\text{grad}\,\phi$$

$$\text{div}\,(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{rot}\,\vec{a} - \vec{a}\cdot\text{rot}\,\vec{b}$$

$$\text{rot}\,(\phi\vec{a}) = \phi\,\text{rot}\,\vec{a} - \vec{a}\times\text{grad}\,\phi$$

$$\text{rot}\,(\vec{a}\times\vec{b}) = (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b} + \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a}$$

Darstellung in krummlinigen, orthogonalen Koordinaten (u,v,w)

Vektorfeld $\vec{a}(\vec{r}) = a_u(\vec{r})\,\vec{e}_u + a_v(\vec{r})\,\vec{e}_v + a_w(\vec{r})\vec{e}_w$

metrische Koeffizienten: $g_u = \left|\frac{\partial\vec{r}}{\partial u}\right| \quad , \quad g_v = \left|\frac{\partial\vec{r}}{\partial v}\right| \quad , \quad g_w = \left|\frac{\partial\vec{r}}{\partial w}\right|$

$$\text{grad}\,\phi(\vec{r}) = \vec{e}_u\,\frac{1}{g_u}\frac{\partial\phi}{\partial u} + \vec{e}_v\,\frac{1}{g_v}\frac{\partial\phi}{\partial v} + \vec{e}_w\,\frac{1}{g_w}\frac{\partial\phi}{\partial w}$$

$$\text{div}\,\vec{a}(\vec{r}) = \frac{1}{g_u g_v g_w}\left(\frac{\partial}{\partial u}\left(g_v g_w a_u\right) + \frac{\partial}{\partial v}\left(g_w g_u a_v\right) + \frac{\partial}{\partial w}\left(g_u g_v a_w\right)\right)$$

$$\text{rot}\,\vec{a}(\vec{r}) = \frac{\vec{e}_u}{g_v g_w}\left(\frac{\partial}{\partial v}\left(g_w a_w\right) - \frac{\partial}{\partial w}\left(g_v a_v\right)\right) + \frac{\vec{e}_v}{g_w g_u}\left(\frac{\partial}{\partial w}\left(g_u a_u\right) - \frac{\partial}{\partial u}\left(g_w a_w\right)\right) + \frac{\vec{e}_w}{g_u g_v}\left(\frac{\partial}{\partial u}\left(g_v a_v\right) - \frac{\partial}{\partial v}\left(g_u a_u\right)\right)$$

$$\Delta\phi\left(\vec{r}\right) = \nabla^2\phi\left(\vec{r}\right) = \frac{1}{g_u g_v g_w}\left[\frac{\partial}{\partial u}\left(\frac{g_v g_w}{g_u}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{g_w g_u}{g_v}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{g_u g_v}{g_w}\frac{\partial\phi}{\partial w}\right)\right]$$

Differentialoperatoren in Kugelkoordinaten (r, ϑ, φ)

Basisvektoren in kartesischer Darstellung:

$$\vec{e}_r = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \quad \vec{e}_\vartheta = \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix} \quad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

metrische Koeffizienten: $g_r = 1 \qquad g_\vartheta = r \qquad g_\varphi = r\sin\vartheta$

$$\text{grad}\,\phi(\vec{r}) = \vec{e}_r \frac{\partial\phi}{\partial r} + \vec{e}_\vartheta \frac{1}{r}\frac{\partial\phi}{\partial\vartheta} + \vec{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial\phi}{\partial\varphi}$$

$$\text{div}\,\vec{a}(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\, a_\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi}$$

$$\text{rot}\,\vec{a}(\vec{r}) = \frac{\vec{e}_r}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\, a_\varphi\right) - \frac{\partial a_\vartheta}{\partial\varphi}\right) + \frac{\vec{e}_\vartheta}{r\sin\vartheta}\left(\frac{\partial a_r}{\partial\varphi} - \sin\vartheta\frac{\partial}{\partial r}\left(r a_\varphi\right)\right) + \frac{\vec{e}_\varphi}{r}\left(\frac{\partial}{\partial r}\left(r a_\vartheta\right) - \frac{\partial a_r}{\partial\vartheta}\right)$$

$$\begin{aligned}\Delta\phi(\vec{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \frac{1}{\sin\vartheta}\frac{\partial^2\phi}{\partial\varphi^2}\right) \\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\phi\right) + \frac{1}{r^2}\left(\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{\tan\vartheta}\frac{\partial\phi}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2\phi}{\partial\varphi^2}\right)\end{aligned}$$

Differentialoperatoren in Zylinderkoordinaten (r, φ, z)

Basisvektoren in kartesischer Darstellung:

$$\vec{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \quad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \quad \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

metrische Koeffizienten: $g_r = 1 \qquad g_\varphi = r \qquad g_z = 1$

$$\text{grad}\,\phi(\vec{r}) = \vec{e}_r \frac{\partial\phi}{\partial r} + \frac{\vec{e}_\varphi}{r}\frac{\partial\phi}{\partial\varphi} + \vec{e}_z \frac{\partial\phi}{\partial z}$$

$$\text{div}\,\vec{a}(\vec{r}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r a_r\right) + \frac{1}{r}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_z}{\partial z}$$

$$\text{rot}\,\vec{a}(\vec{r}) = \vec{e}_r\left(\frac{1}{r}\frac{\partial a_z}{\partial\varphi} - \frac{\partial a_\varphi}{\partial z}\right) + \vec{e}_\varphi\left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right) + \frac{\vec{e}_z}{r}\left(\frac{\partial}{\partial r}\left(r a_\varphi\right) - \frac{\partial a_r}{\partial\varphi}\right)$$

$$\Delta\phi(\vec{r}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$$