Tameness in non-archimedean geometry through model theory (after Hrushovski-Loeser)

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Outline

Introduction

A review of the model theory of ACVF and stable domination

The space $\hat{V}$ of stably dominated types

Topological considerations in $\hat{V}$

Strong deformation retraction onto a $\Gamma$-internal subset
  $\Gamma$-internality
  The curves case
  GAGA for connected components

Transfer to Berkovich spaces and applications
Valued fields: basics and notation

Let $K$ be a field and $\Gamma = (\Gamma, 0, +, <)$ an ordered abelian group. A map $\text{val} : K \rightarrow \Gamma_{\infty} = \Gamma \cup \{\infty\}$ is a valuation if it satisfies

1. $\text{val}(x) = \infty$ iff $x = 0$;
2. $\text{val}(xy) = \text{val}(x) + \text{val}(y)$;
3. $\text{val}(x + y) \geq \min\{\text{val}(x), \text{val}(y)\}$.

(Here, $\infty$ is a distinguished element $> \Gamma$ and absorbing for $+$.)

- $\Gamma = \Gamma_K$ is called the value group.
- $\mathcal{O} = \mathcal{O}_K = \{x \in K \mid \text{val}(x) \geq 0\}$ is the valuation ring, with (unique) maximal ideal $m = m_K = \{x \mid \text{val}(x) > 0\}$;
- $\text{res} : \mathcal{O} \rightarrow k = k_K := \mathcal{O}/m$ is the residue map, and $k_K$ is called the residue field.
The valuation topology

Let $K$ be a valued field with value group $\Gamma$.

- For $a \in K$ and $\gamma \in \Gamma$ let $B_{\geq \gamma}(a) := \{x \in K \mid \text{val}(x - a) \geq \gamma\}$ be the closed ball of (valuative) radius $\gamma$ around $a$.
- Similarly, one defines the open ball $B_{> \gamma}(a)$.
- The open balls form a basis for a topology on $K$, called the valuation topology, turning $K$ into a topological field.
- Both the 'open' and the 'closed' balls are clopen sets in the valuation topology. In particular, $K$ is totally disconnected.
- Let $V$ be an algebraic variety defined over $K$. Using the product topology on $K^n$ and gluing, one defines the valuation topology on $V(K)$ (also totally disconnected).
Fields with a (complete) non-archimedean absolute value

Assume that $K$ is a valued field such that $\Gamma_K \leq \mathbb{R}$.

- $|\cdot|: K \to \mathbb{R}_{\geq 0}$, $|x| := e^{-\text{val}(x)}$, defines an absolute value.
- $(K, |\cdot|)$ is non-archimedean, and any field with a non-archimedean absolute value is obtained in this way.
- $(K, |\cdot|)$ is called **complete** if it is complete as a metric space, i.e. if every Cauchy sequence has a limit in $K$.

Examples of complete non-archimedean fields

- $\mathbb{Q}_p$ (the field $p$-adic numbers), and any finite extension of it
- $\hat{\mathbb{Q}}_p$ (the $p$-adic analogue of the complex numbers)
- $k((t))$, with the $t$-adic absolute value ($k$ any field)
- $k$ with the trivial absolute value ($|x| = 1$ for all $x \in k^\times$)
Non-archimedean analytic geometry

- For $K$ a complete non-archimedean field, one would like to do analytic geometry over $K$ similarly to the way one does analytic geometry over $\mathbb{C}$, with a 'nice' underlying topological space.
- There exist various approaches to this, due to Tate (rigid analytic geometry), Raynaud, Berkovich, Huber etc.

Berkovich's approach: **Berkovich (analytic) spaces** (late 80's)

- provide spaces endowed with an actual topology (not just a Grothendieck topology), in which one may consider paths, singular (co-)homology etc.;
- are obtained by **adding points** to the set of naive points of an analytic / algebraic variety over $K$;
- have been used with great success in many different areas.
Berkovich spaces in a glance

We briefly describe the Berkovich analytification (as a topological space) $V^\text{an}$ of an affine algebraic variety $V$ over $K$.

- Let $K[V]$ be the ring of regular functions on $V$. As a set, $V^\text{an}$ equals the set of multiplicative seminorms $\cdot |$ on $K[V]$ ($|fg| = |f| \cdot |g|$ and $|f + g| \leq \max(|f|, |g|)$) which extend $| \cdot |_K$.

- $V(K)$ may be identified with a subset of $V^\text{an}$, via $a \mapsto | \cdot |_a$, where $|f|_a := |f(a)|_K$.

- Note $V^\text{an} \subseteq \mathbb{R}^K[V]$. The topology on $V^\text{an}$ is defined as the induced one from the product topology on $\mathbb{R}^K[V]$.

Remark

Let $(L, | \cdot |_L)$ be a normed field extension of $K$, and let $b \in V(L)$. Then $b$ corresponds to a map $\varphi : K[V] \to L$, and $| \cdot |_b \in V^\text{an}$, where $|f|_b = |\varphi(f)|_L$. Moreover, any element of $V^\text{an}$ is of this form.
A glimpse on the Berkovich affine line

Example
Let $V = \mathbb{A}^1$, so $K[V] = K[X]$.

- For any $r \in \mathbb{R}_{\geq 0}$, we have $\nu_{0,r} \in \mathbb{A}^{1,an}$, where
  
  $$|\sum_{i=0}^{n} c_i X^i|_{\nu_{0,r}} := \max_{0 \leq i \leq n} (|c_i|_K \cdot r^i) .$$

- $\nu_{0,0}$ corresponds to $0 \in \mathbb{A}^1(K)$, and $\nu_{0,1}$ to the Gauss norm.

- The map $r \mapsto \nu_{0,r}$ is a continuous path in $\mathbb{A}^{1,an}$.

- In fact, the construction generalises suitably, showing that $\mathbb{A}^{1,an}$ is contractible.
Topological tameness in Berkovich spaces

Berkovich spaces have excellent general topological properties, e.g. they are **locally compact** and **locally path-connected**.

Using deep results from algebraic geometry, various **topological tameness** properties had been established, e.g.:

- Any compact Berkovich space is **homotopic to a (finite) simplicial complex** (Berkovich);
- Smooth Berkovich spaces are **locally contractible** (Berkovich).
- If $V$ is an algebraic variety, ’semi-algebraic’ subsets of $V^{an}$ have **finitely many connected components** (Ducros).
Hrushovski-Loeser’s work: main contributions

Foundational

- They develop ‘non-archimedean (rigid) algebraic geometry’, constructing a ’nice’ space $\hat{V}$ for an algebraic variety $V$ over any valued field $K$,
  - with no restrictions on the value group $\Gamma_K$;
  - no need to work with a complete field $K$.

- Entirely new methods: the geometric model theory of ACVF is shown to be perfectly suited to address topological tameness (combining stability and $o$-minimality).

Applications to Berkovich analytifications of algebraic varieties

They obtain strong topological tameness results for $V^{an}$,
- without smoothness assumption on the variety $V$, and
- avoiding heavy tools from algebraic geometry.
Valued fields as first order structures

- There are various choices of languages for valued fields.
- $\mathcal{L}_{\text{div}} := \mathcal{L}_{\text{rings}} \cup \{ \text{div} \}$ is a language with only one sort $\text{VF}$ for the valued field.
- A valued field $K$ gives rise to an $\mathcal{L}_{\text{div}}$-structure, via
  $$x \div y \iff \text{val}(x) \leq \text{val}(y).$$
- $\mathcal{O}_K = \{ x \in K : 1 \div x \}$, so $\mathcal{O}_K$ is $\mathcal{L}_{\text{div}}$-definable
  $\Rightarrow$ the valuation is encoded in the $\mathcal{L}_{\text{div}}$-structure.
- $\text{ACVF}$: theory of alg. closed non-trivially valued fields
QE in algebraically closed valued fields

Fact (Robinson)

The theory ACVF has QE in $\mathcal{L}_{\text{div}}$. Its completions are given by $\text{ACVF}_{p,q}$, for $(p, q) = (\text{char}(K), \text{char}(k))$.

Corollary

1. In ACVF, a set is definable iff it is semi-algebraic, i.e., a finite boolean combination of sets given by conditions of the form $f(\bar{x}) = 0$ or $\text{val}(f(\bar{x})) \leq \text{val}(g(\bar{x}))$, where $f, g$ are polynomials.

2. Definable sets in 1 variable are (finite) boolean combinations of singletons and balls.

3. ACVF is NIP, i.e., there is no formula $\varphi(\bar{x}, \bar{y})$ and tuples $(\bar{a}_i)_{i \in \mathbb{N}}, (\bar{b}_J)_{J \subseteq \mathbb{N}}$ (in some model) such that $\varphi(\bar{a}_i, \bar{b}_J)$ iff $i \in J$. 
A variant: valued fields in a three-sorted language

Let $\mathcal{L}_{k,\Gamma}$ be the following 3-sorted language, with sorts $\text{VF}$ for the valued field, $\Gamma_\infty$ and $k$:

- Put $\mathcal{L}_{\text{rings}}$ on $K = \text{VF}$, $\{0, +, <, \infty\}$ on $\Gamma_\infty$ and $\mathcal{L}_{\text{rings}}$ on $k$;
- $\text{val} : K \to \Gamma_\infty$, and
- $\text{RES} : K \to k$ as additional function symbols.

A valued field $K$ is naturally an $\mathcal{L}_{k,\Gamma}$-structure, via

$$\text{RES}(x, y) := \begin{cases} \text{res}(xy^{-1}), & \text{if } \text{val}(x) \geq \text{val}(y) \neq \infty; \\ 0 \in k, & \text{else.} \end{cases}$$
ACVF in the three-sorted language

Fact
ACVF eliminates quantifiers in $\mathcal{L}_{k,\Gamma}$.

Corollary
In ACVF, the following holds:

1. $\Gamma$ is a pure divisible ordered abelian group: any definable subset of $\Gamma^n$ is $\{0, +, <\}$-definable (with parameters from $\Gamma$). In particular, $\Gamma$ is o-minimal.

2. $k$ is a pure ACF: any definable subset of $k^n$ is $\mathcal{L}_{\text{rings}}$-definable.

3. $k \perp \Gamma$, i.e. every definable subset of $k^m \times \Gamma^n$ is a finite union of rectangles $D \times E$.

4. Any definable function $f : K^n \rightarrow \Gamma_\infty$ is piecewise of the form $f(\overline{x}) = \frac{1}{m}[\text{val}(F(\overline{x})) - \text{val}(G(\overline{x}))]$, for $F, G \in K[\overline{x}]$ and $m \geq 1$. 
A description of 1-types over models of ACVF

Let $K \preceq \mathbb{U} \models ACVF$, with $\mathbb{U}$ suff. saturated. A $K$-(type-)definable subset $B \subseteq \mathbb{U}$ is a generalised ball over $K$ if $B$ is equal to one of the following:

- a singleton $\{a\} \subseteq K$;
- a closed ball $B_{\geq \gamma}(a)$ ($a \in K$, $\gamma \in \Gamma_K$);
- an open ball $B_{> \gamma}(a)$ ($a \in K$, $\gamma \in \Gamma_K$);
- a (non-empty) intersection $\bigcap_{i \in I} B_i$ of $K$-definable balls $B_i$ with no minimal $B_i$;
- $\mathbb{U}$.

Fact

By QE, we have $S_1(K) \xrightarrow{1:1} \{\text{generalised balls over } K\}$, given by

- $p = \text{tp}(t/K) \mapsto \text{Loc}(t/K) := \bigcap b$, where $b$ runs over all generalised balls over $K$ containing $t$;
- $B \mapsto p_B \mid K$, where $p_B \mid K$ is the generic type in $B$ expressing $x \in B$ and $x \notin b'$ for any $K$-def. ball $b' \subsetneq B$. 
Context

- $\mathcal{L}$ is some language (possibly many-sorted);
- $T$ is a complete $\mathcal{L}$-theory with QE;
- $\mathbb{U} \models T$ is a fixed universe (i.e. very saturated and homogeneous);
- all models $M$ (and all parameter sets $A$) we consider are small, with $M \preceq \mathbb{U}$ (and $A \subseteq \mathbb{U}$).
 Imaginary Sorts and Elements

- Let $E$ is a definable equivalence relation on some $D \subseteq_{def} \mathbb{U}^n$. If $d \in D(\mathbb{U})$, then $d/E$ is an imaginary in $\mathbb{U}$.
- If $D = \mathbb{U}^n$ for some $n$ and $E$ is $\emptyset$-definable, then $U^n/E$ is called an imaginary sort.
- Recall: Shelah’s $eq$-construction is a canonical way to pass from $\mathcal{L}, M, T$ to $\mathcal{L}^{eq}, M^{eq}, T^{eq}$, adding a new sort (and a quotient function) for each imaginary sort.
- Given $\varphi(x, y)$, let $E_\varphi(y, y') := \forall x[\varphi(x, y) \leftrightarrow \varphi(x, y')]$. Then $b/E_\varphi$ may serve as a code $\llceil W \rrceil$ for $W = \varphi(\mathbb{U}, b)$.

Example

Consider $K \models ACVF$ (in $\mathcal{L}_{div}$).
- $k, \Gamma \subseteq K^{eq}$, i.e. $k$ and $\Gamma$ are imaginary sorts.
- More generally, $B^o, B^{cl} \subseteq K^{eq}$ (the set of open / closed balls).
Elimination of imaginaries

Definition (Poizat)

The theory $T$ eliminates imaginaries if every imaginary element $a \in U^{eq}$ is interdefinable with a real tuple $b \in U^n$.

Examples of theories which eliminate imaginaries

1. $T^{eq}$ (for an arbitrary theory $T$)
2. ACF (Poizat)
3. The theory $DOAG$ of non-trivial divisible ordered abelian groups (more generally every o-minimal expansion of $DOAG$)

Fact

ACVF does not eliminate imaginaries in the 3-sorted language $L_{k,\Gamma}$ (Holly), even if sorts for open and closed balls $B^o$ and $B^{cl}$ are added (Haskell-Hrushovski-Macpherson).
The geometric sorts

- $s \subseteq K^n$ is a **lattice** if it is a free $\mathcal{O}$-submodule of rank $n$;
- for $s \subseteq K^n$ a lattice, $s/\mathfrak{m}s$ is a definable $n$-dimensional $k$-vector space.

For $n \geq 1$, let

$$ S_n := \{ \text{lattices in } K^n \} , $$

$$ T_n := \bigcup_{s \in S_n} s/\mathfrak{m}s. $$

**Fact**

1. $S_n$ and $T_n$ are imaginary sorts, $S_1 \cong \Gamma$ (via $a\mathcal{O} \mapsto \text{val}(a)$), and also $k = \mathcal{O}/\mathfrak{m} \subseteq T_1$.

2. $S_n \cong \text{GL}_n(K)/\text{GL}_n(\mathcal{O})$; for $T_n$, there is a similar description as a finite union of coset spaces.
Classification of Imaginaries in ACVF

\[ G = \{ \text{VF} \} \cup \{ S_n, n \geq 1 \} \cup \{ T_n, n \geq 1 \} \] are the geometric sorts.

Let \( \mathcal{L}_G \) be the (natural) language of valued fields in \( G \).

Theorem (Haskell-Hrushovski-Macpherson 2006)

ACVF eliminates imaginaries down to geometric sorts, i.e. the theory ACVF considered in \( \mathcal{L}_G \) has EI.

Convention

From now on, by ACVF we mean any completion of this theory, considered in the geometric sorts.

Moreover, any theory \( T \) we consider will be assumed to have EI.
The notion of a definable type

Definition

Let $M \models T$ and $A \subseteq M$. A type $p(\bar{x}) \in S_n(M)$ is called \textit{A-definable} if for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ there is an $\mathcal{L}_A$-formula $d_p \varphi(\bar{y})$ such that

$$\varphi(\bar{x}, \bar{b}) \in p \iff M \models d_p \varphi(\bar{b}) \quad (\text{for every } \bar{b} \in M)$$

We say $p$ is \textbf{definable} if it is definable over some $A \subseteq M$. The collection $(d_p \varphi)_\varphi$ is called a \textbf{defining scheme} for $p$.

Remark

If $p \in S_n(M)$ is definable via $(d_p \varphi)_\varphi$, then the same scheme gives rise to a (unique) type over any $N \succ M$, denoted by $p \mid N$. 
Definable types: first properties

- *(Realised types are definable)*
  Let $\bar{a} \in M^n$. Then $\text{tp}(\bar{a}/M)$ is definable.  
  (Take $d_p \varphi(y) = \varphi(\bar{a}, y)$.)

- *(Preservation under algebraic closure)*
  If $\text{tp}(\bar{a}/M)$ is definable and $\bar{b} \in \text{acl}(M \cup \{\bar{a}\})$, then $\text{tp}(\bar{b}/M)$ is definable, too.

- *(Transitivity)* Let $\bar{a} \in N$ for some $N \supseteq M$, $A \subseteq M$. Assume
  - $\text{tp}(\bar{a}/M)$ is $A$-definable;
  - $\text{tp}(\bar{b}/N)$ is $A \cup \{\bar{a}\}$-definable.
  Then $\text{tp}(\bar{a}\bar{b}/M)$ is $A$-definable.

We note that the converse of this is false in general.
Definable 1-types in $\sigma$-minimal theories

Let $T$ be $\sigma$-minimal (e.g. $T = \text{DOAG}$) and $M \models T$.

- Let $p(x) \in S_1(M)$ be a non-realised type.
- Recall that $p$ is determined by the cut $C_p := \{ d \in M \mid d < x \in p \}$.
- Thus, by $\sigma$-minimality, $p(x)$ is definable $\iff d_p \varphi(y)$ exists for $\varphi(x, y) := x > y$
  $\iff C_p$ is a definable subset of $M$
  $\iff C_p$ is a rational cut
- e.g. in case $C_p = M$, $d_p \varphi(y)$ is given by $y = y$;
- in case $C_p = ] - \infty, \delta]$, $d_p \varphi(y)$ is given by $y \leq \delta$
  ($p(x)$ expresses: $x$ is "just right" of $\delta$; this $p$ is denoted by $\delta^+$).
Definable 1-types in ACVF

Fact

Let $K \models ACVF$ and $p = \text{tp}(t/K) \in S_1(K)$. TFAE:

1. $\text{tp}(t/K)$ is definable;
2. $\text{Loc}(t/K)$ is definable (and not just type-definable).

Proof.

If $\text{tp}(t/K)$ is definable, then the set of $K$-definable balls containing $t$ is definable over $K$, so is its intersection. $(2) \Rightarrow (1)$ is clear. □

For $t \not\in K$, letting $L = K(t)$, we get three cases:

- $L/K$ is a residual extension, i.e. $k_L \supseteq k_K$. Then $t$ is generic in a closed ball, so $p$ is definable.

  [Indeed, replacing $t$ by $at + b$, WMA $\text{val}(t) = 0$ and $\text{res}(t) \not\in k_K$, so $t$ is generic in $\mathcal{O}$.]
Definable 1-types in ACVF (continued)

- $L/K$ is a **ramified** extension, i.e. $\Gamma_L \nsubseteq \Gamma_K$. Up to a translation $\text{WMA } \gamma = \text{val}(t) \notin \Gamma(K)$.

  $p$ is definable $\iff$ the cut def. by $\text{val}(t)$ in $\Gamma_K$ is rational.

  [Indeed, $p$ is determined by $p_\Gamma := \text{tp}_{\text{DOAG}}(\gamma/\Gamma_K)$, so $p$ is definable $\iff p_\Gamma$ is definable.]

- $L/K$ is an **immediate** extension, i.e. $k_K = k_L$ and $\Gamma_K = \Gamma_L$. Then $p$ is not definable.

  [Indeed, in this case, letting $B := \text{Loc}(t/K)$, we get $B(K) = \emptyset$. In particular, $B$ is not definable.]
Definability of types in ACF

Proposition

In ACF, all types over all models are definable.

Proof.
Let $K \models ACF$ and $p \in S_n(K)$.
Let $I(p) := \{ f(\bar{x}) \in K[\bar{x}] \mid f(\bar{x}) = 0 \text{ is in } p \} = (f_1, \ldots, f_r)$.
By QE, every formula is equivalent to a boolean combination of polynomial equations. Thus, it is enough to show:

For any $d$ the set of (coefficients of) polynomials $g(\bar{x}) \in K[\bar{x}]$ of degree $\leq d$ such that $g \in I_p$ is definable. This is classical. \[ \square \]

Remark

The above result is a consequence of the stability of ACF. In fact, it characterises stability.
Products of definable types

- Assume $p = p(x)$ and $q = q(y)$ are $A$-definable types.
- There is a unique $A$-definable type $p \otimes q$ in variables $(x, y)$, constructed as follows: Let $b \models q \upharpoonright A$ and $a \models p \upharpoonright Ab$. Then
  $$p \otimes q \upharpoonright A = \text{tp}(a, b/A).$$
- The $n$-fold product $p \otimes \cdots \otimes p$ is denoted by $p^{(n)}$.

Remark

1. $\otimes$ is associative.
2. $\otimes$ is in general not commutative, as is shown by the following:
   Let $p(x)$ and $q(y)$ both be equal to $0^+$ in $\text{DOAG}$. Then
   $$p(x) \otimes q(y) \vdash x < y,$$
   whereas
   $$q(y) \otimes p(x) \vdash y < x.$$
3. In a stable theory, $\otimes$ corresponds to the non-forking extension, so $\otimes$ is in particular commutative.
The stable part

Let $T$ be given and $A \subseteq U$ a parameter set.
Recall that an $A$-definable set $D$ is stably embedded if every definable subset of $D^n$ is definable with parameters from $D(U) \cup A$.

Definition

- The stable part over $A$, denoted $St_A$, is the multi-sorted structure with a sort for each $A$-definable stable stably embedded set $D$ and with the full induced structure (from $\mathcal{L}_A$).
- For $\bar{a} \in U$, set $St_A(\bar{a}) := dcl(A\bar{a}) \cap St_A$.

Fact

$St_A$ is a stable structure.
The stable part in ACVF

Consider ACVF in $\mathcal{L}_G$. Given $A$, we denote by $\text{VS}_{k,A}$ the many sorted structure with sorts $s/m_s$, where $s \in S_n(A)$ for some $n$.

**Fact (HHM)**

Let $D$ be an $A$-definable set. TFAE:

1. $D$ is stable and stably embedded.
2. $D$ is $k$-internal, i.e. there is a finite set $F \subseteq U$ such that $D \subseteq \text{dcl}(k \cup F)$
3. $D \subseteq \text{dcl}(A \cup \text{VS}_{k,A})$
4. $D \perp \Gamma$ (def. subsets of $D^m \times \Gamma^n$ are finite unions of rectangles)

**Corollary**

Up to interdefinability, $\text{St}_A$ is equal to $\text{VS}_{k,A}$. In particular, if $A = K \models \text{ACVF}$, then $\text{St}_A$ may be identified with $k$. 
Stable domination (in ACVF)

Idea: a stably dominated type is 'generically' controlled by its stable part.

To ease the presentation and avoid technical issues around base change, we will restrict the context and work in ACVF.

Definition
Let $p$ be an $A$-definable type. We say $p$ is stably dominated if for $\bar{a} \models p \upharpoonright A$ and every $B \supseteq A$ such that

$$\text{St}_A(\bar{a}) \downarrow \text{St}_A(B) \ (\text{in the stable structure } \text{St}_A = \text{VS}_{k,A}),$$

we have $tp(\bar{a}/A) \cup tp(\text{St}_A(\bar{a})/\text{St}_A(B)) \vdash tp(\bar{a}/B)$.

(We will then also say that $p \upharpoonright A = tp(\bar{a}/A)$ is stably dominated.)

Fact

The above does not depend on the choice of the set $A$ over which $p$ is defined, so the notion is well-defined.
Stably dominated types inherit many nice properties from stable theories. Here is one:

**Fact**

If $p$ is stably dominated type and $q$ an arbitrary definable type, then $p \otimes q = q \otimes p$. In particular, $p$ commutes with itself, so any permutation of $(a_1, \ldots, a_n) \models p^{(n)} \mid A$ is again realises $p^{(n)} \mid A$.

**Examples**

1. The generic type of $O$ is stably dominated.
   Indeed, let $a \models p_O \mid K$ and $K \subseteq L$. Then $St_K(a) \downarrow_K St_K(L)$ just means that $\text{res}(a) \not\in k^\text{alg}_L$, forcing $a \models p_O \mid L$.

2. The generic type of $m$ is not stably dominated.
   Indeed, we have $p_m(x) \otimes p_m(y) \vdash \text{val}(x) < \text{val}(y)$, whereas $p_m(y) \otimes p_m(x) \vdash \text{val}(x) > \text{val}(y)$.

3. On $\Gamma^m_\infty$, only the realised types are stably dominated.
Characterisation of stably dominated types in ACVF

**Definition**
Let $p$ be a definable type. We say $p$ is **orthogonal to** $\Gamma$ (and we denote this by $p \perp \Gamma$) if for every model $M$ over which $p$ is defined, letting $\bar{a} \models p \mid M$, one has $\Gamma(M) = \Gamma(M\bar{a})$.

**Remark**
Equivalently, in the defintion we may require the property to hold only for some model $M$ over which $p$ is defined.

**Proposition**
Let $p$ be a definable type in ACVF. **TFAE:**

1. $p$ is stably dominated.
2. $p \perp \Gamma$.
3. $p$ commutes with itself, i.e., $p(x) \otimes p(y) = p(y) \otimes p(x)$. 
Stably dominated types in ACVF: some closure properties

- **Realised types are stably dominated.**

- **Preservation under algebraic closure:**
  Suppose $tp(\bar{a}/A)$ is stably dominated for some $A = acl(A)$, and let $\bar{b} \in acl(A\bar{a})$. Then $tp(\bar{b}/A)$ is stably dominated, too.
  In particular, if $p$ is stably dominated on $X$ and $f : X \to Y$ is definable, then $f_*(p)$ is stably dominated on $Y$.

- **Transitivity:**
  If $tp(\bar{a}/A)$ and $tp(\bar{b}/A\bar{a})$ are both stably dominated, then $tp(\bar{a}\bar{b}/A)$ is stably dominated, too.

The converse of this is false in general. (See the examples below.)
Examples of stably dominated types in ACVF

- The generic type of a closed ball is stably dominated.
- The generic type of an open ball is not stably dominated.
- It follows that if $K \models \text{ACVF}$ and $K \subseteq L = K(\bar{a})$ with $\text{tr.deg}(L/K) = 1$, then $\text{tp}(\bar{a}/K)$ is stably dominated iff $\text{tr.deg}(k_L/k_K) = 1$.
- If $\text{tr.deg}(L/K) = \text{tr.deg}(k_L/k_K)$, then $\text{tp}(\bar{a}/K)$ is stably dominated.
- There are more complicated stably dominated types: for every $n \geq 1$, there is $K \subseteq L = K(\bar{a})$ such that
  - $\text{tr.deg}(L/K) = n$,
  - $\text{tr.deg}(k_L/k_K) = 1$, and
  - $\text{tp}(\bar{a}/K)$ is stably dominated.
A valued field $K$ is **maximally complete** if it has no proper immediate extension.

When working over a parameter set $A$, it is often useful to pass to a maximally complete $M \models ACVF$ containing $A$, mainly due to the following important result.

**Theorem (Haskell-Hrushovski-Macpherson)**

Let $M$ be a maximally complete model of $ACVF$, and let $\bar{a}$ be a tuple from $\mathcal{U}$. Then $tp(\bar{a}/M, \Gamma(M\bar{a}))$ is stably dominated.

**Remark**

In abstract terms, the theorem states that $ACVF$ is **metastable** (over $\Gamma$), with metastability bases given by maximally complete models.
Uniform definability of types

Fact

1. Let $T$ be stable and $\varphi(x, y)$ a formula. Then there is a formula $\chi(y, z)$ such that for every type $p(x)$ (over a model) there is $b$ such that $d_p \varphi(y) = \chi(y, b)$.

2. The same result holds in ACVF if we restrict the conclusion to the collection of stably dominated types.

Proof.
For every formula $\varphi(x, y)$ there is $n \geq 1$ such that whenever $p$ is stably dominated and $A$-definable and $(a_0, \ldots, a_{2n}) \models p^{(2n+1)} | A$, then for any $b \in U$, the majority rule holds, i.e.,

$$\varphi(x, b) \in p \iff U \models \bigvee_{i_0 < \cdots < i_n} \varphi(a_{i_0}, b) \land \cdots \land \varphi(a_{i_n}, b).$$
Prodefinable sets

Definition

A prodefinable set is a projective limit \( D = \lim_{i \in I} D_i \) of definable sets \( D_i \), with def. transition functions \( \pi_{i,j} : D_i \to D_j \) and \( I \) some small index set. (Identify \( D(U) \) with a subset of \( \prod D_i(U) \).)

We are only interested in countable index sets \( \Rightarrow \) WMA \( I = \mathbb{N} \).

Example

1. (Type-definable sets) If \( D_i \subseteq U^n \) are definable sets, \( \bigcap_{i \in \mathbb{N}} D_i \) may be seen as a prodefinable set: WMA \( D_{i+1} \subseteq D_i \), so the transition maps are given by inclusion.

2. \( U^\omega = \lim_{i \in \mathbb{N}} U^i \) is naturally a prodefinable set.
Some notions in the prodefinable setting

Let $D = \lim_{i \in I} D_i$ and $E = \lim_{j \in J} E_j$ be prodefinable.

- There is a natural notion of a **prodefinable map** $f : D \to E$ [if given by a compatible system of maps $f_j : D \to E_j$, each $f_j$ factoring through some component $D_{i(j)}$]

- $D$ is called **strict prodefinable** if it can be written as a prodefinable set with surjective transition functions.

- $D$ is called **iso-definable** if it is in prodefinable bijection with a definable set.

- $X \subseteq D$ is called **relatively definable** if there is $i \in I$ and $X_i \subseteq D_i$ definable such that $X = \pi_i^{-1}(X_i)$. 

The set of definable types as a prodefinable set ($T$ stable)

- Assume $T$ is stable with El (e.g. $T = \text{ACF}_p$)
- For any $\varphi(x, y)$ fix $\chi_{\varphi}(y, z)$ s.t. for any definable type $p(x)$ we have $d_p\varphi(y) = \chi_{\varphi}(y, b)$ for some $b = \lceil d_p\varphi \rceil$.
- For $X$ definable, let $S_{\text{def},X}(A)$ be the $A$-definable types on $X$.

Proposition

1. There is a prodefinable set $D$ such that $S_{\text{def},X}(A) = D(A)$ naturally. (Identify $p | U$ with the tuple $(\lceil d_p\varphi \rceil)_{\varphi}$).
2. If $Y \subseteq X$ is definable, $S_{\text{def},Y}$ is relatively definable in $S_{\text{def},X}$.
3. The subset of $S_{\text{def},X}$ corresponding to the set of realised types is relatively definable and isodefinable. (It is $\cong X(U)$.)
Strict pro-definability and nfcp

Problem
Let \( D_{\varphi,\chi} = \{ b \in U \mid \chi(y, b) \text{ is the } \varphi\text{-definition of some type} \} \).
Then \( D_{\varphi,\chi} \) is not always definable.

Fact
In \( \text{ACF} \), all \( D_{\varphi,\chi} \) are definable. More generally, for a stable theory \( T \) this is the case iff \( T \) is nfcp.

Corollary
1. If \( T \) is stable and nfcp (e.g. \( T = \text{ACF} \)), then \( S_{\text{def},X} \) is strict pro-definable.
2. If \( C \) is a curve definable over \( K \models \text{ACF} \), then \( S_{\text{def},C} \) is iso-definable.
3. \( S_{\text{def},\mathbb{A}^2} \) is not iso-definable in \( \text{ACF} \): the generic types of the curves given by \( y = x^n \) cannot be separated by finitely many \( \varphi\)-types.
The set of stably dominated types as a prodefinable set

For $X$ an $A$-definable set in ACVF, we denote by $\hat{X}(A)$ the set of $A$-definable stably dominated types on $X$.

**Theorem**

*Let $X$ be $C$-definable. There exists a strict $C$-prodefinable set $D$ such that for every $A \supseteq C$, we have a canonical identification $\hat{X}(A) = D(A)$.*

Once the theorem is established, we will denote by $\hat{X}$ the prodefinable set representing it.
Proof of the theorem.

For notational simplicity, we will assume $C = \emptyset$.

- Let $f : X \to \Gamma_\infty$ be definable (with parameters) and let $p \in \hat{X}(U)$. Then $f_*(p)$ is stably dominated on $\Gamma_\infty$, so is a realised type $x = \gamma$. We will denote this by $f(p) = \gamma$.

- Now let $f : W \times X \to \Gamma_\infty$ be $\emptyset$-definable, $f_w := f(w, -)$. Then there is a set $S$ and a function $g : W \times S \to \Gamma_\infty$, both $\emptyset$-definable, such that for every $p \in \hat{X}(U)$, the function
  \[
  f_p : W \to \Gamma_\infty, \quad w \mapsto f_w(p)
  \]
  is equal to $g_s = g(s, -)$ for a unique $s \in S$.

This follows from
- uniform definability of types for stably dominated types, and
- elimination of imaginaries in ACVF (in $\mathcal{L}_G$).
End of the proof

Choose an enumeration \( f_i : W_i \times X \to \Gamma_\infty \ (i \in \mathbb{N}) \) of the functions as above (with corresponding \( g_i : W_i \times S_i \to \Gamma_\infty \)).

Then \( p \mapsto c(p) := \{(s_i)_{i \in \mathbb{N}} \mid f_i, p = g_i, s_i \text{ for all } i\} \) defines an injection of \( \hat{X} \) into \( \prod_i S_i \).

The strict prodefinable set we are aiming for is \( D = c(\hat{X}) \).

Let \( I \subseteq \mathbb{N} \) be finite and \( \pi_I(D) = D_I \subseteq \prod_{i \in I} S_i \). We finish by the following two facts:

1. \( D_I \) is type-definable. (This gives prodefinability of \( D \).)
   [This is basically compactness and QE.]
2. \( D_I \) is a union of definable sets.
   [This uses \( \text{St}_A = \text{VS}_{k,A} \), and these are 'uniformly' nfcp.]

\( \Rightarrow \) the \( D_I \) are definable, proving strict prodefinability of \( D \).
Some definability properties in $\hat{X}$

- **Functoriality:**
  For any definable $f : X \to Y$, we get a prodefinable map $\hat{f} : \hat{X} \to \hat{Y}$.

- **Passage to definable subsets:**
  If $Y$ is a definable subset of $X$, then $\hat{Y} \subseteq \hat{X}$ is a relatively definable subset.

- **Simple points:**
  The set of realised types in $\hat{X}$, in natural bijection with $X(\mathbb{U})$, is iso-definable and relatively definable in $\hat{X}$.
  Elements of $\hat{X}$ corresponding to realised types will be called **simple** points.
Isodefinability in the case of curves

Theorem

Let $C$ be an algebraic curve. Then $\hat{C}$ is iso-definable.

Proof.

- WMA $C$ is smooth and projective, $C \subseteq \mathbb{P}^n$. Let $g = \text{genus}(C)$.
- In $K(\mathbb{P}^1) = K(X)$, any element is a product of linear polynomials in $X$. The following consequence of Riemann-Roch gives a generalisation of this to arbitrary genus: There exists an $N$ ($N = 2g + 1$ is enough) s.t. any non-zero $f \in K(C)$ is a product of functions of the form $(g/h) \upharpoonright C$, where $g, h \in K[X_0, \ldots, X_n]$ are homogeneous of degree $N$.
- Thus any valuation on $K(C)$ is determined by its values on a definable family of polynomials, proving iso-definability.
Isodefinability in the case of curves (continued)

From now on, we will write $\mathcal{B}^{cl}$ for the set of closed balls including singletons (closed balls of radius $\infty$).

Examples

1. If $C = \mathbb{A}^1$, the isodefinability of $\widehat{C}$ is clear, as then $\widehat{\mathbb{A}^1} = \mathcal{B}^{cl}$ (which is a definable set).

2. $\widehat{\mathcal{O}^2}$ is not isodefinable. Indeed, let $p_{\mathcal{O}}$ be the generic of $\mathcal{O}$, and $p_n(x, y) \in \widehat{\mathcal{O}^2}$ be given by $p_{\mathcal{O}}(x) \cup \{y = x^n\}$.

No definable family of functions to $\Gamma_\infty$ allows to separate all the $p_n$’s, as $\text{val}(f(p_n)) = \text{val}(f(p_{\mathcal{O}}(x) \otimes p_{\mathcal{O}}(y)))$ for all $f \in K[X, Y]$ of degree $< n$.

Remark

For $X \subseteq K^n$ definable, $\widehat{X}$ is iso-definable iff $\dim(X) \leq 1$.

(Here, $\dim(X)$ denotes the algebraic dimension of $X^{\text{Zar}}$.)
Prodefinable topological spaces

Definition
Let $X$ be (pro-)definable over $A$. A topology $\mathcal{T}$ on $X(U)$ is said to be $A$-definable if

1. there are $A$-definable families $\mathcal{W}^i = (W^i_b)_{b \in U}$ (for $i \in I$) of (relatively) definable subsets of $X$ such that
2. the topology on $X(U)$ is generated by the sets $(W^i_b)$, where $i \in I$ and $b \in U$.

We call $(X, \mathcal{T})$ a (pro-)definable space.

Remark
1. Given a (pro-)definable space $(X, \mathcal{T})$ (over $A$) and $A \subseteq M \preceq U$, the $M$-definable open sets from $\mathcal{T}$ define a topology on $X(M)$.
2. The inclusion $X(M) \subseteq X(U)$ is in general not continuous.
Examples of definable topologies

1. If $M$ is o-minimal, then $M^n$ equipped with the product of the order topology is a definable space.

2. Let $V$ be an algebraic variety over $K \models ACVF$. Then the valuation topology on $V(K)$ is definable.

3. The Zariski topology on $V(K)$ is a definable topology.

Remark

- The topologies in examples (1) and (2) are definably generated, in the sense that a single family of definable open sets generates the topology. (There is even a definable basis of the topology in both cases.)

- The Zariski topology in (3) is not definably generated, unless $\dim(V) \leq 1$. 
Tameness in non-archimedean geometry through model theory (after Hrushovski-Loeser)

The space $\hat{V}$ of stably dominated types

Definable topologies and the topology on $\hat{V}$

$\hat{V}$ as a prodefinable space

Given an algebraic variety $V$ defined over $K \models ACVF$, we will define a definable topology on $\hat{V}$, turning it into a prodefinable space, the Hrushovski-Loeser space associated to $V$.

The construction of the topology is done in several steps:

- We will give an explicit construction in the case $V = \mathbb{A}^n$.
- If $V$ is affine, $V \subseteq \mathbb{A}^n$ a closed embedding, we give $\hat{V}$ the subspace topology inside $\widehat{\mathbb{A}}^n$.
- The case of an arbitrary $V$ done by gluing affine pieces: if $V = \bigcup U_i$ is an open affine cover, $\hat{V} = \bigcup \hat{U}_i$ is an open cover.
- Let $X \subseteq V$ be a definable subset of the variety $V$. Then we give $\hat{X}$ the subspace topology inside $\hat{V}$.

Subsets of $\hat{V}$ of the form $\hat{X}$ will be called semi-algebraic.
The topology on $\widehat{\mathbb{A}}^n$

Recall that any definable function $f : X \to \Gamma_\infty$ canonically extends to a map $f : \widehat{X} \to \Gamma_\infty$ (given by the composition $\widehat{X} \xrightarrow{\hat{f}} \widehat{\Gamma}_\infty \xrightarrow{\sim} \Gamma_\infty$).

**Definition**

We endow $\widehat{\mathbb{A}}^n(U)$ with the topology generated by the (so-called *pre-basic open*) sets of the form

$$\{ a \in \widehat{\mathbb{A}}^n | \text{val}(F(a)) < \gamma \} \text{ or } \{ a \in \widehat{\mathbb{A}}^n | \text{val}(F(a)) > \gamma \},$$

where $F \in U[x_1, \ldots, x_n]$ and $\gamma \in \Gamma(U)$.

**Remark**

1. *The topology is the coarsest one such that for all polynomials $F$, the map $\text{val} \circ F : \widehat{\mathbb{A}}^n \to \Gamma_\infty$ is continuous.*
   (Here, $\Gamma_\infty$ is considered with the order topology.)

2. *It has a basis of open semialgebraic sets.*
Proposition

The topology on \( \hat{V} \) is pro-definable, over the same parameters over which \( V \) is defined.

Proof.

- By our construction, it is enough to show the result for \( V = \mathbb{A}^n \).

- For any \( d \), the pre-basic open sets defined by polynomials of degree \( \leq d \) form a definable family of relatively definable subsets of \( \hat{\mathbb{A}}^n \).
Relationship with the order topology

- For a closed ball $b$, let $p_b$ be the generic type of $b$. The map

$$\gamma : \Gamma^m_\infty \to \widehat{\mathbb{A}}^m, (t_1, \ldots, t_m) \mapsto p_{B \geq t_1}(0) \otimes \cdots \otimes p_{B \geq t_m}(0)$$

is a definable homeomorphism onto its image, where $\Gamma^m_\infty$ is endowed with the (product of the) order topology.

- Let $f = \text{id} \times (\text{val}, \ldots, \text{val}) : V \times \mathbb{A}^m \to V \times \Gamma^m_\infty$. On $\widehat{V} \times \Gamma^m_\infty$ we put the topology induced by $\hat{f}$, i.e.

$U \subseteq \widehat{V} \times \Gamma^m_\infty$ is open iff $\hat{f}^{-1}(U)$ is open in $\widehat{V} \times \mathbb{A}^m$.

Fact

$\Gamma^m_\infty = \Gamma^m_\infty$. Moreover, the map $\widehat{V} \times \Gamma^m_\infty \to \widehat{V} \times \Gamma^m_\infty = \widehat{V} \times \Gamma^m_\infty$ is a homeomorphism, where $\Gamma^m_\infty$ is endowed with the order topology.
Example (The topology on $\hat{\mathbb{A}^1}$)

- Recall that $\hat{\mathbb{A}^1} = \mathcal{B}^{cl}$ as a set.
- A semialgebraic subset $\hat{X} \subseteq \hat{\mathbb{A}^1}$ is open iff $X$ is a finite union of sets of the form $\Omega \setminus \bigcup_{i=1}^{n} F_i$, where
  - $\Omega$ is an open ball or the whole field $K$;
  - the $F_i$ are closed sub-balls of $\Omega$.
- $\hat{m}$ and $\hat{m} \setminus \{0\}$ are open, with closure equal to $\hat{m} \cup \{p_{\Omega}\}$, a definable closed set which is not semi-algebraic.
- $\{p_b \mid \text{rad}(b) > \alpha\}$ ($\alpha \in \Gamma$) is def. open and non semi-algebraic.
- The topology is definably generated by the family $\{\Omega \setminus F\}_{\Omega,F}$.
- There is no definable basis for the topology.

Fact

*For any curve $C$, the topology on $\hat{C}$ is definably generated.*

[This follows from the proof of iso-definability of $\hat{C}$.]
First properties of the topological space $\hat{V}$

Fact

Let $V$ be an algebraic variety defined over $M \models ACVF$.

1. The topology on $\hat{V}(M)$ is Hausdorff.
2. The subset $V(M)$ of simple points is dense in $\hat{V}(M)$.
3. The induced topology on $V(M)$ is the valuation topology.

Proof.

We will assume that $V$ is affine, say $V \subseteq \mathbb{A}^n$.

For (1), let $p, q \in \hat{V}(M)$ with $p \neq q$. There is $F(\bar{x}) \in K[\bar{x}]$ such that $\text{val}(F(p)) \neq \text{val}(F((q)))$, say $\text{val}(F(p)) < \alpha < \text{val}(F((q)))$, where $\alpha \in \Gamma(M)$. Then $\text{val}(F(\bar{x})) < \alpha$ and $\text{val}(F(\bar{x})) > \alpha$ define disjoint open sets in $\hat{V}$, one containing $p$, the other containing $q$.

(2) and (3) follows from the fact that there is a basis of the topology given by semialgebraic open sets.
The $v+g$-topology

- Let $V$ be a variety and $X \subseteq V$ definable. We say
  - $X$ is $v$-open (in $V$) if it is open for the valuation topology;
  - $X$ is $g$-open (in $V$) if it is given (inside $V$) by a positive Boolean combination of Zariski constructible sets and sets defined by strict valuation inequalities $\text{val}(F(\bar{x})) < \text{val}(G(\bar{x}))$;
  - $X$ is $v+g$-open (in $V$) if it is $v$-open and $g$-open.
- We say $X \subseteq V \times \Gamma_\infty^m$ is $v$-open iff its pullback to $V \times \mathbb{A}^m$ is. (Similarly for $g$-open and $v+g$-open.)

Remark

The $g$-open and the $v+g$-open sets do not give rise to a definable topology. Indeed, $\mathcal{O}$ is not $g$-open, but $\mathcal{O} = \bigcup_{a \in \mathcal{O}} a + m$, so it is a definable union of $v+g$-open sets.
Why consider the \(v\)-topology and the \(g\)-topology?

- With the two topologies \((v\text{ and } g)\), one may separate continuity issues related to very different phenomena in \(\Gamma_\infty\), namely
  - the **behaviour near** \(\infty\) (captured by the \(v\)-topology) and
  - the **behaviour near** \(0 \in \Gamma\) (captured by the \(g\)-topology).
- It is e.g. easier to check continuity separately.
- \(v+g\)-topology on \(V \leftrightarrow\) topology on \(\hat{V}\) (see on later slides)

**Exercise**

- The \(v\)-topology on \(\Gamma_\infty\) is discrete on \(\Gamma\), and a basis of open neighbourhoods at \(\infty\) is given by \(\{(\alpha, \infty], \alpha \in \Gamma\}\).
- The \(g\)-topology on \(\Gamma_\infty\) corresponds to the order topology on \(\Gamma\), with \(\infty\) isolated.
- Thus, the \(v+g\)-topology on \(\Gamma_\infty\) is the order topology.
Limits of definable types in (pro-)definable spaces

**Definition**
Let $p(x)$ a definable type on a pro-definable space $X$. We say $a \in X$ is a **limit** of $p$ if $p(x) \models x \in W$ for every $U$-definable neighbourhood $W$ of $a$.

**Remark**
If $X$ is Hausdorff space, then limits are unique (if they exist), and we write $a = \lim(p)$.

**Example**
Let $M$ be an o-minimal structure and $\alpha \in M$. Then $\alpha = \lim(\alpha^+)$. 
Describing the closure with limits of definable types

Proposition

Let $X$ be prodefinable subset of $\hat{V} \times \Gamma^m$. 

1. If $X$ is closed, then it is closed under limits of definable types, i.e. if $p$ is a definable type on $X$ such that $\lim(p)$ exists in $\hat{V} \times \Gamma^m$, then $\lim(p) \in X$.

2. If $a \in \operatorname{cl}(X)$, there is a def. type $p$ on $X$ such that $a = \lim(p)$. Thus, $X$ closed under limits of definable types $\Rightarrow$ $X$ closed.

Example

Recall that $\operatorname{cl}(\mathfrak{m} \setminus \{0\}) = \hat{\mathfrak{m}} \cup \{p_\emptyset\}$.

- Let $q_{0^+}$ be the (definable) type giving the generic type in the closed ball of radius $\epsilon \models 0^+$ around $0$. Then $p_\emptyset = \lim(q_{0^+})$.
- Similarly, $0 \models B_{\geq \infty}(0) = \lim(q_{\infty^-})$. 
Definable compactness

Definition
A (pro-)definable space $X$ is said to be **definably compact** if every definable type on $X$ has a limit in $X$.

Remark
*In an o-minimal structure $M$, this notion is equivalent to the usual one, i.e. a definable subset $X \subseteq M^n$ is definably compact iff it is closed and bounded.*
Lemma (The key to the notion of definable compactness)

Let $f : X \to Y$ be a surjective (pro-)definable map between (pro-)definable sets (in ACVF). Then the induced maps $f_{\text{def}} : S_{\text{def}, X} \to S_{\text{def}, Y}$ and $\hat{f} : \hat{X} \to \hat{Y}$, are surjective, too.

Corollary

Assume $f : \hat{V} \times \Gamma_m^\infty \to \hat{W} \times \Gamma_n^\infty$ is definable and continuous, and $X \subseteq \hat{V} \times \Gamma_m^\infty$ is a pro-definable and definably compact subset. Then $f(X)$ is definably compact.

Proof of the corollary.

- By the lemma, any definable type $p$ on $f(X)$ is of the form $f_* q = f_{\text{def}}(q)$ for some definable type $q$ on $X$.
- As $X$ is definably compact, there is $a \in X$ with $\lim(q) = a$.
- By continuity of $f$, we get $\lim(p) = f(a)$.
Bounded subsets of algebraic varieties

Definition

- Let $V \subseteq \mathbb{A}^m$ be a closed subvariety. We say a definable set $X \subseteq V$ is bounded (in $V$) if $X \subseteq cO^m$ for some $c \in K$.

- For general $V$, $X \subseteq V$ is called bounded (in $V$) if there is an open affine cover $V = \bigcup_{i=1}^n U_i$ and $X_i \subseteq U_i$ with $X_i$ bounded in $U_i$ such that $X = \bigcup_{i=1}^n X_i$.

- $X \subseteq V \times \Gamma_m$ is said to be bounded (in $V \times \Gamma_m$) if its pullback to $V \times \mathbb{A}^m$ is bounded in $V \times \mathbb{A}^m$.

- Finally, we say that a pro-definable subset $X \subseteq \hat{V}$ is bounded (in $\hat{V}$) if there is $W \subseteq V$ bounded such that $X \subseteq \hat{W}$.

Fact

The notion is well-defined (i.e. independent of the closed embedding into affine space and of the choice of an open affine cover).
Examples

1. $X \subseteq \Gamma_{\infty}$ is bounded iff $X \subseteq [\gamma, \infty]$ for some $\gamma \in \Gamma$.

2. $\mathbb{P}^n$ is bounded in itself, so every $X \subseteq \mathbb{P}^n$ is bounded.
   Indeed, if $\mathbb{A}^n \cong U_i$ is the affine chart given by $x_i \neq 0$ and $U_i(\mathcal{O}) \subseteq U_i$ corresponds to $\mathcal{O}^n \subseteq \mathbb{A}^n$, then we may write $\mathbb{P}^n = \bigcup_{i=0}^n U_i(\mathcal{O})$.

3. $\mathbb{A}^1$ is bounded in $\mathbb{P}^1$ and unbounded in itself, so the notion depends on the ambient variety.
A characterisation result for definable compactness

**Theorem**

Let $X \subseteq \hat{V} \times \Gamma_m$ be pro-definable. TFAE:

1. $X$ is definably compact.
2. $X$ is closed and bounded.

To illustrate the methods, we will prove that if $X \subseteq \hat{V} \times \Gamma_m$ is bounded, then any definable type on $X$ has a limit in $\hat{V} \times \Gamma_m$.

**Corollary**

Let $W \subseteq V \times \Gamma_m$.

1. $\hat{W}$ is closed in $\hat{V} \times \Gamma_m$ iff $W$ is $v+g$-closed in $V \times \Gamma_m$.
2. $\hat{W}$ is definably compact iff $W$ is a $v+g$-closed and bounded subset of $V \times \Gamma_m$. 
Some further applications of the characterisation result

The results below are analogous to the complex situation.

Corollary

A variety $V$ is complete iff $\hat{V}$ is definably compact.

Proof.

- By Chow’s lemma, if $V$ is complete there is $f : V' \to V$ surjective with $V'$ projective. This proves one direction.

- For the other direction, use that every variety is an open Zariski dense subvariety of a complete variety.

Corollary

If $f : V \to W$ is a proper map between algebraic varieties, then $\hat{f} : \hat{V} \to \hat{W}$ as well as $\hat{f} \times \text{id} : \hat{V} \times \Gamma^m \to \hat{W} \times \Gamma^m$ are closed maps.
Proof that definable types on bounded sets have limits

**Lemma**

Let $p$ be a definable type on a bounded subset $X \subseteq \mathcal{V} \times \Gamma^m$. Then $\lim(p)$ exists in $\mathcal{V} \times \Gamma^m$.

**Proof.**

- First we reduce to the case where $\mathcal{V} = \mathbb{A}^n$ and $m = 0$.

- Let $K \models \text{ACVF}$ be maximally complete, with $p$ $K$-definable, $d \models p \upharpoonright K$ and $a \models p_d \upharpoonright Kd$, where $p_d$ is the type coded by $d$.

- As $p_d \perp \Gamma$, we have $\Gamma_K \subseteq \Gamma(K(d)) = \Gamma(K(d, a)) =: \Delta$.

  Let $\Delta_0 := \{ \delta \in \Delta \mid \exists \gamma \in \Gamma_K : \gamma < \delta \}$.

- $p$ definable $\Rightarrow$ for $\delta \in \Delta_0$, $\text{tp}(\delta/\Gamma_K)$ is definable and has a limit in $\Gamma_K \cup \{\infty\}$. 
End of the proof

(Recall: $\Delta_0 := \{ \delta \in \Delta \mid \exists \gamma \in \Gamma_K : \gamma < \delta \})$

- We get a retraction $\pi : \Delta_0 \to \Gamma_K \cup \{ \infty \}$ preserving $\leq$ and $+$.
- $\mathcal{O}' := \{ b \in K(a) \mid \text{val}(b) \in \Delta_0 \}$ is a valuation ring on $K(a)$.
- As $K \subseteq \mathcal{O}'$, putting $\tilde{\text{val}}(x + m') := \pi(\text{val}(x))$, we get a valued field extension $\tilde{K} = \mathcal{O}'/m' \supseteq K$, with $\Gamma_{\tilde{K}} = \Gamma_K$.
- The coordinates of $a$ lie in $\mathcal{O}'$, by the boundedness of $X$.
- Consider the tuple $\tilde{a} := a + m' \in K'$.
  - Then $r = \text{tp}(a'/K)$ is stably dominated as $\Gamma(Ka') = \Gamma(K)$ and $K$ is maximally complete.
  - One checks that $r = \lim(p)$. (Indeed, one shows $f(r) = \lim(f_*(p))$ for every $f = \text{val} \circ F$, where $F \in K[\overline{x}]$.)
Γ-internal subsets of $\widehat{V}$

Convention
*From now on, all varieties are assumed to be quasi-projective.*

Definition
A subset $Z \subseteq \widehat{V} \times \Gamma_{\infty}^m$ is called $\Gamma$-internal if

- $Z$ is iso-definable and
- there is a surjective definable $f : D \subseteq \Gamma_{\infty}^n \rightarrow Z$.

Remark
*If we drop in the definition the iso-definability requirement, we get the weaker notion called $\Gamma$-parametrisability.*

Fact
*Let $f : C \rightarrow C'$ be a finite morphism between algebraic curves. Assume that $Z \subseteq \widehat{C}$ is $\Gamma$-internal. Then $\widehat{f}^{-1}(Z)$ is $\Gamma$-internal.*
Topological witness for $\Gamma$-internality

**Proposition**

Let $Z \subseteq \hat{V} \times \Gamma^m$ be $\Gamma$-internal. Then there is an injective continuous definable map $f : Z \hookrightarrow \Gamma^n_\infty$ for some $n$. If $Z$ is definably compact, such an $f$ is a homeomorphism.

The question is more delicate if one wants to control the parameters needed to define $f$. Here is the best one can do:

**Proposition**

Suppose that in the above, both $V$ and $Z$ are $A$-definable, where $A \subseteq VF \cup \Gamma$. Then there is a finite $A$-definable set $w$ and an injective continuous $A$-definable map $f : Z \hookrightarrow \Gamma^w_\infty$.

**Example**

Let $A = \mathbb{Q} \subseteq VF$, $V$ given by $X^2 - 2 = 0$. Then $\hat{V}$ is $\Gamma$-internal, with a non-trivial Galois action, so cannot be $\mathbb{Q}$-embedded into $\Gamma^n_\infty$. 
Generalised intervals

We say that $I = [o_I, e_I]$ is a **generalised closed interval** in $\Gamma_\infty$ if it is obtained by concatenating a finite number of closed intervals $I_1, \ldots, I_n$ in $\Gamma_\infty$, where $<_{I_i}$ is either given by $<_{\Gamma_\infty}$ or by $>_{\Gamma_\infty}$.

**Remark**

- *The absence of the multiplication in $\Gamma_\infty$ makes it necessary to consider generalised intervals.*
- *E.g., there is a definable path $\gamma : I \rightarrow \hat{\mathbb{P}^1}$ with $\gamma(o_I) = 0$ and $\gamma(e_I) = 1$, but only if we allow generalised intervals in the definition of a path.*
Definable homotopies and strong deformation retractions

Definition

Let \( I = [o_I, e_I] \) be a generalised interval in \( \Gamma_\infty \) and let \( X \subseteq V \times \Gamma^m_\infty, \ Y \subseteq W \times \Gamma^m_\infty \) be definable sets.

1. A continuous pro-definable map \( H : I \times \hat{X} \to \hat{Y} \) is called a **definable homotopy** between the maps \( H_o, H_e : \hat{X} \to \hat{Y} \), where \( H_o \) corresponds to \( H \upharpoonright_{\{o_I\} \times \hat{X}} \) (similarly for \( H_e \)).

2. We say that the definable homotopy \( H : I \times \hat{X} \to \hat{X} \) is a **strong deformation retraction** onto the set \( \Sigma \subseteq \hat{X} \) if
   - \( H_0 = \text{id}_{\hat{X}} \),
   - \( H \upharpoonright_{I \times \Sigma} = \text{id}_{I \times \Sigma} \),
   - \( H_e(\hat{X}) \subseteq \Sigma \), and
   - \( H_e(H(t, a)) = H_e(a) \) for all \((t, a) \in I \times \hat{X} \).

We added the last condition, as it is satisfied by all the retractions we will consider.
The standard homotopy on $\hat{\mathbb{P}}^1$

- We represent $\mathbb{P}^1(U)$ as the union of two copies of $\mathcal{O}(U)$, according to the two affine charts w.r.t. $u$ and $\frac{1}{u}$, respectively.

- In this way, unambiguously, any element of $\hat{\mathbb{P}}^1$ corresponds to the generic type $p_{B \geq s}(a)$ of a closed ball of val. radius $s \geq 0$.

**Definition**

The **standard homotopy** on $\hat{\mathbb{P}}^1$ is defined as follows:

$$
\psi : [0, \infty] \times \hat{\mathbb{P}}^1 \to \hat{\mathbb{P}}^1, \ (t, p_{B \geq s}(a)) \mapsto p_{B \geq \min(s, t)}(a)
$$

**Lemma**

The map $\psi$ is continuous. Viewing $[0, \infty]$ as a (generalised) interval with $o_I = \infty$ and $e_I = 0$, $\psi$ is a strong deformation retraction of $\hat{\mathbb{P}}^1$ onto the singleton set $\{p_{\emptyset}\}$. 
A variant: the standard homotopy with stopping time $D$

- $\mathbb{P}^1(U)$ has a tree-like structure: any two elements $a, b \in \mathbb{P}^1(U)$ are the endpoints of a unique segment, i.e. a subset of $\mathbb{P}^1$ definably homeomorphic to a (generalised) interval in $\Gamma_\infty$.

- Given $D \subseteq \mathbb{P}^1$ finite, let $C_D$ be the convex hull of $D \cup \{p_\mathcal{O}\}$ in $\mathbb{P}^1$, i.e. the image of $[0, \infty] \times (D \cup \{p_\mathcal{O}\})$ under $\psi$.

- $C_D$ is closed in $\mathbb{P}^1$ and $\Gamma$-internal, and the map $\tau : \mathbb{P}^1 \to \Gamma_\infty$, $\tau(b) := \max\{t \in [0, \infty] \mid \psi(t, b) \in C_D\}$ is continuous.

**Lemma**

*Consider the standard homotopy with stopping time $D$,*

$$\psi_D : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \quad (t, b) \mapsto \psi(\max(\tau(b), t), b).$$

*Then $\psi_D$ defines a strong deformation retraction of $\mathbb{P}^1$ onto $C_D$.***
A strong deformation retraction for curves

Theorem

Let $C$ be an algebraic curve. Then there is a strong deformation retraction $H : [0, \infty] \times \hat{C} \to \hat{C}$ onto a $\Gamma$-internal subset $\Sigma \subseteq \hat{C}$.

Sketch of the proof.

- WMA $C$ is projective.
- Choose $f : C \to \mathbb{P}^1$ finite and generically étale.
- Idea: one shows that there is $D \subseteq \mathbb{P}^1$ finite such that $\psi_D : [0, \infty] \times \hat{\mathbb{P}^1} \to \hat{\mathbb{P}^1}$ 'lifts' (uniquely) to a strong deformation retraction $H : [0, \infty] \times \hat{C} \to \hat{C}$, i.e., such that $H \circ \hat{f} = \psi_D \circ (\text{id} \times \hat{f})$ holds.
Outward paths on finite covers of $\mathbb{A}^1$

Definition

- A standard outward path on $\hat{\mathbb{A}}^1$ starting at $a = p_{B \geq s}(c)$ is given by $\gamma : (r, s] \rightarrow \hat{\mathbb{A}}^1$ (for some $r < s$) such that $\gamma(t) = p_{B \geq t}(c)$.
- Let $f : C \rightarrow \mathbb{A}^1$ be a finite map. An outward path on $\hat{C}$ starting at $x \in \hat{C}$ (with respect to $f$) is a continuous definable map $\gamma : (r, s] \rightarrow \hat{C}$ for some $r < s$ such that
  - $\gamma(s) = x$ and
  - $\hat{f} \circ \gamma$ is a standard outward path on $\hat{\mathbb{A}}^1$.

Lemma

Let $f : C \rightarrow \mathbb{A}^1$ be a finite map. Then, for every $x \in \hat{C}$, there exists at least one and at most $\deg(f)$ many outward paths starting at $x$ (with respect to $f$).
Finiteness of outward branching points

- Let \( f : C \rightarrow \mathbb{A}^1 \) be a finite map, \( d = \text{deg}(f) \).
- Note that for all \( x \in \hat{\mathbb{A}}^1 \), we have \( |\hat{f}^{-1}(x)| \leq d \).
- We say \( y \in \hat{C} \) is outward branching (for \( f \)) if there is more than one outward path on \( \hat{C} \) starting at \( y \). In this case, we also say that \( \hat{f}(y) \in \hat{\mathbb{A}}^1 \) is outward branching.

**Key lemma**

*The set of outward branching points (for \( f \)) is finite.*
End of the proof

Suppose $f : C \to \mathbb{P}^1$ is finite and generically étale.

By the key lemma, there is $D \subseteq \mathbb{P}^1$ finite such that

- $f$ is étale above $\mathbb{P}^1 \setminus D$;
- $C_D$ contains all outward branching points, with respect to the maps restricted to the two standard affine charts.

**Lemma**

*Under the above assumptions, the map $\psi_D : [0, \infty] \times \hat{\mathbb{P}}^1 \to \hat{\mathbb{P}}^1$ lifts (uniquely) to a strong deformation retraction $H : [0, \infty] \times \hat{C} \to \hat{C}$.*
Example

Consider the elliptic curve $E$ given by the affine equation $y^2 = x(x - 1)(x - \lambda)$, where $\text{val}(\lambda) > 0$ (in char $\neq 2$).

Let $f : E \to \mathbb{P}^1$ be the map to the $x$-coordinate.

- $f$ is ramified at $0, 1, \lambda$ and $\infty$.
- Using Hensel’s lemma, one sees that the fiber of $\hat{f}$ above $x \in \mathbb{A}^1$ has two elements iff $x$ is neither in the segment joining $0$ and $\lambda$, nor in the one joining $1$ and $\infty$.
- Thus, for $B = B_{\geq \text{val}(\lambda)(0)}$, the point $p_B$ is the unique outward branching point on the affine line corresponding to $x \neq \infty$.
- On the affine line corresponding to $x \neq 0$, $p_0$ is the only outward branching point.
- We may thus take $D = \{0, \lambda, 1, \infty\}$.
- If $H$ is the unique lift of $\psi_D$, then $H$ defines a retraction of $\hat{E}$ onto a subset of $\hat{E}$ which is homotopic to a circle.
Definable connectedness

Definition

Let $V$ be an algebraic variety and $Z \subseteq \hat{V}$ strict pro-definable.

- $Z$ is called **definably connected** if it contains no proper non-empty clopen strict pro-definable subset.

- $Z$ is called **definably path-connected** if any two points $z, z' \in Z$ are connected by a definable path.

The following lemma is easy.

Lemma

1. $Z$ definably path-connected $\Rightarrow$ $Z$ definably connected

2. For $X \subseteq V$ definable, $\hat{X}$ is definably connected iff $X$ does not contain any proper non-empty v+g-clopen definable subset.

3. If $\hat{V}$ is definably connected, then $V$ is Zariski-connected.
GAGA for connected components

- For $X \subseteq V$ definable, we say $\hat{X}$ has **finitely many connected components** if $X$ admits a finite definable partition into $v+g$-clopen subsets $Y_i$ such that $\hat{Y}_i$ is definably connected.

- The $\hat{Y}_i$ are then called the **connected components** of $\hat{X}$.

**Theorem**

Let $V$ be an algebraic variety.

- $\hat{V}$ is definably connected iff $V$ is Zariski connected.
- $\hat{V}$ has finitely many connected components, which are of the form $\hat{W}$, for $W$ a Zariski connected component of $V$. 
Proof of the theorem: reduction to smooth projective curves

**Lemma**

Let $V$ be a smooth variety and $U \subseteq V$ an open Zariski-dense subvariety of $V$. Then $\hat{V}$ has finitely many connected components if and only if $\hat{U}$ does. Moreover, in this case there is a bijection between the two sets of connected components.

We assume the lemma (which will be used several times).

- WMA $V$ is Zariski-connected.
- WMA $V$ is irreducible.
- Any two points $v \neq v' \in V$ are contained in an irreducible curve $C \subseteq V$. This uses Chow’s lemma and Bertini’s theorem.

⇒ WMA $V = C$ is an irreducible curve.
- WMA $C$ is projective (by the lemma) and smooth (passing to the normalisation $\tilde{C} \to C$)
The case of a smooth projective curve $C$

We have already seen:

$\hat{C}$ retracts to a $\Gamma$-internal (PL) subspace $S \subseteq \hat{C}$

$\Rightarrow$ $\hat{C}$ has finitely many connected components (all path-connected)

- If $g(C) = 0$, $C \cong \mathbb{P}^1$, so $\hat{C}$ is contractible (thus connected).
- If $g(C) = 1$, $C \cong E$, where $E$ is an elliptic curve.
  - $(E(\mathbb{U}), +)$ acts on $\hat{E}(\mathbb{U})$ by definable homeomorphisms;
  - this action is transitive on simple points (which are dense).

$\Rightarrow E(\mathbb{U})$ acts transitively on the (finite!) set of connected components of $\hat{E}$.

$\Rightarrow \hat{E}$ is connected, since $E(\mathbb{U})$ is divisible.
The case of a smooth projective curve $C$, with $g(C) \geq 2$.

- Let $\widehat{C}_0, \ldots, \widehat{C}_{n-1}$ be the connected components of $\widehat{C}$.

- For $I = (i_1, \ldots, i_g) \in n^g$, $C_I := C_{i_1} \times \cdots \times C_{i_g}$ is a $v+g$-clopen subset of $C^g$, and $\widehat{C}_I$ is definably connected.

- Thus, $\widehat{C}^g$ has $n^g$ connected components. If $n \geq 2$, $\widehat{C}^g$ as well as $\widehat{C}^g/S_g$ has finitely many ($>1$) connected components.

- Recall: $C^g/S_g$ is birational to the Jacobian $J = \text{Jac}(C)$ of $C$.

- Using the lemma twice, we see that $\widehat{J}$ has finitely many ($>1$) connected components. (Both $C^g/S_g$ and $J$ are smooth.)

- But, as before, $(J(\mathbb{U}), +)$ is a divisible group acting transitively on the set of connected components of $\widehat{J}$. Contradiction!
The main theorem of Hrushovski-Loeser (a first version)

Theorem

Suppose $A = K \cup G$, where $K \subseteq VF$ and $G \subseteq \Gamma_\infty$. Let $V$ be a quasiprojective variety and $X \subseteq V \times \Gamma_n^{\infty}$ an $A$-definable subset. Then there is an $A$-definable strong deformation retraction $H : I \times \hat{X} \to \hat{X}$ onto a ($\Gamma$-internal) subset $\Sigma \subseteq \hat{X}$ such that $\Sigma$ $A$-embeds homeomorphically into $\Gamma_\infty^w$ for some finite $A$-definable $w$.

Corollary

Let $X$ be as above. Then $\hat{X}$ has finitely many definable connected components. These are all semi-algebraic and path-connected.

Proof.

Let $H$ and $\Sigma$ be as in the theorem. By $o$-minimality, $\Sigma$ has finitely many def. connected components $\Sigma_1, \ldots, \Sigma_m$. The properties of $H$ imply that $H_e^{-1}(\Sigma_i) = \hat{X}_i$, where $X_i = H_e^{-1}(\Sigma_i) \cap X$. \qed
The main theorem of Hrushovski-Loeser (general version)

**Theorem**

Let $A = K \cup G$, where $K \subseteq \mathbf{VF}$ and $G \subseteq \Gamma_{\infty}$. Assume given:

1. a quasiprojective variety $V$ defined over $K$;
2. an $A$-definable subset of $X \subseteq V \times \Gamma_{\infty}^m$;
3. a finite algebraic group action on $V$ (defined over $K$);
4. finitely many $A$-definable functions $\xi_i : V \to \Gamma_{\infty}$.

Then there is an $A$-definable strong deformation retraction $H : I \times \hat{X} \to \hat{X}$ onto a ($\Gamma$-internal) subset $\Sigma \subseteq \hat{X}$ such that

- $\Sigma$ $A$-embeds homeomorphically into $\Gamma_{\infty}^w$ for some finite $A$-definable $w$;
- $H$ is equivariant w.r.t. to the algebraic group action from (3);
- $H$ respects the $\xi_i$ from (4), i.e. $\xi(H(t, v)) = \xi(v)$ for all $t, v$. 

Some words about the proof of the main theorem

- The proof is by induction on $d = \dim(V)$, fibering into curves.

- The fact that one may respect extra data (the functions to $\Gamma_\infty$ and the finite algebraic group action) is essential in the proof, since these extra data are needed in the inductive approach.

- In going from $d$ to $d + 1$, the homotopy is obtained by a concatenation of four different homotopies.

- Only standard tools from algebraic geometry are used, apart from Riemann-Roch (used the proof of iso-definability of $\hat{C}$).

- Technically, the most involved arguments are needed to guarantee the continuity of certain homotopies. There are nice specialisation criteria (both for the $v$- and for the $g$-topology) which may be formulated in terms of 'doubly valued fields'.
Berkovich spaces slightly generalised

A type $p = \text{tp}(\bar{a}/A) \in S(A)$ is said to be almost orthogonal to $\Gamma$ if $\Gamma(A) = \Gamma(A\bar{a})$.

- Let $F$ a valued field s.t. $\Gamma_F \leq \mathbb{R}$.
- Set $\mathbb{F} = (F, \mathbb{R})$, where $\mathbb{R} \subseteq \Gamma$.
- Let $V$ be a variety defined over $F$, and $X \subseteq V \times \Gamma^m_\infty$ an $F$-definable subset.
- Let $B_X(\mathbb{F}) = \{p \in S_X(\mathbb{F}) \mid p \text{ is almost orthogonal to } \Gamma\}$.
- In a similar way to the Berkovich and the HL setting, one defines a topology on $B_X(\mathbb{F})$.

Fact

If $F$ is complete, then $B_V(\mathbb{F})$ and $V^{an}$ are canonically homeomorphic. More generally, $B_{V \times \Gamma^m_\infty}(\mathbb{F}) = V^{an} \times \mathbb{R}^m_\infty$. 
Passing from $\hat{X}$ to $B_X(F)$

Given $F = (F, \mathbb{R})$ as before, let $F^{\text{max}} \models \text{ACVF}$ be maximally complete such that

- $F \subseteq (F^{\text{max}}, \mathbb{R})$;
- $\Gamma_{F^{\text{max}}} = \mathbb{R}$, and
- $k_{F^{\text{max}}} = k_F^{\text{alg}}$.

Remark

By a result of Kaplansky, $F^{\text{max}}$ is uniquely determined up to $F$-automorphism by the above properties.

Lemma

The restriction of types map $\pi : \hat{X}(F^{\text{max}}) \to S_X(F)$, $p \mapsto p|_F$ induces a surjection $\pi : \hat{X}(F^{\text{max}}) \twoheadrightarrow B_X(F)$.

Remark

There exists an alternative way of passing from $\hat{X}$ to $B_X(F)$, using imaginaries (from the lattice sorts).
The topological link to actual Berkovich spaces

Proposition

1. The map $\pi : \hat{X}(F^{\text{max}}) \rightarrow B_X(F)$ is continuous and closed. In particular, if $F = F^{\text{max}}$, it is a homeomorphism.

2. Let $X$ and $Y$ be $F$-definable subsets of some $V \times \Gamma_m^\infty$, and let $g : \hat{X} \rightarrow \hat{Y}$ be continuous and $F$-prodefinable.

Then there is a (unique) continuous map $\tilde{g} : B_X(F) \rightarrow B_Y(F)$ such that $\pi \circ g = \tilde{g} \circ \pi$ on $\hat{X}(F^{\text{max}})$.

3. If $H : I \times \hat{X} \rightarrow \hat{X}$ is a strong deformation retraction, so is $\tilde{H} : I(\mathbb{R}_\infty) \times B_X(F) \rightarrow B_X(F)$.

4. $B_X(F)$ is compact iff $\hat{X}$ is definably compact.

Remark

The proposition applies in particular to $V^{\text{an}}$. 
The main theorem phrased for Berkovich spaces

**Theorem**

Let $V$ be a quasiprojective variety defined over $F$, and let $X \subseteq V \times \Gamma^m_\infty$ be an $F$-definable subset.

Then there is a strong deformation retraction

$$H : I(\mathbb{R}_\infty) \times B_X(F) \to B_X(F)$$

onto a subspace $Z$ which is homeomorphic to a finite simplicial complex.
Topological tameness for Berkovich spaces I

Theorem (Local contractibility)

Let $V$ be quasi-projective and $X \subseteq V \times \Gamma_m^\infty \mathbb{F}$-definable. Then $B_X(\mathbb{F})$ is locally contractible, i.e. every point has a basis of contractible open neighbourhoods.

Proof.

- There is a basis of open sets given by 'semi-algebraic' sets, i.e., sets of the form $B_{X'}(\mathbb{F})$ for $X' \subseteq X \mathbb{F}$-definable.
- So it is enough to show that any $a \in B_X(\mathbb{F})$ is contained in a contractible subset.
- Let $H$ and $Z$ be as in the theorem, and let $H_e(a) = a' \in Z$. As $Z$ is a finite simplicial complex, it is locally contractible, so there is $a' \subseteq W$ with $W \subseteq Z$ open and contractible.
- The properties of $H$ imply that $H_e^{-1}(W)$ is contractible.
Topological tameness for Berkovich spaces II

Here is a list of further tameness results:

**Theorem**

1. If $V$ quasiprojective and $X \subseteq V \times \Gamma^m$ vary in a definable family, then there are only finitely many homotopy types for the corresponding Berkovich spaces. (We omit a more precise formulation.)

2. If $B_X(F)$ is compact, then it is homeomorphic to $\lim_{\leftarrow i \in I} Z_i$, where the $Z_i$ form a projective system of subspaces of $B_X(F)$ which are homeomorphic to finite simplicial complexes.

3. Let $d = \dim(V)$, and assume that $F$ contains a countable dense subset for the valuation topology. Then $B_V(F)$ embeds homeomorphically into $\mathbb{R}^{2d+1}$ (Hrushovski-Loeser-Poonen).
References


