UN PRINCIPE D'AX-KOCHEN-ERSHOV IMAGINAIRE

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ABSTRACT. We study interpretable sets in henselian and σ-henselian valued fields with value group elementarily equivalent to $\mathbb{Q}$ or $\mathbb{Z}$. Our first result is an Ax-Kochen-Ershov type principle for weak elimination of imaginaries in finitely ramified characteristic zero henselian fields — relative to value group imaginaries and residual linear imaginaries. We extend this result to the valued difference context and show, in particular, that existentially closed equicharacteristic zero multiplicative difference valued fields eliminate imaginaries in the geometric sorts.

On the way, we establish some auxiliary results on separated pairs of characteristic zero henselian fields and on imaginaries in linear structures which are also of independent interest.

1. Introduction

In his seminal work "Une théorie de Galois imaginaire" [Poi83], Poizat introduced the idea that the classification of certain abstract constructions of model theory — namely interpretable sets or Shelah's imaginaries — could play an important role in our comprehension of specific structures. The classification of definable sets, in the guise of quantifier elimination results, has historically been used as a central ingredient in many applications of model theory. But the development of more sophisticated model theoretic tools, in particular stability theory, naturally took place in the larger category of quotients of definable sets by definable equivalence relations, *i.e.*

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interpretable sets. Shelah concretised this idea with his eq construction that formally makes every interpretable set definable.

However, these interpretable sets immediately escape the realm of well understood and classified objects, complicating the possibility of applying those new tools in specific examples, in particular from algebra. Poizat's idea was that these interpretable sets should also be classified, and he did so in algebraically closed fields and in differentially closed fields. In both cases, he showed that they are all definably isomorphic to definable sets, *i.e.* the categories of definable and interpretable sets are equivalent — we say that these structures *eliminate imaginaries*. This property later became an essential feature in model-theoretic applications, *e.g.* to diophantine geometry and algebraic dynamics.

The question of elimination of imaginaries also has a very geometric flavour: given a definable family of sets $X \subseteq Y \times Z$, one wishes to find a definable function $f : Z \to W$ such that for all $z_1, z_2 \in Z$, $X_{z_1} := \{y \in Z : (y, z_1) \in X\} = X_{z_2}$ if and only if $f(y_1) = f(y_2)$ — in other words, one wishes to find a canonical parametrisation of this family where each set appears exactly once. We refer the reader to [TZ12, Section 8.4] for details on these notions and constructions.

Elimination of imaginaries results were then proved for numerous structures, but it was not until work of Haskell, Hrushovski and Macpherson [HHM06] that valued fields entered the picture. However, in this case the situation is more complex. Indeed, the field itself does not eliminate imaginaries, as both the value group and the residue field are interpretable but not isomorphic to a definable set. Nevertheless, one can add certain well understood interpretable sets, the *geometric sorts*. These sorts consist of the field $\mathbf{K}$ and, for all $n \in \mathbb{Z}_{>0}$, of the space $\mathbf{S}_n := \mathrm{GL}_n(\mathbf{K})/\mathrm{GL}_n(\mathcal{O})$ of free rank n $\mathcal{O}$-submodules of $\mathbf{K}^n$, where $\mathcal{O}$ denotes the valuation ring, and of the space $\mathbf{T}_n := \bigcup_{s \in \mathbf{S}_n} s/\mathfrak{m}s$ where $\mathfrak{m} \subseteq \mathcal{O}$ is the unique maximal ideal — *cf.* Section 2.1 for a precise definition of the geometric language. The main result of [HHM06] states that the theory ACVF of algebraically closed non-trivially valued fields eliminates imaginaries in the geometric sorts: given a definable family of sets $X \subseteq Y \times Z$, there exists a definable function $f : Z \to W$, with W a product of geometric sorts, such that for all $z_1, z_2 \in Z$, $X_{z_1} = X_{z_2}$ if and only if $f(z_1) = f(z_2)$ — equivalently the category of sets interpretable in an algebraically closed valued field is equivalent to the category of sets definable in its geometric sorts. One may not overestimate the impact of this result, as it opened the way for the development of geometric model theory in the context of valued fields. A beautiful illustration of the power of these new methods is the work by Hrushovski and Loeser on topological tameness in non-archimedean geometry [HL16].

This result was later extended to other valued fields: real closed valued fields [Mel06], separably closed valued fields of finite imperfection degree [HKR18]. or p-adic fields and their ultraproducts [HMR18] — which allowed to uniformise and extend Denef's result on the rationality of certain zeta functions to interpretable sets. The question remained whether a general principle underlined all these results and such a principle was conjectured in the early 2000's by Hrushovski. The present paper establishes such a principle for a large class of henselian fields, which covers most of the examples considered in applications, and extends it to valued fields with operators.

At this level of generality, one cannot expect elimination in the geometric sorts. Indeed, the residue field and the value group can be arbitrary and might not themselves eliminate imaginaries as is the case in all the results cited above. However, a fundamental idea of the model theory of valued fields, the so-called Ax-Kochen-Ershov principle, is that the model theory of a henselian equicharactersitic zero field should be controlled by its value group and residue field. This principle takes its name from the result of Ax and Kochen [AK65] and independently Ershov [Ers65] that this is indeed the case for elementary equivalence, but this phenomenon has also been observed with respect to numerous other aspects of valued fields, from model theoretic tameness (starting with [Del81]) to motivic integration [HK06].

It is thus tempting to conjecture that, beyond the geometric sorts, imaginaries in equicharacteristic zero henselian fields only arise from the value group and the residue field. This is essentially true up to non trivial torsors of the residue field. One is then naturally led to consider the $\mathbf{k}$-linear imaginaries: given an interpretable set X in the two sorted theory of vector spaces $(\mathbf{k}, \mathbf{V})$, one can consider the collection of this interpretable set in all $\mathbf{k}$-vector spaces arising from $\mathcal{O}$-lattices $s \in \mathbf{S}_n$:

$$\mathbf{T}_{n,X} \coloneqq \bigsqcup_{s \in \mathbf{S}_n} X^{(\mathbf{k}, s/\mathfrak{m}s)}$$

Note that if $X = \mathbf{V}$, $\mathbf{T}_{n,X} = \mathbf{T}_n$ and if $X = \{0\}$, $\mathbf{T}_{n,X} \cong \mathbf{S}_n$. In general, the $\mathbf{T}_{n,X}$ consist exactly of those interpretable sets that admit surjections $\mathbf{T}_n \to \mathbf{T}_{n,X} \to \mathbf{S}_n$. We write $\mathbf{k}^{\mathrm{leq}} \coloneqq \bigsqcup_{n,X} \mathbf{T}_{n,X}$.

Before we state our main results, let us address two technical points. The first is that our results also work in unramified mixed characteristic, at the cost of also considering the higher residue rings $\mathbf{R}_\ell \coloneqq \mathcal{O}/\ell\mathfrak{m}$, for $\ell \in \mathbb{Z}_{\geqslant 1}$, which often play a crucial role in this situation. These rings also come with their linear imaginaries and hence we define $\mathbf{T}_{n,\ell,X} \coloneqq \bigsqcup_{s \in \mathbf{S}_n} X^{(\mathbf{R}_\ell, s/\ell\mathfrak{m}s)}$ and $\mathbf{R}^{\mathrm{leq}} \coloneqq \bigsqcup_{n,\ell,X} \mathbf{T}_{n,\ell,X}$. The second point is that eliminating imaginaries often splits in two distinct problems: describing quotients under the action of finite symmetric groups (in other words finding canonical parameters for finite sets) and classifying interpretable sets up to one-to-finite correspondences: given a definable family of sets $X \subseteq Y \times Z$, one wishes to find a one-to-finite definable correspondence $F : Z \to W$ such that for all $z_1, z_2 \in Z$, $X_{z_1} = X_{z_2}$ if and only if $F(z_1) = F(z_2)$. This latter property is usually referred to as weak elimination of imaginaries and will be the main focus of this paper.

Our first main result is the following Ax-Kochen-Ershov principle for elimination of imaginaries:

Theorem A (Theorem 6.1.1). *Let T be a $\mathbf{\Gamma}$-$\mathbf{R}$-enrichment of* Hen_0 *(with or without angular components), such that:*

- $(\mathbf{C_\Gamma})$ *T has definably complete value group;*
- $(\mathbf{FR})$ *for every $\ell \in \mathbb{Z}_{>0}$, the interval $[0, \mathrm{v}(\ell)]$ is finite and $\mathbf{k}$ is perfect;*
- $(\mathbf{I_k})$ *the residue field $\mathbf{k}$ is infinite;*
- $(\mathbf{E}^\infty_\mathbf{k})$ *the induced theory on $\mathbf{k}$ eliminates $\exists^\infty$.*

Then T weakly eliminates imaginaries in $\mathbf{K} \cup \mathbf{\Gamma}^{\mathrm{eq}} \cup \mathbf{R}^{\mathrm{leq}}$.

Angular components are (compatible) mulitplicative morphisms $\mathrm{ac}_n : \mathbf{K}^\times \to \mathbf{R}_n^\times$ extending the residue map on $\mathcal{O}^\times$. They are often considered in the model theoretic study of valued fields, in particular in the Cluckers-Loeser treatment of motivic integration [CL08].

It is worth noting that PRES = $\mathrm{Th}(\mathbb{Z})$ and DOAG = $\mathrm{Th}(\mathbb{Q})$ are the only complete theories of pure ordered abelian groups which are definably complete. As both PRES and DOAG eliminate imaginaries, we thus get $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathrm{eq}}$ under the assumptions of Theorem A in case $\mathbf{\Gamma}$ is not enriched.

As a corollary, Theorem A yields that if F is a field of characteristic 0 which eliminates $\exists^\infty$, the theories of the valued fields $F((t))$ and $F((t^\mathbb{Q}))$ (with or without angular components) weakly eliminate imaginaries in the sorts $(\mathbf{K} \cup \mathbf{k}^{\mathrm{leq}})$ — noting that $\mathbf{\Gamma} \cong \mathbf{S}_1$ may be identified with a sort in $\mathbf{k}^{\mathrm{leq}}$. In the particular case that F is (of characteristic 0 and) algebraically closed, real closed or pseudofinite, using results of Hrushovski on linear imaginaries, we deduce that $F((t))$ and $F((t^\mathbb{Q}))$ (with or without angular components) eliminate imaginaries in the geometric sorts, after naming some constants in the pseudofinite and in the real closed case (see Corollary 6.1.5 for the precise statement), thus obtaining an absolute elimination result in these cases.

All the results with angular component, even in the algebraically closed (equicharacteristic zero) case are new. Without angular components, this provides alternate proofs of Mellor's result [Mel06] for $\mathrm{Th}(\mathbb{R}((t^\mathbb{Q}))$ and Hrushovski-Martin-Rideau's result [HMR18] for $\mathrm{Th}(F((t))$ where F is pseudofinite of characteristic 0. Independent work of Vicaria [Vic21] also yields the case of $\mathrm{Th}(\mathbb{C}((t))$, although her work also applies to more general value groups.

In mixed characteristic, the main example covered by Theorem A is $W(\mathbb{F}_p^a)$, the fraction field of the ring of Witt vectors with coefficients in $\mathbb{F}_p^a$, and more generally $W(F)$ for any perfect field F of characteristic p which eliminates $\exists^\infty$. Thus, Theorem A provides an important step towards proving that the imaginaries of $W(\mathbb{F}_p^a)$ are classified by the geometric sorts as well.

The second main result of this paper concerns valued difference fields, *i.e.* valued fields with automorphism compatible with the valuation. Quantifier elimination (and hence an Ax-Kochen-Ershov principle for elementary equivalence) has been proved for various classes. First for isometries in [Sca03; BMS07; AvdD10], then for *ω-increasing* automorphism — for every $x \in \mathcal{O}$, $\mathrm{v}(\sigma(x)) > \mathbb{Z} \cdot \mathrm{v}(x)$ — in [Azg10; Hru]. Both of these contexts were

subsumed in later work of Kushik [Pal12] on *multiplicative* automorphisms where the automorphism acts as multiplication by some element of an ordered field (*cf.* Definition 2.3.5 for precise definitions). Finally Durhan and Onay [DO15] proved that these results hold without any hypothesis on the automorphism.
Our second result focuses on the multiplicative setting where we prove an absolute elimination of imaginaries result for the respective model-companions:

Theorem B (Theorem 6.2.1)**.** *The theory* $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$ *eliminates imaginaries in the geometric sorts.*

Since the isometric case and the ω-increasing case correspond, respectively, to the asymptotic theory of $\mathbb{C}_p$ with an isometric lifting of the Frobenius and to $\mathbb{F}_p((t))^{\mathrm{a}}$ with the Frobenius, an immediate corollary of these results is a uniform elimination of imaginaries for large p in these structures.
The proofs of both theorems follow the same general strategy and many technical results are shared between the two. The proof consists of three largely independent steps (Sections 3 to 5).
In stable theories every type p — a maximal consistent set of definable sets — is definable, that is, for every definable $X \subseteq Y \times Z$, the set $\{z \in Z : X_z \in p\}$ is definable. This was used in many proofs of weak elimination of imaginaries in the stable context to reduce the problem of finding canonical parameters for set to finding canonical parameters for types; which, counter-intuitively maybe, is a simpler problem. In [Hru14], Hrushovski formalised the idea that even in an unstable context, this reduction could also prove useful, provided definable were dense: over any algebraically closed imaginary set of parameters, any definable set contains a definable type.
The first step of the proof consists in proving such density results. But the above statement cannot hold in the full generality of henselian equicharacteristic zero fields since it might already fail in the residue field. We prove however that, under certain hypotheses, *quantifier free* definable types are dense, *cf.* Theorem 3.1.1. This result does not apply to discrete valued fields since the family of intervals contains arbitrarily large finite sets. In Theorem 3.1.3, we do however prove that the density of invariant types holds in this context.
The proof improves on similar results in [Rid19] and the general idea is the same. In arity one, we look for a minimal finite set of balls covering the given definable set. The general case proceeds by fibration in relative arity one and by considering germs of functions into the space of (finite sets of) balls instead of actual balls. This fibration process is where most of the technical assumptions of Theorem A are used, in particular the elimination of $\exists^\infty$ in the residue field.
In contexts with an absolute elimination of quantifiers, *e.g.* [Rid19; HKR18], this first density result (and the implicit computation of canonical bases)

suffices to conclude that weak elimination of imaginaries holds. In equicharacteristic zero henselian (and σ-henselian) fields, types come with more information than quantifier free types; an information that mostly lives in the short exact sequence $1 \to \mathbf{k}^\times \to \mathbf{RV}^\times := \mathbf{K}^\times/(1+\mathfrak{m}) \to \mathbf{\Gamma}^\times \to 0$. The second step of our proof, Theorem 4.1.1, consists in showing that quantifier free invariant types have invariant completions over $\mathbf{RV}$ (and $\mathbf{k}$-vector spaces) — this generalises to mixed characteristic by considering the higher residue rings.

By quantifier elimination relative to $\mathbf{RV}$, this step reduces to, given an invariant type, computing canonical generators of the structure generated by (realisations of) the type in $\mathbf{RV}$. Note that in the conclusion of Theorem 4.1.1 the types considered are invariant over definable sets which are of the same size as the model. We do however show various folklore results implying that this a well behaved notion when these sets are stably embedded.

The third step consists in studying imaginaries in $\mathbf{RV}$, which is left as a black box in the previous steps. We show, in the spirit of [HK06, Section 3.3], that the imaginaries in the short exact sequence $1 \to \mathbf{k}^\times \to \mathbf{RV}^\times \to \mathbf{\Gamma}^\times \to 0$ come essentially from $\mathbf{k}$ and $\mathbf{\Gamma}$. But our result is in a sense orthogonal to the one of Hrushovski and Kazhdan since we require $\mathbf{RV}$ to be a pure (in the sense of model theory) extension of $\mathbf{k}$ and $\mathbf{\Gamma}$, which can be both arbitrarily enriched, whereas [HK06] has strong hypotheses on $\mathbf{k}$ and $\mathbf{\Gamma}$ and no hypothesis on $\mathbf{RV}$.

Theorem 5.1.4 is the first version of a series of such reductions of increasing complexity so as to cover the various cases that we require, the ultimate version, Variant 5.2.2 allowing controlled torsion in $\mathbf{\Gamma}$, auxiliary sorts on both the $\mathbf{k}$ and $\mathbf{\Gamma}$ sides and considering not one but a projective system of short exact sequences.

These three steps put together allow us to prove relative results like Theorem A. However, absolute results like Theorem B or Corollary 6.1.5 require one last ingredient: the classification of imaginaries in collections of vector spaces — linear structures in the terminology of [Hru12]. Our contribution consists in a twisted version, *cf.* Fact 2.4.8.(4), of Hrushovski's result on ACF_0-linear structures with flags and roots endowed with an automorphism (the final step to prove Theorem B) and a version, Proposition 2.4.20, for real closed fields.

The plan of the paper is as follows. In Section 2, we provide some preliminary results on separated pairs of valued fields (in the sense of Baur), on valued difference fields and on linear structures. Section 3 is devoted to the proof of the two density results for definable (resp. invariant) types mentioned above. The fact that invariant quantifier free types are invariant over $\mathbf{RV}$ (and $\mathbf{R}_n$-modules), is established in Section 4. In Section 5, we prove the results about imaginaries in certain (enriched) short exact sequences of modules. Finally, in Section 6, we put everything together and prove our main results, in particular Theorem A and Theorem B.

2. Preliminaries

2.1. **The languages of valued fields.** Any valued field (K, v) can be considered as a structure in the language $\mathcal{L}_{\text{div}}$ with one sort $\mathbf{K}$ for the valued field, the ring language and a binary relation $x|y$ interpreted as $v(x) \leqslant v(y)$. This language owes its widespread use to the following result, essentially due to Robinson [Rob77]:

Fact 2.1.1. *The $\mathcal{L}_{\text{div}}$-theory* ACVF *of algebraically closed non trivially valued fields eliminates quantifiers.*

We will write $\mathcal{L}_0 := \mathcal{L}_{\text{div}}$ and throughout this paper, notations with an index 0 — like dcl_0, acl_0 or tp_0 — will refer to the quantifier free $\mathcal{L}_0$-structure; or equivalently the structure induced by any model of ACVF containing the valued field under consideration.
Given a valued field, seen as an $\mathcal{L}_0$-structure, we will denote by $\mathcal{O} := \{x \in \mathbf{K} : v(x) \geqslant 0\}$ its valuation ring, $\mathfrak{m} := \{x \in \mathbf{K} : v(x) > 0\}$ its maximal ideal, $\mathbf{\Gamma} := \mathbf{K}/\mathcal{O}^\times$ its value group, $\mathbf{k} := \mathcal{O}/\mathfrak{m}$ its residue field and $\mathrm{v} : \mathbf{K} \to \mathbf{\Gamma}$ and $\text{res} : \mathcal{O} \to \mathbf{k}$ the canonical projections — we will write $\mathbf{\Gamma}^\times$ for $\mathrm{v}(\mathbf{K}^\times) \subseteq \mathbf{\Gamma}$. More generally, for every $n \in \mathbb{Z}_{>0}$, we write $\mathbf{R}_n := \mathcal{O}/n\mathfrak{m}$. Let also $\text{res}_n : \mathcal{O} \to \mathbf{R}_n$ be the canonical projection, $\mathbf{R}_\infty$ the (pro-definable) set $\varprojlim_n \mathbf{R}_n$ and $\text{res}_\infty : \mathcal{O} \to \mathbf{R}_\infty$ the natural map. Note that, working in a sufficiently saturated model, $\mathbf{R}_\infty \cong \mathcal{O}/\mathfrak{m}_\infty$, where $\mathfrak{m}_\infty := \{x \in \mathbf{K} : \mathrm{v}_\infty(x) > \Delta_\infty\}$ and $\Delta_\infty \leqslant \mathbf{\Gamma}$ is the convex subgroup generated by $\mathrm{v}(\text{char}(\mathbf{k}))$, in mixed characteristic, and $\Delta_\infty = 0$, otherwise. It is a valuation ring whose fraction field is naturally identified with the residue field $\mathbf{k}_\infty$ associated to the the (equicharacteristic) valuation $\mathrm{v}_\infty : \mathbf{K} \to \mathbf{\Gamma} \to \mathbf{\Gamma}/\Delta_\infty$. We also define $\mathbf{R} := \bigsqcup_{n>0} \mathbf{R}_n$.
Although most of the present paper is rather insensitive to the choice of language for valued fields — or, rather, we work in $\mathcal{L}_0^{\text{eq}}$ — we will at times need to work in certain languages tailored for specific elimination results. The first of them is the Haskell-Hrushovski-Macpherson geometric language. For every $n \in \mathbb{Z}_{>0}$, let $\mathbf{S}_n \cong \text{GL}_n(\mathbf{K})/\text{GL}_n(\mathcal{O})$ be the (interpretable) set of rank n free sub-$\mathcal{O}$-modules of $\mathbf{K}^n$, and $\mathbf{T}_n := \bigcup_{s \in \mathbf{S}_n} s/\mathfrak{m}s$. Let $\mathbf{S} := \bigcup_n \mathbf{S}_n$, $\mathbf{T} := \bigcup_n \mathbf{T}_n$ and $\boldsymbol{\mathcal{G}} := \mathbf{K} \cup \mathbf{S} \cup \mathbf{T}$. Note that $\text{GL}_n(K)$ naturally acts transitively on $\mathbf{T}_n$ over its action on $\mathbf{S}_n$. We also denote by $s_n : \text{GL}_n(\mathbf{K}) \to \mathbf{S}_n$, $t_n : \text{GL}_n(\mathbf{K}) \to \mathbf{T}_n$ and $\tau_n : \mathbf{T}_n \to \mathbf{S}_n$ the canonical projections.
These interpretable sets (and the so called *geometric language* of which they are the sorts) were introduced to classify imaginaries in ACVF:

Fact 2.1.2 ([HHM06, Theorem 1.0.1])**.** *The theory* ACVF *eliminates imaginaries in $\boldsymbol{\mathcal{G}}$.*

The second language that we will use allows for a description of definable sets in certain henselian fields. For every $n \in \mathbb{Z}_{>0}$, let $\mathbf{RV}_n := \mathbf{K}/(1+n\mathfrak{m})$. Let $\text{rv}_n : \mathbf{K} \to \mathbf{RV}_n$ and $\text{rv}_{n,m} : \mathbf{RV}_n \to \mathbf{RV}_m$ denote the canonical projections. The set $\mathbf{RV}_n$ is naturally endowed with a multiplication and the trace of addition which we denote, in Krasner's hyperfield manner: $\zeta \oplus \xi := \{\text{rv}(x+y) :$

$\mathrm{rv}_n(x) = \zeta$ and $\mathrm{rv}_n(y) = \xi\} \subseteq \mathbf{RV}_n$. We say that $\zeta \oplus \xi$ is well-defined when $\zeta \oplus \xi = \{\chi\}$ is a singleton, and we often write $\zeta \oplus \xi = \chi$ in that case.
Note that for any two disjoint balls b_1 and b_2, in some valued field (K, v), and any $a_i, c_i \in b_i$, $\mathrm{rv}_1(a_1 - a_2) = \mathrm{rv}_1(c_1 - c_2)$. We will denote by $\mathrm{rv}_1(b_1 - b_2)$ this common value. If $b_1 \cap b_2 \neq \varnothing$, by convention, $\mathrm{rv}_1(b_1 - b_2) = 0$.
We denote by $\mathbf{RV}_\infty$ the (pro-definable) set $\varprojlim_n \mathbf{RV}_n$ and $\mathrm{rv}_\infty : \mathbf{K} \to \mathbf{RV}_\infty$ denotes the natural map. Note that $\mathbf{RV}_\infty \cong \mathbf{K}/(1 + \mathfrak{m}_\infty)$. We also denote $\mathbf{RV} := \bigsqcup_n \mathbf{RV}_n$.
Let $\mathcal{L}_{\mathbf{RV}}$ be the language with sorts $\mathbf{K}$ and $\mathbf{RV}_n$, for all $n \in \mathbb{Z}_{>0}$, the ring structure on $\mathbf{K}$, multiplication and $\oplus$ on each $\mathbf{RV}_n$ and the maps rv_n and $\mathrm{rv}_{n,m}$. Let $\mathcal{L}_{\mathbf{RV}_1}$ be its restriction to the sorts $\mathbf{K}$ and $\mathbf{RV}_1$.
We say that a valued field is:

- *algebraically maximal* if it does not admit non trivial immediate algebraic extensions;
- *Kaplansky* if $\mathrm{v}(\mathbf{K})$ is p-divisible and any finite extension of $\mathbf{k}$ has degree prime to p, where $p = \mathrm{char}(\mathbf{k})$ if it is positive and $p = 1$ otherwise;
- *finitely ramified* if for any $n \in \mathbb{Z}_{>0}$ the interval $[0, \mathrm{v}(n)]$ is finite.

Note that a finitely ramified field is algebraically maximal if and only if it is henselian.
The following quantifier elimination results are due, respectively, to Basarab [Bas91, Theorem A] in characteristic zero and Delon [Del82, Théorème 3.1] in positive characteristic:

Fact 2.1.3.
- *Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathbf{RV}}$ and T an $\mathcal{L}$-theory containing the theory* Hen_0 *of henselian characteristic zero fields. Then T eliminates field quantifiers.*
- *Let $\mathcal{L}$ be an $\mathbf{RV}_1$-enrichment of $\mathcal{L}_{\mathbf{RV}_1}$ and T an $\mathcal{L}$-theory containing the theory of equicharacteristic p algebraically maximal Kaplansky fields, for some fixed $p > 0$. Then T eliminates field quantifiers.*

Convention 2.1.4. Throughout this paper, if M is an $\mathcal{L}$-structure, X is $\mathcal{L}(M)$-definable and $A \subseteq M$, then $X(A)$ denotes $X \cap A$. There are too many structures at play to not be explicit as to which definable or algebraic closures we want to consider.

2.2. **Separated pairs of valued fields.** In this section, we will gather some results about separated pairs of valued fields, in particular concerning pure stable embeddedness of the residue field and value group pairs in specific contexts. In equicharacteristic zero, most of the results below follow from work of Leloup [Lel90]; and from work of Rioux [Rio17], in unramified mixed characteristic.
Recall that an extension L/K of valued fields is called *separated* if every finite-dimensional K-vector subspace of L admits a K-*valuation basis*, i.e., a K-basis $(b_1, \ldots, b_n)$ which is *valuation independent* over K: for any $a_1, \ldots, a_n \in K$ one has $\mathrm{v}(\sum a_i b_i) = \min \mathrm{v}(a_i b_i)$. Also, for fields $K \subseteq L, K' \subseteq U$, we write $L \downarrow^{\mathrm{ld}}_K K'$ if L and K' are linearly disjoint over K.

Definition 2.2.1. Let $K \subseteq L, K' \subseteq U$ be valued fields.

- We say L and K' are $\mathbf{\Gamma k}$-*independent* over K, denoted by $L \downarrow_K^{\mathbf{\Gamma k}} K'$, if $\mathrm{res}(L) \downarrow_{\mathrm{res}(K)}^{\mathrm{ld}} \mathrm{res}(K')$ and $\mathrm{v}(L) \cap \mathrm{v}(K') = \mathrm{v}(K)$.
- Assume that L/K is separated. Then L is said to be *valuatively disjoint* from K' over K, denoted by $L \downarrow_K^{vd} K'$, if whenever a tuple $(b_1, \ldots, b_n)$ from L is valuation independent over K, it is valuation independent over K'.

Fact 2.2.2. *Let $K \subseteq L, K' \subseteq U$ be valued fields with L/K separated and $L \downarrow_K^{\mathbf{\Gamma k}} K'$. Set $L' := LK'$. Then the following hold:*

(1) $L \downarrow_K^{vd} K'$ — in particular, $L \downarrow_K^{\mathrm{ld}} K'$;
(2) L'/K' is separated;
(3) $\mathrm{res}(L') = \mathrm{res}(L)\mathrm{res}(K')$ and $\mathrm{v}(L') = \mathrm{v}(L) + \mathrm{v}(K')$;
(4) If $L_1 \subseteq U$ and $f : L \cong L_1 \subseteq U$ is an isomorphism over $K \cup k_L \cup \Gamma_L$, then f extends (uniquely) to an isomorphism $f' : L' \cong L_1 K'$ over K'.

Proof. This is shown by adapting the proof of the result for K maximally valued from [HHM08, Proposition 12.11]. □

2.2.1. *Reduction to* $\mathbf{RV}$. Most of this paper will be concerned with characteristic zero finitely ramified fields, however, for future reference, we will state and prove certain results, mostly regarding pairs, in all characteristics, as the arguments are essentially identical.

Notation. Assume that $\mathcal{L}$ is a multisorted language, $\mathcal{L}_{\mathbf{P}}$ is the associated language of pairs and M is a pair of $\mathcal{L}$-structures considered as an $\mathcal{L}_{\mathbf{P}}$-structure. If $\mathbf{D}$ is a sort in $\mathcal{L}$, we write $\mathbf{PD}(M)$ for the set of elements in M of sort $\mathbf{D}$ which lie in the substructure singled out by $\mathbf{P}$. We denote $\mathbf{P}(M)$ the whole $\mathcal{L}$-substructure singled out by $\mathbf{P}$.

Let $T^\star$ be a theory of separated pairs $(M, \mathbf{P}(M))$ in the language $\mathcal{L}^{\mathrm{hyb}}_{\mathbf{RV},\mathbf{P}}$ consisting of $\mathcal{L}_{\mathbf{RV}_1,\mathbf{P}}$ enriched with $\mathcal{L}_{\mathbf{RV}}$ on $\mathbf{P}$. We assume that M eliminates fields quantifiers in $\mathcal{L}_{\mathbf{RV}_1}$ and $\mathbf{P}(M)$ eliminates fields quantifiers in $\mathcal{L}_{\mathbf{RV}}$. Note that we do not assume that $\mathbf{PRV}_1$ is stably embedded in $\mathbf{RV}_1$.
By a hybrid $\mathbf{RV}$-structure, we mean a structure (elementarily equivalent to) $(\mathbf{RV}_1(M), \mathbf{PRV}(M))$, where $M \vDash T^\star$ — with the restriction of the $\mathcal{L}^{\mathrm{hyb}}_{\mathbf{RV},\mathbf{P}}$-structure. We also denote $\mathbf{RV}^{\mathrm{hyb}}$ the set of sorts $\mathbf{RV}_1 \cup \bigcup_n \mathbf{PRV}_n$.

Lemma 2.2.3. *Let $M \preccurlyeq N$ be hybrid $\mathbf{RV}$-structures. Then $\mathbf{k}(M) \downarrow_{\mathbf{Pk}(M)}^{\mathrm{ld}} \mathbf{Pk}(N)$ and $\mathbf{\Gamma}(M) \cap \mathbf{P\Gamma}(N) = \mathbf{P\Gamma}(M)$.*

Proof. Immediate from the elementarity of the extension. □

Let M be a hybrid $\mathbf{RV}$-structure. If M is of mixed characteristic we assume that $\mathbf{P}(M)$ is finitely ramified with perfect residue field. Then $\mathbf{PR}_\infty$ is a complete mixed characteristic discrete valuation ring with perfect residue field $\mathbf{Pk}$. It is therefore a finite extension of $\mathrm{W}(\mathbf{Pk})$ of degree $\mathrm{v}(p)$, where $\mathrm{W}(l)$ denotes the fraction field of the ring of Witt vectors over l, *cf.* [Ser68, Ch. II]. In the lemma that follows (and more generally when we refer to "the relevant constants"), in mixed characteristic, we require constants for Witt-coefficients (in $\mathbf{Pk}$) of the minimal polynomial of some choice of uniformizer in $\mathbf{PR}_\infty$.

Lemma 2.2.4. *Assume that $\mathbf{RV}_1(M)$ is divisible, or that $\mathbf{P\Gamma}(M)$ is a pure subgroup of $\mathbf{\Gamma}(M)$. Then in any $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment of M, $\mathbf{k}$ and $\mathbf{\Gamma}$ are purely stably embedded and orthogonal. Moreover, the theory of M is determined by the theories of the $\mathbf{k}$-pair and of the $\mathbf{\Gamma}$-pair.*

Proof. We may assume that M is $\aleph_1$-saturated. Then the short exact sequence of abelian groups

$$1 \to \mathbf{PR}_\infty^\times(M) \to \mathbf{PRV}_\infty^\times(M) \to \mathbf{P\Gamma}^\times(M) \to 0$$

is split and induces coherent splittings of the sequences $1 \to \mathbf{PR}_n^\times(M) \to \mathbf{PRV}_n^\times(M) \to \mathbf{P\Gamma}^\times(M) \to 0$. It follows form the assumptions that the splitting of $1 \to \mathbf{Pk}^\times(M) \to \mathbf{PRV}_1^\times(M) \to \mathbf{P\Gamma}^\times(M) \to 0$ extends to a splitting of the sequence $1 \to \mathbf{k}^\times(M) \to \mathbf{RV}_1^\times(M) \to \mathbf{\Gamma}^\times(M) \to 0$.
Note that the additional structure on $\mathbf{RV}^{\mathrm{hyb}}$, beyond the abelian structure, is given by $\oplus$ and some $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment. But $\oplus$ can be defined using the ring structure on $\mathbf{k}$ and the $\mathbf{PR}_n$. Since the $\mathbf{PR}_n$ are $\varnothing$-interpretable in $\mathbf{Pk}$, it follows that, if we add the sections, $\mathbf{RV}^{\mathrm{hyb}}$ is a $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment of the product of $\mathbf{k}$ and $\mathbf{\Gamma}$. The result follows. □

Let now $M, N \vDash T^\star$, where we suppose that N is $|M|^+$-saturated, and let $A \leqslant M$ and $f : A \to N$ be some embedding.

Definition 2.2.5. We say that:

1. A is *good* if $\mathbf{PK}(A) \leqslant \mathbf{K}(A)$ is a separated extension of valued fields with
$$\mathbf{K}(A) \downarrow^{\mathbf{\Gamma k}}_{\mathbf{PK}(A)} \mathbf{PK}(M),$$
2. f is *good* if $A \leqslant M$ and $f(A) \leqslant N$ are good and $f_{\mathbf{RV}^{\mathrm{hyb}}}$ is elementary — for the $\mathcal{L}^{\mathrm{hyb}}_{\mathbf{RV},\mathbf{P}}\Big|_{\mathbf{RV}^{\mathrm{hyb}}}$-structure.

Proposition 2.2.6. *Assume f is a good embedding. Then f extends to a good embedding $g : M \to N$.*

Proof. We proceed step by step.
Step 1. *We may extend f to a good map defined on $A \cup \mathbf{RV}^{\mathrm{hyb}}(M)$ — and thus assume that $\mathbf{RV}^{\mathrm{hyb}}(A) = \mathbf{RV}^{\mathrm{hyb}}(M)$.*
Indeed, this follows from the fact that $f_{\mathbf{RV}^{\mathrm{hyb}}}$ is elementary and that the only structure involving both $\mathbf{K}$ and $\mathbf{RV}^{\mathrm{hyb}}$ are maps from $\mathbf{K}$ to $\mathbf{RV}^{\mathrm{hyb}}$.

Step 2. *We may extend f to a good map defined on (the substructure generated by) $A \cup \mathbf{P}(M)$ — and thus assume that $\mathbf{PK}(A) = \mathbf{PK}(M)$.*
Indeed, by $\mathbf{K}$-quantifier elimination in the $\mathcal{L}_{\mathbf{RV}}$-theory of the small valued field $\mathbf{P}(M)$, the map $f|_{\mathbf{PK} \cup \mathbf{PRV}}$ extends to an (elementary) $\mathcal{L}_{\mathbf{RV}}$-embedding

$g : \mathbf{P}(M) \to \mathbf{P}(N)$. As $\mathbf{K}(A) \downarrow^{\mathbf{\Gamma k}}_{\mathbf{PK}(A)} \mathbf{PK}(M)$ and $\mathbf{RV}^{\text{hyb}}(A) = \mathbf{RV}^{\text{hyb}}(M)$, by Fact 2.2.2, $f \cup g$ induces a good embedding of $A \cup \mathbf{P}(M)$ into N.

Step 3. *We may extend f to a good embedding of M into N.*
Indeed, by $\mathbf{K}$-quantifier elimination in the $\mathcal{L}_{\mathbf{RV}_1}$-theory of the valued field M, the map f extends to an (elementary) $\mathcal{L}_{\mathbf{RV}_1}$-embedding $\tilde{f} : M \to N$. By Lemma 2.2.3, we get $\tilde{f}(\mathbf{K}(M)) \downarrow^{\mathbf{\Gamma k}}_{f(\mathbf{PK}(M))} \mathbf{PK}(N)$, so in particular $\tilde{f}(\mathbf{K}(M)) \downarrow^{\text{ld}}_{f(\mathbf{PK}(M))} \mathbf{PK}(N)$ by Fact 2.2.2(1), showing that $\tilde{f}$ is an $\mathcal{L}_{\mathbf{RV}_1,\mathbf{P}}$-embedding, with image a good substructure of N. Thus $\tilde{f}$ is a good embedding, since f was already defined on the whole of $\mathbf{RV}^{\text{hyb}}(M)$. □

Corollary 2.2.7. *The theory $T^\star$ is complete relative to $\mathbf{RV}^{\text{hyb}}$, and $\mathbf{RV}^{\text{hyb}}$ is purely stably embedded in $T^\star$, i.e., the induced structure is that of a hybrid $\mathbf{RV}$-structures.*
This even holds resplendently, for any $\mathbf{RV}^{\text{hyb}}$-enrichment of the pair of valued fields.

Proof. Assume that $M, N \vDash T^\star$ are models with $\mathbf{RV}^{\text{hyb}}(M) \equiv \mathbf{RV}^{\text{hyb}}(N)$. The isomorphism of prime substructures is easily seen to be a good embedding. It follows by Proposition 2.2.6, and a back and forth argument, that it is, in fact, elementary — *i.e.* $M \equiv N$.
Similarly, if $M \preccurlyeq N$ — in particular a good substructure — and $f : M \to N$ is an elementary embedding — in particular a good embedding — inducing the identity on $\mathbf{RV}^{\text{hyb}}(M)$, then it remains a good embedding — and hence an elementary one — when extended by the identity on $\mathbf{RV}^{\text{hyb}}(N)$; in other words, $\mathbf{RV}^{\text{hyb}}$ is stably embedded. Finally any $\mathcal{L}^{\text{hyb}}_{\mathbf{RV},\mathbf{P}}\big|_{\mathbf{RV}^{\text{hyb}}}$-elementary map on $\mathbf{RV}^{\text{hyb}}$ is good and hence $\mathcal{L}^{\text{hyb}}_{\mathbf{RV},\mathbf{P}}$-elemetary, so $\mathbf{RV}^{\text{hyb}}$ is pure. □

Remark 2.2.8. If $\mathbf{PRV}$ is (purely) stably embedded in $\mathbf{RV}^{\text{hyb}}$, we can can also deduce that $\mathbf{PK}$ is (purely) stably embedded.

Corollary 2.2.9. *Let $M \vDash T^\star$ be of equicharacteristic or of mixed characteristic with $\mathbf{P}(M)$ finitely ramified with prefect residue field — with relevant constants. Assume that $\mathbf{RV}_1(M)$ is divisible (which is the case for example if $M \vDash$ ACVF) or that $\mathbf{P\Gamma}(M)$ is a pure subgroup of $\mathbf{\Gamma}(M)$. Then in any $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment of M, the theory of M is determined by that of $\mathbf{k}$ and $\mathbf{\Gamma}$ and $\mathbf{k}$ and $\mathbf{\Gamma}$ are purely stably embedded and orthogonal.*

Proof. By Corollary 2.2.7 together with Lemma 2.2.4. □

Remark 2.2.10. If $\mathbf{Pk}$ (resp. $\mathbf{P\Gamma}$) is (purely) stably embedded in $\mathbf{k}$ (resp. $\mathbf{\Gamma}$), we can can also deduce that $\mathbf{PK}$ is (purely) stably embedded.

Remark 2.2.11. If we further assume that $\mathbf{P}(M)$ eliminates fields quantifiers in $\mathcal{L}_{\mathbf{RV}_1}$ — for example if it is algebraically closed or algebraically maximal Kaplansky and of equicharacteristic — then all the above results can easily be adapted to pairs of $\mathcal{L}_{\mathbf{RV}_1}$-structures (with no need for the rather exotic hybrid $\mathbf{RV}$-structures).

2.2.2. *Characteristic 0 Laurent series fields.* Let F be a field of characteristic 0, and let $K := F((t))$. In what follows, we are interested in the pair of valued fields (K^{a}, K). Let us first deal with the pair of value groups. Let $\mathcal{L}_{\mathrm{og}}$ be the language of ordered groups and DOAG be the theory of non trivial divisible ordered abelian groups. Let also $\mathcal{L}_{\mathrm{Pres}}$ be the language $\mathcal{L}_{\mathrm{og}}$ enriched with a constant 1 and unary predicates for divisibility by integers. Let PRES be the $\mathcal{L}_{\mathrm{Pres}}$-theory of $\mathbb{Z}$.

Lemma 2.2.12. *Let $T_{\mathbb{Q},\mathbb{Z}}$ be the theory of all structures (Γ, Δ) with $\Gamma \models$ DOAG, $\Delta \models$ PRES and such for any $\gamma \in \Gamma$ there is a largest $\delta =: \lfloor \gamma \rfloor \in \Delta$ with $\delta \leqslant \gamma$, considered in the language $\mathcal{L}_{\mathbb{Q},\mathbb{Z}}$ given by $\mathcal{L}_{\mathrm{og},\mathbf{P}}$ together with $\mathcal{L}_{\mathrm{Pres}}$ on the predicate $\mathbf{P}$ and the function $\lfloor \cdot \rfloor$. Then $T_{\mathbb{Q},\mathbb{Z}}$ eliminates quantifiers and is complete, and $\mathbf{P}$ is purely stably embedded.*

Proof. Let $M = (\Gamma, \Delta)$ and $M' = (\Gamma', \Delta')$ be models of $T_{\mathbb{Q},\mathbb{Z}}$, with M countable and M' $\aleph_1$-saturated. Let $(A, \mathbf{P}(A)) \leqslant M$, $(A', \mathbf{P}(A')) \leqslant M'$ and $f : (A, \mathbf{P}(A)) \cong (A', \mathbf{P}(A'))$. We claim that f extends to an embedding of M into M'. Clearly, we may assume that A is a subgroup of Γ.

Step 1. *We may assume that $\mathbf{P}(A) = \Delta$.*

Indeed, we may extend $f \restriction_{\mathbf{P}(A)}$ to an $\mathcal{L}_{\mathrm{Pres}}$-embedding $f' : \Delta \to \Delta'$, since PRES admits quantifier elimination and Δ' is an $\aleph_1$-saturated model of PRES. As $A \cap \Delta = \mathbf{P}(A)$, f together with f' yield an embedding of abelian groups $\tilde{f} : A + \Delta \hookrightarrow \Gamma'$. The map $\tilde{f}$ preserves the ordering since, up to translation by an element from $\mathbf{P}(A)$, any element of A lies in $[0, 1)$. Now, for any $z \in \Delta$ and $a \in A$ we have $\lfloor a + z \rfloor = \lfloor a \rfloor + z$. So $\tilde{f}$ respects $\lfloor \cdot \rfloor$ (and $\mathbf{P}$) and is thus an $\mathcal{L}_{\mathbb{Q},\mathbb{Z}}$-embedding.

Step 2. *We may assume that A is divisible.*

Indeed, if $n \in \mathbb{Z}_{>0}$ and $a \in A$, letting $z := \lfloor a \rfloor$ one has $\lfloor \frac{a}{n} \rfloor = \frac{z-i}{n}$, where $i \in \{0, \ldots, n-1\}$ is (the unique element) such that $z - i$ is divisible by n in Δ. It follows that that $\tilde{f}(\frac{a}{n}) := \frac{f(a)}{n}$ defines an $\mathcal{L}_{\mathbb{Q},\mathbb{Z}}$-embedding of the divisible hull of A which extends f, as $f(\Delta)$ is relatively divisible in Δ'.

Step 3. *Let $b \in \Gamma \setminus A$. Then f extends to an $\mathcal{L}_{\mathbb{Q},\mathbb{Z}}$-embedding $\tilde{f} : \langle Ab \rangle \hookrightarrow M'$.*

Indeed, replacing b by $b - \lfloor b \rfloor$, we may assume that $0 < b < 1$. By quantifier elimination in DOAG and $\aleph_1$-saturation of M', we find $b' \in \Gamma'$ such that $b \mapsto b'$ defines an extension of f to an embedding of ordered abelian groups $\tilde{f} : \langle Ab \rangle \hookrightarrow \Gamma'$. We claim that $\tilde{f}$ is an $\mathcal{L}_{\mathbb{Q},\mathbb{Z}}$-embedding. It is clear that it is an $\mathcal{L}_{og,\mathbf{P}}$-embedding. Moreover, given $z \in \mathbb{Z} \setminus \{0\}$ and $a \in A$, setting $\delta = \lfloor a \rfloor$, one has $\lfloor zb + a \rfloor = \lfloor zb + a - \delta \rfloor + \delta$, so in order to show that $\tilde{f}$ preserves $\lfloor \cdot \rfloor$, it suffices to show that $\lfloor zb + a \rfloor$ is uniquely determined by a and the cut defined by b over A in case $0 \leqslant a < 1$. This is clear, as $zb + a \in [-|z|, |z| + 1)$ and for any $\delta \in \Delta \cap [-|z|, |z| + 1)$ and $\Diamond \in \{\leqslant, <, \geqslant, >\}$ we have

$$zb + a \Diamond \delta \Leftrightarrow \begin{cases} b \Diamond \frac{\delta - a}{z}, & \text{if } z > 0 \\ \frac{\delta - a}{z} \Diamond b, & \text{if } z < 0, \end{cases} \quad .$$

proving Step 3.

Iterating steps 2 and 3, this proves the claim, yielding quantifier elimination and also completeness, as $(\mathbb{Q},\mathbb{Z})$ is a substructure of any model of $T_{\mathbb{Q},\mathbb{Z}}$. □

In fact, the proof of Lemma 2.2.12 yields the following more general result.

Remark 2.2.13. Let $\mathcal{L}^+ \supseteq \mathcal{L}_{\mathrm{Pres}}$ and $T^+ \supseteq \mathrm{PRES}$ be a complete $\mathcal{L}^+$-theory with quantifier elimination. Then the corresponding expansion $T^+_{\mathbb{Q},\mathbb{Z}}$ of $T_{\mathbb{Q},\mathbb{Z}}$ is complete, eliminates quantifiers, and $\mathbf{P}$ is purely stably embedded with induced structure given by $\mathcal{L}^+$.

As $T_{\mathbb{Q},\mathbb{Z}}$ admits the complete model $(\mathbb{R},\mathbb{Z})$, it is definably complete. Actually, this even holds resplendently, as shows the following corollary.

Corollary 2.2.14. *Assume that the expansion $T^+ \supseteq \mathrm{PRES}$ is definably complete. Then $T^+_{\mathbb{Q},\mathbb{Z}}$ is definably complete.*

Proof. Let $(\Gamma,\Delta) \vDash T^+_{\mathbb{Q},\mathbb{Z}}$ and let $D \subseteq \Gamma$ be a definable subset which is bounded and non-empty. Then $\lfloor D \rfloor$ is a definable subset of Δ which is non-empty and bounded. By assumption, it admits a supremum s in Δ, which is then the maximum of $\lfloor D \rfloor$.
Quantifier elimination in $T^+_{\mathbb{Q},\mathbb{Z}}$ entails that the structure induced on $[s,s+1]$ is *o*-minimal, so $\sup D = \sup D \cap [s,s+1]$ exists in $[s,s+1] \subseteq \Gamma$. □

Let us now consider the residue field. By a classical result of Keisler [Kei64], if F and F' are fields such that $F \equiv F'$, then $(F^{\mathrm{a}},F) \equiv (F'^{\mathrm{a}},F')$. If $F = F^{\mathrm{a}}$ or F^{a} is real closed, then the axiomatization, in $\mathcal{L}_{ring,\mathbf{P}}$, of (F^{a},F) is clear, and $\mathbf{P}$ is stably embedded with induced structure given by F. In case T_f is a complete theory of fields whose models are neither algebraically nor real closed, and $F \vDash T_f$, then the models of the $\mathcal{L}_{ring,\mathbf{P}}$-theory $T_{f,a}$ of (F^{a},F) are precisely the pairs $(M,\mathbf{P}(M))$ of fields such that $M = M^{\mathrm{a}}$ and $\mathbf{P}(M) \vDash T_f$. By [HKR18, Theorem 4.7], if one definably expands the theory, adding relation symbols $\mathbf{ld}_n$ and function symbols $\ell_{n,i}$, the theory $T_{f,a}$ eliminates quantifiers relative to $\mathbf{P}$, even resplendently on $\mathbf{P}$. This yields in particular the following.

Fact 2.2.15. *For $T_f = \mathrm{Th}(F)$ a complete theory of fields (in arbitrary characteristic), the predicate $\mathbf{P}$ is stably embedded in the $\mathcal{L}_{ring,\mathbf{P}}$-theory $T_{f,a} = \mathrm{Th}(F^{\mathrm{a}},F)$, with induced structure given by T_f.*
This even holds resplendently, for any $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and any $\mathcal{L}$-expansion of F.

Lemma 2.2.16. *Let F be some (enriched) field which eliminates $\exists^\infty$. Then the pair (F^{a},F) also eliminates $\exists^\infty$.*

Proof. We may suppose that F is neither algebraically nor real closed, as otherwise the result is clear. By the relative quantifier elimination result [HKR18, Theorem 4.7] already mentioned above, if $(M,\mathbf{P}(M)) \equiv (F^{\mathrm{a}},F)$, and $(M,\mathbf{P}(M)) \preccurlyeq (\mathcal{U},\mathbf{P}(\mathcal{U}))$ then for any $a,b \in \mathcal{U} \smallsetminus (M\mathbf{P}(\mathcal{U}))^{\mathrm{a}}$ we have $tp(a/M) = tp(b/M) =: p_{gen}(x)$, so any element of $\mathcal{U}$ is the sum of two realizations of p_{gen}. By compactness, it follows that for a definable subset $D \subseteq M$ we have $D + D = M$ if and only if $p_{gen}(x) \vdash x \in D$.

Let $\varphi(x,y)$ be a formula with x a single variable. Assume that c is a tuple from M such that $D = \varphi(M,c)$ is infinite and $p_{gen}(x) \nvdash x \in D$. Choose $a \notin M$ such that $\models \varphi(a,c)$. It follows that there is $a' \in (\mathrm{dcl}(Ma) \cap \mathbf{P}(\mathcal{U})) \setminus \mathbf{P}(M)$, so we find an M-definable function $f : D \to \mathbf{P}(M)$ with infinite image. Using Fact 2.2.15 and the assumption that the theory of F eliminates $\exists^\infty$, we conclude. ☐

Fix some characteristic zero field F and let $T^\star_{\mathrm{Laur}}$ be the theory of separated pairs $(M, \mathbf{P}(M))$ with $(\mathbf{k}(M), \mathbf{Pk}(M)) \equiv (F^{\mathrm{a}}, F)$ and $(\mathbf{\Gamma}(M), \mathbf{P\Gamma}(M)) \models T_{\mathbb{Q},\mathbb{Z}}$.

Proposition 2.2.17. *The theory $T^\star_{\mathrm{Laur}}$ is complete, the definable sets* $\mathbf{k}$*,* $\mathbf{\Gamma}$*,* $\mathbf{RV}$*,* $\mathbf{PK}$*,* $\mathbf{Pk}$*,* $\mathbf{P\Gamma}$ *and* $\mathbf{PRV}$ *are all purely stably embedded and* $\mathbf{k}$ *and* $\mathbf{\Gamma}$ *are orthogonal. All these results hold resplendently for* $\mathbf{Pk}$*-*$\mathbf{P\Gamma}$*-enrichments, and even if one adds a coherent system of angular components.*

Proof. It suffices to combine Corollary 2.2.9, Remark 2.2.10 with Fact 2.2.15 and Lemma 2.2.12. ☐

2.2.3. *Finitely ramified fields.* We will now prove analogous statements in mixed characteristic. Let K be a complete mixed characteristic $\mathbb{Z}$-valued field with perfect residue field F. We are interested in the pair of valued fields (K^{a}, K). Let $T^\star_{\mathrm{Witt}}$ be the theory of separated pairs of henselian valued fields — with relevant contants — such that $(\mathbf{k}(M), \mathbf{Pk}(M)) \equiv (F^{\mathrm{a}}, F)$ and $(\mathbf{\Gamma}(M), \mathbf{P\Gamma}(M)) \models T_{\mathbb{Q},\mathbb{Z}}$.

Proposition 2.2.18. *The theory $T^\star_{\mathrm{Witt}}$ is complete, the definable sets* $\mathbf{k}$*,* $\mathbf{\Gamma}$*,* $\mathbf{RV}^{\mathrm{hyb}}$*,* $\mathbf{PK}$*,* $\mathbf{PRV}$*,* $\mathbf{Pk}$ *and* $\mathbf{P\Gamma}$ *are all purely stably embedded, and* $\mathbf{k}$ *and* $\mathbf{\Gamma}$ *are orthogonal. All these results hold resplendently for* $\mathbf{Pk}$*-*$\mathbf{P\Gamma}$*-enrichments, and even if one adds a coherent system of angular components.*

Proof. One may argue as in the proof of Proposition 2.2.17. ☐

2.2.4. *Divisible value group.* Let F be a field. If char(F) = p > 0, assume that F does not admit a finite extension of degree divisible by p (in particular F is perfect). Let $K := F((t^{\mathbb{Q}}))$. In what follows, we are interested in the pair of valued fields (K^{a}, K).
Let $T^\star_{\mathrm{div}}$ be the theory of separated pairs of equicharacteristic algebraically maximal valued fields $(M, \mathbf{P}(M))$ such that $(\mathbf{k}(M), \mathbf{Pk}(M)) \equiv (F^{\mathrm{a}}, F)$ and $\mathbf{P\Gamma}(M) = \mathbf{\Gamma}(M) \models$ DOAG. Note that the Kaplansky conditions are satisfied in this case.

Proposition 2.2.19. *The theory $T^\star_{\mathrm{div}}$ is complete. The definable sets* $\mathbf{k}$*,* $\mathbf{\Gamma}$*,* $\mathbf{RV}$*,* $\mathbf{PK}$*,* $\mathbf{Pk}$ *and* $\mathbf{PRV}$ *are all purely stably embedded and* $\mathbf{k}$ *and* $\mathbf{\Gamma}$ *are orthogonal. All these results hold resplendently for* $\mathbf{Pk}$*-*$\mathbf{\Gamma}$*-enrichments, and even if one adds a coherent system of angular components.*

Proof. One may argue as in the proof of Proposition 2.2.17. ☐

2.3. **Valued difference fields.** Let (K, v, σ) be a valued field with an automorphism.

Definition 2.3.1. (1) For every $P \in K[x_0, \ldots, x_n]$, $a \in K$, $d \in K^{n+1}$ and $\gamma \in \mathbf{\Gamma}(K)^\times$, we say that (P, a, d, γ) is in *σ-Hensel configuration* if

$$\mathrm{v}(P(\nabla a)) > \min_i \mathrm{v}(d_i) + \sigma^i(\gamma),$$

where $\nabla a \coloneqq (\sigma^i(a))_{i \geqslant 0}$, and, for all $x, y \in K$ with $v(x-a), v(y-a) > \gamma$,

$$\mathrm{v}(f(\nabla y) - f(\nabla x) - d \cdot \nabla(y-x)) > \min_i \mathrm{v}(d_i \sigma^i(y-x)).$$

(2) We say that (K, v, σ) is *σ-henselian* if for every (P, a, d, γ) in σ-Hensel configuration, there exists there exists $c \in K(M)$ such that $P(\nabla c) = 0$ and

$$\mathrm{v}(c-a) \geqslant \max_{i, d_i \neq 0} \mathrm{v}(\sigma^{-i}(P(\nabla a)) d_i^{-1}).$$

In [Azg10; Pal12; DO15] an *a priori* weaker notion of σ-henselian fields is considered. They are, in fact, equivalent since both allow to prove Fact 2.3.4 and both hold in maximally complete fields:

Fact 2.3.2. *Assume that for every linear non constant $L \in \mathbf{k}(K)[x_0, \ldots, x_n]$ and $c \in \mathbf{k}(K)$, there exists $a \in \mathbf{k}(K)$ such that $L(\nabla(a)) = c$ and either:*

- *K is maximally complete;*
- *K is complete (rank one).*

Then (K, v, σ) is σ-henselian.

The Newton approximation proof works, *cf.* [Rid17, Proposition 4.14]. Note that at each step the approximation to the root of a σ-Hensel configuration (P, a, d, γ) improves by at least γ, *cf.* [Rid17, Lemma 4.16], and hence, in rank one, completeness suffices.

Remark 2.3.3. Conversely, if (K, v, σ) is σ-henselian, then for every linear non constant $L \in \mathbf{k}(K)[x_0, \ldots, x_n]$, $L(\nabla(\mathbf{k}(K))) = \mathbf{k}(K)$.

Let $\mathcal{L}^\sigma_{\mathbf{RV}}$ be the language $\mathcal{L}_{\mathbf{RV}}$ with two new unary functions $\sigma_{\mathbf{K}} : \mathbf{K} \to \mathbf{K}$ and $\sigma_{\mathbf{RV}} : \mathbf{RV} \to \mathbf{RV}$. The expected quantifier elimination result also holds in characteristic zero σ-henselian fields, by [DO15, Theorem 7.3] in equicharacteristic 0, and by [Rid17, p. 41, Theorem A] in mixed characteristic:

Fact 2.3.4. *Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}^\sigma_{\mathbf{RV}}$ and T an $\mathcal{L}$-theory containing the theory Hen^σ_0 of characteristic zero σ-henselian valued fields. Then T eliminates fields quantifiers.*

Note that it follows from field quantifier elimination that $\mathbf{RV}$ is stably embedded and its induced structure is that of a $\mathbf{\Gamma}$-$\mathbf{k}$-enrichment of the short exact sequence of $\mathbb{Z}[\sigma]$-modules $1 \to \mathbf{k}^\times \to \mathbf{RV}^\times \to \mathbf{\Gamma}^\times \to 0$.
In order to obtain model complete theories, one often restricts the behaviour of the automorphism on the value group, *e.g.* the class of (existentially closed) multiplicative difference valued fields introduced in [Pal12]:

Definition 2.3.5. Let $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$ be the theory of σ-henselian valued fields such that:

(1) for every $P \in \mathbb{Z}[\sigma]$, either $P(\mathbf{\Gamma}_{>0}) = \mathbf{\Gamma}_{>0}$, $P(\mathbf{\Gamma}_{<0}) = \mathbf{\Gamma}_{>0}$ or $P(\mathbf{\Gamma}) = 0$;
(2) we have $(\mathbf{k}, \sigma_{\mathbf{k}}) \vDash \mathrm{ACFA}_0$
(3) the embedding of $\mathbb{Z}[\sigma]$-modules $\mathbf{k}^\times \to \mathbf{RV}$ is pure.

Remark 2.3.6. Two multiplicative behaviours of σ are of particular interest.

(1) The ω-increasing case — *i.e.* for all $x \in \mathcal{O}$ and $n \in \mathbb{Z}_{>0}$, $\mathrm{v}(\sigma(x)) \geqslant n\mathrm{v}(x)$ — studied in [Azg10]. One then gets the asymptotic theory of $(\mathbb{F}_p(t)^{\mathrm{a}}, \mathrm{v}_t, \phi_p)$, where ϕ_p is the Frobenius automorphism.
(2) The isometric case, studied in [BMS07]. In that case, one gets the asymptotic theory of $(\mathbb{C}_p, \mathrm{v}_p, \sigma_p)$, where σ_p is an isometric lift of the Frobenius automorphism on $\mathbf{k}(\mathbb{C}_p) = \mathbb{F}_p^{\mathrm{a}}$.

Both characterisations follow (*e.g.* see [CH14]) from the Ax-Kochen-Ershov principle for σ-henselian fields and Hrushovski's deep result that ACFA_0 is the asymptotic theory of $(\mathbb{F}_p^{\mathrm{a}}, \phi_p)$, *cf.* [Hru].

Fact 2.3.7. *In* $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$, $\mathbf{k}$ *is a stably embedded pure difference field and* $\mathbf{\Gamma}$ *is a stably embedded o-minimal pure ordered* $\mathbb{Z}[\sigma]$*-module and they are orthogonal. These results also hold if one adds a* σ*-equivariant angular component.*

2.4. **Linear structures.** Let us now recall the results of [Hru12] on linear structures. In the case of valued difference fields, we will need "twisted" versions of these results. As it took us a while to get the arguments clear, we decided to spell them out in detail.

2.4.1. *Independent amalgamation.* We will first recall some material from [Hru12, Section 4]. We fix a complete stable theory T in some language $\mathcal{L}$, and we assume that T eliminates quantifiers and imaginaries. Let $\mathcal{U} \vDash T$ be a monster model. Let $\mathcal{C}_T$ be the category of small algebraically closed subsets of $\mathcal{U}$, with $\mathcal{L}$-embeddings (which are $\mathcal{L}$-elementary in T by assumption) as morphisms. Let $\mathbf{3} = \{0, 1, 2\}$, and set $\mathcal{P}(\mathbf{3})^- := \mathcal{P}(\mathbf{3}) \smallsetminus \{\mathbf{3}\}$. We consider $\mathcal{P}(\mathbf{3})^-$ and $\mathcal{P}(\mathbf{3})$ as categories, with inclusion maps as morphisms.

Definition 2.4.1. Let P equal $\mathcal{P}(\mathbf{3})^-$ or $\mathcal{P}(\mathbf{3})$.

(1) A functor $A : P \to \mathcal{C}_T$ is called *independence preserving* if for any $w, w' \in P$ with $w \cup w' \in P$ one has $A(w) \downarrow_{A(w\cap w')} A(w')$ (inside $A(w \cup w')$).[1]
(2) A functor $A : P \to \mathcal{C}_T$ is called *bounded* if for any $\varnothing \neq w \in P$ one has $A(w) = \mathrm{acl}(\bigcup_{i\in w} A(\{i\}))$.
(3) A *3-amalgamation problem* in T is a bounded independence preserving functor $A^- : \mathcal{P}(\mathbf{3})^- \to \mathcal{C}_T$. A *solution* of A^- is a bounded independence preserving functor $A : \mathcal{P}(\mathbf{3}) \to \mathcal{C}_T$ extending A^-.

[1] Here and in what follows, if $w_1 \subseteq w_2$, with inclusion map ι, we often consider $A(w_1)$ as a subset of $A(w_2)$, thus omitting the map $A(\iota)$ in our notation.

It follows from stability and elimination of imaginaries in T that every 3-amalgamation problem has a solution in T, i.e., T has *3-existence*.

Definition 2.4.2. The theory T is said to have *3-uniqueness* if whenever A and A' are solutions of the same 3-amalgamation problem A^- in T, then A and A' are isomorphic over A^-, i.e., there is an $\mathcal{L}$-isomorphism $f : A(\mathbf{3}) \cong A'(\mathbf{3})$ fixing $A^-(w)$ pointwise for every $w \in \mathcal{P}(\mathbf{3})^-$.

Remark 2.4.3. In the terminology of [Hru12], this notion corresponds to 3-uniqueness of T over every parameter set.

Let σ be a new unary function symbol, $\mathcal{L}_\sigma := \mathcal{L} \cup \{\sigma\}$. Consider the category $\widetilde{\mathcal{C}}_T$ of $\mathcal{L}_\sigma$-structures of the form (A, σ), where $A \in \mathcal{C}_T$ and $\sigma \in \mathrm{Aut}_{\mathcal{L}}(A)$, with $\mathcal{L}_\sigma$-embeddings as morphisms.

Definition 2.4.4. Let P equal $\mathcal{P}(\mathbf{3})^-$ or $\mathcal{P}(\mathbf{3})$.

(1) A functor $A : P \to \widetilde{\mathcal{C}}_T$ is called *independence preserving* (*bounded*, respectively), if it is so when composed with the forgetful functor from $\widetilde{\mathcal{C}}_T$ to $\mathcal{C}_T$.
(2) We say that $\widetilde{\mathcal{C}}_T$ has *3-existence* if every bounded independence preserving functor $A^- : \mathcal{P}(\mathbf{3})^- \to \widetilde{\mathcal{C}}_T$ extends to a bounded independence preserving functor $A : \mathcal{P}(\mathbf{3}) \to \widetilde{\mathcal{C}}_T$.

Fact 2.4.5 ([Hru12, Proposition 4.5 and Proposition 4.7])**.** *For T as above, the following are equivalent:*

(1) T has 3-uniqueness.
(2) $\widetilde{\mathcal{C}}_T$ has 3-existence.

Moreover, assuming in addition that TA exists, the above conditions imply:

(3) TA eliminates imaginaries. □

It is easy to see that if *TA* exists and it eliminates bounded hyperimaginaries (e.g., when T is superstable), then (3) is actually equivalent to (1) and (2). We will not use this in our paper.

2.4.2. *Twisted independent amalgamation.* Let $\tau : \mathcal{L} \cong \mathcal{L}'$ be a bijection between two first order languages (sending sorts to sorts, functions symbols to functions symbols consistently with their arity, similarly for constants and relations). Then τ extends naturally to a bijection between the set of $\mathcal{L}$-formulas and the set of $\mathcal{L}'$-formulas. Given an $\mathcal{L}$-formula φ, we denote by φ^τ its image under this map. If T is an $\mathcal{L}$-theory, $T^\tau := \{\varphi^\tau : \varphi \in T\}$ is an $\mathcal{L}'$-theory. Of course, up to changing the names of the symbols using τ, T^τ is the "same" theory as T.
If M is an $\mathcal{L}$-structure, we denote by M^τ the $\mathcal{L}'$-structure with base set M and interpretations $(\Sigma^\tau)^{M^\tau} = \Sigma^M$, for any symbol $\Sigma \in \mathcal{L}$. If N' is an $\mathcal{L}'$-structure, we call an $\mathcal{L}'$-isomorphism $\sigma : M^\tau \cong N'$ a *τ-twisted isomorphism*. Similarly, one defines the notion of a *τ-twisted elementary map* $\sigma : A \to A'$, where $A \subseteq M$ and $A' \subseteq N'$, i.e., one requires that for any $\mathcal{L}$-formula $\varphi(x)$

and any tuple a from A of the right length, one has $M \models \varphi(a)$ if and only if $N' \models \varphi^\tau(\sigma(a))$.

Lemma 2.4.6. *Let T be a complete stable $\mathcal{L}$-theory eliminating quantifiers and imaginaries. Assume that T has 3-uniqueness. Let $A : \mathcal{P}(\mathbf{3}) \to \mathcal{C}_{\mathbf{T}}$ and $A' : \mathcal{P}(\mathbf{3}) \to \mathcal{C}_{\mathbf{T}^\tau}$ be bounded independence preserving functors.*
Then for any coherent system $(\sigma_w)_{w \in \mathcal{P}(\mathbf{3})^-}$ of τ-twisted elementary bijections $\sigma_w : A(w) \to A'(w)$ there exists a τ-twisted elementary bijection $\sigma_{\mathbf{3}} : A(\mathbf{3}) \to A'(\mathbf{3})$ extending σ_w for every $w \in \mathcal{P}(\mathbf{3})^-$.

Proof. The result follows from 3-uniqueness of T^τ, since we may consider A as a functor to $\mathcal{C}_{T^\tau}$, replacing $\mathcal{U} \models T$ by $\mathcal{U}^\tau \models T^\tau$. □

We now consider the special case where $\mathcal{L}' = \mathcal{L}$, τ is a permutation of $\mathcal{L}$ and T is a complete $\mathcal{L}$-theory such that $T = T^\tau$. Let $\widetilde{\mathcal{C}}_T^{(\tau)}$ be the category of $\mathcal{L}_\sigma$-structures (B, σ) with $B \in \mathcal{C}_T$ and $\sigma : B \to B$ a τ-twisted elementary bijection. When T is stable, we use the same terminology as in Definition 2.4.4, for functors $A : P \to \widetilde{\mathcal{C}}_T^{(\tau)}$. Lemma 2.4.6 then yields:

Corollary 2.4.7. *Let T be a complete stable $\mathcal{L}$-theory eliminating quantifiers and imaginaries, and let $\tau : \mathcal{L} \to \mathcal{L}$ be a bijection such that $T^\tau = T$. Assume that T has 3-uniqueness. Then $\widetilde{\mathcal{C}}_T^{(\tau)}$ has 3-existence.* □

Given a complete $\mathcal{L}$-theory T and a permutation τ of $\mathcal{L}$ such that $T^\tau = T$, we let $T_\sigma^{(\tau)}$ be the $\mathcal{L}_\sigma$-theory whose models are of the form (M, σ), where $M \models T$ and where σ is a τ-twisted automorphism of M.
Assume now in addition that T is stable and eliminates quantifiers and imaginaries. Then $T_\sigma^{(\tau)}$ is a $\forall\exists$-theory, and so it has a model-companion if and only if the e.c. models of $T_\sigma^{(\tau)}$ form an elementary class. If this is the case, denote by $T^{(\tau)}A$ the model-companion of $T_\sigma^{(\tau)}$. Then the models of $T^{(\tau)}A$ are precisely the e.c. models of of $T_\sigma^{(\tau)}$. The basic results on TA, due to Chatzidakis and Pillay ([CP98]), generalize to this context in a straight forward manner. We obtain for instance:

Fact 2.4.8. *Let T and τ be as above, and assume that $T^{(\tau)}A$ exists. Then the following holds:*

(1) If $(M, \sigma) \models T^{(\tau)}A$ and $B \subseteq M$ then
$$\mathrm{acl}_{(M,\sigma)}(B) = \mathrm{acl}_\sigma(B) := \mathrm{acl}_M\Big(\bigcup_{z \in \mathbb{Z}} \sigma^z(B)\Big).$$
(2) Quantifier reduction: If $(M_i, \sigma_i) \models T^{(\tau)}A$ and $B_i \subseteq M_i$ for $i = 1, 2$, then $B_1 \equiv_{\mathcal{L}_\sigma} B_2$ if and only if there is an L_σ-isomorphism from $\mathrm{acl}_\sigma(B_1)$ to $\mathrm{acl}_\sigma(B_2)$ sending B_1 to B_2.
(3) $T^{(\tau)}A$ is simple and
$$A \mathop{\downarrow}_E^{T^{(\tau)}A} B \text{ if and only if } \mathrm{acl}_\sigma(A) \mathop{\downarrow}_{\mathrm{acl}_\sigma(E)}^{T} \mathrm{acl}_\sigma(B).$$
If T is superstable, then $T^{(\tau)}A$ is supersimple.

(4) *Assume that* $\widetilde{\mathcal{C}}^{(\tau)}$ *has 3-existence. (Equivalently, in* $T^{(\tau)}A$*, the independence theorem holds over* acl_σ*-closed sets.) Then* $T^{(\tau)}A$ *eliminates imaginaries.*

Proof. The items (1–3) may be proved as in [CP98]. Item (4) is the analog of [Hru12, Proposition 4.7], and it follows directly from a formalization of Hrushovski's argument by Montenegro and the second author (see [MR21, Proposition 1.17]). □

2.4.3. *Linear imaginaries.* Let us now recall some notions from [Hru12, Section 5].

Definition 2.4.9. Let $\mathfrak{t}$ be a theory of fields (possibly with additional structure). A $\mathfrak{t}$-*linear structure* M is an $\mathcal{L}$-structure with a sort $\mathbf{k}$ for a model of $\mathfrak{t}$, and additional sorts V_i ($i \in I$) denoting finite-dimensional $\mathbf{k}$-vector spaces, such that the family $(V_i)_{i\in I}$ is closed under tensor products and duals. Each V_i has (at least) the $\mathbf{k}$-vector space structure. One assumes that $\mathbf{k}$ is stably embedded in M with induced structure given by $\mathfrak{t}$.
We now fix such a $\mathfrak{t}$-linear structure M.

(1) M is said to *have flags* if for any i with $\dim(V_i) > 1$, for some j, k with $\dim(V_j) = \dim(V_i) - 1$, there exists a $\varnothing$-definable exact sequence $0 \longrightarrow V_k \longrightarrow V_i \longrightarrow V_j \longrightarrow 0$. We will call such a short exact sequence a *flag*.
(2) M is said to *have roots* if for any one-dimensional $V = V_i$, and any $m \geqslant 2$, there exists a (one-dimensional) $W = V_j$ and a $\varnothing$-definable $\mathbf{k}$-linear isomorphism $f : W^{\otimes m} \cong V$.

Let us now mention two results from [Hru12]. The proof of the first one is rather elementary, whereas that of the second one is quite involved.

Fact 2.4.10 ([Hru12, Lemma 5.6])**.** *The theory of an* ACF*-linear structure with flags (in any characteristic) eliminates imaginaries.*

The following fact follows from [Hru12, Proposition 5.7] in combination with [Hru12, Proposition 4.3 and Corollary 4.10].

Fact 2.4.11. *Let* T *be the theory of an* ACF_0*-linear structure with flags and roots. Then* T *has 3-uniqueness.*

Our main interest in linear structures stems from the fact that the $\mathbf{k}$-internal sets in a given model of ACVF give rise to such a structure. For every $M \models \mathrm{ACVF}$ and $A \subseteq \mathcal{G}(M)$, we define

$$\mathbf{Lin}_A := \bigsqcup_{\substack{s \in \mathbf{S}(\mathrm{dcl}_0(A)) \\ \ell \in \mathbb{Z}_{>0}}} s/\ell\mathfrak{m}s.$$

In equicharacteristic zero, this is exactly the *stable part* $\bigsqcup_{s\in\mathbf{S}(\mathrm{dcl}_0(A))} s/\mathfrak{m}s$ of [HHM08]. In mixed characteristic, however, this is a more complicated structure since it also consists of (free) $\mathbf{R}_\ell$-modules — and this more complicated structure is actually needed in Section 4. Note that, by Convention 2.1.4

and our choice of representation of the geometric sorts, $\mathbf{Lin}_A(M)$ denotes the set of cosets $c+\ell\mathfrak{m}s$ where $s\in\mathbf{S}(\mathrm{dcl}_0(A))$ has a basis in M and $c\in s(M)$.

Lemma 2.4.12. *Let $M\vDash \mathrm{Hen}_{0,0}$ and $A\subseteq\mathcal{G}(M)$. Then $\mathbf{Lin}_A(M)$, with its $\mathcal{L}_0(A)$-induced structure, is an $\mathrm{Th}(\mathbf{k}(M))$-linear structure with flags. Moreover, if $\mathbf{\Gamma}(M)$ is divisible, then $\mathbf{Lin}_A(M)$ has roots.*

Note that, by convention, $\mathbf{Lin}_A(M)$ consists of the point of the form $c+\ell\mathfrak{m}s$ where $s\in\mathbf{S}(\mathrm{dcl}_0(A))$ has a basis in M and $c\in s(M)$.

Proof. We may assume that $\mathrm{dcl}_0(A)\cap\mathcal{G}(M)\subseteq A$. The fact that the residue field $\mathbf{k}$ is stably embedded in $\mathrm{Hen}_{0,0}$, with induced structure that of a pure field, is well known.
Now let V,W be two sorts from $\mathbf{Lin}_A(M)$, *i.e.* vector spaces over $\mathbf{k}$ of the form $V=a/\mathfrak{m}a$, $W=b/\mathfrak{m}b$ for some $a\in\mathbf{S}_m(A)$ and $b\in\mathbf{S}_n(A)$ — with bases in M. Then $a\otimes_{\mathcal{O}}b$ is canonically isomorphic to an element c from $\mathbf{S}_{m\cdot n}(A)$, so we may identify $V\otimes_{\mathbf{k}}W$ with $c/\mathfrak{m}c$, which is a sort from $\mathbf{Lin}_A(M)$. Similarly, $\check{a}$ can be identified with $\{z\in\mathbf{K}^n:\forall v\in a,\ \sum z_iv_i\in\mathcal{O}\}\in\mathbf{S}_m(A)$, so $\check{V}\cong\check{a}/\mathfrak{m}\check{a}$ is a sort from $\mathbf{Lin}_A$ as well.

Flags: For $a\in\mathbf{S}_n(A)$ define $a_1:=a\cap(\mathbf{K}\times\{(0,\dots,0)\}$. Then the projection onto the first coordinate identifies a_1 with an element of $\mathbf{S}_1(A)$. Let $\pi:a\to\mathbf{K}^{n-1}$ be induced from the projection on the last $n-1$ coordinates. Then

$$0\to\ker(\pi)=a_1\to a\to\pi(a)\to 0$$

is an A-definable exact sequence of free $\mathcal{O}$-modules, and $\pi(a)\in\mathbf{S}_{n-1}(A)$ — this follows from the fact that $\pi(a)$ is a finitely generated torsion free $\mathcal{O}$-submodule of $\mathbf{K}^{n-1}$ of rank $n-1$. Reducing modulo $\mathfrak{m}$, we conclude.

Roots: Assume $\mathbf{\Gamma}(M)$ is divisible. Let $n\geqslant 1$, and let V be a one-dimensional sort from $\mathbf{Lin}_A$. Then $V=\gamma\mathcal{O}/\gamma\mathfrak{m}$ for some $\gamma\in\Gamma(\mathrm{dcl}_0(A))$. Consider $V_n:=\delta\mathcal{O}/\delta\mathfrak{m}$, for $\delta=\frac{\gamma}{n}$. The map

$$x/\gamma\mathfrak{m}\mapsto y/\delta\mathfrak{m}\otimes\ldots\otimes y/\delta\mathfrak{m}:V\to V_n^{\otimes n},$$

where $y^n=x$, is well-defined and an A-definable isomorphism of $\mathbf{k}$-vector spaces defined over A. In particular, $\mathbf{Lin}_A$ has roots. □

The result above holds of the *stable part* $\bigsqcup_{s\in\mathbf{S}(\mathrm{dcl}_0(A))}s/\mathfrak{m}s$ in all characteristic (provided $\mathbf{k}$ is stably embedded).

Corollary 2.4.13. *Let $M\vDash\mathrm{ACVF}_{0,0}$ and $A\subseteq\mathcal{G}(M)$. Then $\mathbf{Lin}_A$ satisfies 3-uniqueness.* □

2.4.4. *Twisted linear imaginaries.* The following result is a special case of Lemma 2.4.15, which we prove below.

Remark 2.4.14. Let $\mathfrak{t}$ be a stable theory of fields, and let T be the theory of a $\mathfrak{t}$-linear structure, such that T eliminates quantifiers. Suppose $\mathfrak{t}A$ exists. Then TA exists and is given by $T_\sigma\cup\mathfrak{t}A$. In particular, this holds for $\mathfrak{t}=\mathrm{ACF}$.

Lemma 2.4.15. *Let* $\mathfrak{t}$ *be a stable theory of fields, and let* T *be the theory of a* $\mathfrak{t}$*-linear structure such that* T *eliminates quantifiers. Let* τ *be a permutation of the language with* $T = T^{\tau}$ *such that* τ *fixes all the formulas on the sort* $\mathbf{k}$*. Suppose* $\mathfrak{t}A$ *exists. Then* $T_{\sigma}^{(\tau)} \cup \mathfrak{t}A$ *is the model-companion of* $T_{\sigma}^{(\tau)}$*. In particular, this holds for* $\mathfrak{t} = \mathrm{ACF}$.

Proof. Let $(M, \sigma) \models T_{\sigma}^{(\tau)}$. Note that for any $N \succcurlyeq_{\mathcal{L}} M$, we have $N = \mathrm{dcl}_{\mathcal{L}}(M\mathbf{k}(N))$. Thus, any extension of σ to a τ-twisted automorphism on N is uniquely determined by its restriction to $\mathbf{k}(N)$. It follows that (M, σ) is an e.c. model of $T_{\sigma}^{(\tau)}$ if and only if $(\mathbf{k}(M), \sigma|_{\mathbf{k}(M)})$ is an e.c. model of $\mathfrak{t}_{\sigma}$. This yields the statement of the lemma. □

Definition 2.4.16. Let $\mathbf{k}$ be a stably embedded sort in a theory T. An A_0-definable set D is said to be *internally* $\mathbf{k}$*-internal* (over A_0), if there is a tuple $d \in D$ and an A_0d-definable surjection $f : Y \to D$, where $Y \subseteq \mathbf{k}^n$ for some n.

Lemma 2.4.17. *Let* $\mathbf{k}$ *be a stably embedded sort in a theory* T*, and let* D *be* A_0*-definable and internally* $\mathbf{k}$*-internal (over* A_0*). Then* $\mathbf{k} \cup D$ *is stably embedded (over* A_0*).*

Proof. Let f be an A_0d-definable surjection as in the definition, with $d \in D$. Taking the preimage under f, one sees that any $\mathcal{U}$-definable subset X of $\mathbf{k}^l \times D^m$ is $\mathbf{k}(\mathcal{U})A_0d$-definable, by stable embeddedness of $\mathbf{k}$. In particular, X is $A_0\mathbf{k}(\mathcal{U})D(\mathcal{U})$-definable, proving stable embeddedness of $\mathbf{k} \cup D$ (over A_0). □

Proposition 2.4.18. *Let* $M \models \mathrm{VFA}_{0,0}^{\mathrm{mult}}$ *and* $A \subseteq \mathcal{G}(M)$*. Then* $\mathbf{Lin}_A$ *is stably embedded in* $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$ *and its* A*-induced structure eliminates imaginaries.*

Proof. Stable embeddedness follows from stable embeddedness of $\mathbf{k}$ — *cf.* Fact 2.3.7 — and the fact that it is internally $\mathbf{k}$-internal (by naming a basis for every sort).
Now, let T be the theory of $\mathbf{Lin}_A(M)$, with its $\mathcal{L}_0(A)$-induced structure. By Fact 2.4.10 and Lemma 2.4.12, T eliminates imaginaries. Let τ be the permutation of $\mathcal{L}_0(A)$ induced by σ. Then τ fixes all the formulas on the sort $\mathbf{k}$, and we have $T^{\tau} = T$. It follows from Corollary 2.4.13 and Fact 2.4.8(4) that $T^{(\tau)}A$ eliminates imaginaries.
Also, $(\mathbf{k}(M), \sigma_{\mathbf{k}})$ is a stably embedded pure model of ACFA and hence $\mathbf{Lin}_A(M) \models T^{(\tau)}A$. Since the A-induced structure on $\mathbf{Lin}_A$ is a definable enrichment of its ACF-linear structure with a twisted automorphism, elimination of imaginaries follows — *e.g.* [Hru12, Lemma 5.4]. □

2.4.5. *Real linear imaginaries.* We conclude these preliminaries with a study of RCF-linear structures.

Definition 2.4.19. An RCF-linear structure with flags is said to be *oriented* if for every sort V of dimension one, the two half lines are $\varnothing$-definable.

Proposition 2.4.20. *Any oriented* RCF*-linear structure with flags eliminates imaginaries.*

Proof. Let us first prove a few preliminary results. Let M be an oriented RCF-linear structure with flags. First, note that if $0 \to W \to V \to U \to 0$ is an $\varnothing$-definable flag, then any translate of W in V is ordered by $a < b$ if $a - b$ is in a fixed half line of W.

Claim 2.4.21. *M is rigid: for every $A \subseteq M$,* $\mathrm{acl}(A) = \mathrm{dcl}(A)$*.*

Proof. Let X be a finite A-definable set. Using tensors, we may assume that X is contained in some sort V. We proceed by induction on $\dim(V)$. Let $0 \to W \to V \to U \to 0$ be an $\varnothing$-definable flag for V. By induction, we may assume that X projects to a singleton $a \in U$, *i.e.* X is contained in a translate of W in V and we can conclude since this translate is totally ordered. □

Claim 2.4.22. *Let $X \subseteq a + W \subseteq V$ be definable for some $\varnothing$-definable flag $0 \to W \to V \to U \to 0$ and some $a \in V$. Then X is coded.*

Proof. Since $a + W$ is definably isomorphic to $\mathbf{k}$ which is o-minimal, X is a finite union of points and intervals and hence it is coded by its (finite) border. □

Let $K = \mathbf{k}(M)^{\mathrm{a}}$ and $K \otimes M$ be the structure whose sorts are the sorts V of M interpreted as $K \otimes_{\mathbf{k}(M)} V(M)$, with the field structure on $\mathbf{k}$, the $\mathbf{k}$-vector space structure on each V and the tensor, dual and flag structure. Then $K \otimes M$ is an ACF-structure with flags and for every $N \preccurlyeq M$, and $a \in M$, $\mathrm{dcl}(Na) \subseteq \mathrm{acl}_0(Na)$ where acl_0 denotes the algebraic closure in $K \otimes M$.
To prove elimination of imaginaries in M, it suffices to code every definable function $f : V \to S$, where S is a sort. Let $0 \to W \to V \to U \to 0$ be an $\varnothing$-definable flag for V. Let us first assume that the domain of f is a translate of W in V. Let F be the Zariski closure of the graph of f in $K \otimes M$ — any choice of basis induces a Zariski topology on V, but this topology is invariant under choice of coordinates. By the above remark, for every $a \in V(A)$, F_a is a finite set containing $f(a)$. By Fact 2.4.10, F has a code in $\mathrm{dcl}_0(\ulcorner f \urcorner) = \ulcorner f \urcorner \cap M$.
Moreover, by compactness and Claim 2.4.21, we find $\ulcorner f \urcorner$-definable maps $(f_i)_{i<n}$ such that for all $a \in M$, $F_a(M) = \{f_i(a) : i < n\}$. Let $X_i = \{x : f(x) = f_i(x)\}$. There remains to code the X_i, but this is done in Claim 2.4.22.
We now proceed by induction on $\dim(V)$. For every $a \in U$, let f_a be the restriction of f to the fiber above a in the flag. By the above, we may assume $g(a) := \ulcorner f_a \urcorner \in M$. By induction, g is coded. But f is $\ulcorner g \urcorner$-definable. □

Proposition 2.4.23. *Let $(K, <, v)$ be an ordered field with a convex valuation and $A \subseteq \boldsymbol{\mathcal{G}}(A)$. Then $\mathbf{Lin}_A(K)$ is oriented.*
In particular, if v is henselian and $\mathbf{k}(K) \vDash$ RCF*, then $\mathbf{Lin}_A(K)$ is stably embedded and its A-induced structure eliminates imaginaries.*

Proof. Dimension one sorts in $\mathbf{Lin}_A$ are of the form $\gamma\mathcal{O}/\gamma\mathfrak{m}$ for some $\gamma \in \mathbf{\Gamma}(K)$. But this quotient inherits the order on $\gamma\mathcal{O}$; so it is oriented. The rest of the proposition follows from Lemma 2.4.12 and Proposition 2.4.20. ☐

Remark 2.4.24. Any $K \equiv \mathbb{R}((t))$ admits exactly two orders, depending on the sign of a choice of uniformizer. Both orders are definable (using a constant for the rv of said uniformizer).

3. C-minimal definable generics

We will now consider generalisations of [Rid19, Theorem 8.7].

Notation. Let $\mathcal{L}_0 = \mathcal{L}_{\text{div}}$ and T_0 be the $\mathcal{L}_0$-theory ACVF. Let $\mathcal{L} \supseteq \mathcal{L}_0$ and T be a (complete) $\mathcal{L}$-theory of valued fields. Let $M \vDash T$ be sufficiently saturated and homogeneous and $M_0 = M^{\text{a}} \vDash T_0$. Note that since ACVF eliminates quantifiers, we will implicitly assume that every $\mathcal{L}_0$-formula is quantifier free. We will denote by $\mathrm{S}^0_x(M)$ the set of (quantifier free) $\mathcal{L}_0(M)$-types (in M_0) in variables x and whenever $\Psi(x;t)$ is a set of $\mathcal{L}_0$-formulas, $\mathrm{S}^\Psi_x(M)$ will denote the set of Ψ-types over M.

3.1. Main results.
In this section we prove the following two density results :

Theorem 3.1.1. *Assume*

- $(\mathbf{C_B})$ *T is definably spherically complete;*
- $(\mathbf{C_\Gamma})$ *T has definably complete value group;*
- $(\mathbf{E^\infty_k})$ *for any $M \vDash T$ and $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq \mathbf{k}$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $|Y_z| < \infty$ implies $|Y_z| \leqslant n$;*
- $(\mathbf{E^\infty_\Gamma})$ *for any $M \vDash T$ and $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq \mathbf{\Gamma}$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $|Y_z| < \infty$ implies $|Y_z| \leqslant n$.*

Then, for every strict pro-$\mathcal{L}(A)$-definable $X \subseteq \mathbf{K}^x$, with x countable and $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$, there exists an $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable $p \in \mathrm{S}^0_x(M)$ consistent with X.

In other terms, there exists $N \succcurlyeq M$ and $a \in X(N)$ such that $\mathrm{tp}_0(a/M)$ is $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable. Recall, *e.g.* [HL16, Section 2.2], that a set is strict pro-definable if it is the limit of a small directed system of definable sets with surjective transition maps. In other terms, it is a $\star$-definable set whose projection on any finite set of variables is definable.

Proof. This is a particular case of Proposition 3.5.1 ☐

Remark 3.1.2.
- Any (non zero) definably complete ordered abelian group Γ is elementarily equivalent to either $\mathbb{Z}$ or $\mathbb{Q}$. Indeed, Γ is necessarily regular as it clearly cannot have a proper non trivial definable convex subgroup. We may thus assume it is a subgroup of $(\mathbb{R}, +, <)$. If it is not elementarily equivalent to $\mathbb{Z}$ or $\mathbb{Q}$, it is a dense non divisible subgroup of $\mathbb{R}$. For any $\gamma \in \Gamma$ non divisible by $n \in \mathbb{Z}_{>0}$, the cut at γ/n yields a counter-example.

- Hypotheses $(\mathbf{C_B})$ and $(\mathbf{C_\Gamma})$ are necessary for the conclusion to hold.
- Hypothesis $(\mathbf{E^\infty_\Gamma})$ does not allow for discrete value groups. However, since the conclusion of the theorem does not hold in characteristic zero non-Archimedean local fields, some hypothesis is needed beyond $(\mathbf{C_B})$, $(\mathbf{C_\Gamma})$ and $(\mathbf{E^\infty_k})$.
- As Theorem 3.1.3 illustrates, by restricting to a mild class of enrichments of $\mathrm{ACVF}_\forall$, one can trade hypothesis $(\mathbf{E^\infty_\Gamma})$ for purely algebraic conditions and a weaker conclusion.

Let Hen_0 be the $\mathcal{L}_0$-theory of characteristic zero henselian valued fields.

Theorem 3.1.3 (*cf.* Corollary 3.5.6)**.** *Let T be a $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment of* Hen_0*, such that:*

- $(\mathbf{C_\Gamma})$ *T has definably complete value group;*
- $(\mathbf{FR})$ *for every $n \in \mathbb{Z}_{>0}$, the interval $[0, \mathrm{v}(n)]$ is finite and $\mathbf{k}$ is perfect;*
- $(\mathbf{I_k})$ *the residue field $\mathbf{k}$ is infinite;*
- $(\mathbf{E^\infty_k})$ *the induced theory on $\mathbf{k}$ eliminates $\exists^\infty$.*

Then, for every strict pro-$\mathcal{L}(A)$-definable $X \subseteq \mathbf{K}^x$, with $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$, there exist an $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant $p \in \mathrm{S}^0_x(M)$ consistent with X.

Note that in this setting $\mathbf{k}$ is stably embedded and there is no ambiguity as to what the induced structure on $\mathbf{k}$ is. So we need not be as explicit.

Remark 3.1.4.

- Contrary to Theorem 3.1.1, Theorem 3.1.3 requires finite ramification in mixed characteristic. Even if Theorem 3.1.3 does not apply to characteristic zero non-Archimedean local fields either, *cf.* the stronger [HMR18, Remark 4.7].
- Under the hypotheses of Theorem 3.1.3, locally, we do find definable types: for any *finite* set $\Psi(x;t)$ of $\mathcal{L}_0$-formulas, we can find an $\mathcal{L}_0(\mathcal{G}(A))$-definable $p \in \mathrm{S}^\Psi_x(M)$ consistent with X, cf. Proposition 3.5.4. This local statement does not hold in characteristic zero non-Archimedean local fields.
- In both theorems, hypothesis $(\mathbf{E^\infty_k})$ is an artefact of our proof. It is necessary to prove certain intermediate results. However, we do not know if it is necessary to prove either theorems. Moreover, these theorems are the only reason hypothesis $(\mathbf{E^\infty_k})$ appears in the imaginary Ax-Kochen-Ershov principle, *cf.* Theorem 6.1.1 .

Given these observations, the following questions are quite natural:

Question 3.1.5.

1. *Can the density of either invariant or definable types — i.e. the conclusion of either theorem — be proved without assuming $(\mathbf{E^\infty_k})$?*
2. *Under the hypotheses of Theorem 3.1.3, can we find an $\mathcal{L}_0(\mathcal{G}(A))$-definable type p?*
3. *Can the hypotheses of Theorem 3.1.3 be weakened to also encompass characteristic zero non-Archimedean local fields?*

3.2. **The uniform arity one case.** Let us first remind some of terminology and results from [Rid19]. We define $\mathbf{B}$ to be the ($\mathcal{L}_0$-definable) set of closed and open balls in models of T_0 — the field itself is the open ball of radius $-\infty$ and points are closed balls of radius $+\infty$. For every $r \in \mathbb{Z}_{>0}$, $\mathbf{B}^{[r]}$ is the set of finite subsets of cardinality at most r of $\mathbf{B}$ of the same type (open or closed) and the same radius and $\mathbf{B}^{[<\infty]} := \bigcup_{r>0} \mathbf{B}^{[r]}$.
For every finite set of balls B, we define $B^{\cup} := \bigcup_{b\in B} b$ and for every finite sets of balls B_1 and B_2, we write $B_1 \leqslant B_2$ if $B_1^{\cup} \subseteq B_2^{\cup}$. For every $b_1, b_2 \in \mathbf{B}$, we also define $d(b_1, b_2) := \inf\{\mathrm{v}(x_1 - x_2) : x_i \in b_i\}$. Note that this is not a metric on the space of balls since $d(b_1, b_1) = \mathrm{rad}(b_1)$, the radius of b_1. Finally, for $B_1, B_2 \in \mathbf{B}^{[r]}$, we define $D(B_1, B_2) := \{d(b_1, b_2) : b_i \in B_i\}$ and we fix an enumeration $\{d_i(B_1, B_2) : 0 \leqslant i < r^2\} = D(B_1, B_2)$, with, for every $i < j$, $d_i(B_1, B_2) < d_j(B_1, B_2)$ or $d_i(B_1, B_2) = d_j(B_1, B_2) = d_{r^2-1}(B_1, B_2)$. The actual enumeration does not actually matter as long as it is chosen uniformly.

Definition 3.2.1 (*cf.* [Rid19, Definition 6.11])**.** Let $\Psi(x; t)$ be a set of $\mathcal{L}_0$-formulas, and $F := (F_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^x \to \mathbf{B}^{[r]}$ be $\mathcal{L}_0$-definable. We say that (Ψ, F) is a *good presentation* if, for every $p \in \mathrm{S}_x^{\Psi}(M)$:

1. the type p decides the following statements:
   - $F_\lambda(x) \,\square \bigcup_{i<r} F_{\mu_i}(x)$ where $\lambda, \mu_i \in \Lambda(M)$ and $\square \in \{=, \subseteq, \subset, \leqslant, <\}$;
   - $F_\lambda^{\cup}(x) = F_{\mu_1}^{\cup}(x) \cap F_{\mu_2}^{\cup}(x)$, where $\lambda, \mu_i \in \Lambda(M)$;
   - $F_\lambda(x)$ is closed;
   - $\mathrm{rad}(F_\lambda(x)) \,\square\, d_i(F_{\mu_1}(x), F_{\mu_2}(x))$, where $\lambda, \mu_i \in \Lambda(M)$, $\square \in \{=, \leqslant\}$ and $i < r^2$;
2. there exists $\lambda, \mu \in \Lambda(M)$ with $F_\lambda(x) = \{\mathbf{K}\}$ and $F_\mu(x) = \varnothing$;
3. F is closed under intersections of realisations over p: for every $\lambda, \mu \in \Lambda(M)$, there exists $\epsilon \in \Lambda(M)$ with $p(x) \vdash F_\lambda^{\cup}(x) \cap F_\mu^{\cup}(x) = F_\epsilon^{\cup}(x)$;
4. F is closed under complements over p: for every $\lambda, \mu \in \Lambda(M)$, with $p(x) \vdash F_\mu(x) \subseteq F_\lambda(x)$, there exists $\epsilon \in \Lambda(M)$ with $p(x) \vdash F_\lambda(x) = F_\mu(x) \sqcup F_\epsilon(x)$;
5. F has large balls over p, that is, for every $\lambda, \mu \in \Lambda(M)$ and $i \in \mathbb{Z}_{\geqslant 0}$:
   - if $p(x) \vdash F_\lambda(x) \neq \mathbf{K}$, there is $\eta \in \Lambda(M)$ such that $p(x) \vdash \mathrm{rad}(F_\eta(x)) = d_i(F_\lambda(x), F_\mu(x)) \wedge$ "$F_\eta(x)$ is closed" $\wedge\, F_\lambda(x) \leqslant F_\eta(x)$;
   - if $p(x) \vdash$ "$F_\lambda(x)$ is open"$\vee \mathrm{rad}(F_\lambda(x)) < d_i(F_\lambda(x), F_\mu(x))$, there is $\eta \in \Lambda(M)$ such that, $p(x) \vdash \mathrm{rad}(F_\eta(x)) = d_i(F_\lambda(x), F_\mu(x)) \wedge$ "$F_\eta(x)$ is open" $\wedge\, F_\lambda(x) \leqslant F_\eta(x)$.

Let $\Delta(xy; s)$ with $|y| = 1$ be a set of $\mathcal{L}_0$-formulas and $G := (G_\omega)_{\omega\in\Omega} : \mathbf{K}^x \to \mathbf{B}^{[\ell]}$ be $\mathcal{L}_0$-definable. We say that (Ψ, F) is a *good presentation for* Δ if (Ψ, F) is a good presentation and every M-instance of Δ is a Boolean combination of M-instances of Ψ and conditions $y \in F_\lambda(x)^{\cup}$ with $\lambda \in \Lambda(M)$. If, moreover, for every $\omega \in \Omega(M)$, there exists $\lambda \in \Lambda(M)$ such that $G_\omega = F_\lambda$. we say that (Ψ, F) is a *good presentation for* (Δ, G).

Lemma 3.2.2 (cf. [Rid19, Proposition 5.14, 5.15, 5.18 and 6.12])**.** *Let $\Delta(xy;s)$ be a finite set of $\mathcal{L}_0$-formulas with $|y| = 1$ and $(G_\omega)_{\omega\in\Omega} : \mathbf{K}^x \to \mathbf{B}^{[\ell]}$ be $\mathcal{L}_0$-definable. Then there exists a finite set of $\mathcal{L}_0$-formulas $\Psi(x;t)$ and an $\mathcal{L}_0$-definable $F := (F_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^x \to \mathbf{B}^{[r]}$ such that (Ψ, F) is a good presentation for (Δ, G).*

Proof. Let us sketch the proof. The existence of Ψ and F that decide all instances of Δ follows by compactness from the Swiss cheese decomposition. Enlarging F, we may assume it contains G and that condition (2) holds. At any point, enlarging Ψ, we may assume that condition (1) holds. Condition (3) is obtained by closing F under intersections. Since the intersection of two balls is either empty or one of these balls, so finitely many intersections suffice, and uniformly so. Condition (4) is obtained by considering the finite Boolean algebra generated by the subsets of any given F. They are generated in (uniformly) finitely many steps. Finally, all the previous properties are preserved when replacing F by all the instances required by (4). ☐

Fix $(\Psi(x;t), F)$ a good presentation and $p \in \mathrm{S}^{\Psi}_x(M)$.

Definition 3.2.3. Let $\lambda \in \Lambda(M)$. We say that F_λ is *irreducible over* p, if for every $\mu \in \Lambda(M)$, $p(x) \vdash F_\mu(x) \subseteq F_\lambda(x)$ implies $p(x) \vdash F_\mu(x) = \varnothing \vee F_\mu(x) = F_\lambda(x)$.

We define $\Lambda_p(M) := \{\lambda \in \Lambda(M) : F_\lambda \text{ irreducible over } p\}$.

Lemma 3.2.4 (cf. [Rid19, Lemma 5.17])**.** *For every $\lambda \in \Lambda(M)$, there exists $(\mu_i)_{i<r} \in \Lambda_p(M)$ with $p(x) \vdash F_\lambda(x) = \bigcup_i F_{\mu_i}(x)$.*

Lemma 3.2.5 (cf. [Rid19, Proposition 5.9])**.** *For every $\lambda, \mu \in \Lambda_p(M)$, we have $p(x) \vdash F_\lambda(x) \cap F_\mu(x) = \varnothing \vee F_\lambda(x) \leqslant F_\mu(x) \vee F_\mu(x) \leqslant F_\lambda(x)$*

Definition 3.2.6. Let $\pi(x)$ be a partial $\mathcal{L}(M)$-type and $A \subseteq M^{\mathrm{eq}}$. We say that π is $\mathcal{L}(A)$*-quantifiable over* $\mathcal{L}$ if, for every $\mathcal{L}$-formula $\varphi(x;t)$, there exists an $\mathcal{L}(A)$-formula $\theta(t)$ such that $\{b \in M^t : \pi(x) \vdash \varphi(x;b)\} = \theta(M)$. When it exists, we write $\forall_\pi x\ \varphi(x;t) := \theta(t)$ and $\exists_\pi x\ \varphi(x;t) = \neg(\forall_p x\ \neg\varphi(x;t))$.

Remark 3.2.7. If, for some set of $\mathcal{L}_0$-formulas Δ, p is a complete $\mathcal{L}(A)$-quantifiable Δ-type over M, then it is $\mathcal{L}(A)$-definable, as a Δ-type. As we will see in Lemma 3.3.1, under certain hypotheses on T and Δ, the converse also holds.

Definition 3.2.8. For every $\mathcal{L}(M)$-definable maps $f, g : X \to Y$ and every partial $\mathcal{L}(M)$-type p concentrating on X, we say that f and g have the same *p-germ*, and we write $[f]_p = [g]_p$, if $p(x) \vdash f(x) = g(x)$.

Remark 3.2.9. A good presentation $(\Psi(x;t), F)$ remains a good presentation as Ψ grows, and irreducibility does not change. So, given a set $\Psi(xy;t)$ of $\mathcal{L}_0$-formulas and an $\mathcal{L}_0$-definable $F := (F_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^x \to \mathbf{B}^{[r]}$, we say that $(\Psi(xy;t), F)$ is a good presentation if there exists $\Phi(x;t) \subseteq \Psi$ such that $(\Phi(x;t), F)$ is a good presentation.

Notation. Let us fix now a good presentation $(\Psi(xy;t),F)$, with $|y|=1$, and $p\in \mathrm{S}^{\Psi}_{xy}(M)$. Let (Ψ,F) be the set of $\mathcal{L}_0$-formulas $\Psi\cup\{y\in F^{\cup}_{\lambda}(x)\}$ in variables xy and parameters $t\lambda$. Let $\mathrm{S}^{\Psi,F}_{xy}(M)$ denote the space of (Ψ,F)-types over M (in M_0). Let $X\subseteq \mathbf{K}^{xy}$ be $\mathcal{L}(A)$-definable and $p\in \mathrm{S}^{\Psi}_{xy}(M)$ be consistent with X.

Definition 3.2.10. Let $E\subseteq\Lambda_p(M)$. We define

$$\begin{aligned}\eta_{E,p}(xy) &:= p(xy)\\ &\cup\{y\in F^{\cup}_{\lambda}(x):\lambda\in\Lambda(M)\wedge\exists\mu\in E\ p(xy)\vdash F_{\mu}(x)\leqslant F_{\lambda}(x)\}\\ &\cup\{y\notin F^{\cup}_{\lambda}(x):\lambda\in\Lambda(M)\wedge\forall\mu\in\Lambda(M)\ F^{\cup}_{\mu}(x)\cap F^{\cup}_{\lambda}(x)\subset F^{\cup}_{\mu}(x)\}.\end{aligned}$$

Lemma 3.2.11 (cf. [Rid19, Proposition 5.12])**.** *Let $E\subseteq\Lambda_p(M)$. Provided $\eta_{E,p}$ is consistent, we have $\eta_{E,p}\in\mathrm{S}^{\Psi,F}_{xy}(M)$.*

Let us now assume that p is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$, where $A=\mathrm{acl}^{\mathrm{eq}}(A)\subseteq M^{\mathrm{eq}}$. Since p is, in particular, $\mathcal{L}(A)$-definable, $\Lambda_p(M)$ is an $\mathcal{L}(A)$-definable set. If $\mathcal{E}\subseteq\Lambda_p$ is $\mathcal{L}(A)$-definable then, when it is consistent, $\eta_{\mathcal{E}(M),p}$ is an $\mathcal{L}(A)$-definable (Ψ,F)-type that we denote $\eta_{\mathcal{E},p}$.
We write $\lambda\leqslant_p\mu$ whenever $\forall_p xy\ F_{\lambda}(x)\leqslant F_{\mu}(x)$. Note that $\leqslant_p$ is an $\mathcal{L}(A)$-definable pre-order whose associated equivalence relation is equality of p-germs. Moreover, restricted to Λ_p, by Lemma 3.2.5, there is a largest element, $\mathbf{K}$, and the $\leqslant_p$-upwards closure of any $\lambda\in\Lambda_p\smallsetminus[\varnothing]_p$ is totally ordered: $(\Lambda_p\smallsetminus[\varnothing]_p,\leqslant_p)$ is a tree.
We can now prove the crucial step in proving Theorems 3.1.1 and 3.1.3: the relative arity one case.

Definition 3.2.12. For every $\lambda,\mu\in\Lambda_p$, let $\lambda\trianglelefteq\mu$ hold whenever $\forall_p xy\ y\in(X_x\cap F^{\cup}_{\lambda}(x))\to y\in F^{\cup}_{\mu}(x)$.

Note that $\trianglelefteq$ depends on both p and X but since it will only be used in the following pages, we opted for lighter notation. The relation $\trianglelefteq$ is an $\mathcal{L}(A)$-definable preorder on Λ_p and we denote $\equiv$ the associated equivalence relation. Since $\leqslant_p$ refines $\trianglelefteq$ on $\Theta_p:=\Lambda_p\smallsetminus(\varnothing/{\equiv})$, this is also a tree with root $\mathbf{K}/\equiv$ and $\equiv$-classes are $\leqslant_p$-convex.

Lemma 3.2.13 (cf. [Rid19, Claim 8.4])**.** *For every $\lambda\in\Theta_p$, if $\eta_{\lambda/\equiv,p}$ is not consistent with X, then $\lambda/{\equiv}$ has finitely many $\trianglelefteq$-daughters $(\mu_i/{\equiv})_{0\leqslant i<n}\in\mathrm{acl}^{\mathrm{eq}}(A\ulcorner\lambda/{\equiv}\urcorner)$ in $\Theta_p/{\equiv}$. Moreover, $n\geqslant 2$ and $p(xy)\vdash y\in X_x\cap F^{\cup}_{\lambda}(x)\to\bigvee_{i<n}y\in F^{\cup}_{\mu_i}(x)$.*

Proof. By compactness, there exist $(\nu_i)_{0\leqslant i<m}\in\Theta_p(M)$ such that $\nu_i\vartriangleleft\lambda$ and $p(xy)\vdash y\in X_x\cap F^{\cup}_{\lambda}(x)\to\bigvee_{i<m}y\in F^{\cup}_{\nu_i}(x)$. The existence of the μ_i now follows from the facts that any $\mu\trianglelefteq\lambda$ is $\trianglelefteq$-comparable to one of the ν_i and that since the F_{ν_i} are irreducible, the subtree with root λ and leaves $(\nu_i)_{i<m}$ embeds in the lattice of subsets of $\{0,\dots,m-1\}$, which is finite. Finally, if $n=1$, we would have $\lambda\trianglelefteq\mu_0$, contradicting that μ_0 is a daughter of λ. □

Let $\overline{\Theta}_p := \{\lambda \in \Theta_p : \forall_p xy \text{ “}F_\mu(x) \text{ is closed”}\}$ and, for every $\lambda \in \Lambda$, let $Y_\lambda := \{F_\mu \in \overline{\Theta}_p : \forall_p xy \operatorname{rad}(F_\mu(x)) = \operatorname{rad}(F_\lambda(x))\}$.

Lemma 3.2.14. *One of the following holds:*

- *There exists a $\lambda \in \Theta_p$ such that $\lambda/{\equiv} \in A$ and $\eta_{\lambda/\equiv,p}$ is consistent with X.*
- *For every $n \in \mathbb{Z}_{\geqslant 0}$, there exists $\lambda/{\equiv} \in A$ such that $[Y_\lambda]_p$ is finite of cardinality larger than n, where $[Y_\lambda]_p := \{[F_\mu]_p : \mu \in Y_\lambda\}$.*

Proof. Assume that X is consistent with no $\eta_{\lambda/\equiv,p}$, where $\lambda/{\equiv} \in A$. Then, by Lemma 3.2.13, $\Theta_p/\equiv$ admits an initial finitely strictly branching discrete tree — that is every, element has at least two daughters — with every branch infinite. Note that, for every $\lambda \in \Theta_p$ with $\lambda <_p \mathbf{K}$, by the large ball property, there is $\mu \in \Lambda$ with $\lambda \leqslant_p \mu$ and $\forall_p xy \text{ “}F_\mu(x) \text{ is closed”} \wedge \operatorname{rad}(F_\mu(x)) = \operatorname{rad}(F_\lambda(x))$. We may assume that F_μ is irreducible over p. Then $\lambda = \mu$ or λ is the unique $\leqslant_p$-daugther of μ. Note also that, by the large ball property, $\overline{\Theta}_p \cap \mathbf{K}/{\equiv} \neq \varnothing$. It follows that $\overline{\Theta}_p/\equiv$ also admits an initial finitely branching discrete tree, denoted Ξ_p, with every branch infinite.

Note that, for any two $\mu, \nu \in Y_\lambda$, since $\forall_p xy \operatorname{rad}(F_\mu(x)) = \operatorname{rad}(F_\nu(x))$, we have that $[F_\mu]_p = [F_\nu]_p$ implies $\mu \equiv \nu$, which implies that $\forall_p xy\ F^\cup_\mu(x) \cap F^\cup_\nu(x) \neq \varnothing$, which, by irreducibility, implies that $[F_\mu]_p = [F_\nu]_p$, so these three statements are equivalent. In particular, the identity induces a bijection between $[Y_\lambda]_p$ and $Y_\lambda/{\equiv}$.

We now build, by induction, $\lambda_i \in \Lambda$ such that $Y_{\lambda_i}/{\equiv} \subseteq \Xi_p$ and $|Y_{\lambda_i}/{\equiv}| = |[Y_{\lambda_i}]_p|$ is finite and strictly increasing. Start with any $\lambda_0 \in \overline{\Theta}_p \cap \mathbf{K}/{\equiv}$. Then $Y_{\lambda_0} = [F_\lambda]_p$ and $Y_{\lambda_0}/{\equiv} = \lambda_0/{\equiv} = \mathbf{K}/{\equiv}$. If λ_i is built, let $(\mu_j)_{j<m}$ enumerate all the $\trianglelefteq$-daughters — in Ξ_p — of the elements in $Y_{\lambda_i}/{\equiv}$. Let j_0 be such that, for all j, $\forall_p xy \operatorname{rad}(F_{\mu_{j_0}}(x)) \leqslant \operatorname{rad}(F_{\mu_j}(x))$. For every $\nu \in Y_{\mu_{j_0}}$, by the large ball property, we find $\lambda \in Y_{\lambda_i}$ such that $\nu \leqslant_p \lambda$. Since $\forall_p xy \operatorname{rad}(F_\nu(x)) = \operatorname{rad}(F_{\mu_{j_0}}(x)) \leqslant \operatorname{rad}(F_{\mu_j}(x))$, we cannot have $\nu \triangleleft \mu_j$ and hence $\nu/{\equiv}$ is either in $Y_{\lambda_i}/{\equiv}$ or it is one of the $\mu_j/{\equiv}$. So $Y_{\mu_{j_0}}/{\equiv} \subseteq \Xi_p$ is finite. Furthermore, for every μ_j, by the large ball property, there exists $\nu \in Y_{\mu_{j_0}}$ such that $\mu_j \leqslant_p \nu$. It follows that, for every element of $Y_{\lambda_i}/{\equiv}$, either it or all of its (more than one) daughters appear in $Y_{\mu_{j_0}}$. In particular, all the sisters of $\mu_{j_0}/{\equiv}$ appear, and hence $|Y_{\mu_{j_0}}/{\equiv}| > |Y_{\lambda_i}/{\equiv}|$. Thus, we can choose $\lambda_{i+1} = \mu_{j_0}$. □

We can now eliminate the second option in Lemma 3.2.14 by imposing a uniform bound on the size of finite instances of $(Y_\lambda)_{\lambda\in\Lambda}$:

Corollary 3.2.15. *Assume:*

($\mathbf{E}^\infty_{p,\overline{F}}$) *for every $\mathcal{L}(M)$-definable family $(Y_z)_z$ of subsets of $[F_{\Lambda_p}]_p := \{[F_\lambda]_p : \lambda \in \Lambda_p\}$ such that for all z and $[F_\lambda]_p, [F_\mu]_p \in Y_z$, $p(xy) \vdash$ “$F_\mu(x)$ is closed”$\wedge \operatorname{rad}(F_\lambda(x)) = \operatorname{rad}(F_\mu(x))$, there exists $n \in \mathbb{Z}_{>0}$ such that, for all z, $|Y_z| < \infty$ implies $|Y_z| \leqslant n$.*

Then there exists an $\mathcal{L}(A)$-definable $\mathcal{E} \subseteq \Lambda_p$ such that $\eta_{\mathcal{E},p}$ is consistent with X. □

However the family $[Y_\Lambda]_p$ is not any definable family in $[F_{\Lambda_p}]_p$. It has certain geometric properties that reflect that of X. In particular, with further hypotheses on X, we can dispense with $(\mathbf{E}^{\infty}_{p,\overline{F}})$ altogether.

We now wish to apply the construction above in the pair (M_0, M) which is naturally an $\mathcal{L}_{0,P}$ structure enriched with the $\mathcal{L}$-structure on M. To be precise and avoid an unnecessary conflict of notation:

Notation. Let $\mathcal{L}_1$ be some enrichment of $\mathcal{L}_0$ and T_1 some $\mathcal{L}_1$-theory of valued fields. In the following lemma, we apply the above with T the theory of the pair $M := (M_1^{\mathrm{a}}, M_1)$, where $M_1 \vDash T_1$ is sufficiently saturated and homogeneous, in the language $\mathcal{L} := \mathcal{L}_{\mathrm{P}}$ consisting of $\mathcal{L}_{0,P}$ structure enriched with the $\mathcal{L}_1$-structure on $\mathbf{P}$ — so $M_0 = M_1^{\mathrm{a}}$.

Let us now introduce some useful terminology from [CHR]:

Definition 3.2.16. Fix $n \in \mathbb{Z}_{>0} \cap \mathbf{K}^\times(M)$. For any ball b, we define $b[n] := \{a + n^{-1}(a - c) : a, c \in b\}$. It is a ball of radius $\mathrm{rad}(b) - \mathrm{v}(n)$ around b, open if b is, closed otherwise. For a set of balls B, we set $B[n] := \{b[n] : b \in B\}$.

(1) An $\mathcal{L}(M_1)$-definable set $X \subseteq \mathbf{K}(M_1)$ is *n-prepared* by some finite set $C \subseteq \mathbf{K}(M_0)$ if for every ball $b \in \mathbf{B}(M_0)$ with $b[n] \cap C = \emptyset$, either $b \cap X(M_1) = \emptyset$ or $b \cap X(M_1) = b(M_1)$.
(2) We say that some $\mathcal{L}_0(M_1)$-definable $G : \mathbf{K}^n \to \mathbf{K}^{[r]}$ *n-prepares* $X \subseteq \mathbf{K}^{n+1}(M_1)$ if, for every $x \in \mathbf{K}(M_1)^n$, $G(x)$ n-prepares X_x.
(3) We say that $X \subseteq \mathbf{K}^{n+1}(M_1)$ is *n-prepared* by F if there exists $\lambda \in \Lambda(M_1)$ such that F_λ n-prepares X.

Remark 3.2.17. By field quantifier elimination (*cf.* Fact 2.1.3), if M_1 is a pure henselian field of characteristic zero, any $\mathcal{L}(M_1)$-definable $X \subseteq \mathbf{K}$ is p^ℓ-prepared, for some ℓ, by the finite set of roots of polynomials that appear in the (field quantifier free) definition of X, where p is either 1 or the residue characteristic when it is positive.

Let also $A_1 = \mathrm{acl}_1^{\mathrm{eq}} \subseteq M_1^{\mathrm{eq}}$, $X \subseteq \mathbf{K}^{xy}(M_1)$ be $\mathcal{L}_1(A_1)$-definable, $A = A_P = \mathrm{acl}^{\mathrm{eq}}(A)$ and $p \in \mathrm{S}^{\Psi}_{xy}(M_0)$ be $\mathcal{L}_{\mathrm{P}}(A_P)$-definable and consistent with X.

Lemma 3.2.18. *Let $\Phi(x;t) \subseteq \Psi$ be such that (Φ, F) is a good presentation for Ψ. Assume that there is some $n \in \mathbb{Z}_{>0} \cap \mathbf{K}^\times(M)$ such that:*

$(\mathbf{P}^{F,n}_X)$ *the set X is n-prepared by F;*
$(\mathbf{FR}_n)$ *the interval $[0, \mathrm{v}(n)] \subseteq \mathbf{\Gamma}(M_1)$ is finite and $\mathbf{k}(M_1)$ is perfect;*
$(\mathbf{I_k})$ *the residue field $\mathbf{k}(M_1)$ is infinite.*

Then, there exists an $\mathcal{L}_{\mathrm{P}}(A_P)$-definable $\mathcal{E} \subseteq \Lambda_p(M_0)$ such that $\eta_{\mathcal{E},p}|_{M_0}$ is consistent with X.

Proof. Let ρ be such that $F_\rho(x)$ n-prepares X_x for all x. By Lemma 3.2.14, applied in $M = (M_1^{\mathrm{a}}, M_1)$, either the conclusion of the lemma holds, or we can find $\lambda/\equiv \in A$ with $|[Y_\lambda]_p| > r$. Then some element of Y_λ, say F_λ, does

not contain any point of $F_\rho(x)$. Replacing λ with any $\mu \trianglelefteq \lambda$ in Ξ_p such that $|[\mu,\lambda]| \geqslant |[0,\mathrm{v}(n)]|$, we may further assume that $F_\rho(x) \cap F_\lambda(x)[n]^\cup = \varnothing$ — where, for any $B \in \mathbf{B}^{[<\infty]}$, $B[n] := \{b[n] : b \in B\}$ — and that $\lambda \triangleleft \mathbf{K}$.
By Lemma 3.2.13, we have $p(xy) \vdash y \in X_x \cap F_\lambda^\cup(x) \to \bigvee_{i<n} y \in F_{\mu_i}^\cup(x)$, where the $(\mu_i/\equiv)_{i<n}$ are the daughters of $\lambda/\equiv$. By compactness, there exists some $\psi(xy) \in p$ such that $q := p|_\Phi \vdash \forall y\ \psi(xy) \wedge y \in X_x \cap F_\lambda^\cup(x) \to \bigvee_{i<n} y \in F_{\mu_i}^\cup(x)$. Since (Φ, F) is a good presentation for Ψ, there are κ and $(\mu_i)_{n\leqslant i<m} \in \Lambda_p$ with $\vDash \psi(xy) \leftrightarrow y \in F_\kappa^\cup(x) \smallsetminus (\bigcup_{n\leqslant i<m} F_{\mu_i}^\cup(x))$. In particular, $p(xy) \vdash y \in F_\kappa^\cup(x)$. It follows that $\kappa \equiv \mathbf{K}$ and hence $\lambda \trianglelefteq \kappa$. So, we have

$$q \vdash F_\lambda^\cup(x) \cap X_x \subseteq \bigcup_i F_{\mu_i}^\cup(x).$$

Since $\lambda \not\equiv \varnothing$, there exists $ac \vDash p$ in some $N_1 \succcurlyeq M_1$ such that $c \in X_a \cap F_\lambda^\cup(a)$. Let $b_0 \in \mathbf{B}(N_1^{\mathrm{a}})$ be the ball of $F_\lambda(a)$ containing c. Since $F_\rho(a) \cap b_0[n] = \varnothing$, we have $b_0 \cap X_a(N_1) = b_0(N_1)$. It follows that $b_0(N_1) \subseteq \bigcup_i F_{\mu_i}^\cup(a)$. By construction, $b_0(N_1)$ is not covered by any single ball in $\bigcup_i F_{\mu_i}(a)$. So $\{\mathrm{v}(x-y) : x, y \in b_0(N_1)\}$ has a minimal element (realised by some x, y in distinct balls of $\bigcup_i F_{\mu_i}(a)$). Let $b \in \mathbf{B}(N_1)$ be the smallest closed ball containing $b_0(N_1)$. Then $b(N_1) = b_0(N_1)$ is covered by finitely many of its maximal open subballs, contradicting hypothesis $(\mathbf{I_k})$. ☐

3.3. **Quantifiable types.** To use the above constructions in an induction, we need a number of results on quantifiable types. The first one is that $\eta_{\mathcal{E},p}$ is itself quantifiable when p is. For any finite set B of balls, let $\mathbf{R}_B$ be the set of maximal open subballs of the balls $b \in B$ and $\mathrm{res}_B : B^\cup \to \mathbf{R}_B$ be the projection.

Lemma 3.3.1 (cf. [Rid19, Corollary 6.9])**.** *Let $(\Psi(xy;t), (F_\lambda(x))_{\lambda\in\Lambda})$ be a good presentation. Let $p \in \mathrm{S}_{xy}^\Psi(M)$ be $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$, where $A \subseteq M^{\mathrm{eq}}$. Assume:*

- $(\mathbf{E}_{p,\mathring{F}}^\infty)$ *for any $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq [F_\Lambda]_p$ and $\lambda \in \Lambda(M)$ such that, for all z and $[F_\mu]_p \in Y_z$, $p(xy) \vdash F_\mu(x) \subseteq \mathbf{R}_{F_\lambda(x)}$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that, for all z, $|Y_z| < \infty$ implies $|Y_z| \leqslant n$.*

Then, any $\mathcal{L}(A)$-definable $q \in \mathrm{S}_{xy}^{\Psi,F}(M)$ containing p is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$.

Proof. Let $\mathcal{E} := \{\lambda \in \Lambda_p : q(xy) \vdash y \in F_\lambda^\cup(x)\}$. Then $\mathcal{E}$ is $\mathcal{L}(A)$-definable and $q = \eta_{\mathcal{E},p}$. If $\mathcal{E}$ does not have an $\leqslant_p$-minimal element which is closed, for any $\mathcal{L}$-formula $\varphi(xy;s)$ and $e \in M^s$, $q(xy) \vdash \varphi(xy;e)$ if and only if there exists $\lambda \in \mathcal{E}(M)$ and $\mu \in \Lambda_p(M)$ with $\lambda <_p \mathcal{E}$ and $q(xy) \vdash y \in F_\lambda^\cup(x) \smallsetminus F_\mu^\cup(x) \to \varphi(xy;e)$ — cf. [Rid19, Proposition 6.4]. So let us assume that $\mathcal{E}$ has an $\leqslant_p$-minimal element λ_0 which consists of closed balls. If $p(xy) \vdash \mathrm{rad}(F_{\lambda_0}(x)) = +\infty$, then $q(xy) \vdash \varphi(xy;e)$ if and only if $p(xy) \vdash F_{\lambda_0}^\cup(x) \to \varphi(xy;e)$ — cf. [Rid19, Proposition 6.6]. If $p(xy) \vdash \mathrm{rad}(F_{\lambda_0}(x)) \neq +\infty$, let

$$Y_s := \{\mu \in \Lambda_p : \forall_p xy\ F_\mu(x) \subseteq \mathbf{R}_{F_\lambda(x)} \wedge \exists_p xy\ \varphi(xy;s) \wedge y \in F_\mu^\cup(x)\}.$$

Let n be a uniform bound on the cardinality of finite $[Y_s]_p$, as in $(\mathbf{E}_{p,\mathring{F}}^\infty)$.

Claim 3.3.2. *For every $e \in M^s$, q is consistent with $\varphi(xy;e)$ if and only if, for every $(\mu_i)_{i<n} \in \Lambda_p$, with $\forall_p xy F_{\mu_i}(x) \in \mathbf{R}_{F_{\lambda_0(x)}}$, $\exists_p xy\ \varphi(xy;e) \wedge y \in F^{\cup}_{\lambda_0}(x) \smallsetminus \bigcup_{i<n} F^{\cup}_{\mu_i}(x)$.*

Proof. Assume q is not consistent with $\varphi(xy;e)$, then, by compactness, there exists $(\mu_i)_{i<m} \in \Lambda_p$ such that $\mu_i <_p \mathcal{E}$ and $\forall_p xy\ \varphi(xy;e) \wedge y \in F^{\cup}_{\lambda_0}(x) \to \bigvee_{i<m} y \in F^{\cup}_{\mu_i}(x)$. By the large ball property, we may assume $\forall_p xy\ F_{\mu_i}(x) \in \mathbf{R}_{F_{\lambda_0(x)}}$. Choosing a minimal m, we may also assume that, $\exists_p xy\ \varphi(xy;e) \wedge y \in F^{\cup}_{\mu_i}(x)$. In particular, $\mu_i \in Y_e$.
By definition of Y_s, for every $\mu \in Y_e(M)$, we find $ac \models p$ such that $\varphi(ac;e)$ and $c \in F^{\cup}_{\mu}(a) \subseteq F^{\cup}_{\lambda_0}(a)$. So there is an i such that $c \in F^{\cup}_{\mu_i}(a)$. By irreducibility, $F_{\mu}(a) = F_{\mu_i}(a)$. It follows that $[Y_e]_p$ is finite and thus $m \leqslant |[Y_e]_p| \leqslant n$. □

Since $q \vdash \varphi(xy;e)$ if and only if q is not consistent with $\neg\varphi(xy;e)$, Claim 3.3.2 allows us to conclude. □

We also need a better understanding of the interpretable set $[F_\Lambda]_p$. Note that it is, a priori, $\mathcal{L}(M)$-interpretable which is exactly the kind of sets elimination of imaginaries aims at describing. However, if p happens to be the restriction to M of a global $\mathcal{L}_0(M)$-definable type, then $[F_\Lambda]_p$ naturally embeds in an $\mathcal{L}_0(M)$-interpretable set. The goal of the following lemmas is to give (necessary) hypotheses under which any definable p verifies that condition. Valued vector spaces will play an important role:

Definition 3.3.3. Let (K,v) be a valued field and V be a K-vector space. A *valuation on V* is a map $v : V \to X$ where X is an ordered set with a maximal element ∞ and an action $+$ of Γ, respecting the order, such that:

- $v(0) = \infty$;
- for all $x,y \in V$, $v(x+y) \geqslant \min\{v(x),v(y)\}$;
- for all $a \in K$ and $x \in V$, $v(a\cdot x) = \mathrm{v}(a) + v(x)$.

We say that a family $(x_i)_i \in V$ is *separating* if for every almost everywhere zero $(a_i)_i \in K$, $v(\sum_i a_i x_i) = \min_i(\mathrm{v}(a_i) + v(x_i))$.

The following lemma owes much to Johnson's computation of the canonical basis of definable types in ACVF, *cf.* [Joh20, Section 5.2]. Let $\varphi_d(x;yz) := \mathrm{v}(\sum_{|I|<d} y_I x^I) \geqslant \mathrm{v}(\sum_{|I|<d} z_I x^I)$.

Proposition 3.3.4. *Assume:*

($\mathbf{C}_V$) *every $\mathcal{L}(M)$-definable valuation v on $\mathbf{K}^n$ has a separating basis;*
($\mathbf{C}_\Gamma$) *T has definably complete value group.*

Then, for every $A = \mathrm{dcl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, $\mathcal{L}(A)$-definable $p \in \mathrm{S}_{\varphi_d}(M)$ and algebraic extension $K(M) \leqslant L$, any $q \in \mathrm{S}_{\varphi_d}(L)$, extending p and finitely satisfiable in M, is $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable.

Proof. For every field F with $K = \mathbf{K}(M) \leqslant F \leqslant L$, we define a valuation v on the F-vector space $V_d(F) := F[x]_{\leqslant d}$ of polynomials in variables x over F of degree at most d by $v(P(x)) \leqslant v(Q(x))$ if $\mathrm{v}(P(x)) \leqslant \mathrm{v}(Q(x)) \in q$. The valuation v on $V_d(K)$ is $\mathcal{L}(A)$-definable. By hypothesis ($\mathbf{C}_V$), it has a separating basis $(P_i)_i \in V_d(K)$.

Claim 3.3.5. *$(P_i)_i$ is a separating basis of $V_d(L)$ over L.*

Proof. We may assume that $K \leqslant L$ is finite. By [Joh21, Remark 2.7], the valuation on L (interpreted in K) is then $\mathcal{L}(M)$-definable and hence, by hypothesis $(\mathbf{C}_V)$, also has a separating basis $(c_j)_j \in L$ over K. Let $b_i = \sum_j b_{i,j} c_j \in L$, where $b_{i,j} \in K$. If $v(\sum_j (\sum_i b_{i,j} P_i) c_j) > \min_j v(\sum_i b_{i,j} P_i) + \mathrm{v}(c_j)$, since q is finitely satisfiable in M, we have $v(\sum_j (\sum_i b_{i,j} P_i(a)) c_j) > \min_j v(\sum_i b_{i,j} P_i(a)) + \mathrm{v}(c_j)$ for some $a \in M$, contradicting that $(c_j)_j$ is separating over K. So

$$\begin{aligned} v(\sum_i b_i P_i) &= v(\sum_j (\sum_i b_{i,j} P_i) c_j) \\ &= \min_j v(\sum_i b_{i,j} P_i) + \mathrm{v}(c_j) \\ &= \min_{i,j} \mathrm{v}(b_{i,j}) + \mathrm{v}(c_j) + v(P_i) \\ &\leqslant \min_i \mathrm{v}(b_i) + v(P_i) \\ &\leqslant v(\sum_i b_i P_i) \end{aligned}$$ □

We now define the L-Archimedean equivalence on $v(V_d(L))$: $v(P) \sim_L^\infty v(Q)$ holds if there exists $c \in L^\times$ such that $-\mathrm{v}(c) + v(P) \leqslant v(Q) \leqslant \mathrm{v}(c) + v(P)$. One can check that $|v(V_d(L))/\sim_L^\infty| \leqslant |v(V_d(L))/\mathrm{v}(L)| \leqslant \dim_L(V_d(L)) + 1 < \infty$. We also define the L-infinitesimal equivalence on $v(V_d(L))$: $v(P) \sim_L^0 v(Q)$ holds if for every $\gamma \in \mathrm{v}(L)_{>0}$ we have $-\gamma + v(P) < v(Q) < \gamma + v(P)$. Note that two elements of the same $\mathrm{v}(L)$-orbit cannot be $\sim_L^0$-equivalent unless they are equal. It follows that $\sim_L^0$-classes are finite.
Let C be any K-Archimedean class and $\overline{C}$ denote its upwards closure. Then $V_C := v^{-1}(\overline{C}) \leqslant V_d(K)$ is a sub-K-vector space.

Claim 3.3.6 ([Joh20, Observation 4.3])**.** *V_C has a basis of elements in A.*

Proof. Some coordinatewise projection $V_C \subseteq K^l \to K^m$ restricts to an isomorphism on V_C. The preimage of the standard basis of K^m then has the required properties. □

Let C_0 be the successor of C in $V_d(K)/\sim_K^\infty$. Since $V_{C_0} \subset V_C$, any basis of V_C has an element outside V_{C_0}. In particular $(V_C \smallsetminus V_{C_0})(A) \neq \varnothing$ and we find $\gamma_C \in C(A) \neq \varnothing$. Then the whole (finite) K-infinitesimal class of γ_C is in A. Let i be such that $v(P_i) \in C$. By $(\mathbf{C}_\Gamma)$, the set $\{\gamma \in \mathrm{v}(K) : \gamma + v(P_i) \leqslant \gamma_C\}$ has a supremum γ_i. Multiplying P_i by some constant $c \in K$ with $\mathrm{v}(c) = -\gamma_i$, we may assume that $\gamma_i = 0$ in which case $v(P_i)$ is K-infinitesimally close to $\gamma_C \in A$ and hence is itself in A. Since every $\mathrm{v}(K)$-orbit is contained in some K-Archimedean class, we now have that for any i, $v(P_i) \in A$ and for any j, if $v(P_i) \sim_K^\infty v(P_j)$, then $v(P_i) \sim_K^0 v(P_j)$.
Note that, since $\mathrm{v}(L)$ is in the convex hull of $\mathrm{v}(K)$, $\sim_L^\infty$ extends $\sim_K^\infty$. Also, if $v(K)$ is dense, then $\sim_L^0$ extends $\sim_K^0$. However, if $\mathrm{v}(K)$ is discrete then $\sim_K^0$

reduces to equality. In particular, we also have that, if $v(P_i) \sim_L^\infty v(P_j)$, then $v(P_i) \sim_L^0 v(P_j)$.
For every i, d, let $M_{i,d}(L) = \{P \in V_d(L) : v(P) \geqslant v(P_i)\}$. Note that, by Claim 3.3.5, $\sum \lambda_j P_j \in M_{i,d}(L)$ if and only if:

- $\lambda_j = 0$ for every j with $v(P_j) < v(P_i)$ and $v(P_j) \not\sim_K^\infty v(P_i)$;
- $\lambda_j \in \mathfrak{m}$ for j with $v(P_j) < v(P_i)$ and $v(P_j) \sim_K^0 v(P_i)$;
- $\lambda_j \in \mathcal{O}$ for j with $v(P_i) \leqslant v(P_j)$ and $v(P_j) \sim_K^0 v(P_i)$.

So $M_{i,d}$ is (quantifier free) $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable. Since q is $\mathcal{L}_0(\bigcup_{i,d} \ulcorner M_{i,d} \urcorner)$-definable, it is indeed $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable. □

Remark 3.3.7. Any definably complete ordered abelian group is elementarily equivalent to either $\mathbb{Q}$, $\mathbb{Z}$ or 0.

If $p \in \mathrm{S}^0(M)$ the existence (and uniqueness) of such a q follows, on general grounds, from the finite satisfiability of p:

Lemma 3.3.8. *Let $p \in \mathrm{S}^0(M)$ be finitely satisfiable in M. Then any two realisations of p have the same $\mathcal{L}_0(\mathrm{acl}_0(M))$-type. In particular, the unique extension of p to $\mathrm{acl}_0(M)$ is finitely satisfiable in M.*

Proof. Fix any $c \in \mathrm{acl}_0(M)$, $\varphi(xy)$ an $\mathcal{L}_0$-formula and $\psi(y)$ an $\mathcal{L}_0(M)$-formula that algebrises c. Then $\forall y\ [\psi(y) \to (\varphi(x_1 y) \leftrightarrow \varphi(x_2 y))]$ defines an $\mathcal{L}_0(M)$-definable equivalence relation with finitely many classes. Then the E-class of any $a \in N \succcurlyeq M$ realising p has an element $e \in M$. It follows that $p(x) \vdash xEe$. In particular $p(x) \vdash \varphi(xc)$ whenever $\varphi(ec)$ holds.
Let q be the unique extension of p to $\mathrm{acl}_0(M)$. Then $p \vdash q$ and hence q is finitely satisfiable in M. □

Following [Bau82], we can prove that $(\mathbf{C}_V)$ follows from definable spherical completeness. Up to definability, this is a standard result. But we include its proof, on the one hand, for the sake of completeness, and, on the other, to show that the proof can indeed been done definably.

Lemma 3.3.9. *Assume:*

$(\mathbf{C_B})$ *T is definably spherically complete : any $\mathcal{L}(M)$-definable chain of balls has a non-empty intersection.*

Then any (finite dimensional) $\mathcal{L}(M)$-interpretable valued $\mathbf{K}$-vector space (V, v) has a separating basis.

Proof. Let us proceed by induction on $n + 1 := \dim(V)$. In particular, we may assume that we have found a separating family $(y_i)_{0 \leqslant i < n} \in V$.

Claim 3.3.10. *For every $x \in V$, $\{v(x - \lambda y) : \lambda \in \mathbf{K}^n\}$ has a maximal element.*

Proof. For every $\lambda, \mu \in \mathbf{K}^n$, we have $v(x - \mu y) \geqslant v(x - \lambda y) =: \gamma$ if and only if $\min_i\{\mathrm{v}(\mu_i - \lambda_i) + v(y_i)\} = v((\mu - \lambda)y) \geqslant \gamma$. For every $i < n$ and $\lambda \in \mathbf{K}^n$, let $B_{i,\lambda} := \{\mu \in \mathbf{K} : v(x - \lambda_{\neq i} y_{\neq i} - \mu y_i) \geqslant v(x - \lambda y)\} = \{\mu \in \mathbf{K} : \mathrm{v}(\mu - \lambda_i) + v(y_i) \geqslant v(x - \lambda y)\}$. They form a chain for inclusion.
If there is a minimal B_{i,λ_0}, pick any $\lambda_i \in B_{i,\lambda_0}$. If there is no minimal $B_{i,\lambda}$, for every $\gamma \in \mathrm{v}(\mathbf{K})$, let $b_{i,\gamma}$ be the closed ball of radius γ containing some

$B_{i,\lambda}$, if it exists, or $\mathbf{K}$ otherwise. Since the chain of $B_{i,\lambda}$ does not have a minimal element, any $B_{i,\lambda}$ contains a $b_{i,\gamma}$ that itself contains a $B_{i,\mu}$. By definable spherical completeness, we find $\lambda_i \in \bigcap_\gamma b_{i,\gamma} = \bigcap_\lambda B_{i,\lambda}$. Then λ_i has the property that, for any $\mu \in \mathbf{K}^n$, $v(x-\mu y) \leqslant v(x - \mu_{\neq i} y_{\neq i} - \lambda_i y_i)$. It follows that $v(x-\lambda y)$ is maximal. □

Let x be linearly independent from the y_i. By the Claim 3.3.10, we may assume that $v(x) = \max\{v(x-\lambda y) : \lambda \in \mathbf{K}^n\}$. Then, for every $\mu \in \mathbf{K}$ and $\lambda \in \mathbf{K}^n$, we have $v(\mu x + \lambda y) \leqslant v(\mu x)$ and thus, $v(\mu x+\lambda y) = \min\{v(\mu x), v(\lambda y)\} = \min_i\{v(\mu x), v(\lambda_i y_i)\}$. □

3.4. **Counting germs.** The last ingredient in this section is to reduce the (seemingly horrendous) hypotheses $(\mathbf{E}^\infty_{p,\mathring{F}})$ and $(\mathbf{E}^\infty_{p,\overline{F}})$ to something more tractable. We start by generalizing [Rid19, Section 7]. Let $M \vDash T$, $M_0 := M^{\mathrm{a}} \vDash T_0$ and Q be either $\mathbf{\Gamma}$ or $\mathbf{k}$.

Lemma 3.4.1. *Let $(f_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^n \to Q$ be an $\mathcal{L}_0$-definable family and $c \in \mathbf{K}(N_0)^n$, where $N_0 \succcurlyeq M_0$. There exists an $\mathcal{L}_0(M)$-definable family $(g_\rho)_{\rho\in R} : \mathbf{K}^n \to Q^{[<\infty]}$, with $R \subseteq Q^m$, such that for all $\lambda \in \Lambda(M_0)$, there exists $\rho \in R(M_0)$ with $f_\lambda(c) \in g_\rho(c)$.*
In particular, if $p \in \mathrm{S}^0(M_0)$ is $\mathcal{L}_0(M)$-definable, there is a $\mathcal{L}_0(M)$-definable finite-to-one map $\{[f_\lambda]_p : \lambda \in \Lambda(M_0)\} \to Q^m$, for some $m \in \mathbb{Z}_{>0}$.

Proof. We start with the non-uniform version of the result:

Claim 3.4.2. *For every $N_0 \vDash$ ACVF, $A \leqslant \mathbf{K}(N_0)$ and finite tuple $c \in \mathbf{K}(N_0)$, there exists a finite tuple $a \in A$, such that $Q(\mathrm{acl}_0(Ac)) \subseteq \mathrm{acl}_0(Q(A)ac)$.*

Proof. If $|c| = 1$, we find $a_0 \in \mathrm{acl}_0(A)$ with $Q(\mathrm{acl}_0(Ac)) = \mathrm{acl}_0(Q(A(c))) \subseteq \mathrm{acl}_0(Q(\mathrm{acl}_0(A))ca_0) = \mathrm{acl}_0(Q(A)ca_0)$, by the characterization of transcendental 1-types in ACVF. If $a \in A$ is such that $a_0 \in \mathrm{acl}_0(a)$, we indeed have $Q(\mathrm{acl}_0(Ac)) \subseteq \mathrm{acl}_0(Q(A)ac)$. If $c = de$ with $|e| = 1$, we proceed by induction: $Q(\mathrm{acl}_0(Ade)) \subseteq \mathrm{acl}_0(Q(\mathrm{acl}_0(Ad))be) \subseteq \mathrm{acl}_0(Q(A)acbe)$, with $a, b \in A$. □

By Claim 3.4.2, and compactness in a saturated model of the pair (N_0, M_0), there exists an $\mathcal{L}_0(M_0)$-definable g as above. The union of its conjugates over M has the same properties and is $\mathcal{L}_0(M)$-definable.
Now, if $p \in \mathrm{S}^0(M_0)$ is $\mathcal{L}_0(M)$-definable, then for any $\lambda \in \Lambda(M_0)$, let $Y_\lambda := \{\rho : \forall_p x\ f_\lambda(x) \in g_\rho(x)\}$ and $h([f_\lambda]_p) := \ulcorner Y_\lambda \urcorner \in Q^m$. Note that $p(x) \vDash f_\mu(x) \in \bigcap_{\rho\in Y_\mu} g_\rho(x)$, so h is finite-to-one. □

Lemma 3.4.3. *Assume:*

$(\mathbf{E}^\infty_{\mathbf{k}})$ *for any $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq \mathbf{k}$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that, for all z, $|Y_z| < \infty$ implies $|Y_z| \leqslant n$;*

Then, for every $\mathcal{L}_0(M)$-definable $p \in \mathrm{S}^0_x(M_0)$ and $\mathcal{L}_0$-definable $(F_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^x \to \mathbf{B}^{[r]}$, $(\mathbf{E}^\infty_{p,\mathring{F}})$ holds, uniformly in λ.

Proof. The core of the proof is the following almost internality result:

Claim 3.4.4. *For every $\lambda \in \Lambda(M)$, there exists an $\mathcal{L}_0(M)$-definable finite-to-one map $g_\lambda : X_\lambda := \{[F_\mu]_p : \mu \in \Lambda(M_0)$ and $p(x) \vdash$ "$F_\mu(x)$ are maximal open balls of $F_\lambda(x)$"$\} \to \mathbf{k}^m$, for some $m \in \mathbb{Z}_{>0}$.*

Proof. Let $c \vDash p$. Working in $M(c)^{\mathrm{a}}$ and then taking unions of Galois conjugates, we may find $G_i(c) \in \mathbf{B}^{[<\infty]}$, for $i := 1, 2$, two $\mathcal{L}_0(Mc)$-definable sets picking at least one maximal open balls in each of the ball of F_λ and such that $G_1(c) \cap G_2(c) = \varnothing$. For every μ with $[F_\mu]_p \in X_\lambda$, let $f_\mu(x) := \{(b - b_1)/(b_2 - b_1) : b \in F_\mu(x),\ b_i \in G_i(x)$ and b, b_1, b_2 are in the same ball of $F_\lambda(x)\} \in \mathbf{k}^{[<\infty]}$. Note that $F_\mu(c) \in \mathrm{acl}_0(f_\mu(c)G_1(c)G_2(c))$. Using symmetric functions, we identify $\mathbf{k}^{[<\infty]}$ with some $\mathbf{k}^n$.
By Lemma 3.4.1, we find an $\mathcal{L}_0(M)$-definable finite to one map $h : \{[f_\mu]_p : [F_\mu]_p \in X_\lambda\} \to \mathbf{k}^m$. Then $[F_\mu]_p \in \mathrm{acl}_0([G_1]_p[G_1]_p[f_\mu]_p) \subseteq \mathrm{acl}_0(Mh(\mu))$ and $h(\mu) \in \mathrm{dcl}_0(M[f_\mu]_p) \subseteq \mathrm{dcl}_0(M[F_\mu]_p)$. □

Since $\mathbf{k}(\mathrm{dcl}_0(M))$ is the perfect closure of $\mathbf{k}(M)$, by compactness, composing with a power of the Frobenius automorphism, we may assume that $g_\lambda(X_\lambda(M)) \subseteq \mathbf{k}(M)$ and that g_λ in uniform in λ. The (uniform) bound in $(\mathbf{E}^\infty_{p,\mathring{F}})$ now follows from $(\mathbf{E}^\infty_{\mathbf{k}})$. □

Lemma 3.4.5. *Assume $(\mathbf{E}^\infty_{\mathbf{k}})$ and:*

$(\mathbf{E}^\infty_{\mathbf{\Gamma}})$ *for any $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq \mathbf{\Gamma}$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that, for all z, $|Y_z| < \infty$ implies $|Y_z| \leqslant n$.*

Then for every $\mathcal{L}_0(M)$-definable $p \in \mathrm{S}^0_x(M_0)$, $\mathcal{L}_0$-definable $(F_\lambda)_{\lambda\in\Lambda} : \mathbf{K}^x \to \mathbf{B}^{[r]}$, and $\mathcal{L}(M)$-definable $(Y_z)_z \subseteq [F_\Lambda]_p$, there exists $n \in \mathbb{Z}_{>0}$ such that $|Y_z| < \infty$ implies $|Y_z| \leqslant n$.

In particular, $(\mathbf{E}^\infty_{p,\overline{F}})$ holds.

Proof. Let $r_\lambda(x) := \mathrm{rad}(F_\lambda(x))$. By Lemma 3.4.1, there exists an $\mathcal{L}_0(M)$-definable finite-to-one map $g : [r_\Lambda]_p \to \Gamma^m$. Composing by division by a fixed integer, we may assume that $g([r_\Lambda]_p(M)) \subseteq \Gamma(M)$. It now follows that there is a bound on finite $\{[r_\lambda]_p : [F_\lambda]_p \in Y_z\}$. So, cutting each Y_z in finitely many pieces (and getting rid of the infinite ones), we may assume that $[\mathrm{rad}(F_\lambda)]_p$ is constant and the balls are of the same type, as $[F_\lambda]_p$ ranges through Y_z. Similarly, we may assume that the set of distances between balls in $F_\lambda(x)$ and $F_\mu(x)$, with $[F_\lambda]_p, [F_\lambda]_p \in Y_z$ has size bounded by some integer k. We now proceed by induction on k.
Let $\gamma_z(x)$ be the smallest such distance, $G_{\lambda,z}(x)$ be the set of closed balls of radius $\gamma_z(x)$ around $F_\lambda(x)$ and $Z_z := \{[G_{\lambda,z}]_p : [F_\lambda]_p \in Y_z\}$. Then the set of distances between balls in Z_z has size at most $k - 1$ and we find a bound by induction. In particular, removing some more infinite Y_z, we find an $H_z : \mathbf{K}^n \to \mathbf{B}^{[<\infty]}$ such that every maximal open ball of $H_z(x)$ contains at most one ball of $F_\lambda(x)$ as $[F_\lambda]_p$ varies through Y_z. The bound now follows from Lemma 3.4.3. □

3.5. **The higher arity case.** We can now proceed with the induction:

Proposition 3.5.1. *Assume* $(\mathbf{C}_V)$, $(\mathbf{C}_\mathbf{\Gamma})$, $(\mathbf{E}^\infty_\mathbf{k})$ *and* $(\mathbf{E}^\infty_\mathbf{\Gamma})$. *Let* $X \subseteq \mathbf{K}^x$ *be strict pro-*$\mathcal{L}(A)$*-definable, where* $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$ *and* x *is countable. Let* $\Delta(x;t)$ *be a finite set of* $\mathcal{L}_0$*-formulas,* $p \in \mathrm{S}^\Delta_x(M)$ *be* $\mathcal{L}(A)$*-quantifiable over* $\mathcal{L}$ *and consistent with* X *and* $z \subseteq x$. *Then, there exists an* $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$*-definable* $q \in \mathrm{S}^0_z(M_0)$ *such that* $q|_M$ *is consistent with* p *and* X.

Proof. We proceed by induction on $|z|$. In particular, we may assume that for any set $\Delta(x;t)$ of $\mathcal{L}_0$-formulas, $p \in \mathrm{S}^\Delta_x(M)$ which is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$ and finite strictly smaller $w \subset z$, $p|_w$ can be extended to an $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable $q \in \mathrm{S}^0_w(M_0)$.

Claim 3.5.2. *Let* $\Delta(x;t)$ *be a finite set of* $\mathcal{L}_0$*-formulas,* $p \in \mathrm{S}^\Delta_x(M)$ *be* $\mathcal{L}(A)$*-quantifiable over* $\mathcal{L}$ *and consistent with* X, *finite* $z \subseteq x$ *and* $\Phi(z;s)$ *be a finite set of* $\mathcal{L}_0$*-formulas. Then, there exists a finite set* $\Theta(z;t)$ *containing* Φ *and* $q \in \mathrm{S}^{\Delta,\Theta}_x(M)$ *which is* $\mathcal{L}(A)$*-quantifiable over* $\mathcal{L}$ *and consistent with* p *and* X.

Proof. We proceed by induction on $|z|$. Assume $z = wy$ with $|y| = 1$. By Lemma 3.2.2, we find a finite good presentation $(\Psi(w;t), F(w))$ for Φ. By induction, we find $\Xi(w;u) \supseteq \Psi$ and $q \in \mathrm{S}^{\Delta,\Xi}_x(M)$ which is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$ and consistent with p and X. Since $w \subset z$, as stated in the first paragraph of the proof, $q|_w$ extends to a complete $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable $\mathcal{L}_0(M_0)$-type.

By Corollary 3.2.15 and Lemma 3.4.5, we now find an $\mathcal{L}(A)$-definable $r \in \mathrm{S}^{\Delta,\Xi,F}_x(M)$ which is consistent with q and X. By Lemmas 3.3.1 and 3.4.3, r is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$. □

Let $(\varphi_i(z;t_i))_{i\in\omega}$ enumerate all $\mathcal{L}_0$-formulas. By Claim 3.5.2, we find Θ_i containing φ_i and $q_i \in \mathrm{S}^{\Delta,\Theta_{\leqslant i}}_z(M)$, which is $\mathcal{L}(A)$-quantifiable over $\mathcal{L}$ and consistent with $p \cup \bigcup_{j<i} q_j$ and X. Then $\bigcup_i q_i \in \mathrm{S}^0_z(M)$ is $\mathcal{L}(A)$-definable and consistent with p and X. By Proposition 3.3.4 and Lemma 3.3.8, q extends to a complete $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable $\mathcal{L}_0(M_0)$-type. □

This result is already non-trivial when $X = \mathbf{K}^x$ and $T = \mathrm{ACVF}$:

Corollary 3.5.3. *Let* $\Psi(x;t)$ *be a set of* $\mathcal{L}_0$*-formulas and* $A = \mathrm{acl}_0(A) \leqslant M_0$. *Any* $\mathcal{L}_0(A)$*-quantifiable* $p \in \mathrm{S}^\Psi_x(M_0)$ *can be extended to an* $\mathcal{L}_0(A)$*-definable* $q \in \mathrm{S}^0_x(M_0)$. □

If we do not assume $(\mathbf{E}^\infty_\mathbf{\Gamma})$, and try to replace the use of Corollary 3.2.15 by that of Lemma 3.2.18, the above induction fails. We can, nevertheless, recover a local version of the result:

Proposition 3.5.4. *Let* $n \in \mathbb{Z}_{>0} \cap \mathbf{K}^\times(M)$ *and* $X \subseteq \mathbf{K}^x$ *be* $\mathcal{L}(A)$*-definable, where* $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$. *Assume* $(\mathbf{C}_V)$ *in both* M *and the pair* (M_0, M), $(\mathbf{C}_\mathbf{\Gamma})$, $(\mathbf{E}^\infty_\mathbf{k})$, $(\mathbf{I}_\mathbf{k})$, $(\mathbf{FR}_n)$. *Also assume that:*

$(\mathbf{P}^n_{\pi(X)})$ *for every projection* $Y \subseteq \mathbf{K}^{zy}$ *of* X, *with* $|y| = 1$, $N \succcurlyeq M$ *and* $a \in N^y$, Y_a *is* n*-prepared by some finite* $\mathcal{L}_0(Ma)$*-definable set* $C \subseteq \mathbf{K}$.

Then, for every finite set $\Psi(x,t)$ *of* $\mathcal{L}_0$*-formulas, there exists an* $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$*-definable* $p \in \mathrm{S}^{\Psi}_x(M)$ *consistent with* X.

Proof. Let $A_P = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_\mathrm{P}}(A)$. We say that $\Theta(zy,s)$, where $|y| = 1$, is a hereditarily good presentation if Θ is of the form $\Phi(z,t) \cup \{y \in F_\lambda(z)\}$ for some good presentation (Φ, F) where Φ is itself an hereditarily good presentation.

Claim 3.5.5. *Let* $\Theta(x,s)$ *be a hereditarily good presentation and* $q \in \mathrm{S}^{\Theta}_x(M_0)$ *be* $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A_P))$*-definable. Then for* $\mathcal{L} \in \{\mathcal{L}_0, \mathcal{L}_\mathrm{P}\}$, q *is* $\mathcal{L}(\boldsymbol{\mathcal{G}}(A_P))$*-quantifiable over* $\mathcal{L}$ *— in particular it has a complete* $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A_P))$*-definable extension to* $\mathrm{S}^0_x(M_0)$.

Proof. We proceed by induction on $|x|$. Since $(\mathbf{E}^\infty_\mathbf{k})$ holds both in M_0 and, by Lemma 2.2.16, in the pair (M_0, M), quantifiability follows from Lemmas 3.3.1 and 3.4.3 — applied respectively to M_0 and to the pair (M_0, M). The existence of a complete definable extension follows by Corollary 3.5.3 applied in M_0. □

We now prove, by induction on $x = zy$, the existence of $p \in \mathrm{S}^{\Psi}_x(M_0)$ which is $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A_P))$-definable and consistent with X. By compactness, there exists $(G_\omega)_{\omega\in\Omega} : \mathbf{K}^z \to \mathbf{K}^{[<\infty]}$ such that the family $(X_z)_z$ is n-prepared by G. Let $d \in \mathbb{Z}_{>0}$ bound the degree of any polynomial appearing in Ψ and G. By Lemma 3.2.2, and induction, we find a finite hereditarily good presentation $(\Theta(z,s), F(z))$ for $\varphi_d(x, uv) := \mathrm{v}(\sum_{|I|<d} u_I x^I) \geqslant \mathrm{v}(\sum_{|I|<d} v_I x^I)$. By induction, there exists $q \in \mathrm{S}^{\Theta}_z(M_0)$ which is $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A_P))$-definable and consistent with X. By Claim 3.5.5, q is $\mathcal{L}_\mathrm{P}(\boldsymbol{\mathcal{G}}(A_P))$-quantifiable over $\mathcal{L}_\mathrm{P}$. By Lemma 3.2.18, there exists an $\mathcal{L}_\mathrm{P}(A_P)$-definable $p \in \mathrm{S}^{\Theta,F}_x(M_0)$ consistent with X. By hypothesis, $(\mathbf{C}_V)$ holds in (M_0, M), and so does $(\mathbf{C}_\Gamma)$, by Corollary 2.2.14. By Proposition 3.3.4, it follows that $p|_{\varphi_d}$ is $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A_P))$-definable — and hence so is $p|_\Psi$.
Let $a \in A_P$ be the canonical basis of $p|_{\varphi_d}$. Since $M_0 = M^\mathrm{a} \subseteq \mathrm{acl}_0(M)$, we have $a \in \mathrm{acl}_0(M)$. Let $c \in \mathrm{dcl}_0(M)$ be the code of the finite $\mathcal{L}_0(M)$-orbit of a — which is included in its finite $\mathcal{L}_\mathrm{P}(A)$-orbit. Let f be $\mathcal{L}_0$-definable such that $c \in f(M)$ and $e \in M^{\mathrm{eq}}$ be the code of $f^{-1}(c)$. The $\mathcal{L}_\mathrm{P}(A)$-orbit of c consists of finite subsets of the $\mathcal{L}_\mathrm{P}(A)$-orbit of a and is therefore finite. Hence, so is the $\mathcal{L}(A)$-orbit of e; *i.e.* $e \in \mathrm{acl}^{\mathrm{eq}}(A) = A$. It follows that $p|_{\varphi_d, M} \subseteq \bigcap_{\sigma\in\mathrm{Aut}(M_0/M)} \sigma(p|_{\varphi_d})$ is $\mathcal{L}(A)$-definable. By Proposition 3.3.4, it is in fact $\mathcal{L}_0(\boldsymbol{\mathcal{G}}(A))$-definable — and hence so is $p|_{\Psi,M}$. □

This local result does imply the existence of a global invariant type:

Corollary 3.5.6. *Let* $n \in \mathbb{Z}_{>0} \cap \mathbf{K}^\times(M)$ *and* $X \subseteq \mathbf{K}^x$ *be strict pro-*$\mathcal{L}(A)$*-definable, where* $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$. *Assume* $(\mathbf{C}_V)$ *in both* M *and the pair* (M_0, M), $(\mathbf{C}_\Gamma)$, $(\mathbf{E}^\infty_\mathbf{k})$, $(\mathbf{I}_\mathbf{k})$, $(\mathbf{FR}_n)$ *and:*

$(\mathbf{P}^n_{\pi(X)})$ *for every projection $Y \subseteq \mathbf{K}^{zy}$ of X onto finitely many coordinates, with $|y| = 1$, every $N \succcurlyeq M$ and every $a \in \mathbf{K}^z(N)$, Y_a is n^ℓ-prepared by some $\mathcal{L}_0(Ma)$-definable set $C \subseteq \mathbf{K}(N)$, for some $\ell \in \mathbb{Z}_{\geqslant 0}$.*

Then there exists an $\mathrm{Aut}(M/\mathcal{G}(A))$*-invariant $p \in \mathrm{S}^0_x(M)$ consistent with X.*

Note that $(\mathbf{FR}_n)$ implies $(\mathbf{FR}_{n^\ell})$ for every $\ell \in \mathbb{Z}_{\geqslant 0}$. Also if M is a finitely ramified henselian field, $(\mathbf{C}_V)$ holds in both M and (M_0, M), since both theories have maximally complete models, and $(\mathbf{P}^n_{\pi(X)})$ holds by Remark 3.2.17.

Proof. For every (finite) set $\Psi(x,t)$ of $\mathcal{L}_0$-formulas, the set of $p \in \mathrm{S}^0_x(M)$ which are consistent with X and whose Ψ-type is $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant is closed. It is non-empty, by Proposition 3.5.4. By compactness, the intersection of all these sets, which coincides with the set of $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant $p \in \mathrm{S}^0_x(M)$ consistent with X, is also non-empty. □

Remark 3.5.7. If T is a $\mathbf{k}$-$\mathbf{\Gamma}$-enrichment of Hen_0, then the pair (M_0, M) is elementarily equivalent to one where both $\mathbf{K}$ and $\mathbf{PK}$ are maximally complete. Hence $(\mathbf{C_B})$ — and therefore $(\mathbf{C}_V)$— holds both in M and in the pair (M_0, M).

4. Invariant completions

Notation. In this section. let T be an $\mathbf{RV}$-enrichment of the theory of characteristic zero henselian fields.

4.1. Main results.

Our goal in this section is to describe the behaviour of global $\mathcal{L}$-types whose underlying $\mathcal{L}_0$-type is invariant. A crucial point is that Fact 2.1.3 can be reformulated in the following manner: for every $A \leqslant M \vDash T$,

$$\mathrm{tp}_0(A) \cup \mathrm{tp}(\mathbf{RV}(A)) \vdash \mathrm{tp}(A).$$

Therefore, the main point of this section is to better understand $\mathrm{tp}(\mathbf{RV}(A))$ and then deduce properties of $\mathrm{tp}(A)$. In particular, we will show that $\mathbf{RV}(A)$ is generated by a small canonical set. This will allow us to conclude that a global type whose underlying quantifier free type is invariant is itself invariant over $\mathbf{RV}$ (*cf.* Corollary 4.3.17). However, a better control of the parameters requires more auxiliary sorts. Recall that

$$\mathbf{Lin}_A := \bigsqcup_{\substack{s \in \mathbf{S}(\mathrm{dcl}_0(A)) \\ \ell \in \mathbb{Z}_{>0}}} s/\ell\mathfrak{m}s.$$

Theorem 4.1.1. *Assume*

$(\mathbf{I_k})$ *the residue field $\mathbf{k}$ is infinite.*

Let $M \preccurlyeq N \vDash T$ sufficiently saturated and homogeneous, $A \subseteq \mathcal{G}(M)$ and $a \in \mathbf{K}(N)$ such that $\mathrm{tp}_0(a/M)$ *is* $\mathrm{Aut}(M/A)$*-invariant. Then the type* $\mathrm{tp}(a/M)$ *is* $\mathrm{Aut}(M/A\mathbf{RV}(M)\mathbf{Lin}_A(M))$*-invariant.*

4.2. **Invariance and stably embedded sets.** Note that we consider invariance over large subsets of our model — that happen to be the points of some stably embedded definable sets. This gives rise to some subtle issues and two notions of invariance.

Definition 4.2.1. Let M be an $\mathcal{L}$-structure, $C \subseteq M$, D be a (ind-)$\mathcal{L}$-definable set and p be a partial $\mathcal{L}(M)$-type. We say that p:

- is $\mathrm{Aut}(M/C)$-*invariant* if for every $\sigma \in \mathrm{Aut}(M/C)$, p and $\sigma(p)$ are equivalent.
- has $\mathrm{Aut}(M/C)$-*invariant* D-*germs* if it is $\mathrm{Aut}(M/C)$-invariant and so is the p-germ of every $\mathcal{L}(M)$-definable map $f : p \to D^{\mathrm{eq}}$;
- is $\mathrm{Aut}(M/D)$-*invariant* if it has $\mathrm{Aut}(M/D(M))$-invariant D-germs.

We will only apply these notions for M saturated, C equal to the M-points of a stably embedded (ind-)$\mathcal{L}$-definable set, D stably embedded and p a complete Δ-type for some set of $\mathcal{L}$-formulas Δ.

Remark 4.2.2.
(1) A type might be $\mathrm{Aut}(M/D(M))$-invariant but not $\mathrm{Aut}(M/D)$-invariant. Indeed, let b be a closed ball in some $M \vDash$ ACVF without any $\mathrm{acl}(\ulcorner b \urcorner \mathbf{RV}_1(M))$-definable subballs. Then any two $a_1, a_2 \in b(M)$ have the same type over $\mathrm{acl}(\ulcorner b \urcorner \mathbf{RV}_1(M))$. However, for every $x \in b$, $\mathrm{rv}_1(x - a_1) = \mathrm{rv}_1(x - a_2)$ implies that the a_i are in the same maximal open subball of b. It follows that the generic of b over M is $\mathrm{Aut}(M/\ulcorner b \urcorner)$-invariant but not $\mathrm{Aut}(M/\ulcorner b \urcorner \mathbf{RV}_1)$-invariant.
(2) A type $p \in \mathrm{S}(M)$ is $\mathrm{Aut}(M/C)$-invariant if and only if for every $a \vDash p$ in a sufficiently homogeneous $N \succcurlyeq M$, any $\sigma \in \mathrm{Aut}(M/C)$ extends to an element of $\mathrm{Aut}(N/Ca)$.
(3) On the other hand, a type $p \in \mathrm{S}(M)$ has $\mathrm{Aut}(M/C)$ invariant D-germs, where D is stably embedded, if and only if for every $a \vDash p$ in a sufficiently saturated $N \succcurlyeq M$, any $\sigma \in \mathrm{Aut}(M/C)$ extends to an element of $\mathrm{Aut}(N/CD(N)a)$ — *cf.* the proof of Lemma 4.2.4.
(4) The space of types with $\mathrm{Aut}(M/C)$-invariant D-germs is closed: for any $\sigma \in \mathrm{Aut}(M/C)$ and $\mathcal{L}(M)$-definable map $f : p \to D^{\mathrm{eq}}$, no types in the open set "$[f]_p \neq [f^\sigma]_p$" has $\mathrm{Aut}(M/C)$-invariant D-germs.

Let us now recall the following folklore result on stable embeddedness, which is implicit in [CH99, Appendix] and states that we can recover the usual characterisation of types and hence of definable closure (equivalently internality) from invariance over a stably embedded definable set:

Lemma 4.2.3. *Let M be saturated, D be (ind-)$\mathcal{L}$-definable stably embedded and $e \in M$. We have:*

- *for every $e' \equiv_{D(M)} e$, there exists $\sigma \in \mathrm{Aut}(M/D(M))$ such that $e' = \sigma(e)$;*
- *if $\sigma(e) = e$ for every $\sigma \in \mathrm{Aut}(M/D(M))$, then $e \in \mathrm{dcl}(D(M))$.*

Proof.
- It suffices to prove that the set of partial elementary isomorphisms with domain $AD(M)$ fixing $D(M)$, where $A \subseteq M$ is

smaller than $|M|$, has the back and forth property. So let f be such a morphism and $a \in M$ be some element. By stable embeddedness of D, there exists $E \subseteq D(M)$ of size at most $|A|$ such that $\mathrm{tp}(a/AE) \vdash \mathrm{tp}(a/AD(M))$. Then $f_\star \mathrm{tp}(a/AE)$ is realised in M by some b. Then $\mathrm{tp}(b/f(A)E) \vdash \mathrm{tp}(b/f(A)D(M))$ and hence f extends by sending a to b.

- Let us now consider $e' \in M$ such that $e \equiv_{D(M)} e'$, then by the previous point, we can find $\sigma \in \mathrm{Aut}(M/D(M))$ such that $e' = \sigma(e) = e$. Since D is stably embedded, there exists a small $A \subseteq D(M)$ such that, $\mathrm{tp}(e/A) \vdash \mathrm{tp}(e/D(M))$. So both types have a single realisation in M, *i.e.* $e \in \mathrm{dcl}(A) \subseteq \mathrm{dcl}(D(M))$. □

One advantage of the stronger notion of invariance is transitivity:

Lemma 4.2.4. *Let $M \preccurlyeq N$ be $\mathcal{L}$-structures with N saturated and sufficiently large, $C \subseteq M$ (potentially large), D be an (ind-)$\mathcal{L}$-definable stably embedded set, $p \in \mathrm{S}(M)$ have $\mathrm{Aut}(M/C)$-invariant D-germs, $a \vDash p$ in N and $q \in \mathrm{S}(N)$ be $\mathrm{Aut}(N/CD(N)a)$-invariant. Then $q|_M$ is $\mathrm{Aut}(M/C)$-invariant.*
If, moreover, q has $\mathrm{Aut}(N/CD(N)a)$-invariant E-germs, for some (ind-)$\mathcal{L}$-definable set E, then $q|_M$ has $\mathrm{Aut}(M/C)$-invariant E-germs.

Proof. Let τ be the partial $\mathcal{L}$-elementary isomorphism $M(a) \to M(a)$ induced by $\sigma \in \mathrm{Aut}(M/C)$. Since σ fixes the germs of every definable map from p to D^{eq}, τ induces the identity on $D^{\mathrm{eq}}(\mathrm{dcl}(M(a)))$. Since D is stably embedded, *cf.* Lemma 4.2.3, τ extends to an element of $\mathrm{Aut}(N/CD(N)a)$, also denoted τ, which thus fixes q. It follows that $q|_M$ is fixed by $\tau|_M = \sigma$. If, moreover, q has $\mathrm{Aut}(N/CD(N)a)$-invariant E-germs, then $\sigma = \tau|_M$ fixes the $q|_M$-germ of any $\mathcal{L}(M)$-definable function into E^{eq}. □

The core of our proof of Theorem 4.1.1 is the following variation on transitivity:

Lemma 4.2.5. *Let $M \preccurlyeq N \vDash T$, $C \subseteq M$ potentially large, $a \in \mathbf{K}^x(N)$ a (potentially infinite) tuple and $\rho : \mathbf{K}^x \to \mathbf{RV}$ be pro-$\mathcal{L}_0(M)$-definable. Assume that $\mathrm{rv}_\infty(M(a)) \subseteq \mathrm{dcl}_0(C\rho(a))$ and that $p := \mathrm{tp}_0(a/M)$ and $[\rho]_p$ are $\mathrm{Aut}(M/C)$-invariant. Then $\mathrm{tp}(a/M)$ has $\mathrm{Aut}(M/C)$-invariant $\mathbf{RV}$-germs.*

Proof. Pick $\sigma \in \mathrm{Aut}(M/C)$. Let $N_0 \vDash \mathrm{ACVF}$ containing N be saturated and sufficiently large. Let $\tau : M(a) \to M(a)$ be the $\mathcal{L}_0$-isomorphism induced by σ. Note that $\tau(\rho(a)) = \rho^\sigma(a) = \rho(a)$ and hence $\tau|_{(\mathrm{rv}_\infty(M(a)))}$ is the identity. We may thus extend τ to a partial elementary map which is the identity on $\mathbf{RV}(N_0)$. So, by stable embeddedness of $\mathbf{RV}$, τ extends to some element of $\mathrm{Aut}(N_0/C\mathbf{RV}(N_0)a)$, also denoted τ. By Fact 2.1.3, $\mathrm{tp}(Ma) = \mathrm{tp}(\sigma(M)a)$, *i.e.* $\sigma(p) = p$. Moreover, any $\mathcal{L}(Ma)$-definable $X \subseteq \mathbf{RV}^n$ is $\mathcal{L}(\mathrm{rv}_\infty(M(a)))$-definable and hence $X(N) = \tau(X(N)) = X^\tau(N)$, equivalently, σ fixes the p-germ of any $\mathcal{L}(M)$-definable function into $\mathbf{RV}^{\mathrm{eq}}$. □

4.3. **Computing leading terms.** In view of Lemma 4.2.5, given any A-invariant type $\mathrm{tp}_0(a/M)$, we want to find a pro-$\mathcal{L}_0(M)$-definable map ρ such that $\rho(a)$ generates $\mathrm{rv}_\infty(M(a))$ and $[\rho]_p$ is $\mathrm{Aut}(M/A)$-invariants. When $A \preccurlyeq M$ and A is sufficiently large, this is done in Corollary 4.3.16. As previously stated, for general small A, dealing with closed balls forces us to also consider maps into certain A-definable $\mathbf{k}$-vector spaces. The goal then becomes to build a "nice" model of T containing A and proceed by transitivity.
The technical core of the proof consists in a generalisation to relative arity one of the classical description of 1-types in henselian fields, cf. Lemmas 4.3.8, 4.3.10 and 4.3.13.

Let us start with three leading term computations that we will need later.

Lemma 4.3.1. *Let* $M \vDash \mathrm{ACVF}$*,* $L \leqslant K = \mathbf{K}(M)$*,* $\mathrm{rv}_1(L) \leqslant R \leqslant \mathrm{rv}_1(K)$*,* $b \in \mathbf{B}(K)$*,* $g \in \mathbf{K}(\mathrm{dcl}_0(LR))$*,* $c \in K$ *and* $P := \prod_{i<d}(x - e_i) \in \mathbf{K}(\mathrm{dcl}_0(LR))[x]$*.*

1. *If* $c, g \in b$ *and* $e_i \notin b$*, for all* i*, then* $\mathrm{rv}_1(P(c)) = \mathrm{rv}_1(P(g)) \in \mathrm{dcl}_0(R)$*.*
2. *If* $c \notin b$ *and* $g, e_i \in b$*, for all* i*. then* $\mathrm{rv}_1(P(c)) = \mathrm{rv}_1(c-g)^d$*.*
3. *Assume that* b *is closed, that* $c, g, e_i \in b$*, for all* i*, and that the maximal open subball of* b *around* c *does not contain* g *nor any* e_i*. Then*
$$\mathrm{rv}_1(P(c)) = \bigoplus_{i\leqslant d} \mathrm{rv}_1(P_i(g))\mathrm{rv}_1(c-g)^i \in \mathrm{dcl}_0(R\mathrm{rv}_1(c-g)),$$
where $P(y+x) = \sum_i P_i(y)x^i$*. In particular, the sum is well-defined.*

In fact, computation (2) is an easy particular case of computation (3) — consider the smallest closed ball containing c, g and the e_i.

Proof. Let us first prove (1). We have $\mathrm{rv}_1(P(c)) = \prod_i \mathrm{rv}_1(c-e_i) = \prod_i \mathrm{rv}_1(g-e_i) = \mathrm{rv}_1(P(g)) \in \mathbf{RV}_1(\mathrm{dcl}_0(LR)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_1(L)R) = \mathrm{dcl}_0(R)$, where the inclusion follows from quantifier elimination for ACVF in $\mathcal{L}_{\mathbf{RV}}$. As for (2), we have $\mathrm{rv}_1(P(c)) = \prod_i \mathrm{rv}_1(c-e_i) = \mathrm{rv}_1(c-g)^d$. Finally, in the case of (3), let $Q(x) = P((c-g)x+g)/(c-g)^d$. The roots $(e_i-g)/(c-g)$ of Q are in $\mathcal{O}$. Thus $Q \in \mathcal{O}[x]$, $Q_i(0) = P_i(g)(c-g)^{i-d} \in \mathcal{O}$ and $\mathrm{v}(Q(1)) = 0$. We have:
$$\begin{aligned}\mathrm{rv}_1(P(c)) &= \mathrm{rv}_1(c-g)^d\mathrm{res}(Q(1))\\ &= \mathrm{rv}_1(c-g)^d(\sum_i \mathrm{res}(Q_i(0)))\\ &= \mathrm{rv}_1(c-g)^d(\bigoplus_i \mathrm{rv}_1(Q_i(0)))\\ &= \bigoplus_{i\leqslant d} \mathrm{rv}_1(P^{(i)}(g))\mathrm{rv}_1(c-g)^i\\ &\in \mathbf{RV}_1(\mathrm{dcl}_0(LR\mathrm{rv}_1(c-g)))\\ &\subseteq \mathrm{dcl}_0(R\mathrm{rv}_1(c-g)),\end{aligned}$$
where the third equality follows from the fact that $\mathrm{v}(\sum_i Q_i(0)) = \mathrm{v}(Q(1)) = 0 \leqslant \min_i\{\mathrm{v}(Q_i(0))\} \leqslant \mathrm{v}(\sum_i Q_i(0))$. ☐

Remark 4.3.2. In mixed characteristic, we will be applying this result to the least equicharacteristic zero coarsening, yielding a computation for rv_∞ and not just rv_1.

Essentially every computation of leading terms reduces to the above cases by the following lemma.

Lemma 4.3.3. *Let $M \vDash \mathrm{ACVF}$, $L \leqslant K = \mathbf{K}(M)$, $c \in K$ and $\mathrm{rv}_\infty(L) \leqslant R = \mathbf{RV}(\mathrm{dcl}_0(R)) \leqslant \mathrm{rv}_\infty(K)$. The following are equivalent:*

(1) $\mathrm{rv}_\infty(L(c)) \subseteq R$;

(2) for every monic irreducible $P \in \mathbf{K}(\mathrm{dcl}_0(LR))$, $\mathrm{rv}_\infty(P(c)) \in R$.

Moreover if $P \in \mathbf{K}(\mathrm{dcl}_0(LR))[x]$ is irreducible, then its roots are either all inside or outside any $B \in \mathbf{B}^{[<\infty]}(\mathrm{dcl}_0(LR))$.

Proof. By (1), $\mathrm{rv}_\infty(P(c)) \in \mathbf{RV}(\mathrm{dcl}_0(LRc)) \subseteq \mathbf{RV}(\mathrm{dcl}_0(\mathrm{rv}_\infty(L(c))R)) = R$. The converse is a consequence of the fact that rv_∞ is a multiplicative morphism and any polynomial over L is a product of (an element of L and) monic irreducible polynomials over $\mathbf{K}(\mathrm{dcl}_0(LR))$.
As for the moreover statement, let $Q = \prod_{e \in B^\cup}(x-e) \in \mathbf{K}(\mathrm{dcl}_0(LR))[x]$ where the e range over the roots of P (with multiplicity). If P is irreducible, then $Q = P$ or $Q = 1$. ☐

One last important ingredient — also ubiquitous in the development of motivic integration, *e.g.* [HK06] — is the fact that, in characteristic zero, finite sets of points (and of balls in equicharacteristic zero) can be canonically parametrised by $\mathbf{RV}$. Recall the definition of $b[n]$ and $B[n]$ from Definition 3.2.16.

Lemma 4.3.4. *For every $r \in \mathbb{Z}_{>0}$, there exists $m \in \mathbb{Z}_{>0}$ such that for every characteristic zero valued field L and $B \in \mathbf{B}^{[r]}(L)$ with $|B[m]| = r$, there exists an $\mathcal{L}_0(\ulcorner B \urcorner)$-definable injection $\nu : B \to \mathbf{RV}^n$.*

Proof. We proceed by induction on r. If $r = 1$, take $m = 1$ and ν to be constant equal to $1 \in \mathbf{RV}_1$. If $|B| > 1$, we may assume that $|B[m]| = r$ for all m, the lemma will follow by compactness. Also, assuming that $L \vDash \mathrm{ACVF}$ is sufficiently saturated and homogeneous, it suffices to find an $\mathrm{Aut}(L/\ulcorner B \urcorner)$-invariant injection $\nu : B \to (\mathbf{RV}_\infty)^n$. Indeed, since B is finite some projection to $\mathbf{RV}^n$ is already injective and it must be definable. Finally, let $\gamma := \max\{\mathrm{v}(b_1 - b_2) : b_i \in B \text{ distinct}\}$. Since $|B[m]| = r$ for all m, $\gamma < \mathrm{rad}(B) + \mathrm{v}(\mathbb{Z})$ and B can be injected in the set of open v_∞-balls of radius γ/Δ_∞. So we may assume that the residue characteristic of L is zero. Let B' be the set of closed balls of radius γ around the balls of B. By construction, we have $|B'| < |B| = r$. For every $b' \in B'$, let $B_{b'} := \{b \in B : b \subseteq b'\}$. Note that, by hypothesis, $\mathrm{res}_{b'}(b) \in \mathbf{R}_{b'} = \{\text{maximal open subballs of } b'\}$ uniquely determines b inside B. Let $c_{b'} \in \mathbf{R}_{b'}$ denote the average of the $\mathrm{res}_{b'}(b)$ as b ranges over $B_{b'}$. By induction, we find an $\mathcal{L}_0(\ulcorner B \urcorner)$-definable injection $\mu : \{c_{b'} : b' \in B'\} \to \mathbf{RV}^n$. For every $b \in B$, let $\nu(b) := (\mathrm{rv}_1(b - c_{b'}), \mu(c_{b'}))$ where $b \subseteq b' \in B'$. Then $\nu : B \to \mathbf{RV}^{n+1}$ is an $\mathcal{L}_0(\ulcorner B \urcorner)$-definable injection. ☐

Lemma 4.3.5. *For every $r \in \mathbb{Z}_{>0}$ there exists an $m \in \mathbb{Z}_{>0}$ such that for every characteristic zero valued field L and every $B \in \mathbf{B}^{[r]}(L)$ with $|B[m]| = r$, there exists $g \in \mathbf{K}^{[r]}(L)$ with exactly one point inside each ball of $B[m]$.*

Proof. Let us start with a weaker version of the result:

Claim 4.3.6. *For every $b \in \mathbf{B}(\mathrm{acl}_0(L))$, $b(\mathrm{acl}_0(L)) \neq \varnothing$.*

Proof. We may assume that L is algebraically closed. If $v(L) \neq 0$, $L \models$ ACVF and hence, by model completeness, $b(L) \neq \varnothing$. If $\mathrm{v}(L) = 0$, $\mathrm{rad}(b) \in \mathbf{\Gamma}(\mathrm{dcl}_0(L)) = \{0\}$. If $0 \in b$, we are done. Otherwise, $\mathrm{v}(b) = 0$ and $b \subseteq \mathcal{O}$. So b is open and it is (interdefinable with) a residue element. But $\mathbf{k}(\mathrm{acl}_0(L)) = \mathrm{res}(L)$ and thus $b(L) \neq \varnothing$. ☐

Claim 4.3.7. *For every $B \in \mathbf{B}^{[r]}(\mathrm{dcl}_0(L))$, there exists $m \in \mathbb{Z}_{>0}$ and $g \in \mathbf{K}^{[r]}(\mathrm{dcl}_0(L))$ such that, if $|B[m]| = r$, there is exactly one point of g inside each ball of $B[m]$.*

Proof. We may assume that B is irreducible over L. For every $b \in B$, let $d \in b(\mathrm{acl}_0(L))$. Let D be $\mathcal{L}_0(L)$-definable and irreducible over L containing d and let $g_b \in \mathrm{acl}_0(L)$ be the average of $D \cap b$. Then $g_b \in b[m]$, where $m := |D \cap b|$. Let g be finite $\mathcal{L}_0(L)$-definable set irreducible over L containing g_b. Since $|B[m]| = r = |B|$, we get $g \cap b[m] = \{g_b\}$. By irreducibility, each ball of $B[m]$ contains exactly one element of g. ☐

The lemma follows by compactness. ☐

Let $\mathbf{B}_x^{[r]}$ denote the (ind-$\mathcal{L}_0$-definable) set of $\mathcal{L}_0$-definable maps $F : \mathbf{K}^x \to \mathbf{B}^{[r]}$ and $\mathbf{B}_x^{[<\infty]}$ denote the (ind-$\mathcal{L}_0$-definable) set $\bigcup_r \mathbf{B}_x^{[r]}$. Similarly we denote $\mathbf{K}_x^{[r]}$ the (ind-$\mathcal{L}_0$-definable) set of $\mathcal{L}_0$-definable maps $F : \mathbf{K}^x \to \mathbf{K}^{[r]}$ and $\mathbf{K}_x^{[<\infty]}$ the (ind-$\mathcal{L}_0$-definable) set $\bigcup_r \mathbf{K}_x^{[r]}$.

Notation. We fix $M \preccurlyeq N \models T$, $A \subseteq M^{\mathrm{eq}}$, $a \in \mathbf{K}^x(N)$ a potentially infinite tuple and $c \in \mathbf{K}(N)$ a single element. Assume that $p(xy) = \mathrm{tp}_0(ac/M)$ is $\mathrm{Aut}(M/A)$-invariant and let $q := \mathrm{tp}_0(a/M)$. For every $F, G \in \mathbf{B}_x^{[<\infty]}(M)$, we write $F \leqslant_q G$ if $q(x) \vdash F^{\cup}(x) \subseteq G^{\cup}(x)$. Finally, let $E := \{F \in \mathbf{B}_x^{[<\infty]}(M) : p(x,y) \vdash y \in F^{\cup}(x)\}$.

In the following Lemmas 4.3.8, 4.3.10 and 4.3.13, we will describe how $\mathbf{RV}(M(ac))$ is generated depending on the shape of E.

Lemma 4.3.8 (Finite sets). *Assume E has a least element $f \in \mathbf{K}_x^{[r]}$ for $\leqslant_q$. Then, there exists a pro-$\mathcal{L}_0(M)$-definable map $\rho : \mathbf{K}^{xy} \to \mathbf{RV}^n$, whose p-germ is $\mathrm{Aut}(M/A)$-invariant such that $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\rho(ac))$.*

Proof. Let $\rho(ac) = \nu_a(c)$, where $\nu_a : f(a) \to \mathbf{RV}^n$ is the $\mathcal{L}_0(f(a))$-definable injection of Lemma 4.3.4. By invariance of p, for every $\sigma \in \mathrm{Aut}(M/A)$, $c \in f^\sigma(a) \cap f(a)$. By minimality, we have $f^\sigma(a) = f(a)$ and hence $[f]_q$ — and thus $[\rho]_p$ — is $\mathrm{Aut}(M/A)$-invariant. Moreover, since $c \in \mathrm{dcl}_0(Ma\rho(ac))$, we have $\mathrm{rv}_\infty(M(ac)) \subseteq \mathbf{RV}(\mathrm{dcl}_0(Ma\rho(ac))) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(Ma)\rho(ac))$. ☐

We now assume that $E \cap \mathbf{K}_x^{[<\infty]} = \emptyset$.

Lemma 4.3.9. *There exists a pro-$\mathcal{L}_0(M)$-definable map $\nu : \mathbf{K}^{xy} \to \mathbf{RV}^{\xi}$, with ξ potentially infinite, whose p-germ is* $\mathrm{Aut}(M/A)$*-invariant such that, for every $F \in E$, for some $m \in \mathbb{Z}_{>0}$ uniformly bounded in $|F(a)|$, the ball $b \in F(a)[m]$ containing c is $\mathcal{L}_0(Ma\nu(ac))$-definable, and $\nu(ac) \in \mathrm{acl}_0(Ma)$.*

Proof. For every $r \in \mathbb{Z}_{>0}$, let $m_r \in \mathbb{Z}_{>0}$ be as in Lemma 4.3.4. Let $F \in E \cap \mathbf{B}_x^{[r]}$ be irreducible over q and such that $|F[m_r]| = r$ — if such an F does not exist let $\nu_r(x) = 1$. By irreducibility, for every $G \in E \cap \mathbf{B}_x^{[r]}$ with $G \leqslant_q F$, every ball in $F(a)$ contains exactly one ball of $G(a)$. In particular, neither $\gamma = \max\{\mathrm{v}(b_1 - b_2) : b_i \in F(a) \text{ distinct}\}$ nor $B_r(a)$, the set of open balls of radius $\gamma + \mathrm{v}(m_r)$ around balls of $F(a)$, depend on the choice of F. It follows that $[B_r]_q$ is $\mathrm{Aut}(M/A)$-invariant. By construction, inclusion induces an injection $F(a) \to B_r(a)$ and that $|B_r(a)| = |B_r(a)[m_r]|$.
Ss in Lemma 4.3.4, let $\nu_r : B_r(a) \to \mathbf{RV}^n$ be an $\mathcal{L}_0(\ulcorner B_r(a) \urcorner)$-definable injection. Let $\nu_r(ac) = \nu_r(b)$ where $c \in b \in B_r(a)$. Note that $\nu_r(ac) \in \mathrm{acl}_0(Ma)$. The element of $F(a)$ containing c is uniquely determined by b, and hence by $\nu_r(ac)$; and $[\nu_r]_p$ is $\mathrm{Aut}(M/A)$-invariant by construction.
Let us now fix any $F \in E$ that we can assume irreducible. Let $M = \max\{m_s : s \leqslant |F(a)|\}$. The sequence $|F(a)[M^k]| \geqslant 1$ is decreasing, bounded by $|F(a)|$ and hence, there exists $k \leqslant |F(a)|$ such that $|F(a)[M^k][M]| = |F(a)[M^{k+1}]| = |F(a)[M^k]|$. Let $r := |F[M^k]|$. By the previous paragraphs and the choice of M, the ball $b \in F(a)[M^k]$ containing c is $\mathcal{L}_0(Ma\nu_r(ac))$-definable. It follows that $\nu = (\nu_r)_{r \geqslant 1}$ has the required properties. ☐

We now wish to consider the case where, either E induces a strict intersection in the least equicharacteristic zero coarsening v_∞ (case (1)), or c is generic over Ma in a finite set of open v_∞-balls (case (2)):

Lemma 4.3.10 (Open and strict balls)**.** *Assume that one of the following holds:*

(1) for all $F \in E$, there exists $G \in E$ with $G[m] <_q F$, for any $m \in \mathbb{Z}_{>0}$;
(2) there exists an $r \in \mathbb{Z}_{>0}$ such that for every $F \in E$ and $m \in \mathbb{Z}_{>0}$, there exists an open $G \in E \cap \mathbf{B}_x^{[r]}$ with $G[m] \leqslant_q F$.

Then, there is a pro-$\mathcal{L}_0(M)$-definable map $\rho : \mathbf{K}^{xy} \to \mathbf{RV}^{\xi}$ whose p-germ is $\mathrm{Aut}(M/A)$*-invariant and such that* $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\rho(ac))$.

Note that the cases (1) and (2) are not mutually exclusive.

Proof. Let $\nu : \mathbf{K}^{xy} \to \mathbf{RV}^{\xi}$ be as in Lemma 4.3.9.

Claim 4.3.11. *For every $F \in E$, the ball $b \in F(a)$ containing c is in $\mathrm{dcl}_0(Ma\nu(ac))$.*

Proof. Assume that there exist $G \in E$ with $G[m] \leqslant_q F$, for every $m \in \mathbb{Z}_{>0}$. Then, by Lemma 4.3.9 applied to G, the ball $b' \in G(a)[m]$ containing c is $\mathcal{L}_0(Ma\nu(ac))$-definable. The claim follows since $b' \subseteq b$.

Otherwise, by case (2), we can find a minimal r such that for every $F \in E$ and $m \in \mathbb{Z}_{>0}$, there exists an open $G \in E \cap \mathbf{B}_x^{[r]}$ with $G[m] \leqslant_q F$. Then for m sufficiently large, depending on r, by Lemma 4.3.9, the ball $b' \in G[m]$ containing c is $\mathcal{L}_0(Ma\nu(ac))$-definable. The claim follows since $b' \subseteq b$. □

If there does not exist $g \in \mathbf{K}_x^{[<\infty]}(M)$ such that $\varnothing <_q g \leqslant_q E$, let $\rho(ac) = \nu(ac)$. If such a g exists, we may assume that it is irreducible and, then, the cardinality of the $F \in E$ irreducible over q is bounded by $|g(a)|$. Let $F \in E$ be irreducible over q of maximal cardinality r and let $b(ac) \in F(a)$ contain c. Note that the partition of $g(a)$ induced by F, and in particular $h(ac) \coloneqq g(a) \cap b(ac)$, does not depend on F. Moreover, since there is some (irreducible) $G \in E \cap \mathbf{B}_x^{[r]}$ with $G[|h(ac)|] \leqslant_q F$, the average of $h(ac)$ is in $b(ac)$. So, replacing $h(ac)$ by its average, we may assume that $h(ac)$ is a singleton. Then, $h(ac) \in \mathrm{dcl}_0(g(a)b(ac)) \subseteq \mathrm{dcl}_0(Ma\nu(ac))$ by Claim 4.3.11. Let $\rho(ac) = (\nu(ac), \mathrm{rv}_\infty(c - h(ac)))$.
For every $\sigma \in \mathrm{Aut}(M/A)$, applying the previous argument to g and $F^{\sigma^{-1}} \in E$, we have $h(ac) \in b^{\sigma^{-1}}(ac)$, and hence, by invariance of p, $h^\sigma(ac) \in b(ac)$. Note that the smallest (closed) ball containing $h(ac)$ and $h^\sigma(ac)$ is algebraic over Ma and let $D(a)$ be its finite orbit over Ma. We have $D \leqslant_q E$. If $D[m] \in E$, for some $m \in \mathbb{Z}_{>0}$, then there is $G \in E$ open (irreducible) such that $G[m] \leqslant_q D[m]$ and hence $D >_q G \in E$. But then either $h(ac) \notin G(ac)$ or $h^\sigma(ac) \notin G$. By invariance of p, we may assume that $h(ac) \notin G(ac)$, contradicting the fact that $h(ac)$ does not depend on the choice of F. Thus $D(a)[m] \notin E$ and so $c \notin D(a)[m]^\cup$. It follows that $\mathrm{rv}_\infty(c - h(ac)) = \mathrm{rv}_\infty(c - h^\sigma(ac))$. We have just proved the $\mathrm{Aut}(M/A)$-invariance of $[\rho]_p$.
Now, to prove that $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\rho(ac))$, by Lemma 4.3.3, it suffices to prove that, for every irreducible $P \in \mathbf{K}(\mathrm{dcl}_0(Ma\nu(ac)))[x]$, $\mathrm{rv}_\infty(P(c)) \in \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\rho(ac))$. Recall that, by Lemma 4.3.9, $\nu(ac) \in \mathrm{acl}_0(Ma)$. Let $z(a)$ be the finite set, irreducible over $M(a)$, containing the set Z of roots of P. If $z \leqslant_q E$, then, as above c avoids $d(ac)[m]$, where $d(ac)$ is the smallest closed ball around $Z \cup \{h(ac)\}$. By Lemma 4.3.1.(2), taking into account Remark 4.3.2, $\mathrm{rv}_\infty(P(c)) = \mathrm{rv}_\infty(c - h(ac))^d \in \mathrm{dcl}_0(\rho(ac))$. Otherwise, there is some $F \in E$ such that $Z \cap F(a)^\cup = \varnothing$. Then, by Lemma 4.3.1.(1), $\mathrm{rv}_\infty(P(c)) \in \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\nu(ac))$. □

Remark 4.3.12. There are actually two distinct possible behaviours in Lemma 4.3.10:

- If there does does not exist $g \in \mathbf{K}_x^{[<\infty]}$ such that $g \leqslant_q F$ for every $F \in E$, then $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\nu(ac)) \subseteq \mathrm{acl}_0(\mathrm{rv}_\infty(M(a)))$;
- if such a g exists, then $\mathrm{v}(M(ac)) \nsubseteq \mathrm{acl}_0(\mathrm{v}(M(a)))$.

The last remaining case to consider is when c is generic over Ma in some closed v_∞-ball. For every $B \in \mathbf{B}^{[<\infty]}$, we define $\mathbf{R}_{B,m} \coloneqq \{b' \subseteq B^\cup : b'$ open ball of radius $\mathrm{rad}(B) + \mathrm{v}(m)\}$ and $\mathbf{R}_{B,\infty} = \varprojlim_m \mathbf{R}_{B,m}$. For every $x \in B^\cup$, let $\mathrm{res}_{B,m}(x)$ denote the unique element of $\mathbf{R}_{B,m}$ containing x and $\mathrm{res}_{B,\infty} : B^\cup \to \mathbf{R}_{B,\infty}$ be the induced map.

Lemma 4.3.13 (Closed balls)**.** *Assume that there exists an* $F \in E$ *such that for every* $g \in \mathbf{K}_x^{[<\infty]}(M)$ *with* $g \leqslant_q F$, $c \notin \mathrm{res}_{F(a),\infty}(g(a))^{\cup}$. *Let* $b \in F(a)$ *contain* c, $\xi \in \mathbf{RV}^n$ *such that* $b \in \mathrm{dcl}_0(Ma\xi)$ *and* $G \in \mathrm{res}_{b,\infty}(\mathrm{dcl}_0(Ma\xi))$. *Then* $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\xi\mathrm{rv}_\infty(\mathrm{res}_{F(a),\infty}(c) - G(a\xi)))$.

In later applications of this lemma, we will take $\xi = \nu(ac)$ as given by Lemma 4.3.9.

Proof. Note that for any $m \in \mathbb{Z}_{>0}$, the hypothesis on F remains true of $F[m]$. So, replacing F by some $F[m]$, with $|F[m](a)|$ minimal, we may assume that $|F[m](a)|$ is constant. By Lemma 4.3.5, we can now find $f \in \mathbf{K}_x^{[<\infty]}(M)$ such that $f(a)$ has exactly one point in every ball of $F(a)$. By hypothesis, $c \notin \mathrm{res}_{F(a),\infty}(f(a))$. Let $h \in \mathrm{dcl}_0(Ma\xi)$ denote the unique element of $f(a) \cap b$. Since $\mathrm{rad}(F(a))/\Delta_\infty = \mathrm{v}_\infty(c-h) = \mathrm{v}_\infty(c - G(a,\xi)) \leqslant \mathrm{v}_\infty(G(a,\xi) - h)$, we have $\mathrm{rv}_\infty(c-h) = \mathrm{rv}_\infty(\mathrm{res}_{F(a),\infty}(c) - G(a\xi)) \oplus \mathrm{rv}_\infty(G(a\xi)-h)$. So it suffices to prove that $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\xi\mathrm{rv}_\infty(c-h))$.
By Lemma 4.3.3, it further suffices to prove that, for every irreducible $P \in \mathbf{K}(\mathrm{dcl}_0(Ma\xi))[x]$, $\mathrm{rv}_\infty(P(c)) \in \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\xi\mathrm{rv}_\infty(c-h))$. If, for every $m \in \mathbb{Z}_{>0}$, no root of P is in $b[m]$, then, by Lemma 4.3.1.(1), $\mathrm{rv}_\infty(P(c)) \in \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\xi)$. Otherwise, let $m \in \mathbb{Z}_{>0}$ be such that every root of P is in $b[m]$. Since $\mathbf{K}(\mathrm{acl}_0(Ma\xi)) \subseteq \mathrm{acl}_0(Ma)$, let $z(a)$ be finite irreducible over $M(a)$ containing the roots of P. By hypothesis, $c \notin \mathrm{res}_{F(a)[m]}(z(a))$. By Lemma 4.3.1.(3), $\mathrm{rv}_\infty(P(c)) \in \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\mathrm{rv}_\infty(c-h))$. □

Notation. Let $\widehat{A} \subseteq \mathbf{K}(M)$ contain a realisation of every $\mathcal{L}(A)$-type and assume that M is sufficiently saturated and homogeneous.

We can now wrap up the relative arity one case:

Proposition 4.3.14. *There exists a pro-*$\mathcal{L}_0(\widehat{A})$*-definable map* $\rho : \mathbf{K}^{xy} \to \mathbf{RV}^{\xi}$ *such that* $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\rho(ac))$.

Proof. Note first that any $\mathrm{Aut}(M/A)$-invariant p-germ of $\mathcal{L}_0(M)$-definable functions is represented by an $\mathcal{L}_0(\widehat{A})$-definable function — it suffices to consider a realisation in $\widehat{A}$ of the type of the parameters over A.
If $E \cap \mathbf{K}_x^{[<\infty]} \neq \varnothing$, we apply Lemma 4.3.8. So let us assume that $E \cap \mathbf{K}_x^{[<\infty]} = \varnothing$. If for every $F \in E$, there exists $G \in E$ with $G[m] <_q F$, for every $m \in \mathbb{Z}_{>0}$, we are in case (1) of Lemma 4.3.10 and we can conclude. So we may assume that there exists F such that for every $G \in E$, $F \leqslant_q G[m]$, for some $m \in \mathbb{Z}_{>0}$. If there exists $g \in \mathbf{K}_x^{[<\infty]}$ with $g \leqslant_q F$ and $c \in \mathrm{res}_{F(a),\infty}(g(a))$, then, for all $H \in E$ and $n \in \mathbb{Z}_{>0}$, $H \leqslant_q F[m]$, for some $m \in \mathbb{Z}_{>0}$. Let $G(a) := \mathrm{res}_{F(a),mn}(g(a))$, then $c \in G[n] \leqslant_q H$ and hypothesis (2) of Lemma 4.3.10 holds with $r = |g|$.
So we may assume that no such g exist, *i.e.* the hypotheses of Lemma 4.3.13 hold. As previously, we may assume that $|F[m]|$ is constant. Let ν be as in Lemma 4.3.9; we may assume that ν is $\mathcal{L}_0(\widehat{A})$-definable. Let $b(ac) \in F(a)$ contain c.

Claim 4.3.15. *There exists* $G(ac) \in \mathrm{res}_{b(a)[m],\infty}(\mathrm{dcl}_0(\widehat{A}a\nu(ac)))$*, for some* $m \in \mathbb{Z}_{>0}$.

Proof. By construction of $\widehat{A}$, there exists $\sigma \in \mathrm{Aut}(M/A)$ such that F^σ is $\mathcal{L}_0(\widehat{A})$-definable. By $\mathrm{Aut}(M/A)$-invariance of p, we have $F^\sigma \in E$ and hence $F^\sigma \leqslant_q F[m]$, for some $m \in \mathbb{Z}_{>0}$. So, up to replacing F by F^σ, we may assume F is $\mathcal{L}_0(\widehat{A})$-definable. By Lemma 4.3.5, and replacing F by some $F[m]$, we find $g \in \mathbf{K}_x^{[<\infty]}(\widehat{A})$ with exactly one element in each ball of F. It then suffices to consider the only element of $\mathbf{k}_{F(a),\infty}(g(a))$ contained in $b(ac)$. □

By Lemma 4.3.13, $\mathrm{rv}_\infty(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(a))\nu(ac)\mathrm{rv}_\infty(\mathrm{res}_{F(a),\infty}(c) - G(ac))) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(Ma)\widehat{A}ac)$. □

Corollary 4.3.16. *There exists a pro-*$\mathcal{L}_0(\widehat{A})$*-definable map* $\rho : \mathbf{K}^x \to \mathbf{RV}^\xi$*, such that* $\mathrm{rv}_\infty(M(a)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(a))$.

Proof. We proceed by induction on an enumeration of a. The induction step is Proposition 4.3.14 and the limit case is trivial. □

Corollary 4.3.17. *The type* $\mathrm{tp}(a/M)$ *is* $\mathrm{Aut}(M/\widehat{A}\mathbf{RV})$*-invariant.*

Proof. It follows from Corollary 4.3.16 and Lemma 4.2.5. □

4.4. **Invariant resolutions.**

Notation. Let $M \preccurlyeq N \vDash T$ both be sufficiently saturated and homogeneous and $A \subseteq \boldsymbol{\mathcal{G}}(M)$.

By transitivity, there remains to build a sufficiently saturated model containing A whose type is invariant.

Lemma 4.4.1. *Assume that* $A \subseteq \mathbf{K}(M)$ *and let* $R \subseteq \mathbf{RV}(M)$*. There exists* $C \subseteq \mathbf{K}(N)$ *and a pro-*$\mathcal{L}_0(M)$*-definable map* $\rho : \mathbf{K}^{|C|} \to \mathbf{RV}^\xi$ *such that* $R \subseteq \mathrm{rv}_\infty(A(C)) \subseteq \mathrm{dcl}_0(AR)$*,* $q := \mathrm{tp}_0(C/M)$ *and* $[\rho]_q$ *are* $\mathrm{Aut}(M/AR)$*-invariant and* $\mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$.

Proof. We proceed by induction on an enumeration of R. Assume the property holds of R for some C and ρ and pick any $\zeta \in \mathbf{RV}_\infty(M)$. If $\zeta \in \mathrm{rv}_\infty(\mathrm{acl}_0(AC))$, let $c \in \mathbf{K}(\mathrm{acl}_0(AC))$ be such that $\mathrm{rv}_\infty(c) = \zeta$. Let D be a minimal finite $\mathcal{L}_0(AC\zeta)$-definable set containing c. Replacing c by the average of D, we may assume that $c \in \mathrm{dcl}_0(AC\zeta) \subseteq \mathrm{dcl}_0(MC) \subseteq N$. We have $R\zeta \subseteq \mathrm{rv}_\infty(A(Cc)) \subseteq \mathbf{RV}(\mathrm{dcl}_0(AC\zeta)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(A(C))\zeta) \subseteq \mathrm{dcl}_0(AR\zeta)$, $\mathrm{tp}_0(Cc/M)$ is $\mathrm{Aut}(M/AR\zeta)$-invariant and, since $c \in \mathrm{dcl}_0(MC) = M(C)^{\mathrm{h}}$, $\mathrm{rv}_\infty(M(Cc)) = \mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$.
If $\zeta \notin \mathrm{rv}_\infty(\mathrm{acl}_0(AC))$, let $c \in N$ be generic in $\mathrm{rv}_\infty^{-1}(\zeta)$ over M. Then $p := \mathrm{tp}_0(Cc/M)$ is $\mathrm{Aut}(M/AR\zeta)$-invariant. By Lemma 4.3.10, we find a pro-$\mathcal{L}_0(M)$-definable map $\rho' : \mathbf{K}^{|C|+1} \to \mathbf{RV}_\infty^\xi$ such that $\mathrm{rv}_\infty(M(Cc)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(C))\rho'(Cc)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C)\rho'(Cc))$ and whose p-germ is $\mathrm{Aut}(M/AR\zeta)$-invariant. Moreover, no root of any $P \in \mathbf{K}(\mathrm{dcl}(AC))[x]$ is in $\mathrm{rv}_\infty^{-1}(\zeta)$. For any $g \in \mathrm{rv}_\infty^{-1}(\zeta)$, by Lemma 4.3.1.(1), $\mathrm{rv}_\infty(P(c)) = \mathrm{rv}_\infty(P(g))$

does not depend on g and is thus in $\mathbf{RV}(\mathrm{dcl}_0(AC\zeta)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(AC)\zeta) \subseteq \mathrm{dcl}_0(AR\zeta)$. By Lemma 4.3.3, $\mathrm{rv}_\infty(A(Cc)) \subseteq \mathrm{dcl}_0(AR\zeta)$. □

Corollary 4.4.2. *Assume that $A \subseteq \mathbf{K}(M)$. There exists $A \subseteq C \preccurlyeq N$ and a pro-$\mathcal{L}_0(M)$-definable map $\rho : \mathbf{K}^{|C|} \to \mathbf{RV}^{\xi}$, such that $p := \mathrm{tp}_0(C/M)$ is $\mathrm{Aut}(M/A\mathbf{RV}(M))$-invariant, $[\rho]_p$ is $\mathrm{Aut}(M/A\mathbf{RV}(M))$-invariant and $\mathrm{rv}_\infty(MC) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$.*

In particular, by Lemma 4.2.5, $\mathrm{tp}(C/M)$ is $\mathrm{Aut}(M/A\mathbf{RV})$-invariant.

Proof. Let $A \subseteq M_1 \preccurlyeq M$, with M_1 small. Applying Lemma 4.4.1 to $\mathrm{rv}_\infty(M_1)$, we find $C \subseteq N$ and ρ such that $\mathrm{rv}_\infty(M_1) \subseteq \mathrm{rv}_\infty(A(C)) \subseteq \mathrm{dcl}_0(A\mathrm{rv}_\infty(M_1)) \cap \mathrm{rv}_\infty(N) \subseteq \mathrm{rv}_\infty(M_1)$, $p := \mathrm{tp}_0(C/M)$ and $[\rho]_p$ are $\mathrm{Aut}(M/A\mathrm{rv}_\infty(M_1))$-invariant and $\mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$. By replacing C with $\mathbf{K}(\mathrm{dcl}_0(AC))$, we may assume $A \subseteq C = C^{\mathrm{h}}$. Since $\mathrm{rv}_\infty(C) = \mathrm{rv}_\infty(A(C)) = \mathrm{rv}_\infty(M_1) \preccurlyeq \mathrm{rv}_\infty(N)$ and C is a characteristic zero henselian field, it follows from Fact 2.1.3 that $C \preccurlyeq N$. □

Corollary 4.4.3. *Assume that $A \subseteq \mathbf{K}(M)$. There exists $\widehat{A} \preccurlyeq N$ containing a realisation of every $\mathcal{L}(A)$-type such that $\mathrm{tp}(\widehat{A}/M)$ is $\mathrm{Aut}(M/A\mathbf{RV})$-invariant.*

Proof. Corollary 4.4.2 implies that, for every $\mathcal{L}(A)$-definable set X, there exists an $\mathrm{Aut}(M/A\mathbf{RV})$-invariant type concentrating on X — since the C from Corollary 4.4.2 is a model containing A. Since the set of $\mathrm{Aut}(M/A\mathbf{RV})$-invariant types is closed, it follows by compactness that any $\mathcal{L}(A)$-type has an $\mathrm{Aut}(M/A\mathbf{RV})$-invariant extension. The corollary follows by the standard construction relying on transitivity, Lemma 4.2.4 — for the limit steps, note that $\mathrm{Aut}(M/A\mathbf{RV})$-invariance is finitary: $\mathrm{tp}(c/M)$ is $\mathrm{Aut}(M/A\mathbf{RV})$-invariant if and only if for every finite $c_0 \subseteq c$, $\mathrm{tp}(c_0/M)$ is $\mathrm{Aut}(M/A\mathbf{RV})$-invariant. □

Recall that, by Convention 2.1.4, $\mathbf{Lin}_A(M)$ denotes the set of cosets $c + \ell\mathfrak{m}s$ where $s \in \mathbf{S}(\mathrm{dcl}_0(A))$ has a basis in M and $c \in s(M)$.

Lemma 4.4.4. *Assuming:*

($\mathbf{I_k}$) *the residue field $\mathbf{k}$ is infinite in models of T,*

there exists $C \subseteq \mathbf{K}(N)$ and an $\mathcal{L}_0(A)$-definable map $\rho : \mathbf{K}^{|C|} \to \mathbf{Lin}_A^{\xi}$ such that, for all $n \in \mathbb{Z}_{>0}$, $\mathbf{S}_n(A) \subseteq s_n(C)$, $\mathrm{tp}_0(C/M)$ is $\mathcal{L}_0(A)$-definable and $\mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\mathbf{Lin}_A(M)\rho(C))$.
In mixed characteristic, we may further assume that $\bigcup_{n>0} \mathbf{T}_n(A) \subseteq \mathrm{dcl}_0(C)$.

Proof. Fix $s \in \mathbf{S}_n(A)$ and let $\beta \in \mathrm{GL}_n(M)$ be a basis of s. Then, any $\alpha \models \beta \cdot (\eta_{\mathcal{O}}|_M)^{\otimes n^2}$ — which is realised in N by ($\mathbf{I_k}$) — where $\eta_{\mathcal{O}}$ is the generic (quantifier free) type of $\mathcal{O}$, is a basis of s. Note that $\mathrm{tp}_0(\alpha/M) = \beta \cdot \eta_{\mathcal{O}}^{\otimes n^2}$ only depends on s and is indeed $\mathcal{L}_0(A)$-definable. Let $\overline{\alpha}$, respectively $\overline{\beta}$, be the basis of $s/\mathfrak{m}_\infty s$ — seen as a pro-definable set— induced by α, respectively β. The matrix of $\mathbf{k}_\infty$-coefficients of α in the basis β is $\mathrm{res}_\infty(\beta^{-1} \cdot$

α) where $\beta^{-1}\cdot\alpha \vDash (\eta_{\mathcal{O}}|_M)^{\otimes n^2}$. It follows that $\mathrm{rv}_\infty(M(\alpha)) = \mathrm{rv}_\infty(M(\beta^{-1}\cdot\alpha)) \subseteq \mathrm{dcl}_0(\mathbf{RV}(M)\mathrm{res}_\infty(\beta^{-1}\cdot\alpha)) \subseteq \mathrm{dcl}_0(\mathbf{RV}(M)\overline{\beta}\overline{\alpha})$. The first part of the statement follows by iterating the above construction independently for every $s \in \mathbf{S}_n(A)$.

Let us now assume that we are in mixed characteristic. For every $c \vDash \eta_{\mathfrak{m}}|_M$, $\mathrm{rv}_\infty(M(Cc)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M(C))\mathrm{res}_\infty(c))$. This also holds for all maximal open subballs of $\mathcal{O}$. So, enlarging C further, we may also assume that $\mathbf{k}(\mathrm{dcl}_0(AC)) \cap M \subseteq \mathrm{res}(C)$. Then, for every $e \in \mathbf{T}_n(A)$, $s = \tau_n(e) \in \mathbf{S}(A)$ has a basis in C and every coordinate of e in that basis is the residue of an element of C. It then follows that $e \in t_n(C)$. □

Lemma 4.4.5. *In equicharacteristic zero, assuming that, for all $n \in \mathbb{Z}_{>0}$, $\tau_n(\mathbf{T}_n(A)) \subseteq s_n(\mathbf{K}(A))$. Then there exists $C \subseteq \mathbf{K}(N)$ and an $\mathcal{L}_0(M)$-definable map $\rho : \mathbf{K}^{|C|} \to \mathbf{RV}^{\xi}$ such that $A \subseteq \mathrm{dcl}_0(C)$, $q := \mathrm{tp}_0(C/M)$ is $\mathrm{Aut}(M/A)$-invariant, $[\rho]_q$ is $\mathrm{Aut}(M/A)$-invariant and $\mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$.*

Proof. For every $e \in \mathbf{T}_n(A)$, by hypothesis, $s := \tau(e)$ has a basis in $\mathbf{K}(A)$. It follows that $s/\mathfrak{m}s$ also has a basis of $\mathrm{dcl}_0(A)$-points and hence is $\mathcal{L}_0(A)$-definably isomorphic to $\mathbf{k}^n$. By Lemma 4.4.1 applied to $R := \mathbf{k}(\mathrm{dcl}_0(A)) \cap M \subseteq \mathrm{dcl}_0(A)$, we find $\mathbf{K}(A) \subseteq C \leqslant \mathbf{K}(N)$ such that $\mathbf{k}(\mathrm{dcl}_0(A)) \cap M \subseteq \mathrm{res}(C)$, $q := \mathrm{tp}_0(C/M)$ is $\mathrm{Aut}(M/A)$-invariant, $[\rho]_q$ is $\mathrm{Aut}(M/A)$-invariant and $\mathrm{rv}_\infty(M(C)) \subseteq \mathrm{dcl}_0(\mathrm{rv}_\infty(M)\rho(C))$. Then, we have $A = \mathbf{K}(A) \cup \bigcup_n \mathbf{S}_n(A) \cup \bigcup_n \mathbf{T}_n(A) \subseteq \mathrm{dcl}_0(C)$. □

Corollary 4.4.6. *Assume that $(\mathbf{I_k})$ holds. Then there exists $C \subseteq \mathbf{K}(N)$ such that $\mathrm{tp}_0(C/M)$ has $\mathrm{Aut}(M/A\mathbf{RV}(M)\mathbf{Lin}_A(M))$-invariant $\mathbf{RV}$-germs and $A \subseteq \mathrm{dcl}_0(C)$.*

Proof. In mixed characteristic, this follows immediately from Lemmas 4.2.5 and 4.4.4. In residue characteristic zero, it follows from Lemmas 4.4.4 and 4.4.5 and transitivity, *cf.* Lemma 4.2.4. □

Proof of Theorem 4.1.1. By Corollary 4.4.6, we find $C \subseteq \mathbf{K}(N)$ such that $A \subseteq \mathrm{dcl}_0(C)$ and $\mathrm{tp}(C/M)$ has $\mathrm{Aut}(M/A\mathbf{RV}(M)\mathbf{Lin}_A(M))$-invariant $\mathbf{RV}$-germs. Let $M \preccurlyeq M_1 \preccurlyeq N$ be sufficiently saturated and homogeneous and contain C. By Corollary 4.4.3, we find $\widehat{C} \preccurlyeq N$ containing a realisation of every $\mathcal{L}(C)$-type such that $\mathrm{tp}(\widehat{C}/M_1)$ is $\mathrm{Aut}(M_1/C\mathbf{RV})$-invariant. Let $M_1 \preccurlyeq M_2 \preccurlyeq N$ be sufficiently saturated and homogeneous and contain $\widehat{C}$.
Let $p := \mathrm{tp}_0(a/M)$, which is $\mathrm{Aut}(M/A)$-invariant by assumption.

Claim 4.4.7. $\mathrm{tp}(a/M) \cup p|_{M_2}$ *is consistent.*

Proof. Let $\varphi(x,m)$ be some $\mathcal{L}(M)$-formula such that $N \vDash \varphi(a,m)$ and let $\psi(x,d) \in p|_{M_2}$. Let $\sigma \in \mathrm{Aut}(N/Amd)$ be such that $\sigma(d) \in M$. Then $N \vDash \psi(a,\sigma(d))$ and hence $\sigma^{-1}(a) \vDash \psi(x,d) \wedge \varphi(x,m)$. □

Let $a' \vDash \mathrm{tp}(a/M) \cup p|_{M_2}$. Then $\mathrm{tp}_0(a'/M_2) = p|_{M_2}$ is $\mathrm{Aut}(M_2/C)$-invariant and thus, by Corollary 4.3.17, $\mathrm{tp}(a'/M_2)$ is $\mathrm{Aut}(M_2/\widehat{C}\mathbf{RV})$-invariant. Since

$\mathrm{tp}(\widehat{C}/M_1)$ is $\mathrm{Aut}(M_1/C\mathbf{RV})$-invariant, by transitivity, *cf.* Lemma 4.2.4, $\mathrm{tp}(a'/M_1)$ is $\mathrm{Aut}(M_1/C\mathbf{RV})$-invariant. By transitivity, since $\mathrm{tp}(C/M)$ has $\mathrm{Aut}(M/A\mathbf{RV}(M)\mathbf{Lin}_A(M))$-invariant $\mathbf{RV}$-germs, $\mathrm{tp}(a/M) = \mathrm{tp}(a'/M)$ is $\mathrm{Aut}(M/A\mathbf{RV}(M)\mathbf{Lin}_A(M))$-invariant. □

Let us conclude this section by relating Theorem 4.1.1 to imaginaries:

Proposition 4.4.8. *Let $T \supseteq \mathrm{Hen}_0$ be an $\mathcal{L}$-theory, such that:*

- **(D)** *for every strict pro-$\mathcal{L}(A)$-definable $X \subseteq \mathbf{K}^x$, with $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$, there exist an $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant $p \in \mathrm{S}^0_x(M)$ consistent with X;*
- **($\mathbf{Q_K}$)** *for every tuple $a \in \mathbf{K}(M)$, with $M \vDash T$, $\mathrm{tp}_1(f(a)) \vdash \mathrm{tp}(a)$, where $f : \mathbf{K} \to \mathbf{K}^x$ is pro-$\mathcal{L}$-definable and $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}$ such that $\mathcal{L}_1$ is an $\mathbf{RV}$-enrichment of $\mathcal{L}_0$;*
- **($\mathbf{I_k}$)** *the residue field $\mathbf{k}$ is infinite;*
- **(SE)** $\mathbf{RV}$ *and $\mathbf{R} = \bigcup_\ell \mathcal{O}/\ell\mathfrak{m}$ are stably embedded.*

Let $M \vDash T$, $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}^{\mathrm{eq}}(e)$. Then

$$e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{K}(A) \cup (\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A)).$$

Proof. Let $M \vDash T$ be saturated and sufficiently large, $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}^{\mathrm{eq}}(e)$. Then $e = g(a)$ for some $\mathcal{L}$-definable map g and tuple $a \in \mathbf{K}(M)$. Let $Y = g^{-1}(e)$ and $X = f(Y)$, which is a strict pro-$\mathcal{L}(A)$-definable set. By **(D)**, there exists an $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant $p \in \mathrm{S}^0_x(M)$ consistent with X. We may assume that $f(a) \vDash p$. By Theorem 4.1.1, $\mathrm{tp}_1(f(a)/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A)\mathbf{RV}(M)\mathbf{Lin}_{\mathcal{G}(A)}(M))$-invariant. By **($\mathbf{Q_K}$)**, $\mathrm{tp}(a/M)$ — and hence $e \in M^{\mathrm{eq}}$ — is also $\mathrm{Aut}(M/\mathcal{G}(A)\mathbf{RV}(M)\mathbf{Lin}_{\mathcal{G}(A)}(M))$-invariant.
Since $\mathbf{Lin}_{\mathcal{G}(A)}$ is a collection of free $\mathcal{O}/\ell\mathfrak{m}$-modules, $\mathbf{Lin}_{\mathcal{G}(A)}$ and, in fact $\mathbf{Lin}_{\mathcal{G}(A)} \cup \mathbf{RV}$, is stably embedded. By Lemma 4.2.3, $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathcal{G}(A) \cup \mathbf{RV}(M) \cup \mathbf{Lin}_{\mathcal{G}(A)}(M))$ — *i.e.* $e = h(c)$ for some $\mathcal{L}(\mathcal{G}(A))$-definable map h and tuple $c \in \mathbf{RV}^m(M) \times \mathbf{Lin}^n_{\mathcal{G}(A)}(M)$. Let $Z = h^{-1}(e)$. Then $\ulcorner Z \urcorner \in (\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A)$ and

$$\begin{aligned} e \in \mathrm{dcl}^{\mathrm{eq}}(\mathcal{G}(A)\ulcorner Z \urcorner) &\subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathcal{G}(A) \cup (\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A)) \\ &\subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{K}(A) \cup (\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A)), \end{aligned}$$

since $\mathcal{G}(A) \smallsetminus \mathbf{K}(A) \subseteq \mathbf{Lin}^{\mathrm{eq}}_{\mathcal{G}(A)}(A)$. □

5. Imaginaries in short exact sequences

In this section we will establish results which yield a relative understanding of imaginaries in certain pure short exact sequences of modules.

5.1. **The core case.** We start with a well known lemma. We include a proof for convenience.

Lemma 5.1.1. *Let D and C be stably embedded (ind-)definable sets in $D \cup C$, such that $D \perp C$.*

(1) *Assume that both D and C (considered with the full induced structures) weakly eliminate imaginaries. Then $D \cup C$ weakly eliminates imaginaries.*
(2) *Assume that both D and C eliminate imaginaries and that in C one has* $\mathrm{dcl} = \mathrm{acl}$. *Then $D \cup C$ eliminates imaginaries.*

Proof. (1) Let $X \subseteq D^m \times C^n$ be a definable subset. Since $D \perp C$, the equivalence relation $\sim$ on C^n, given by $c \sim c' :\Leftrightarrow X_c = X_{c'}$, has finitely many equivalence classes $Z_1 = c_1/\sim, \dots, Z_k = c_k/\sim$. As $\sim$ is $\ulcorner X \urcorner$-definable and C is stably embedded, the Z_i are all $C^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$- definable.
For $i = 1, \dots, k$, set $Y_i = X_{c_i} \subseteq D^m$, which is $D^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-definable since D is stably embedded. As D and C weakly eliminate imaginaries, there are finite tuples $d \in D(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$ and $c \in C(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$ such that the Y_i are all d-definable and the Z_i are all c-definable. Thus $X = \bigcup_{i=1}^{k}(Y_i \times Z_i)$ is dc-definable, so in particular $D(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))C(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable.
(2) The assumptions on C yield $C^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)) \subseteq \mathrm{dcl}^{\mathrm{eq}}(C(\ulcorner X \urcorner))$, and so the sets $Z_1, \dots Z_k$ are c-definable for some $c \in C(\ulcorner X \urcorner)$. In particular the Y_i are then all $\ulcorner X \urcorner$-definable, thus $D^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable. By elimination of imaginaries in D, we find $d \in D(\ulcorner X \urcorner)$ such that all Y_i are d-definable. We now finish as in (1). □

Fact 5.1.2. *Let G be a group, and let $H_1, \dots, H_N$ be subgroups of G. Then the left cosets of the H_i form a pre-basis of closed sets for a noetherian topology on G. Moreover, setting $H_I := \bigcap_{i \in I} H_i$ for $I \subseteq \{1, \dots, N\}$, the irreducible closed sets for this topology are precisely the left cosets of those H_I with the property that any proper subgroup of H_I of the form H_J is of infinite index in H_I.*

Proof. This is an easy consequence of Neumann's Lemma. □

Let R be an integral domain, $\mathcal{L} \supseteq \mathcal{L}_{R-\mathrm{mod}}$ and M an $\mathcal{L}$-expansion of an infinite torsion free R-module. Let $Z \subseteq M^n$ be an $\mathcal{L}(M)$-definable set. We set $\dim_R(Z) := \max\{\dim_R(c/M) : c \in Z(N)\}$, where $N \succcurlyeq M$ is sufficiently saturated and $\dim_R(a/B)$ denotes the $Q(R)$-linear dimension of a over B, for $Q(R)$ the field of fractions of R.

Lemma 5.1.3. *In the above situation, assume* $\dim_R(Z) \leqslant r$. *Then there are definable sets $C_1, \dots, C_s \subseteq M^n$ with the following properties:*
(1) $\bigcup_{i=1}^{s} C_i$ *is $\ulcorner Z \urcorner$-definable.*
(2) *Each C_i is* $\mathrm{acl}^{\mathrm{eq}}(\ulcorner Z \urcorner)$*-definable.*
(3) *Each C_i is of the form $\gamma_i + H_i$, where H_i is a definable R-submodule of M^n given by a condition $L_i x' = m x''$, for some splitting of the variables $x = x'x''$ with $|x'| = r$ and $|x''| = n - r$, matrix $L_i \in R^{(n-r)\times r}$ and $m \in R \setminus \{0\}$. (In particular,* $\dim_R(C_i) = r$ *for all i.)*
(4) $Z \subseteq \bigcup_{i=1}^{s} C_i$

Moreover, if $\mathcal{Z}$ is a definable family such that $\dim_R(\mathcal{Z}_b) \leqslant r$ *for all b, there are finitely many R-submodules $H_1, \dots, H_N$ as in the statement such that for any b the C_i may be chosen among the cosets of the H_k.*

Proof. By compactness there are sets $C_1, \dots, C_N$ with $C_i = \delta_i + H_i$ as in (3) such that $Z \subseteq \bigcup_{i=1}^N C_i$. Note that all H_i are $\emptyset$-definable subgroups.
By Fact 5.1.2, the cosets of the H_i form a pre-basis of closed sets for a noetherian topology on M^n. In particular, in this topology there exists a smallest closed subset W of M^n containing Z, and this W is clearly $\ulcorner Z \urcorner$-definable. The (finitely many) irreducible components of W are then all $\mathrm{acl}^{\mathrm{eq}}(\ulcorner Z \urcorner)$-definable. If W_j is such an irreducible component, it is of the form $W_j = \gamma_j + \bigcap_{i \in I_j} H_i$ (where $I_j \neq \emptyset$ in case $r < n$, since $Z \subseteq \bigcup_{i=1}^N \delta_i + H_i$ by assumption). As the H_i are $\emptyset$-definable, it is easy to see that if we replace each component $W_j = \gamma_j + \bigcap_{i \in I_j} H_i$ by $W_j' = \bigcup_{i \in I_j} \gamma_j + H_i$, then the union of all W_j' is $\ulcorner Z \urcorner$-definable and each coset $\gamma_j + H_i$ occurring in this union is $\mathrm{acl}^{\mathrm{eq}}(\ulcorner Z \urcorner)$-definable.
The moreover part follows by compactness. ☐

Theorem 5.1.4. *Let R be an integral domain and M be*

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

a short exact sequence of R-modules, in an $\mathbf{A}$-$\mathbf{C}$-enrichment $\mathcal{L}$ of the pure (in the sense of model theory) three sorted sequence of R-modules. Assume the following properties hold:

1. *$\mathbf{A}$ is a pure submodule of $\mathbf{B}$ (in the sense of module theory),*
2. *$\mathbf{C}$ is torsion free,*
3. *for any $l \in R \setminus \{0\}$, the quotient $\mathbf{C}/l\mathbf{C}$ is finite and the preimage in $\mathbf{B}$ of any coset $c + l\mathbf{C}$ contains an element which is algebraic over $\emptyset$.*

Let $e \in M^{\mathrm{eq}}$. Then, setting $E := \mathrm{acl}^{\mathrm{eq}}(e)$ and $\Delta := \mathbf{C}(E)$, we have

$$e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{C}^{\mathrm{eq}}(E)\mathbf{B}_{\Delta}^{\mathrm{eq}}(E)),$$

where $\mathbf{B}_\Delta$ denotes the union of all $\mathbf{B}_\delta$ for $\delta \in \Delta$.

We will prove a a slight generalization of Theorem 5.1.4, namely the following Theorem 5.1.5.

Theorem 5.1.5. *Let $\tilde{R}$ be a ring and $R = \tilde{R}/I$ an integral domain, with I a finitely generated ideal. Let M be*

$$0 \to \tilde{\mathbf{A}} \to \tilde{\mathbf{B}} \to \tilde{\mathbf{C}} \to 0$$

a short exact sequence of $\tilde{R}$-modules, in an $\tilde{\mathbf{A}}$-$\tilde{\mathbf{C}}$-enrichment $\mathcal{L}$ of the pure (in the sense of model theory) three sorted sequence of $\tilde{R}$-modules. Let $\mathbf{A} = \{a \in \tilde{\mathbf{A}} : Ia = (0)\}$, and let $\mathbf{B}$ and $\mathbf{C}$ be the $\tilde{R}$-submodules of $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ defined similarly. Consider $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as R-modules in the natural way.
Assume the following properties hold:

1. *$\tilde{\mathbf{A}}$ is a pure $\tilde{R}$-submodule of $\tilde{\mathbf{B}}$ (in the sense of module theory),*
2. *$\mathbf{C} = \tilde{\mathbf{C}}$,*
3. *$\mathbf{C}$ is a torsion free R-module,*
4. *for any $l \in R \setminus \{0\}$, the quotient $\mathbf{C}/l\mathbf{C}$ is finite and the preimage in $\tilde{\mathbf{B}}$ of any coset $c + l\mathbf{C}$ contains an element which is algebraic over $\emptyset$.*

Let $e \in M^{\mathrm{eq}}$. Then, setting $E := \mathrm{acl}^{\mathrm{eq}}(e)$ and $\Delta := \mathbf{C}(E)$, we have

$$e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{C}^{\mathrm{eq}}(E)\tilde{\mathbf{B}}^{\mathrm{eq}}_{\Delta}(E)),$$

where $\tilde{\mathbf{B}}_{\Delta}$ denotes the union of all $\tilde{\mathbf{B}}_{\delta}$ for $\delta \in \Delta$.

Proof. Denote $\iota : \tilde{\mathbf{A}} \to \tilde{\mathbf{B}}$ and $v : \tilde{\mathbf{B}} \to \tilde{\mathbf{C}}$ the structural maps.
Note that in particular $\tilde{\mathbf{A}} = \tilde{\mathbf{B}}_0 \subseteq \tilde{\mathbf{B}}_{\Delta}$. By (1), $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ are (purely) stably embedded in T, with $\tilde{\mathbf{A}} \perp \tilde{\mathbf{C}}$. Indeed, replacing M by a sufficiently saturated extension, the purity assumption (1) entails that the short exact sequence of $\tilde{R}$-modules is split, and adding a splitting yields an expansion in which $\tilde{\mathbf{B}}$ is just the product structure of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$. Thus the claimed properties hold in an expansion in which $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ are unchanged, so the same properties already hold in M. As $\tilde{\mathbf{B}}_{\Delta}$ is internally $\tilde{\mathbf{A}}$-internal, it follows from Lemma 2.4.17 that $\tilde{\mathbf{B}}_{\Delta}$ is stably embedded in T (over Δ), and one has $\tilde{\mathbf{B}}_{\Delta} \perp \tilde{\mathbf{C}}$.
Since I is finitely generated, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are definable, and it follows from purity of the initial exact sequence that the induced sequence of R-modules

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

is also exact.
Let $X \subseteq \tilde{\mathbf{B}}^n$ be $\mathcal{L}(M)$-definable with $\ulcorner X \urcorner = \mathrm{dcl}^{\mathrm{eq}}(e)$. Assume $\dim_R(v(X)) = r$. Up to passing to some subset of X which is definable over $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner) = E$, we may assume, using Lemma 5.1.3, that there are $L \in R^{(n-r)\times r}$, $m \in R \smallsetminus \{0\}$ and $\delta'' \in \mathbf{C}^{n-r}$ such that for $x = x'x''$ with $|x'| = r$, we have $mv(x'') = Lv(x') + \delta''$ for any $x = x'x'' \in X$. We now consider the (fiberwise) action of $(\mathbf{A}^r, +)$ on $\tilde{\mathbf{B}}^n$ given by

$$a \cdot (x', x'') := (x' + ma, x'' + La).$$

Claim 5.1.6. *There is $l \in R \smallsetminus \{0\}$ such that*

$$\dim_R(\{z \in v(X) : X_z \textit{ is not } l\mathbf{A}^r\textit{-invariant}\}) < r.$$

Proof. Let $\pi(z)$ be the partial type expressing that $z \in v(X)$ and that $\dim_R(z/M) = r$, and let $\tau(y)$ be the partial type expressing that $y \in l\mathbf{A}^r$ for any $l \in R \smallsetminus \{0\}$. Given a solution $(c, a) \vDash \pi(z) \cup \tau(y)$ in $N \succcurlyeq M$, with $c = c'c''$. Then there is an R-linear map $\theta : \mathbf{C}(N) \to \mathbf{A}(N)$ which is trivial on $\mathbf{C}(M)$ and such that $\theta(c') = ma$. Indeed, such a map θ may be found as the restriction of an R-linear map from $\mathbf{C}(N) \otimes_R Q(R)$ to $\mathbf{A}(N)$, by choosing a coherent sequence of integer divisors of a.
For $b \in \tilde{\mathbf{B}}(N)$, let $\rho(b) := b + \theta(v(b))$. Then ρ is an $\tilde{R}$-module-automorphism of $\tilde{\mathbf{B}}(N)$ which induces the identity on $\tilde{\mathbf{B}}(M) \cup \tilde{\mathbf{A}}(N) \cup \mathbf{C}(N)$. It follows from the assumptions that $\rho \in \mathrm{Aut}_{\mathcal{L}}(N/M)$.
In particular, for any $b \in X_c$, we have $\mathrm{tp}(b/M) = \mathrm{tp}(\rho(b)/M)$ and so $\rho(b) \in X_c$. On the other hand, as $c'' = (Lc' + \delta'')/m = Lc'/m + \delta''/m$ and $v(b) = (c', c'')$, using $L \circ \theta = \theta \circ L$ we compute

$$\theta(v(b)) = (\theta(c'), \theta(c'')) = (\theta(c'), \theta(Lc'/m) + \theta(\delta''/m)) = (ma, La),$$

from which it follows that

$$\rho(b) = b + \theta(v(b)) = (b' + \theta(c'), b'' + \theta(c'')) = (b' + ma, b'' + La) = a \cdot (b', b'').$$

Thus X_c is invariant under the action by $\bigcap_{l \in R \setminus \{0\}} l\mathbf{A}^r$. Claim 5.1.6 now follows by compactness. □

Fix $l \in R \setminus \{0\}$ as in the claim. Arguing by induction on r, we may assume that X_z is $l\mathbf{A}^r$-invariant for any $z \in v(X)$. In addition, using assumption (4), we may suppose that $v(X) \subseteq ml\mathbf{C}^n$. Indeed, there are only finitely many cosets of $ml\mathbf{C}^n$, all $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$-definable, so we may assume $X \subseteq v^{-1}(W)$ for a coset W of $ml\mathbf{C}^n$. Replacing X by $X - h$ for some $h \in W \cap \mathrm{acl}(\varnothing)$ if necessary, we may assume $W = ml\mathbf{C}^n$. Let $a \in \tilde{\mathbf{A}}^r$ and $c \in \mathbf{C}^n$. If there exist $b_1 = b_1'b_1'' \in X_c$ and $b_0' \in \mathbf{B}^r$ such that $b_1' = a + mlb_0'$, we set

$$Y_{a,c} := \{b'' - lLb_0' : (b_1', b'') \in X\} = X_{(b_1')} - lLb_0',$$

where $X_{(b_1')}$ denotes the fiber $\{b'' \in \tilde{\mathbf{B}}^{n-r} : (b_1', b'') \in X\}$. Else we set $Y_{a,c} := \varnothing$. Let us first show that in the first case, $Y_{a,c}$ does not depend on the choice of b_1 and b_0'. Indeed, if $d_1 = d_1'd_1'' \in X_c$ and $d_0' \in \mathbf{B}^r$ are such that $d_1' = a + mld_0'$, then $b_1' - d_1' = ml(b_0' - d_0')$, so $mlv(b_0' - d_0') = 0$, thus $v(b_0' - d_0') = 0$, i.e., $b_0' - d_0' \in \mathbf{A}^r$. Set $a_0' := l(b_0' - d_0')$. For $d'' \in X_{(d_1')}$, the invariance of X_c under $l\mathbf{A}^r$ then yields

$$a_0' \cdot (d_1', d'') = (d_1' + ma_0', d'' + La_0') = (b_1', d'' + La_0') \in X_c,$$

so $d'' + lL(b_0' - d_0') = d'' + La_0' \in X_{(b_1')}$, showing that $X_{(d_1')} - lLd_0' \subseteq X_{(b_1')} - lLb_0'$. By symmetry, we get the other inclusion $X_{(b_1')} - lLb_0' \subseteq X_{(d_1')} - lLd_0'$, thus $X_{(b_1')} - lLb_0' = X_{(d_1')} - lLd_0'$.
Let $\delta'' = (\delta_i'')_{1 \leqslant i \leqslant n-r}$ and let $b' = a + mlb_0'$ be as in the definition of $Y_{a,c}$. Then for $y = b'' - lLb_0' \in Y_{a,c}$, we compute

$$mv(y) = mv(b'') - mlLv(b_0') = (Lv(b') + \delta'') - Lv(b') = \delta'',$$

yielding $Y_{a,c} \subseteq \tilde{\mathbf{B}}_{\delta''/m} = \prod_{i=1}^{n-r} \tilde{\mathbf{B}}_{\delta_i''/m}$. It follows that

$$Y \subseteq (\tilde{\mathbf{B}}_{\delta''/m} \times \tilde{\mathbf{A}}^r) \times \mathbf{C}^n.$$

As $\delta''/m \in \mathbf{C}(E) = \Delta$ and $\tilde{\mathbf{B}}_\Delta \perp \mathbf{C}$, using Lemma 5.1.1(1), we finish the proof of the theorem once the following claim is established.

Claim 5.1.7. *$\ulcorner X \urcorner$ and $\ulcorner Y \urcorner$ are interdefinable.*

It is clear by construction that Y is $\ulcorner X \urcorner$-definable. For the converse, we will use that we have reduced to the case where $v(X) \subseteq ml\mathbf{C}^n$. We may thus reconstruct X from Y as follows:

$$d = d'd'' \in X \Leftrightarrow \exists a \in \tilde{E}^r \, \exists d_0' \in \mathbf{B}^r : d' = a + mld_0' \text{ and } d'' \in Y_{a,v(d)} + lLd_0'$$

This yields the claim. □

5.2. **Some variants.** We will now state two variants of Theorem 5.1.5, tailor made for our applications to (enriched) henselian valued fields.

Variant 5.2.1. *Let $\mathcal{L}$ be a multisorted language, $\mathcal{A} \sqcup \{\tilde{\mathbf{B}}\} \sqcup \mathcal{C}$ a partition of the sorts of $\mathcal{L}$ and let $\tilde{\mathbf{A}} \in \mathcal{A}$ and $\tilde{\mathbf{C}} \in \mathcal{C}$. Let $\tilde{R}$ be a ring and $R = \tilde{R}/I$ an integral domain, with I a finitely generated ideal. Let*

$$0 \to \tilde{\mathbf{A}} \to \tilde{\mathbf{B}} \to \tilde{\mathbf{C}} \to 0 \tag{5.1}$$

be a short exact sequence of $\tilde{R}$-modules. Let M be an $\mathcal{L}$-structure which is an $\mathcal{A}$-$\mathcal{C}$-enrichment of the pure (in the sense of model theory) sequence of $\tilde{R}$-modules (5.1)*. Assume that the properties (1)-(4) from the statement of Theorem 5.1.5 hold.*
Let $e \in M^{\mathrm{eq}}$. Then, setting $E := \mathrm{acl}^{\mathrm{eq}}(e)$ and $\Delta := \mathbf{C}(E)$, we have

$$e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{C}^{\mathrm{eq}}(E) \cup (\mathcal{A} \cup \tilde{\mathbf{B}}_{\Delta})^{\mathrm{eq}}(E)),$$

where $\tilde{\mathbf{B}}_{\Delta}$ denotes the union of all $\tilde{\mathbf{B}}_{\delta}$ for $\delta \in \Delta$.

Proof. The proof is a slight variation of the proof of Theorem 5.1.5. Let us indicate the necessary adaptations.
Firstly, it follows from the assumptions that $\mathcal{A}$ and $\mathcal{C}$ are (purely) stably embedded with $\mathcal{A} \perp \mathcal{C}$. Thus, by Lemma 2.4.17, $(\mathcal{A} \cup \tilde{\mathbf{B}}_{\Delta}) \perp \mathcal{C}$ and $\mathcal{A} \cup \tilde{\mathbf{B}}_{\Delta}$ is stably embedded.
Given $e \in M^{\mathrm{eq}}$, we choose an $\mathcal{L}(M)$-definable set $X \subseteq \mathbf{A}' \times \mathbf{C}' \times \tilde{\mathbf{B}}^n$ with $e = \ulcorner X \urcorner$, where $\mathbf{A}'$ is a finite product of sorts from $\mathcal{A}$ and $\mathbf{C}'$ is a finite product of sorts from $\mathcal{C}$. For $(a', c') \in \mathbf{A}' \times \mathbf{C}'$, let

$${}_{(a',c')}X \subseteq \tilde{\mathbf{B}}^n$$

be the fiber over (a', c'). Performing the same reductions as in the proof of Theorem 5.1.5, by compactness, we may assume that there is an $\varnothing$-definable set

$$Y \subseteq \mathbf{A}' \times \mathbf{C}' \times \tilde{\mathbf{B}}_{\delta} \times \mathbf{C}^n$$

for some finite tuple $\delta \in \Delta$ such that for any $(a', c') \in \mathbf{A}' \times \mathbf{C}'$, $\ulcorner {}_{(a',c')}X \urcorner$ and $\ulcorner {}_{(a',c')}Y \urcorner$ are interdefinable, so in particular, $\ulcorner X \urcorner$ and $\ulcorner Y \urcorner$ are interdefinable. The result then follows from Lemma 5.1.1, since $(\mathcal{A} \cup \tilde{\mathbf{B}}_{\Delta}) \perp \mathcal{C}$. □

The second variant is designed for applications to henselian valued fields in mixed characteristic.

Variant 5.2.2. *Let $\mathcal{L}$ be a multisorted language, $\mathcal{A} \sqcup \{\tilde{\mathbf{B}}_n : n \in \mathbb{N}\} \sqcup \mathcal{C}$ a partition of the sorts of $\mathcal{L}$. For any $n \in \mathbb{N}$, let $\tilde{\mathbf{A}}_n \in \mathcal{A}$, and let $\tilde{C} \in \mathcal{C}$. Let $\tilde{R}$ be a ring and $R = \tilde{R}/I$ an integral domain, with I a finitely generated ideal. Let $\tilde{\mathbf{A}} = (\tilde{\mathbf{A}}_n)_{n \in \mathbb{N}}$, $\tilde{\mathbf{B}} = (\tilde{\mathbf{B}}_n)_{n \in \mathbb{N}}$ be projective systems of $\tilde{R}$-modules with surjective transition functions, let $\mathbf{C} = (\tilde{\mathbf{C}}_n)_{n \in \mathbb{N}}$ be the projective system with $\tilde{\mathbf{C}_n} = \tilde{\mathbf{C}}$ for all n and identical transition functions. Let*

$$0 \to \tilde{\mathbf{A}} \to \tilde{\mathbf{B}} \to \tilde{\mathbf{C}} \to 0 \tag{5.2}$$

be a short exact sequence of projective systems of $\tilde{R}$-modules.

Let M be an $\mathcal{L}$-structure which is an $\mathcal{A}$-$\mathcal{C}$-enrichment of the pure (in the sense of model theory) sequence of projective systems of $\tilde{R}$-modules (5.2). *Assume that for very $n \in \mathbb{N}$ the exact sequence $0 \to \tilde{\mathbf{A}}_n \to \tilde{\mathbf{B}}_n \to \tilde{\mathbf{C}} \to 0$ satisfies the properties (1)-(4) from the statement of Theorem 5.1.5.*
Let $e \in M^{\mathrm{eq}}$. Then, setting $E \coloneqq \mathrm{acl}^{\mathrm{eq}}(e)$ and $\Delta \coloneqq \mathbf{C}(E)$, we have

$$e \in \mathrm{dcl}^{\mathrm{eq}}(\tilde{\mathcal{C}}^{\mathrm{eq}}(E)(\mathcal{A} \cup \tilde{\mathbf{B}}_\Delta)^{\mathrm{eq}}(E)),$$

where $\tilde{\mathbf{B}}_\Delta$ denotes the union of all $(\tilde{\mathbf{B}_n})_\delta$ for $\delta \in \Delta$ and $n \in \mathbb{N}$.

Proof. Let us first show that $\mathcal{A}$ and $\mathcal{C}$ are (purely) stably embedded in M such that $\mathcal{A} \perp \mathcal{C}$. For this, given $N \in \mathbb{N}$, we consider the structure M_N given by restricting M to the sorts $\mathcal{A} \sqcup \{\tilde{\mathbf{B}}_m : m \leqslant N\} \sqcup \mathcal{C}$. For $m \leqslant N$ we denote by $p_{N,m}$ the structural map from $\tilde{\mathbf{B}}_N$ to $\tilde{\mathbf{B}}_m$ and by $q_{N,m}$ the one from $\tilde{\mathbf{A}}_N$ to $\tilde{\mathbf{A}}_m$. For any $m \leqslant N$, the sequence $\tilde{S}_m$ of $\tilde{R}$-modules

$$0 \to \tilde{\mathbf{A}}_m \to \tilde{\mathbf{B}}_m \to \tilde{\mathbf{C}} \to 0$$

is interpretable in the sequence $\tilde{S}_N$ once a predicate for $\ker(q_{N,m}) \leqslant \tilde{\mathbf{A}}_N$ is added. Thus, M_N may be seen as an $\mathcal{A}$-$\mathcal{C}$-enrichment of $\tilde{S}_N$
As in the previous proofs, it follows that $\mathcal{A} \cup \tilde{\mathbf{B}}_\Delta$ is stably embedded in M, with $(\mathcal{A} \cup \tilde{\mathbf{B}}_\Delta) \perp \tilde{\mathbf{C}}$. Given $e \in M^{\mathrm{eq}}$, we choose $X \subseteq \mathbf{A}' \times \mathbf{C}' \times \tilde{\mathbf{B}}_k^n$ $\mathcal{L}(M)$-definable with $e = \ulcorner X \urcorner$, where $\mathbf{A}'$ is a finite product of sorts from $\mathcal{A}$, $\mathbf{C}'$ is a finite product of sorts from $\mathcal{C}$ and $k \in \mathbb{N}$. Let $N \geqslant k$ be such that X may be defined using formulas involving only variables from sorts in $\mathcal{A} \cup \mathcal{C} \cup \{\tilde{\mathbf{B}}_i : i \leqslant N\}$. Since, for $N \geqslant m$, $\tilde{S}_m$ is interpretable in an $\mathcal{A}$-$\mathcal{C}$-enrichment of $\tilde{S}_N$, we may conclude with Variant 5.2.1. □

5.3. **Imaginaries in RV.** Recall that in a finitely ramified henselian valued field, the projective system of short exact sequences:

$$1 \to \mathbf{R}_n^\times \to \mathbf{RV}_n^\times \to \mathbf{\Gamma}^\times \to 0$$

is stably embedded with the induced structure a $\mathbf{\Gamma}$-$\mathbf{R}$-enrichment of the pure short exact sequence of abelian groups. Thus Variant 5.2.2 applies and yields the following elimination of imaginaries:

Proposition 5.3.1. *Let M be a $\mathbf{\Gamma}$-$\mathbf{R}$-enriched finitely ramified henselian field, $A \subseteq \boldsymbol{\mathcal{G}}(M)$, $e \in (\mathbf{RV} \cup \mathbf{Lin}_A)^{\mathrm{eq}}(M)$ and $E = \mathrm{acl}^{\mathrm{eq}}(e)$. Assume that:*

- *for every $n, \ell \in \mathbb{Z}_{>0}$, $\mathbf{\Gamma}/\ell\mathbf{\Gamma}$ is finite and the preimage in $\mathbf{RV}_n$ of any coset of $\ell\mathbf{\Gamma}$ contains an element which is algebraic over $\varnothing$.*

Then $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{\Gamma}^{\mathrm{eq}}(E) \cup (\mathbf{Lin}_A \cup \mathbf{RV}_{\mathbf{\Gamma}(E)})^{\mathrm{eq}}(E))$.

In particular, for $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, $(\mathbf{RV} \cup \mathbf{Lin}_{\boldsymbol{\mathcal{G}}(A)})^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{\Gamma}^{\mathrm{eq}}(A) \cup \mathbf{Lin}^{\mathrm{eq}}_{\boldsymbol{\mathcal{G}}(A)}(A))$.

Proof. We apply Variant 5.2.2 with $R = \tilde{R} = \mathbb{Z}$. Since $\mathbf{\Gamma}$ is ordered is it a torsion free $\mathbb{Z}$-module, so (1) - (3) hold and (4) holds by hypothesis. □

These result also apply with an automorphism:

Proposition 5.3.2. *Let* $M \vDash \mathrm{VFA}^{\mathrm{mult}}_{0,0}$, $A \subseteq \mathcal{G}(M)$, $e \in (\mathbf{RV} \cup \mathbf{Lin}_A)^{\mathrm{eq}}(M)$ *and* $E = \mathrm{acl}^{\mathrm{eq}}(e)$. *Then* $e \in \mathrm{dcl}^{\mathrm{eq}}(A\mathbf{\Gamma}(E)\mathbf{RV}_{\mathbf{\Gamma}(E)}(E)\mathbf{Lin}_A(E))$.

In particular, for $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, $(\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathcal{G}(A))$.

Proof. We apply Variant 5.2.1 with $R = \mathbb{Z}[\sigma]$ and $I := \{P \in \mathbb{Z}[\sigma] : P(\mathbf{\Gamma}) = 0\}$. Hypothesis (1) holds by assumption. Hypothesis (2) and (3) hold by multiplicativity: if $c \in \mathbf{\Gamma}_{>0}$ and $P \in \mathbb{Z}[\sigma]$ are such that $P(c) = 0$, then for all $c \in \mathbf{\Gamma}$, $P(c) = 0$ and $P \in I$. Finally hypothesis (4) holds by divisibility.
So $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{\Gamma}^{\mathrm{eq}})(E) \cup (\mathbf{Lin}_A \cup \mathbf{RV}_{\mathbf{\Gamma}(E)})^{\mathrm{eq}}(E)$. But $\mathbf{\Gamma}$ is an ordered vector field over (the field of fraction of) $\mathbb{Z}[\sigma]/I$, so it eliminates imaginaries. Also, by Proposition 2.4.18, $\mathbf{Lin}_A \cup \mathbf{RV}_{\mathbf{\Gamma}(E)}$ weakly eliminate imaginaries. So $\mathbf{\Gamma}^{\mathrm{eq}}(E) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{\Gamma}(E))$ and

$$(\mathbf{Lin}_A \cup \mathbf{RV}_{\mathbf{\Gamma}(E)})^{\mathrm{eq}}(E) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{Lin}_A(E)\mathbf{RV}_{\mathbf{\Gamma}(E)}(E)).$$

The result follows. □

6. Imaginaries in valued fields

6.1. **The henselian case.** Let $\mathcal{L}_{\mathrm{mod}}$ be the two sorted language of $\mathbf{A}$-modules $\mathbf{V}$ and FMod_n be the two sorted $\mathcal{L}_{\mathrm{mod}}$-theory of free $\mathbf{A}$-modules of rank $n \in \mathbb{Z}_{>0}$, where $\mathbf{A}$ is a unitary ring. For every $\mathcal{L}_{\mathrm{mod}}$-definable set $X \subseteq \mathbf{V}^2$ (in FMod_n), we define the equivalence relation E_X on V by vE_Xw if $X_v = X_w$. For every $n, \ell \in \mathbb{Z}_{>0}$, we define $\mathbf{T}_{n,\ell,X} := \bigsqcup_{s \in \mathbf{S}_n} (\mathbf{V}/E_X)^{(\mathbf{R}_\ell, s/\ell\mathfrak{m}s)}$, the collection of these $\mathcal{L}_{\mathrm{mod}}$-imaginaries interpreted in $(\mathbf{R}_\ell, s/\ell\mathfrak{m}s) \vDash \mathrm{FMod}_n$ as s varies in $\mathbf{S}_n$. Let $\mathbf{R}^{\mathrm{leq}}$ denote $\bigsqcup_{n,\ell,X} \mathbf{T}_{n,\ell,X}$, the $\mathbf{R}$-linear imaginaries. We can now prove our imaginary Ax-Kochen-Ershov principle. Let $\mathrm{Hen}_0^{\mathrm{ac}}$ denote the $\mathbf{RV}$-enrichment of Hen_0 with a compatible system of angular components $\mathrm{ac}_n : \mathbf{K} \to \mathbf{R}_n^\times$ — we denote this language by $\mathcal{L}_{\mathrm{ac}}$.

Theorem 6.1.1 (Theorem A)**.** *Let T be a $\mathbf{\Gamma}$-$\mathbf{R}$-enrichment of either* Hen_0 *or* $\mathrm{Hen}_0^{\mathrm{ac}}$, *such that:*

- $(\mathbf{C}_{\mathbf{\Gamma}})$ *T has definably complete value group;*
- $(\mathbf{FR})$ *for every $\ell \in \mathbb{Z}_{>0}$, the interval $[0, \mathrm{v}(\ell)]$ is finite and $\mathbf{k}$ is perfect;*
- $(\mathbf{I}_{\mathbf{k}})$ *the residue field $\mathbf{k}$ is infinite;*
- $(\mathbf{E}^\infty_{\mathbf{k}})$ *the induced theory on $\mathbf{k}$ eliminates $\exists^\infty$.*

Then T weakly eliminates imaginaries in $\mathbf{K} \cup \mathbf{\Gamma}^{\mathrm{eq}} \cup \mathbf{R}^{\mathrm{leq}}$.

Proof. We will use Proposition 4.4.8. By Theorem 3.1.3, hypothesis **(D)** holds. Hypothesis $(\mathbf{Q}_{\mathbf{K}})$ holds trivially for $\mathcal{L}_1 = \mathcal{L}$ — and $f = \mathrm{id}$. Also, $\mathbf{RV}$ and $\mathbf{R}$ are stably embedded in characteristic zero henselian fields. Let $M \vDash T$, $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}^{\mathrm{eq}}(e)$. By Proposition 4.4.8, $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{K}(A) \cup (\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A))$.

Claim 6.1.2. $(\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{\Gamma}^{\mathrm{eq}}(A) \cup \mathbf{Lin}^{\mathrm{eq}}_{\mathcal{G}(A)}(A))$

Proof. If $T \supseteq \mathrm{Hen}_0^{\mathrm{ac}}$, then $\mathbf{RV}_n$ is $\mathcal{L}_{\mathrm{ac}}$ isomorphic to $\mathbf{R}_n^\times \times \mathbf{\Gamma}$ and the isomorphisms are compatible as n varies. It follows that $(\mathbf{RV} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}} \subseteq (\mathbf{\Gamma} \cup \mathbf{Lin}_{\mathcal{G}(A)})^{\mathrm{eq}}$, and since $\mathbf{\Gamma}$ and $\mathbf{Lin}_{\mathcal{G}(A)}$ are orthogonal, the claim follows.

If T is a $\mathbf{\Gamma}$-$\mathbf{R}$-enrichment of Hen_0, the claim follows from Proposition 5.3.1. Note that by $(\mathbf{C}_{\mathbf{\Gamma}})$, $\mathbf{\Gamma} \equiv \mathbb{Q}$ or $\mathbf{\Gamma} \equiv \mathbb{Z}$ and hence $\mathbf{\Gamma}/n\mathbf{\Gamma}$ is finite and every coset is represented in $\mathbf{\Gamma}(\mathrm{dcl}^{\mathrm{eq}}(\varnothing))$. □

Claim 6.1.3. $\mathbf{Lin}^{\mathrm{eq}}_{\mathcal{G}(A)}(A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{R}^{\mathrm{leq}}(A))$

Proof. Recall that $\mathbf{Lin}_{\mathcal{G}(A)}$ is stably embedded. It follows that, for every $e \in \mathbf{Lin}^{\mathrm{eq}}_{\mathcal{G}(A)}(A)$, taking tensor products of lattices, we may assume that there exists $n, \ell \in \mathbb{Z}_{>0}$ and $s \in \mathbf{S}_n(A)$ such that e codes some subset X_a of $s/\ell\mathfrak{m}s$ and a a single parameter in $s/\ell\mathfrak{m}s$. Since $s/\ell\mathfrak{m}s$ is definably isomorphic to $\mathbf{R}_\ell^n$ once we name a basis, it follows that X is definable with parameters in the $\mathcal{L}_{\mathrm{mod}}$-structure $(\mathbf{R}_\ell, s/\ell\mathfrak{m}s)$ — so $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{T}_{n,\ell,X}(A))$. □

It follows that $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{K}(A) \cup \mathbf{\Gamma}^{\mathrm{eq}}(A) \cup \mathbf{R}^{\mathrm{leq}}(A))$, which concludes the proof. □

Let $\mathbf{k}^{\mathrm{leq}} := \bigsqcup_{s,n,X} (\mathbf{V}/E_X)^{(\mathbf{k}, s/\mathfrak{m}s)}$.

Corollary 6.1.4. *Let F be a characteristic zero field that eliminates $\exists^\infty$. Then $F((t))$ and $F((t^{\mathbb{Q}}))$ (with or without angular components) weakly eliminate imaginaries in $\mathbf{K} \cup \mathbf{k}^{\mathrm{leq}}$.* □

In certain cases, the elimination of imaginaries in $\mathrm{Th}(\mathbf{k})$-linear structures, *cf.* [Hru12, Lemma 5.6, Theorem 5.10] and Proposition 2.4.20, allows to further reduce this result to the geometric sorts:

Corollary 6.1.5. *Let F be a characteristic zero field that is either algebraically closed, pseudofinite or real closed. Then $F((t))$ and $F((t^{\mathbb{Q}}))$ (with or without angular components) eliminate imaginaries in $\mathcal{G}$ — with constants for a Galois generator if F is pseudo-finite and a constant for a half line of $\mathbf{RV}_{\mathrm{v}(t)}$ in the $\mathbb{R}((t))$ case.* □

Not all of these results are new, although all of the statements with angular components are. The case of $\mathbb{C}((t^{\mathbb{Q}}))$ just amounts to Haskel-Hrushovski-Macpherson's result [HHM06] for ACVF. The case of $\mathbb{R}((t^{\mathbb{Q}}))$ is Mellor's result [Mel06] for RCVF and the case of $F((t))$, with F pseudo finite, is Hrushovski-Martin-Rideau's result [HMR18] for pseudo local fields — marginally improved since we only require constants in $\mathbf{k}$ and not in $\mathbf{K}$.

Corollary 6.1.6. *Let F be a positive characteristic perfect field that eliminates $\exists^\infty$. Then $\mathrm{W}(F)$ (with or without compatible angular components) weakly eliminate imaginaries in $\mathbf{K} \cup \mathbf{R}^{\mathrm{leq}}$.* □

It seems plausible that, if $F \vDash \mathrm{ACF}$, the $\mathbf{R}$-linear imaginaries can also be eliminated, yielding elimination of imaginaries in $\mathcal{G}$ for $\mathrm{W}(\mathbb{F}_p^{\mathrm{a}})$. However, this remains an open problem.

6.2. **The σ-henselian case.** Let us conclude with the description of the imaginaries in $\mathrm{VFA}^{\mathrm{mult}}_{0,0}$:

Theorem 6.2.1 (Theorem B). *The theory* $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$ *(with or without equivariant angular components) eliminates imaginaries in* $\boldsymbol{\mathcal{G}}$.

Proof. Any model of $\mathrm{VFA}_{0,0}^{\mathrm{mult}}$ is elementarily equivalent to a maximally complete one and hence $(\mathbf{C_B})$ holds. By Fact 2.3.7, the structure induced on $\mathbf{\Gamma}$ is *o*-minimal. So $(\mathbf{C_\Gamma})$ and $(\mathbf{E}^\infty_{\mathbf{\Gamma}})$ hold. Finally, $(\mathbf{E}^\infty_{\mathbf{k}})$ holds since ACFA eliminates $\exists^\infty$. By Theorem 3.1.1, $(\mathbf{D})$ holds. Hypothesis $(\mathbf{Q_K})$ follows from Fact 2.3.4, with $\mathcal{L}_1 := \mathcal{L}_{\mathbf{RV}} \cup \{\sigma_{\mathbf{RV}}\}$ and $f(x) := (\sigma^n(c))_{n\in\mathbb{Z}_{\geqslant 0}}$. So, by Proposition 4.4.8, for every $M \vDash \mathrm{VFA}_{0,0}^{\mathrm{mult}}$, $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}^{\mathrm{eq}}(e)$, we have $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{K}(A) \cup (\mathbf{RV} \cup \mathbf{Lin}_{\boldsymbol{\mathcal{G}}(A)})^{\mathrm{eq}}(A))$. By Proposition 5.3.2 — or using the angular components — we have $(\mathbf{RV} \cup \mathbf{Lin}_{\boldsymbol{\mathcal{G}}(A)})^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\boldsymbol{\mathcal{G}}(A))$. ☐

Similar results hold in differential valued fields.

Corollary 6.2.2. *The following two families of difference valued fields, indexed by integer primes* p*:*

1. $K_p := (\mathbb{F}_p(t)^{\mathrm{a}}, \mathrm{v}_t, \phi_p)$, *where* ϕ_p *is the Frobenius automorphism;*
2. $K_p := (\mathbb{C}_p, \mathrm{v}_p, \sigma_p)$, *where* σ_p *is an isometric lift of the Frobenius automorphism on* $\mathbf{k}(\mathbb{C}_p) = \mathbb{F}_p^{\mathrm{a}}$.

uniformly eliminate imaginaries in $\boldsymbol{\mathcal{G}}$ *for large* p*: for every* $\mathcal{L}^\sigma_{\mathbf{RV}}$*-definable sets* $X \subseteq Y \times Z$*, there exists an* $\mathcal{L}^\sigma_{\mathbf{RV}}$*-definable map* $f : Z \to W$*, where* W *is a product of sorts in* $\boldsymbol{\mathcal{G}}$ *and some* $N \in \mathbb{Z}_{\geqslant 0}$ *such that, for every prime* $p > N$ *and* $z_1, z_2 \in Z(K_p)$*,* $f(z_1) = f(z_2)$ *iff and only if* $X_{z_1}(K_p) = X_{z_1}(K_p)$. ☐

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