Introduction to Model Theory

Martin Hils

Équipe de Logique Mathématique, Institut de Mathématiques de Jussieu
Université Paris Diderot – Paris 7

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Outline

First order languages

A first order language $\mathcal{L}$ is given by

- **constant symbols** $\{c_i\}_{i \in I}$;
- **relation symbols** $\{R_j\}_{j \in J}$ (R_j of some fixed arity n_j);
- **function symbols** $\{f_k\}_{k \in K}$ (f_k of some fixed arity n_k);
- a distinguished binary relation "=" for **equality**;
- an infinite set of **variables** $\{v_i \mid i \in \mathbb{N}\}$ (we also use x, y etc.);
- the **connectives** $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and
- the **quantifiers** $\forall, \exists$.

First order languages (continued)

$\mathcal{L}$-formulas are built inductively (in the obvious manner).

Let φ be an $\mathcal{L}$-formula.

- A variable x is **free** in φ if it is not bound by a quantifier.
- φ is called a **sentence** if it contains no free variables.
- We write $\varphi = \varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are among $\{x_1, \ldots, x_n\}$.

In what follows, we will only consider **countable** languages.

First order structures

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is a tuple $\mathcal{M} = (M; c_i^{\mathcal{M}}, R_j^{\mathcal{M}}, f_k^{\mathcal{M}})$, where

- M is a non-empty set, the **domain** of $\mathcal{M}$;
- $c_i^{\mathcal{M}} \in M$, $R_j^{\mathcal{M}} \subseteq M^{n_j}$, and $f_k^{\mathcal{M}} : M^{n_k} \to M$ are **interpretations** of the symbols in $\mathcal{L}$.

To interpret an $\mathcal{L}$-formula φ in $\mathcal{M}$, note that the quantified variables **run over** M.

Let $\varphi(x_1, \dots, x_n)$ and $\overline{a} \in M^n$ be given.

We set $\mathcal{M} \models \varphi(\overline{a})$ if and only if φ holds for $\overline{a}$ in $\mathcal{M}$.

Examples of languages and structures

- $\mathcal{L}_{rings} = \{0, 1, +, -, \cdot\}$ (**language of rings**).
  Any (unitary) ring is naturally an $\mathcal{L}_{rings}$-structure, e.g.
  $\mathcal{C} = (\mathbb{C}; 0, 1, +, -, \cdot)$ and $\mathcal{R} = (\mathbb{R}; 0, 1, +, -, \cdot)$.
  $\varphi \equiv \forall x \exists y\, y \cdot y = x$ is an $\mathcal{L}_{rings}$-formula (even a sentence),
  with $\mathcal{C} \models \varphi$ and $\mathcal{R} \models \neg\varphi$.
- $\mathcal{L}_{\mathrm{oag}} = \{0, +, <\}$ (**language of ordered abelian groups**)
  Let $\mathcal{Z} = (\mathbb{Z}; 0, +, <)$ and $\mathcal{Q} = (\mathbb{Q}; 0, +, <)$.
  Let $\psi(x, y) \equiv \exists z(x < z \wedge z < y)$.
  Then $\mathcal{Q} \models \psi(1, 2)$, $\mathcal{Z} \not\models \psi(1, 2)$ and $\mathcal{Z} \models \psi(0, 2)$.

We will often write M instead of $\mathcal{M}$, if the structure we mean is clear from the context.

First order theories

An $\mathcal{L}$-theory T is a set of $\mathcal{L}$-sentences.

- An $\mathcal{L}$-structure $\mathcal{M}$ is a **model** of T if $\mathcal{M} \models \varphi$ for every $\varphi \in T$. We denote this by $\mathcal{M} \models T$.
- T is called **consistent** if it has a model.

Examples

1. The usual field axioms, in $\mathcal{L}_{rings}$, give rise a theory T_{fields}, with $\mathcal{M} \models T_{fields}$ if and only if $\mathcal{M} = (M; 0, 1, +, -, \cdot)$ is a field.
2. Let $\varphi_n \equiv \forall z_0 \cdots \forall z_{n-1} \exists x \, x^n + z_{n-1}x^{n-1} + \ldots + z_0 = 0$. **ACF**$=T_{fields} \cup \{\varphi_n \mid n \geq 2\}$. (Models are **alg. closed fields**.)
3. There is an $\mathcal{L}_{\mathrm{oag}}$-theory **DOAG** whose models are preciseley the **non-trivial divisible ordered abelian groups**.
4. If $\mathcal{M}$ is an $\mathcal{L}$-structure, $\mathrm{Th}(\mathcal{M}) = \{\varphi \ \mathcal{L}\text{-sentence} \mid \mathcal{M} \models \varphi\}$.

The expressive power of first order logic

Theorem (Compactness Theorem)

Let T be a theory. Suppose that any finite subtheory T_0 of T has a model. Then T has a model.

Corollary

1. *If T has arbitrarily large finite models, it has an infinite model. Thus, there is e.g. no theory whose models are the finite fields.*
2. *If T has an infinite model, it has models of arbitrarily large cardinality. In particular, an infinite $\mathcal{L}$-structure is not determined (up to $\mathcal{L}$-isomorphism) by its theory.*

To prove (1), consider $\psi_n \equiv \exists x_1, \ldots, x_n \bigwedge_{i<j} x_i \neq x_j$, and apply compactness to $T' = T \cup \{\psi_n \mid n \in \mathbb{N}\}$.

Complete theories

Let T be a theory. A sentence ψ is a **consequence** of T, denoted $T \models \psi$, if every model of T is also a model of ψ.

$\mathcal{M}$ and $\mathcal{N}$ are called **elementarily equivalent** if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. We write $\mathcal{M} \equiv \mathcal{N}$.

A consistent theory T is complete if all its models are elementarily equivalent. Alternatively, for every φ, either $T \models \varphi$ or $T \models \neg\varphi$.

Examples

1. $\mathrm{Th}(\mathcal{M})$ is complete, for any structure $\mathcal{M}$.
2. ACF_p is a complete $\mathcal{L}_{rings}$-theory, for $p = 0$ or a prime.
3. DOAG is a complete $\mathcal{L}_{\mathrm{oag}}$-theory.

Definable sets

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A set $D \subseteq M^n$ is said to be definable if there is a formula $\varphi(\overline{x}, \overline{y})$ and parameters $\overline{b}$ from M such that

$$D = \varphi(\mathcal{M}, \overline{b}) := \left\{ \overline{a} \in M^n \mid \mathcal{M} \models \varphi(\overline{a}, \overline{b}) \right\}.$$

If $\overline{b}$ may be taken from $B \subseteq M$, we say D is B-definable.

Convenient to add parameters, passing to $\mathcal{L}_B = \mathcal{L} \cup \{c_b \mid b \in B\}$. Then $\mathcal{M}$ expands naturally to an $\mathcal{L}_B$-structure $\mathcal{M}_B$.

Examples

1. In $\mathbb{R}$, the set $\mathbb{R}_{\geq 0}$ is $\mathcal{L}_{rings}$-definable, as the set of squares.
2. Let $K \models \mathrm{ACF}$, and let $V = V(K) \subseteq K^n$ be an affine variety. Then V is definable in $\mathcal{L}_{rings}$ by a quantifier free formula. More generally, this is the case for every constructible subset of K^n.

Elementary substructures

- $\mathcal{M} \subseteq \mathcal{N}$ is a **substructure** if

$$c^{\mathcal{M}} = c^{\mathcal{N}}, f^{\mathcal{N}} \restriction_{M^n} = f^{\mathcal{M}} \text{ and } R^{\mathcal{N}} \cap M^n = R^{\mathcal{M}}.$$

- We say $\mathcal{M}$ is an **elementary** substructure of $\mathcal{N}$, $\mathcal{M} \preccurlyeq \mathcal{N}$ if for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every tuple $\overline{a} \in M^n$ one has

$$\mathcal{M} \models \varphi(\overline{a}) \text{ iff } \mathcal{N} \models \varphi(\overline{a}).$$

  In other words, the embedding respects all definable sets.

Note: $\mathcal{M} \preccurlyeq \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$.

Quantifier elimination

Definition

A theory T has quantifier elimination (QE) if for every formula $\varphi(\overline{x})$ there is a quantifier free (q.f.) formula $\psi(\overline{x})$ such that

$$T \models \forall \overline{x} \, (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})) \, .$$

Proposition

Let T be a (consistent) theory with QE.

- *In $\mathcal{M} \models T$, every definable set is q.f. definable. Equivalently, projections of q.f. definable sets are q.f. definable.*
- *Let $\mathcal{M}$ and $\mathcal{N}$ be models of T. Then $\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{M} \preccurlyeq \mathcal{N}$. ($T$ is **model complete**).*
- *If any two models of T contain a common substructure, then T is complete.*

Examples of theories with QE

Theorem (Chevalley-Tarski Theorem)

ACF has quantifier elimination.

Corollary

In algebraically closed fields, a set is definable iff it is constructible.

Corollary

ACF_p *is complete and* **strongly minimal**: *in every model* $\mathcal{M} \models \mathrm{ACF}_p$, *every definable subset of* M *is finite or cofinite.*

Remark

Model-completeness *of ACF* $\hat{=}$ *Hilbert's* **Nullstellensatz**.

Example

The theory of the real field $\mathcal{R} = (\mathbb{R}; 0, 1, +, -, \cdot)$ does not have QE. (The set of squares is not q.f. definable.)

Tarski's theorem

Let $\mathcal{L}_{o.rings} = \mathcal{L}_{rings} \cup \{<\}$, and let **RCF** (the **theory of real closed fields**) be the $\mathcal{L}_{o.rings}$-theory whose models are

- **ordered fields** F such that
- every positive element in F is a square in F and
- every polynomial of odd degree over F has a zero in F.

Theorem (Tarski 1951)

RCF is complete (so equal to $Th(\mathbb{R})$) and has QE.

Corollary

The definable sets in RCF are precisely the **semi-algebraic sets** *(sets defined by boolean combinations of polynomial inequalities).*

0-minimal theories

Definition

Let $\mathcal{L} = \{<, \ldots\}$. An $\mathcal{L}$-theory T is *o*-**minimal** if in any $M \models T$, any definable subset of M is a finite union of intervals and points.

Corollary

RCF is an o-minimal theory.

Proof.

Clearly, $p(X) \geq 0$ defines a set of the right form, for p a polynomial. We are done by Tarski's QE result. □

Proposition

1. *DOAG is complete and has QE (in $\mathcal{L}_{\mathrm{oag}}$).*
2. *Definable sets in DOAG are* **piecewise linear** *(given by bool. comb. of linear inequalities). In particular, DOAG is o-minimal.*

The notion of a complete type

Definition

Let $\mathcal{M}$ be a structure and $B \subseteq M$. A set $p(\overline{x})$ of $\mathcal{L}_B$-formulas $\varphi(x_1, \ldots, x_n)$ is a (complete) ***n*-type over B** if

- $p(\overline{x})$ is finitely satisfiable, i.e. for any $\varphi_1, \ldots, \varphi_k \in p$ there is $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi_i(\overline{a})$ for all i;
- $p(\overline{x})$ is maximal with this property.

Example

Let $\mathcal{N} \succcurlyeq \mathcal{M}$. For $\overline{a} \in N^n$, $\mathrm{tp}(\overline{a}/B) := \{\varphi(\overline{x}) \in \mathcal{L}_B \mid \mathcal{N} \models \varphi(\overline{a})\}$ is a complete n-type over B, the **type of $\overline{a}$ over B**.

Lemma

Every complete type p is of the form $p(\overline{x}) = \mathrm{tp}(\overline{a}/B)$.
Such a tuple $\overline{a}$ is called a **realisation of** *p.*

Type Spaces

- For $B \subseteq M$, let $S_n^{\mathcal{M}}(B)$ be the set of complete n-types over B.
- $\mathcal{M} \preccurlyeq \mathcal{N} \Rightarrow S_n^{\mathcal{M}}(B) = S_n^{\mathcal{N}}(B)$ canonically, so we write $S_n(B)$.
- For $\varphi = \varphi(x_1, \ldots, x_n) \in \mathcal{L}_B$, put $U_\varphi = \{p \in S_n(B) \mid \varphi \in p\}$.
  The sets U_φ form a **basis of clopen sets** for a topology on $S_n(B)$, the space of complete n-types over B, a profinite space.

Example (Type spaces in ACF)

Let $K \models \mathrm{ACF}$ and let $K_0 \subseteq K$ be a subfield. Then, by QE,

$$S_n(K_0) \cong \mathrm{Spec}(K_0[x_1, \ldots, x_n]), \text{ via}$$

$$p(\overline{x}) \mapsto \{f(\overline{x}) \in K_0[\overline{x}] \mid f(\overline{x}) = 0 \text{ is in } p\},$$

as types are determined by the polynomial equations they contain.

Space of 1-types in *o*-minimal theories

Let T be *o*-minimal (e.g. $T = \mathrm{DOAG}$ or RCF) and $\mathcal{D} \models T$.

Note $D \hookrightarrow S_1(D)$ naturally, via $d \mapsto \mathrm{tp}(d/D)$.

For $p(x) \in S_1(D) \setminus D$, let $C_p := \{d \in D \mid d < x \text{ is in } p\}$.
The map $p \mapsto C_p$ induces a bijection between

- $S_1(D) \setminus D$ and
- **cuts** in D (viewed as initial pieces).

Hence, we have

$$S_1(D) \overset{1:1}{\longleftrightarrow} D \,\dot\cup\, \{\text{cuts in } (D,<)\}.$$

Saturation

Definition

Let κ be an infinite cardinal. An $\mathcal{L}$-structure $\mathcal{M}$ is **κ-saturated** if for every $B \subseteq M$ with $|B| < \kappa$, every $p \in S_n(B)$ is realised in $\mathcal{M}$.

Remark

It is enough to check the condition for $n = 1$.

Examples

1. $K \models \mathrm{ACF}$ is κ-saturated if and only if $\mathrm{tr.deg}(K) \geq \kappa$.
2. $\mathbb{R} \models \mathrm{RCF}$ is not $\aleph_0$-saturated: the type $p_\infty(x) \in S_1(\emptyset)$ determined by $\{x > n \mid n \in \mathbb{N}\}$ is not realised in $\mathbb{R}$.

Homogeneity

Definition

Let κ be given. An $\mathcal{L}$-structure $\mathcal{M}$ is κ-**homogeneous** if for all $B \subseteq M$ with $|B| < \kappa$ and all $\overline{a}, \overline{b} \in M^n$ with $\mathrm{tp}(\overline{a}/B) = \mathrm{tp}(\overline{b}/B)$ there is $\sigma \in \mathrm{Aut}_B(\mathcal{M})$ s.t. $\sigma(\overline{a}) = \overline{b}$.

Remark

It is enough to check the condition for $n = 1$.

Example

Let $K \models \mathrm{ACF}$. Then K is $|K|$-homogeneous.

Fact

Let κ and $\mathcal{M}$ be given. There exists an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ which is κ-saturated and κ-homogeneous.

The Universe

Let T be complete and κ a very big cardinal.

A universe $\mathcal{U}$ for T is a κ-saturated and κ-homogeneous model.

When working with a universe $\mathcal{U}$,

- "**small**" means "of cardinality $< \kappa$";
- "$\mathcal{M} \models T$" means "$\mathcal{M} \preccurlyeq \mathcal{U}$ and M is small";
- similarly, all parameter sets B are small subsets of U.

We write $\mathcal{U}$ for some **fixed universe** (for T).

Fact

Let D be a definable set in $\mathcal{U}$, and let $B \subseteq U$ be a set of parameters. TFAE:

1. *D is B-definable.*
2. *$\sigma(D) = D$ for all $\sigma \in \mathrm{Aut}_B(\mathcal{U})$.*

Definable and algebraic closure I

Definition

Let $B \subseteq \mathcal{U}$ be a set of parameters and $a \in \mathcal{U}$.

- a is **definable over** B if $\{a\}$ is a B-definable set;
- a is **algebraic over** B if there is a finite B-definable set containing a.
- The **definable closure of** B is given by

$$\mathrm{dcl}(B) = \{a \in \mathcal{U} \mid a \text{ definable over } B\}.$$

- Similarly define $\mathrm{acl}(B)$, the **algebraic closure of** B.

Definable and algebraic closure II

Examples

- In **ACF**, if K denotes the field generated by B, then $\mathrm{dcl}(B) = K^{1/p^\infty}$ and $\mathrm{acl}(B) = K^{alg}$.
- In **DOAG**, $\mathrm{dcl}(B) = \mathrm{acl}(B)$ is the divisible hull of $\langle B \rangle$.
- In **RCF**, $\mathrm{dcl}(B) = \mathrm{acl}(B)$ equals the real closure of the field generated by B.

Fact

1. $a \in \mathrm{dcl}(B)$ *if and only if* $\sigma(a) = a$ *for all* $\sigma \in \mathrm{Aut}_B(\mathcal{U})$
2. $a \in \mathrm{acl}(B)$ *if and only if there is a* **finite set** A_0 *containing* a *which is* **fixed set-wise** *by every* $\sigma \in \mathrm{Aut}_B(\mathcal{U})$.

A criterion for QE

The following criterion is often useful in practice.

We will use it in the context of valued fields.

Theorem

Let T be a theory and κ an infinite cardinal. TFAE:

1. *T has QE.*
2. *Let $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N} \models T$. Assume*
   - *$|M| < \kappa$ and*
   - *$\mathcal{N}$ is κ-saturated.*

   Then $\mathcal{M}$ may be embedded into $\mathcal{N}$ over $\mathcal{A}$.

Valued fields: notations and choice of a language

Let K be a valued field. We use standard notation:

- $\mathrm{val} : K^\times \to \Gamma$ (the **valuation map**)
- $\Gamma = \Gamma_K$ is an ordered abelian group (written additively), plus a distinguised element ∞ ($+$ and $<$ are extended as usual);
- $\mathcal{O} = \mathcal{O}_K \supseteq \mathfrak{m} = \mathfrak{m}_K$;
- $\mathrm{res} : \mathcal{O} \to k = k_K := \mathcal{O}/\mathfrak{m}$ is the **residue map**.
- For $a \in K$ and $\gamma \in \Gamma$ denote $B_{\geq\gamma}(a)$ (resp. $B_{>\gamma}(a)$) the **closed** (resp. **open**) **ball** of radius γ around a.
- K gives rise to an $\mathcal{L}_{\mathrm{div}} = \mathcal{L}_{rings} \cup \{\,\mathrm{div}\,\}$-structure, via
$$x \,\mathrm{div}\, y :\Leftrightarrow \mathrm{val}(x) \leq \mathrm{val}(y).$$
- $\mathcal{O}_K = \{x \in K \,:\, x \,\mathrm{div}\, 1\}$, so $\mathcal{O}_K$ is $\mathcal{L}_{\mathrm{div}}$-definable
$\Rightarrow$ the valuation is encoded in the $\mathcal{L}_{\mathrm{div}}$-structure.

QE in algebraically closed valued fields

ACVF: $\mathcal{L}_{\mathrm{div}}$-theory of **alg. closed non-trivially valued fields**

Theorem (Robinson)

The theory ACVF has QE. Its completions are given by $\mathrm{ACVF}_{p,q}$*, for* $(p, q) = (char(K), char(k))$*.*

Corollary

1. *In ACVF, a set is definable iff it is* **semi-algebraic***, i.e. a boolean combination of sets given by polynomial equations and valuation inequalities.*
2. *In particular, definable sets in 1 variable are (finite) boolean combinations of singletons and balls.*
3. *If* $K_0 \subseteq K \models \mathrm{ACVF}$ *is a subfield, then* $\mathrm{acl}(K_0) = K_0^{alg}$ *and* $\mathrm{dcl}(K_0) = \left(K_0^{1/p^\infty}\right)^h$.

Classification of purely transcendental extensions

For $i = 1, 2$, let $L_i = K(t_i)$ be valued fields, with $t_i \notin K = K^{alg}$.

- (**residual** case) If $\mathrm{val}(t_i) = 0$ and $\mathrm{res}(t_i) \notin k_K$ for $i = 1, 2$, then $t_1 \mapsto t_2$ induces an isomorphism $L_1 \cong_K L_2$.
- (**ramified** case) If $\gamma_i = \mathrm{val}(t_i) \notin \Gamma_K$ for $i = 1, 2$, and γ_1 and γ_2 define the same cut in Γ_K, then $L_1 \cong_K L_2$ via $t_1 \mapsto t_2$.
- (**immediate** case) If there is a pseudo-Cauchy sequence (a_ρ) in K without pseudo-limit in K such that $a_\rho \Rightarrow t_i$ for $i = 1, 2$, then $L_1 \cong_K L_2$ via $t_1 \mapsto t_2$.

The proof of QE in ACVF

We use the criterion.

Let $L, L^* \models \mathrm{ACVF}$, and $A \subseteq L, L^*$ a common $\mathcal{L}_{\mathrm{div}}$-substructre. Assume L is countable and L^* is $\aleph_1$-saturated. We have to show that L embeds into L^* over A.

- WMA $A = K$ is a field. (Easy)
- WMA $K = K^{alg}$. (Extensions of $\mathcal{O}_K$ to K^{alg} are Gal(K)-conj.)
  $\Rightarrow$ Enough to K-embed $K(t)$ into L^*, for $t \notin K = K^{alg}$:
- $K(t)/K$ is either residual, or ramified, or immediate.
- **Residual case**: replacing t by $at + b$ for $a, b \in K$, WMA $\mathrm{val}(t) = 0$ and $\mathrm{res}(t) \notin k = k^{alg}$.
  By saturation $\exists t^* \in \mathcal{O}_{L^*}$ s.t. $\mathrm{res}(t^*) \notin k$, so $t \mapsto t^*$ works.
- The other cases are treated similarly. □

Multi-sorted languages and structures

A multi-sorted language $\mathcal{L}$ is given by

- a non-empty family of **sorts** $\{S_i \mid i \in I\}$;
- **constants** c, where c specifies the sort $S_{i(c)}$ it belongs to;
- **relation symbols** $R \subseteq S_{i_1} \times \cdots \times S_{i_n}$, for $i_1, \ldots, i_n \in I$;
- **function symbols** $f : S_{i_1} \times \cdots \times S_{i_n} \to S_{i_0}$;
- **variables** $(v_j^i)_{j \in \mathbb{N}}$ running over the sort S_i (for every i).

$\mathcal{L}$-formulas are built in the obvious way.

An $\mathcal{L}$-**structure** $\mathcal{M}$ is given by

- non-empty **base sets** $S_i^{\mathcal{M}} = M_i$ for every $i \in I$;
- **interpretations** of the symbols, subject to the sort restrictions, e.g. $c^{\mathcal{M}} \in M_{i(c)}$.

A variant: valued fields in a three-sorted language

Let $\mathcal{L}_{k,\Gamma}$ be the following 3-sorted language, with sorts K, Γ and k:

- Put $\mathcal{L}_{rings}$ on K, $\{0, +, <, \infty\}$ on Γ and $\mathcal{L}_{rings}$ on k;
- $\mathrm{val} : K \to \Gamma$, and
- $\mathrm{RES} : K^2 \to k$ as additional function symbols.

A valued field K is naturally an $\mathcal{L}_{k,\Gamma}$-structure, via

$$\mathrm{RES}(x, y) := \begin{cases} \mathrm{res}(xy^{-1}), \text{ if } \mathrm{val}(x) \geq \mathrm{val}(y) \neq \infty; \\ 0 \in k, \text{ else.} \end{cases}$$

ACVF in the three-sorted language

Theorem

ACVF eliminates quantifiers in $\mathcal{L}_{k,\Gamma}$.

Remark

The proof is similar to the one in the one-sorted context (in $\mathcal{L}_{\mathrm{div}}$).

Corollary

In ACVF, the following holds:

1. Γ *is a* **pure divisible ordered abelian group***: any definable subset of Γ^n is $\{0, +, <\}$-definable (with parameters from Γ).*
2. k *is a* **pure ACF***: any definable subset of k^n is $\mathcal{L}_{rings}$-definable.*

The Ax-Kochen-Eršov principle

Lemma

The class of henselian valued fields is axiomatisable in $\mathcal{L}_{k,\Gamma}$.

Theorem (Ax-Kochen, Eršov)

Let K and K' be henselian valued fields of equicharacteristic 0. Then, the following holds:

1. $K \equiv K'$ *iff* $k \equiv k'$ *and* $\Gamma \equiv \Gamma'$;
2. *if* $K \subseteq K'$, *then* $K \preccurlyeq K'$ *iff* $k \preccurlyeq k'$ *and* $\Gamma \preccurlyeq \Gamma'$.

A general transfer principle

Corollary

For any $\mathcal{L}_{k,\Gamma}$-sentence φ there is $N \in \mathbb{N}$ s.t. for any $p > N$,

$$\mathbb{Q}_p \models \varphi \quad \textit{iff} \quad \mathbb{F}_p((t)) \models \varphi.$$

Idea of the proof.

Else, applying compactness, one may find henselian valued fields K, K' of equicharacteristic 0 with $\Gamma \cong \Gamma' \equiv \mathbb{Z}$ and $k \cong k'$ such that $K \models \varphi$ and $K' \models \neg\varphi$, contradicting the AKE principle. □

Remark

Ever since the **approximate solution** *to* **Artin's Conjecture**, *this kind of transfer principle has shown to be extremely powerful.*

QE in p-adic fields

Let $\mathcal{L}_{\mathrm{Mac}} = \mathcal{L}_{rings} \cup \{P_n \mid n \geq 1\}$, with P_n a new unary predicate.

Any field K gets an $\mathcal{L}_{\mathrm{Mac}}$-structure, letting $P_n(x) \leftrightarrow \exists y\, y^n = x$.

If $K = \mathbb{Q}_p$, then $\mathbb{Z}_p$ is $\mathcal{L}_{\mathrm{Mac}}$-definable in a quantifier-free way:

$$x \in \mathbb{Z}_p \iff \mathbb{Q}_p \models P_2(1 + px^2) \quad \text{(assume } p \neq 2\text{)}$$

Theorem (Macintyre)

$\mathbb{Q}_p$ has QE in $\mathcal{L}_{\mathrm{Mac}}$.

Remark

Along with p-adic cell decompostion, this was used by Denef in his work on p-adic integration, giving **rationality** *results for various* **Poincaré series** *associated to an algebraic variety.*

Angular component maps

A map $\mathrm{ac} : K \to k$ is an angular component if

- $\mathrm{ac}(0) = 0$;
- $\mathrm{ac} \restriction_{K^\times} : K^\times \to k^\times$ is a group homomorphism;
- $\mathrm{val}(x) = 0 \Rightarrow \mathrm{ac}(x) = \mathrm{res}(x)$.

Example

In $K = k((\Gamma))$, mapping an element to its **leading coefficient** defines an angular component map. (This also works in $\mathbb{Q}_p$.)

Fact

1. *Let $s : \Gamma \to K^\times$ be a cross-section (homomorphic section of val). Then $\mathrm{ac}(a) := \mathrm{res}\left(s(a)^{-1}a\right)$ is an angular component.*
2. *If K is an $\aleph_1$-saturated valued field, then K admits a cross-section, so in particular an angular component map.*

Relative QE in Pas' language

Let $\mathcal{L}_{\mathrm{Pas}} = \mathcal{L}_{k,\Gamma} \cup \{\mathrm{ac}\}$, where $\mathrm{ac} : K \to k$.

Let T_{Pas} be the $\mathcal{L}_{\mathrm{Pas}}$-theory of **henselian** valued fields of **equicharacteristic 0** with an angular component map.

Theorem (Pas)

T_{PAS} admits **elimination of field quantifiers***:*

If $\varphi(\overline{x}_f, \overline{x}_\gamma, \overline{x}_r)$ is an $\mathcal{L}_{\mathrm{Pas}}$-formula, with variables $\overline{x}_f, \overline{x}\gamma$ and $\overline{x}_r$ running over the sorts K, Γ and k, respectively, there is an $\mathcal{L}_{\mathrm{Pas}}$-formula $\psi(\overline{x}_f, \overline{x}_\gamma, \overline{x}_r)$ without field quantifiers such that φ and ψ are equivalent modulo T_{Pas}.

Remark

The map ac is not definable in $\mathcal{L}_{k,\Gamma}$. Thus, passing from $\mathcal{L}_{k,\Gamma}$ to $\mathcal{L}_{\mathrm{Pas}}$ leads to more definable sets.

Extensions to valued difference fields

A **valued difference field** is a valued field K together with a distinguished automorphism $\sigma \in \mathrm{Aut}(K)$.

$\Rightarrow$ get induced automorphisms σ_Γ on Γ and σ_{res} on k.

Remark

AKE principles and relative QE in Pas' language have recently been obtained for several classes of valued difference fields:

- *in the* **Witt Frobenius case**, *where* $\sigma_\Gamma = \mathrm{id}$ *(work by Scanlon, Bélair-Macintyre-Scanlon, Azgin-van den Dries);*
- *in the* ω-**increasing case** *(e.g. the non-standard Frobenius), where one has* $\gamma > 0 \Rightarrow \sigma_\Gamma(\gamma) > n\gamma \, \forall \, n \in \mathbb{N}$ *(work by Hrushovski, Azgin).*

Context

- $\mathcal{L}$ is some countable language (possibly many-sorted);
- T is a **complete** $\mathcal{L}$-theory;
- $\mathcal{U} \models T$ is a fixed **universe** (i.e. very saturated and homogeneous);
- all models $\mathcal{M}$ we consider (and all parameter sets A) are **small**, with $\mathcal{M} \preccurlyeq \mathcal{U}$;
- there is a **dominating sort** S_{dom}: for every sort S from $\mathcal{L}$ there is $n \in \mathbb{N}$ and an n-ary function π_S in $\mathcal{L}$,

  $$\pi_S : S_{dom}^n \to S$$

  such that $\pi_S^{\mathcal{U}}$ is surjective.
- E.g., the field sort is a dominating sort for a theory of valued fields considered in $\mathcal{L}_{k,\Gamma}$ (3-sorted).

Imaginary Sorts and Elements

Definition

An imaginary element in $\mathcal{U}$ is an equivalence class d/E, where E is a definable equivalence relation on some $D \subseteq_{def} U^n$ and $d \in D(\mathcal{U})$.

If $D = U^n$ for some n and E is definable without parameters, the set of equivalence classes U^n/E is called an imaginary sort.

Examples of Imaginaries I

Unordered Tuples

- In any theory, the formula

$$(x = x' \wedge y = y') \vee (x = y' \wedge y = x')$$

defines an equiv. relation $(x, y)E_2(x', y')$ on pairs, with

$$(a, b)E_2(a', b') \Leftrightarrow \{a, b\} = \{a', b'\}.$$

Thus, $\{a, b\}$ may be thought of as an imaginary element.

- Similarly, $\{a_1, \ldots, a_n\}$ may be thought of as an imaginary.

Examples of Imaginaries II

A group $(G, \cdot)$ is a definable group in $\mathcal{U}$ if, for some $k \in \mathbb{N}$,

- $G \subseteq_{def} U^k$ and
- $\Gamma = \{(f, g, h) \in G^3 \mid f \cdot g = h\} \subseteq_{def} U^{3k}$.

Example (Cosets)

Let $(G, \cdot)$ be definable group in $\mathcal{U}$, and let $H \leq G$ a definable subgroup of G. Then any coset $g \cdot H$ is an imaginary.

(Note that $gEg' \Leftrightarrow \exists h \in H\, g \cdot h = g'$ is definable.)

Shelah's $\mathcal{M}^{eq}$-Construction

There is a canonical way, due to S. Shelah, of expanding

- $\mathcal{L}$ to a many-sorted language $\mathcal{L}^{eq}$,
- T to a (complete) $\mathcal{L}^{eq}$-theory T^{eq} and
- $\mathcal{M} \models T$ to $\mathcal{M}^{eq} \models T^{eq}$ such that
- $\mathcal{M} \mapsto \mathcal{M}^{eq}$ is an equivalence of categories between $\langle Mod(T), \preccurlyeq \rangle$ and $\langle Mod(T^{eq}), \preccurlyeq \rangle$.

Shelah's $\mathcal{M}^{eq}$-Construction (continued)

For any $\emptyset$-definable equivalence relation E on S_{dom}^n we add

- a new **imaginary sort** S_E (S_{dom} is called the **real sort**), a new function symbol $\pi_E : S_{dom}^n \to S_E$ $\Rightarrow$ obtain $\mathcal{L}^{eq}$;
- axioms stating that π_E is surjective and that its fibres correspond to E-classes $\Rightarrow$ obtain T^{eq};
- the interpretation of π_E and S_E on models $\mathcal{M} \models T$ according to the axioms $\Rightarrow$ obtain $\mathcal{M}^{eq}$.

Existence of codes for definable sets in $\mathcal{U}^{eq}$

Fact

For any definable $D \subseteq \mathcal{U}^n$ there exists $c \in \mathcal{U}^{eq}$ such that $\sigma \in \mathrm{Aut}(\mathcal{U})$ fixes D setwise iff it fixes c.

Proof.

Suppose D is defined by $\varphi(\overline{x}, \overline{d})$. Define an equivalence relation

$$E(\overline{z}, \overline{z}') :\Leftrightarrow \forall \overline{x}(\varphi(\overline{x}, \overline{z}) \leftrightarrow \varphi(\overline{x}, \overline{z}')).$$

Then $c := \overline{d}/E$ serves as a code for D. □

We sometimes write $\ulcorner D \urcorner = \ulcorner \varphi(\overline{x}, \overline{b}) \urcorner$ for this code (it is unique up to interdefinability).

Galois Correspondence in T^{eq}

The definitions of definable / algebraic closure make sense in $\mathcal{U}^{eq}$. We write dcl^{eq} or acl^{eq} to stress that we work in $\mathcal{U}^{eq}$.

- For $B \subseteq \mathcal{U}^{eq}$, any $\sigma \in \mathrm{Aut}_B(\mathcal{U})$ fixes $\mathrm{acl}^{eq}(B)$ setwise.
- $\mathrm{Gal}(B) := \{\sigma \restriction_{\mathrm{acl}^{eq}(B)} \mid \sigma \in \mathrm{Aut}_B(\mathcal{U})\}$ is called the absolute Galois group of B.

Theorem (Poizat)

The map

$$H \mapsto \{a \in \mathrm{acl}^{eq}(B) \mid h(a) = a \; \forall\, h \in H\}$$

induces a bijection between the set of closed subgroups of $\mathrm{Gal}(B)$ *and* $\mathcal{D} = \{A \mid B \subseteq A = \mathrm{dcl}^{eq}(A) \subseteq \mathrm{acl}^{eq}(B)\}$.

Elimination of Imaginaries

Definition (Poizat)

The theory T eliminates imaginaries if every imaginary element $a \in \mathcal{U}^{eq}$ is interdefinable with a real tuple $\overline{b} \in \mathcal{U}^n$.

Fact

- *Suppose that for every $\emptyset$-definable equivalence relation E on $\mathcal{U}^n$ there is an $\emptyset$-definable function*

  $$f : \mathcal{U}^n \to \mathcal{U}^m \text{ (for some } m \in \mathbb{N}\text{)}$$

  such that $E(\overline{a}, \overline{a}')$ if and only if $f(\overline{a}) = f(\overline{a}')$.

  Then T eliminates imaginaries.
- *The converse is true if there are two distinct $\emptyset$-definable elements in $\mathcal{U}$.*

Examples of theories which eliminate imaginaries

1. T^{eq} (for an arbitrary theory T)
2. ACF (Poizat)

   This follows from

   - the existence of a **smallest field of definition** of a variety, and
   - the fact that **finite sets** can be coded using **symmetric functions**, e.g. $\{a, b\}$ is coded by $(a + b, ab)$.
3. RCF (see the following slides)

Theorem (Definable choice in RCF)

Let $R \models \mathrm{RCF}$ and let $(D_a)_{a\in R^k}$ be a definable family of non-empty subsets of R^n. Then there is a definable function $f : R^k \to R^n$ s.t. $f(a) \in D_a \; \forall\, a \in R^k$. Furthermore, if $D_a = D_b$, then $f(a) = f(b)$.

Proof.

Projecting and using induction, it suffices to treat the case $n = 1$. D_a is a finite union of intervals. Let I be the leftmost interval.

- If I is reduced to a point, we let $f(a)$ be this point;
- if $I = R$, let $f(a) = 0$;
- if $Int(I) =]c, +\infty[$, let $f(a) = c + 1$;
- if $Int(I) =] - \infty, c]$, let $f(a) = c - 1$;
- if $Int(I) =]c, d[$, let $f(a) = \frac{c+d}{2}$.

Clearly, this construction is uniform and gives what we want. □

Elimination of imaginaries in RCF and in DOAG

Corollary

The theory RCF eliminates imaginaries.

In proving definable choice, we only used that the theory is an *o*-minimal expansion of DOAG (with some non-zero element named). From this, one may easily infer the following.

Corollary

DOAG eliminates imaginaries. More generally, any o-minimal expansion of DOAG eliminates imaginaries.

Utility of Elimination of Imaginaries

T has EI $\Rightarrow$ many constructions may be done already in T:

- quotient objects are present in $\mathcal{U}$
  (e.g. a definable group modulo a definable subgroup)
  $\Rightarrow$ easier to classify e.g. interpretable groups and fields in $\mathcal{U}$;
- every definable set admits a real tuple as a code
- get a Galois correspondence in T, replacing dcl^{eq}, acl^{eq} by dcl and acl, respectively.

In search for imaginaries in ACVF

Consider $K \models \mathrm{ACVF}$ (in $\mathcal{L}_{\mathrm{div}}$).

- Clearly, k and Γ are imaginary sorts, i.e. $k, \Gamma \subseteq K^{eq}$.
- More generally, $\mathcal{B}^o$ and $\mathcal{B}^{cl}$ (the set of open / closed balls) are imaginary sorts.

Fact

There is no definable bijection between k and a subset of K^n, similarly for Γ instead of k.

Proof idea.

- By QE, any infinite def. subset of K contains an open ball.
- Thus, every infinite definable subset of K^n admits definable maps with infinite image to k as well as to Γ.
- But, using QE in $\mathcal{L}_{k,\Gamma}$, it is easy to see that every definable subset of $k \times \Gamma$ is a finite union of rectancles $D \times E$. □

In search for imaginaries in ACVF (continued)

Question

Does (K, k, Γ) eliminate imaginaries (in $\mathcal{L}_{k,\Gamma}$)?

- The answer is NO (Holly).
- The answer is NO even if in addition $\mathcal{B}^o$ and $\mathcal{B}^{cl}$ are added. (Haskell-Hrushovski-Macpherson)

Sketch: Let $\gamma > 0$ and let b_1, b_2 be generic elements of $\mathcal{O}$.

Let A_i be the set of open balls of radius γ inside $B_{\geq\gamma}(b_i)$. Then A_i is a definable affine space over k.

It can be shown that a generic affine morphism between A_1 and A_2 cannot be coded in $K \cup \mathcal{B}^o \cup \mathcal{B}^{cl}$.

The geometric sorts

- $s \subseteq K^n$ is a lattice if it is a free $\mathcal{O}$-submodule of rank n;
- for $s \subseteq K^n$ a lattice, $s/\mathfrak{m}s \cong_k k^n$.

For $n \geq 1$, let

$$S_n := \{\text{lattices in } K^n\},$$

$$T_n := \dot{\bigcup}_{s \in S_n} s/\mathfrak{m}s.$$

Fact

1. *S_n and T_n are imaginary sorts, $S_1 \cong \Gamma$ (via $a\mathcal{O} \mapsto \mathrm{val}(a)$), and also $k = \mathcal{O}/\mathfrak{m} \subseteq T_1$.*
2. $S_n \cong \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O}) \cong \mathrm{B}_n(K)/\mathrm{B}_n(\mathcal{O})$
3. *There is a similar description of T_n as a finite union of coset spaces.*

Classification of Imaginaries in ACVF

$\mathcal{G} = \{K\} \cup \{S_n,\ n \geq 1\} \cup \{T_n,\ n \geq 1\}$ are the geometric sorts.
Let $\mathcal{L}_{\mathcal{G}}$ be the (natural) language of valued fields in $\mathcal{G}$.

Theorem (Haskell-Hrushovski-Macpherson 2006)

ACVF eliminates imaginaries down to **geometric sorts**, *i.e. the theory ACVF considered in $\mathcal{L}_{\mathcal{G}}$ has EI.*

Using this result, Hrushovski and Martin were able to classify the imaginaries in the p-adics:

Theorem (Hrushovski-Martin 2006)

$\mathbb{Q}_p$ *eliminates imaginaries down to* $\{K\} \cup \{S_n,\ n \geq 1\}$.

Classification of Imaginaries in ACVF (cont'd)

Some consequences of the classification of imaginaries in ACVF:

1. May do **Geometric Model Theory** in valued fields.
2. Development of **stable domination** as a by-product
   $\Rightarrow$ apply methods from stability outside the stable context.
3. There are striking applications outside model theory:
   - in **representation theory** (Hrushovski-Martin);
   - in **non-archimedean geometry** (Hrushovski-Loeser).

The notion of a definable type

- As before, T is a **complete** $\mathcal{L}$-theory;
- $\mathcal{U} \models T$ is very saturated and homogeneous.

Definition

Let $\mathcal{M} \models T$ and $A \subseteq M$. A type $p(\overline{x}) \in S_n(M)$ p is A-definable if for every $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ there is an $\mathcal{L}_A$-formula $d_p\varphi(\overline{y})$ s.t.

$$\varphi(\overline{x}, \overline{b}) \in p \;\Leftrightarrow\; \mathcal{M} \models d_p\varphi(\overline{b}) \quad (\text{for every } \overline{b} \in M)$$

We say p is definable if it is definable over some $A \subseteq M$.

The collection $(d_p\varphi)_\varphi$ is called a defining scheme for p.

Remark

If $p \in S_n(M)$ is definable via $(d_p\varphi)_\varphi$, then the same scheme gives rise to a (unique) type over any $\mathcal{N} \succcurlyeq \mathcal{M}$, denoted by $p \mid N$.

Definable types: first properties

- **(Realised types are definable)**
  Let $\overline{a} \in M^n$. Then $\mathrm{tp}(\overline{a}/M)$ is definable.
  (Take $\mathrm{d}_p\,\varphi(\overline{y}) = \varphi(\overline{a}, \overline{y})$.)

- **(Preservation under definable functions)**
  Let $\overline{b} \in \mathrm{dcl}(M \cup \{\overline{a}\})$, i.e. $f(\overline{a}) = \overline{b}$ for some M-definable function f. Then, if $\mathrm{tp}(\overline{a}/M)$ is definable, so is $\mathrm{tp}(\overline{b}/M)$.

- **(Transitivity)** Let $\overline{a} \in N$ for some $\mathcal{N} \succcurlyeq \mathcal{M}$, $A \subseteq M$. Assume
  - $\mathrm{tp}(\overline{a}/M)$ is A-definable;
  - $\mathrm{tp}(\overline{b}/N)$ is $A \cup \{\overline{a}\}$-definable.

  Then $\mathrm{tp}(\overline{a}\overline{b}/M)$ is A-definable.

  We note that the converse of this is false in general.

Definable 1-types in *o*-minimal theories

Let T be *o*-minimal (e.g. $T = \mathrm{DOAG}$) and $\mathcal{D} \models T$.

- Let $p(x) \in S_1(D)$ be a non-realised type.
- Recall that p is determined by the cut $C_p := \{d \in D \mid d < x \in p\}$.
- Thus, by *o*-minimality, $p(x)$ is definable
  $\Leftrightarrow d_p\varphi(y)$ exists for $\varphi(x, y) := x > y$
  $\Leftrightarrow C_p$ is a definable subset of D
  $\Leftrightarrow C_p$ is a rational cut
- e.g. in case $C_p = D$, $d_p\varphi(y)$ is given by $y = y$;
- in case $C_p =]-\infty, \delta]$, $d_p\varphi(y)$ is given by $y \leq \delta$
  ($p(x)$ expresses: x is "just right" of δ; this p is denoted by δ^+).

Definable 1-types in *o*-minimal theories (cont'd)

Corollary

Let $\mathcal{D} \models \mathrm{DOAG}$ The following are equivalent:

1. $\mathcal{D} \cong (\mathbb{R}, +, <)$;
2. *Any $p \in S_1(D)$ is definable;*
3. *For every $n \geq 1$, any $p \in S_n(D)$ is definable.*

Proof.

1. $\Rightarrow$ 2. Clearly, every cut in $\mathbb{R}$ is rational.

2. $\Rightarrow$ 3. If $p = \mathrm{tp}(a_1, \ldots, a_n/D)$, by QE, p is determined by the 1-types $\mathrm{tp}(a'/D)$, where $a' = \sum_{i=1}^{n} z_i a_i$ for some $z_i \in \mathbb{Z}$.

2. $\Rightarrow$ 1. If $\mathcal{D}$ is non-archimedean, choose $0 < \epsilon << d$.
Then $\{d \in D \mid d < n\epsilon \text{ for some } n \in \mathbb{N}\}$ is an irrational cut. So $\mathcal{D}$ has to be archimedean, and of course equal to its completion. □

Definable 1-types in ACVF

Let $K \models \mathrm{ACVF}$, $K \preccurlyeq L$, $t \in L \setminus K$, and put $p := \mathrm{tp}(t/K)$.

- If $K(t)/K$ is a residual extension, then p is definable.

Proof.
Replacing t by $at + b$, WMA $\mathrm{val}(t) = 0$ and $\mathrm{res}(t) \notin k_K$.

$\Rightarrow$ Enough to guarantee definably that
$\mathrm{val}(X^n + a_{n-1}X^{n-1} + \ldots + a_0) = 0$ is in p for all $a_i \in \mathcal{O}_K$. □

- If $K(t)/K$ is a ramified extension, up to a translation WMA $\gamma = \mathrm{val}(t) \notin \Gamma(K)$.

  p is definable $\Leftrightarrow$ the cut def. by $\mathrm{val}(t)$ in $\Gamma(K)$ is rational.

  (Indeed, p is determined by $p_\Gamma := \mathrm{tp}_{\mathrm{DOAG}}(\gamma/\Gamma(K))$, so p is definable $\Leftrightarrow$ p_Γ is definable.)

Definable 1-types in ACVF (cont'd)

- If $K(t)/K$ is an immediate extension, then p is not definable.

  (There is no smallest K-definable ball containing t. If p were definable, the intersection of all (closed or open) K-definable balls containing t would be definable.)

Corollary

Let $K \models \mathrm{ACVF}$ The following are equivalent:

1. *K is maximally valued and $\Gamma(K) \cong (\mathbb{R}, +, <)$;*
2. *Any $p \in S_1(K)$ is definable;*
3. *For every $n \geq 1$, any $p \in S_n(K)$ is definable.*

Proof.

1. $\Leftrightarrow$ 2. follows from the above. 1. $\Rightarrow$ 3. follows from the detailed analysis of types in ACVF by Haskell-Hrushovski-Macpherson. □

Definability of types in ACF

Proposition

In ACF, all types over all models are definable.

Proof.

Let $K \models ACF$ and $p \in S_n(K)$.

Let $I(p) := \{f(\overline{x}) \in K[\overline{x}] \mid f(\overline{x}) = 0 \in p\} = (f_1, \ldots, f_r)$.

By QE, every formula is equivant to a boolean combination of polynomial equations. Thus, it is enough to show:

For any d the set of (coefficients of) polynomials $g(\overline{x}) \in K[\overline{x}]$ of degree $\leq d$ such that $g \in I_p$ is definable. This is classical. □

Remark

The above result is a consequence of the **stability** *of ACF.*

Equivalent definitions of stability

Definition

A theory T is called stable if there is no formula $\varphi(\overline{x}, \overline{y})$ and tuples $(\overline{a}_i, \overline{b}_i)_{i \in \mathbb{N}}$ (in $\mathcal{U}$) such that $\mathcal{U} \models \varphi(\overline{a}_i, \overline{b}_j) \iff i \leq j$.

Theorem (Shelah)

The following are equivalent:

1. *T is stable.*
2. *There is an infinite cardinal κ such that for every $A \subseteq U$ with $| A | \leq \kappa$ one has $| S_1(A) | \leq \kappa$.*
3. *All types over all models are definable.*

3. $\Rightarrow$ 2. There are $\leq | A^{\mathbb{N}} |$ many A-def. types, so $\kappa = 2^{\aleph_0}$ works.
2. $\Rightarrow$ 1. T unstable $\Rightarrow$ may code cuts in the type space.
1. $\Rightarrow$ 3. More difficult.

Examples of stable theories

- ACF, more generally every strongly minimal theory;
- any theory of abelian groups.

Examples of unstable theories

- Every *o*-minimal theory (e.g. DOAG, RCF);
- the theory of any non-trivially valued field, e.g. ACVF;
- the theory of any pseudofinite field...

Uniform definability of types in stable theories

Theorem

Let T be stable and $\varphi(\overline{x}, \overline{y})$ a formula. Then there is a formula $\chi(\overline{y}, \overline{z})$ such that for every type $p(\overline{x})$ (over a model) there is $\overline{b}$ such that $d_p\varphi(\overline{y}) = \chi(\overline{y}, \overline{b})$.

Problem

Is $D_{\varphi,\chi} = \{\overline{b} \in U \mid \chi(\overline{y}, \overline{b})$ is the φ-definition of some type$\}$ always a definable set?

Fact

For T stable, all $D_{\varphi,\chi}$ are definable iff for every formula $\psi(x, \overline{y})$ (in T^{eq}), there is $N_\psi \in \mathbb{N}$ such that whenever $\psi(\mathcal{U}, \overline{b})$ is finite, one has $|\psi(\mathcal{U}, \overline{b})| \leq N_\psi$.

Corollary

In ACF, the sets $D_{\varphi,\chi}$ are definable.

Prodefinable sets

Definition

A prodefinable set is a projective limit $D = \varprojlim_{i \in I} D_i$ of definable sets D_i, with def. transition functions $\pi_{i,j} : D_i \to D_j$ and I some small index set. (Identify $D(\mathcal{U})$ with a subset of $\prod D_i(\mathcal{U})$.)

We are only interested in **countable** index sets $\Rightarrow$ WMA $I = \mathbb{N}$.

Example

1. (**Type-definable sets**) If $D_i \subseteq U^n$ are definable sets, $\bigcap_{i \in \mathbb{N}} D_i$ may be seen as a prodefinable set: WMA $D_{i+1} \subseteq D_i$, so the transition maps are given by inclusion.
2. $U^\omega = \varprojlim_{i \in \mathbb{N}} U^i$ is naturally a prodefinable set.

Some notions in the prodefinable setting

Let $D = \varprojlim_{i \in I} D_i$ and $E = \varprojlim_{j \in J} E_j$ be prodefinable.

- There is a natural notion of a prodefinable map $f : D \to E$.
- D is called strict prodefinable if it can be written as a prodefinable set with surjective transition functions;
- D is called iso-definable if it is in prodefinable bijection with a definable set.
- $X \subseteq D$ is called relatively definable if there is $i \in I$ and $X_i \subseteq D_i$ definable such that $X = \pi_i^{-1}(X_i)$.

Remark

D is strict pro-definable iff $\pi_i(X) \subseteq D_i$ is definable for every relatively definable X and any i.

The set of definable types as a prodefinable set

Assume:

- T has **EI** and
- **uniform definability of types** (e.g. T stable)

For any $\varphi(\overline{x}, \overline{y})$ fix $\chi_\varphi(\overline{y}, \overline{z})$ such that for any definable type $p(\overline{x})$ we may take $d_p\varphi(\overline{y}) = \chi_\varphi(\overline{y}, \overline{b})$ for some $\overline{b} = \ulcorner d_p\varphi \urcorner$.

$\Rightarrow$ may identify p (more exactly $p \mid U$) with the tuple $(\ulcorner d_p\varphi \urcorner)_\varphi$.

Proposition

1. *With these identifications, the set of definable n-types $S_{def,n}$ is naturally a prodefinable set. Moreover, if $X \subseteq U^n$ is definable, denoting $S_{def,X}(A)$ the set of A-definable types on X, $S_{def,X}$ is a relatively definable subset of $S_{def,n}$.*
2. *If all $D_{\varphi,\chi}$ are definable, then $S_{def,X}$ strict prodefinable.*

The space of types in ACF as a prodefinable set

Corollary

Let V be an algebraic variety. There is a strict prodefinable set D (in ACF) such that for any field K, $S_V(K) \cong D(K)$ naturally.

Proposition

1. *If V is a curve, then S_V is iso-definable.*
2. *If $\dim(V) \geq 2$, then S_V is not iso-definable.*

Proof sketch.

1. is clear, since S_V is the set of realised types (which is always iso-definable) plus a finite number of generic types.

2. If $V = \mathbb{A}^2$, one may show that the generic types of the curves given by $y = x^n$ may not be seperated by finitely many φ-types. The result follows. (The general case reduces to this.) □

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