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Generic automorphisms and green fields

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Abstract

We show that the generic automorphism is axiomatizable in the green field of Poizat (once Morleyized) as well as in the bad fields that are obtained by collapsing this green field to finite Morley rank. As a corollary, we obtain 'bad pseudofinite fields' in characteristic 0. In both cases, we give geometric axioms. In fact, a general framework is presented allowing this kind of axiomatization. We deduce from various constructibility results for algebraic varieties in characteristic 0 that the green and bad fields fall into this framework. Finally, we give similar results for other theories obtained by Hrushovski amalgamation, for example, the free fusion of two strongly minimal theories having the definable multiplicity property. We also close a gap in the construction of the bad field, showing that the codes may be chosen to be families of strongly minimal sets.

1. *Introduction*

For more than two decades now, new and often unexpected stable structures have been constructed using Hrushovski's amalgamation method, starting in 1988 when Hrushovski obtained a strongly minimal theory that violated Zilber's trichotomy conjecture (see [**14**]). This construction is called the *ab initio* case. The *fusion* of two strongly minimal structures having the definable multiplicity property (DMP) (that is, definable Morley degrees) into a single one [**13**] then showed that the realm of strongly minimal theories is vast, even when one only looks at strongly minimal expansions of algebraically closed fields.

Poizat's *bicoloured fields* are expansions of algebraically closed fields by a new predicate. The black fields (where a new subset is added) answer a question of Berline and Lascar about possible ranks of superstable fields [**23**]. The construction of the green fields, algebraically closed fields of characteristic 0 with a proper subgroup of the multiplicative group of the field, requires non-trivial results from algebraic geometry, in order to establish the relevant definability properties needed for the amalgamation construction to work [**24**]. Poizat's green fields are infinite rank analogues of so-called *bad fields*, fields of finite Morley rank with a definable proper infinite subgroup of the multiplicative group. In positive characteristic, bad fields are very unlikely to exist, by a result of Wagner [**28**]. Their absence would have simplified the study of groups of finite Morley rank, in particular that of infinite simple groups of finite Morley rank which according to Cherlin–Zilber's Algebraicity Conjecture should be algebraic groups. Baudisch, Martin Pizarro, Wagner and the author showed in [**3**] that Poizat's green field may be collapsed into a bad field.

The positivity of the predimension is one of the key features of Hrushovski's amalgamation method. Zilber suggested that one interprets this as a *generalized Schanuel condition*, due to the analogy with Schanuel's Conjecture (SC) which asserts that, for any $\mathbb{Q}$-linearly independent tuple $(y_1, \ldots, y_n)$ of complex numbers one has $\operatorname{tr.deg}(y_1, \ldots, y_n, e^{y_1}, \ldots, e^{y_n}/\mathbb{Q}) \geqslant n$. This conjecture is wide open. Ax [**1**] proved a differential version of it. In the case of the green fields, the analogy raised by Zilber is supported by two facts. On the one hand, assuming (SC), Zilber constructs a natural model of the theory of the green field of Poizat, with universe

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the complex numbers, and which has an 'analytic' flavour. On the other hand, Ax's result, or rather a consequence thereof, a weak version of Zilber's *Conjecture on Intersections with* Tori (*weak CIT*), is essentially used in the construction by Poizat. Weak CIT is a finiteness result on intersections of algebraic varieties with cosets of tori in characteristic 0 which allows to control atypical components of such intersections, that is, those having a greater dimension than the expected one.

If T is a stable model-complete theory, then one may build the theory T_σ of models of T with a distinguished automorphism. It is an interesting question to determine whether T_σ admits a model-companion. If it does, then we denote it by TA and say that the generic automorphism is axiomatizable in T. The geometric model theory of TA (paradigmatically that of $ACFA$ in [**6**]) has proved to be a powerful tool when applied to problems in algebraic geometry, number theory and algebraic dynamics (see, for example, [**7**, **8**, **15**, **17**, **25**]). Whether TA exists or not is a test question on how well one definably controls 'multiplicities' in T. Existence of $ACFA$, for example, is an easy consequence of the fact that being irreducible is definable in families of algebraic varieties; the abstract analogue of this for a theory of finite Morley rank is the DMP.

For many structures obtained by Hrushovski amalgamation (when definably expanded to a language in which they become model-complete), it is quite elementary to show that the generic automorphism is axiomatizable, using so-called 'geometric axioms'. However, in the green fields of Poizat and in the bad fields, using just weak CIT one only gains good definable control of dimension and rank. In this vein, there is the result of Evans that the green fields do not have the finite cover property [**10**]. But in order to axiomatize the generic automorphism, we also need a definable control of 'multiplicities'. There are difficulties related to a necessary choice of green roots, and Kummer theory comes into the picture.

The paper is organized as follows. In Section 2, we present a framework for 'geometric axioms' for TA in the case where T is a stable, complete and model-complete theory: we show in Proposition 2.5 that such an axiomatization may be given if T admits a *geometric notion of genericity* (see Definition 2.4). We then review the construction of the green fields of Poizat and of the bad fields, including the relevant uniformity results from algebraic geometry used in the course of the construction (Section 3).

Section 4 is devoted to the proof of a definability result in characteristic 0. We show that being Kummer generic is definable for algebraic varieties V among an algebraic family (Proposition 4.5), where Kummer genericity of V is a property defined in terms of Kummer extensions of the field of rational functions $K(V)$. Definability of Kummer genericity is then used to overcome the difficulties related to the choice of green roots that were mentioned above. This also enables us to close a gap in the construction of the bad field which had been observed by Roche (see Corollary 4.8).

In Section 5, we use the definability of Kummer genericity to prove the main results of the paper, namely that the generic automorphism is axiomatizable in the green field of Poizat (Theorem 5.2) and in the bad fields (Theorem 5.5). From the latter result, passing to the fixed structure, we deduce the existence of 'bad pseudofinite fields' (Corollary 5.6).

Finally, in the last section, we mention existence results of TA for other theories obtained by Hrushovski amalgamation. The common feature is that a geometric notion of genericity may be exhibited in these theories in a straightforward way, using the respective 'base theories'. A full proof is presented in the case of the free fusion of two strongly minimal theories having DMP. The section also includes a brief review of Hrushovski's amalgamation method.

2. *Generic automorphisms of stable theories*

Let T be a complete $\mathcal{L}$-theory, and let $\mathcal{L}_\sigma = \mathcal{L} \cup \{ \sigma\}$, where σ is a new unary function symbol. We consider the $\mathcal{L}_\sigma$-theory T_σ obtained by adding to T axioms expressing that σ is an

$\mathcal{L}$-automorphism of the corresponding model of T. If T is model-complete, it follows that T_σ is an inductive theory, so it has a model-companion (which we denote by TA if it exists) if and only if the class of its existentially closed models is elementary. In this case, we say that *the generic automorphism is axiomatizable in* T, or that TA *exists*.

If T is an arbitrary complete theory, then we say that the generic automorphism is axiomatizable in T if this holds for some expansion by definitions T^* of T, which is model-complete. This does not depend on the choice of T^*, and so we may as well assume that T^* eliminates quantifiers, by taking the Morleyization of T. Hence, we really deal with some kind of relative existence of a model-companion.

If TA exists for some stable theory T, then all its completions are simple (see Fact 2.1), and in general unstable. The reader may consult [**26**] for a survey on simple theories, although we shall make no real use of them in the present paper.

Some notation: in any model (M,σ) of T_σ we denote by $\mathrm{acl}(A)$ the algebraic closure of A in the sense of $M^{\mathrm{eq}} \models T^{\mathrm{eq}}$, and by $\mathrm{acl}_\sigma(A)$ the set $\mathrm{acl}(\bigcup_{z\in\mathbb{Z}} \sigma^z(A))$, a subset of M^{eq} that is easily seen to be closed under (the induced actions on M^{eq} of) σ and σ^{-1}, and algebraically closed in the sense of T^{eq}.

2.1. *Some known results*

If T has the strict order property, then TA does not exist [**19**]. Kikyo and Pillay [**18**] conjectured that the existence of TA implies that T is stable. In the following, we concentrate on stable theories. For the rest of this section, *we assume that* T *is complete, model-complete and stable*. Fact 2.1 lists some basic results shown by Chatzidakis and Pillay [**9**].

FACT 2.1. Let T be a stable complete theory with quantifier elimination, such that TA exists.

(1) The algebraic closure in models of TA is given by acl_σ.

(2) For $A_i \subseteq M_i \models TA$, $i = 1, 2$, one has $A_1 \equiv_{\mathcal{L}_\sigma} A_2$ if and only if $\mathrm{acl}_\sigma(A_1) \cong_{\mathcal{L}_\sigma} \mathrm{acl}_\sigma(A_2)$ (over the map sending A_1 to A_2). In particular, the completions of TA are given by the $\mathcal{L}_\sigma$-isomorphism types of $\mathrm{acl}(\emptyset) = \mathrm{acl}_\sigma(\emptyset)$.

(3) Any completion $\tilde{T}$ of TA is simple (supersimple if T is superstable), and the following characterization of non-forking holds:

$$A \mathop{\downarrow}^{\tilde{T}}_{B} C \Leftrightarrow \mathrm{acl}_\sigma(AB) \mathop{\downarrow}^{T}_{\mathrm{acl}_\sigma(B)} \mathrm{acl}_\sigma(BC).$$

(4) Assume in addition that T eliminates imaginaries and that any algebraically closed set is a model of T. Then any completion of TA eliminates imaginaries, and the definable set $F = \mathrm{Fix}(\sigma) = \{m \in M \mid \sigma(m) = m\}$ is stably embedded in M.

The existence of TA may be considered as a (very nice) property of the initial theory T. Kudaiberganov observed that for a stable theory T, this implies T does not have the finite cover property (that is, is nfcp). Baldwin and Shelah [**2**] gave an abstract characterization of those stable theories T for which TA exists. (It consists of a strengthening of nfcp, is purely in terms of T and uses Δ-types.) In this direction, one may also mention the following result due to Hasson and Hrushovski.

FACT 2.2 [**12**]. Let T be a strongly minimal theory. Then TA exists if and only if T has the DMP.

Recall that a theory T of finite Morley rank has the *DMP* if, for any pair of natural numbers (r,d) and any formula $\varphi(\bar{x},\bar{b})$ with $\mathrm{MRD}(\varphi(\bar{x},\bar{b}))=(r,d)$, there exists $\theta(\bar{z})\in \mathrm{tp}(\bar{b})$, such that $\mathrm{MRD}(\varphi(\bar{x},\bar{b}'))=(r,d)$ whenever $\models\theta(\bar{b}')$. (See [13] for a discussion of the DMP.)

2.2. *A framework for geometric axioms*

The framework we present here allows to unify existing proofs showing that TA exists for specific stable theories T. The common feature of these proofs is the axiomatization of TA in terms of what is called 'geometric axioms'. In a way, we give in what follows a general principle to organize such proofs. Compared with the characterization of stable complete theories in which the generic automorphism is axiomatizable given in [2], the criterion we present is of a more 'geometric' nature, since it brings global considerations into play.

Before we give definitions, let us start with a motivating example. If $(M,\sigma)\models T_\sigma$ and $X\subseteq M^n$ is $\mathcal{L}_M$-definable, say $X=\varphi(M,\bar{b})$ for some $\mathcal{L}$-formula $\varphi(\bar{x},\bar{y})$ and $\bar{b}\in M$, let $X^\sigma=\{\sigma(\bar{c})\in M^n \mid \bar{c}\in X\}=\varphi(M,\sigma(\bar{b}))$. Clearly, $\mathrm{MRD}(X)=\mathrm{MRD}(X^\sigma)$.

Example 2.3 [9]. Let T be a theory of finite Morley rank with DMP and such that MR is additive: for all $\bar{a},\bar{b}$ and C one has $\mathrm{MR}(\bar{a}\bar{b}/C)=\mathrm{MR}(\bar{a}/\bar{b}C)+\mathrm{MR}(\bar{b}/C)$.

Then TA exists. More precisely, if $(M,\sigma)\models T_\sigma$, then (M,σ) is existentially closed (that is, a model of TA) if and only if the following condition holds:

$(*)$ Assume that $X\subseteq M^n$ and $Y\subseteq X\times X^\sigma$ are $\mathcal{L}_M$-definable sets of Morley degree 1, such that if $(\bar{a},\bar{a}')$ is generic in Y (over M), then $\bar{a}$ is generic in X and $\bar{a}'$ is generic in X^σ. Then there exists $\bar{c}\in M^n$, such that $(\bar{c},\sigma(\bar{c}))\in Y$.

If $f:Y\to X$ is a definable function, with $\mathrm{MRD}(Y)=(n,1)$, $\mathrm{MRD}(X)=(m,1)$, then, by the additivity of MR, f maps the generic type of Y to the generic type of X if and only if $\mathrm{MR}(\{\bar{a}\in X\mid \mathrm{MR}(f^{-1}(\bar{a}))=n-m\})=m$. Since MR and MD are definable in T by assumption, this shows that the condition $(*)$ may be expressed in a first-order way.

Let $R_\mathfrak{g}$ be a relation defined on pairs of the form $(p(\bar{x}),\varphi(\bar{x}))$, where $p(\bar{x})\in S(M)$ is a finitary type over some model $M\models T$ and $\varphi(\bar{x},\bar{b})\in p$ is a formula.

(1) If (p,φ) is in $R_\mathfrak{g}$, then we say that p *is generic in* φ. A tuple $\bar{a}\in N\succcurlyeq M$ is *generic in* φ *over* M (where φ is a formula with parameters from the model M) if $p:=\mathrm{tp}(\bar{a}/M)$ is generic in φ, that is, if the pair (p,φ) is in $R_\mathfrak{g}$.

(2) A formula $\psi(\bar{x},\bar{z})$ (without parameters) is called *nice* if, for any $\bar{b}$ with $\psi(\bar{x},\bar{b})\neq\emptyset$ and any model M containing $\bar{b}$, there is a unique type $p\in S(M)$ which is generic in $\psi(\bar{x},\bar{b})$. A formula $\psi(\bar{x},\bar{b})\neq\emptyset$ is called nice if $\psi(\bar{x},\bar{z})$ is nice.

(3) A type $p\in S(M)$ is *nice* if $N(p)\vdash p$ holds, where

$$N(p):=\{\psi(\bar{x},\bar{b}) \text{ nice } \mid p \text{ is generic in } \psi(\bar{x},\bar{b})\}.$$

Definition 2.4. The relation $R_\mathfrak{g}$ is a *geometric notion of genericity* (for T) if the following properties hold:

(1) *Invariance.* $R_\mathfrak{g}$ is invariant under automorphisms.
(2) *Coherence.* Let $p\in S(N)$, $M\preccurlyeq N$ and φ be a nice formula with parameters from M. Then the (unique) generic type of φ over N restricts to the generic type of φ over M.
(3) *Enough nice types.* For every n-type p_0 over some model M, there is (for some m) a nice type $p\in S_{n+m}(M)$, such that $\pi(p)=p_0$, where π is the natural projection $S_{n+m}(M)\to S_n(M)$.
(4) *Definability of generic projections.* Let $\bar{x}\supseteq\bar{x}_1$, and let $\psi(\bar{x},\bar{b})$ and $\varphi(\bar{x}_1,\bar{b}_1)$ be nice formulas, with generic types $p(\bar{x})$ and $p_1(\bar{x}_1)$ in $S(M)$, respectively. Assume that $\models$

$\psi(\bar{x}, \bar{b}) \to \varphi(\bar{x}_1, \bar{b}_1)$ and $\pi_1(p) = p_1$. Then there is $\theta(\bar{z}, \bar{z}_1) \in \mathrm{tp}(\bar{b}\bar{b}_1)$, such that, for all $\bar{b}'\bar{b}'_1 \models \theta(\bar{z}, \bar{z}_1)$, one has $\psi(\bar{x}, \bar{b}') \neq \emptyset$, $\models \psi(\bar{x}, \bar{b}') \to \varphi(\bar{x}_1, \bar{b}'_1)$ and the unique generic type p' of $\psi(\bar{x}, \bar{b}')$ projects onto the unique generic type p'_1 of $\varphi(\bar{x}_1, \bar{b}'_1)$.

In a stable theory the non-forking extension of a stationary type is the unique extension which is invariant under automorphisms. Thus, for a nice formula $\varphi(\bar{x}, \bar{b})$ and $\bar{b} \in M \preccurlyeq N$, the generic type of φ over N is the non-forking extension of the generic type over M.

Here is the result the notion is made for.

Proposition 2.5. *Suppose that T admits a geometric notion of genericity $R_{\mathfrak{g}}$. Then TA exists.*

Proof. We give 'geometric axioms' for TA, using the relation $R_{\mathfrak{g}}$. Let $(M, \sigma) \models T_\sigma$ and $\tilde{p}(\bar{x}, \bar{x}', \bar{x}_r) \in S_{2n+k}(M)$ be a nice type restricting to nice types $p(\bar{x})$ and $p'(\bar{x}')$ in $S_n(M)$, such that $p' = \sigma(p)$. Let $\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b}) \in N(\tilde{p})$, $\varphi(\bar{x}, \bar{b}) \in N(p)$ and thus (by invariance) $\varphi(\bar{x}', \sigma(\bar{b})) \in N(p')$, such that

$$\models \psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b}) \to \varphi(\bar{x}, \bar{b}) \wedge \varphi(\bar{x}', \sigma(\bar{b})).$$

Moreover, let $\theta(\tilde{z}, \bar{z})$ and $\theta'(\tilde{z}, \bar{z}')$ be given by property (4) applied to the pairs of formulas $(\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b}), \varphi(\bar{x}, \bar{b}))$ and $(\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b}), \varphi(\bar{x}', \sigma(\bar{b})))$. Put $\Theta(\tilde{z}, \bar{z}) := \theta(\tilde{z}, \bar{z}) \wedge \theta'(\tilde{z}, \sigma(\bar{z}))$. The corresponding axiom for this choice of formulas is

$$\forall \tilde{z}\bar{z} \exists \bar{x}\, \bar{x}_r [\Theta(\tilde{z}, \bar{z}) \to \psi(\bar{x}, \sigma(\bar{x}), \bar{x}_r, \tilde{z})].$$

We call these axioms the *geometric axioms*. We will show that a model of T_σ is existentially closed if and only if it satisfies the geometric axioms. This is straightforward and will complete the proof.

Let (M, σ) be an existentially closed model of T_σ, and suppose that $(M, \sigma) \models \Theta(\tilde{b}, \bar{b})$. This means that the unique generic type $\tilde{p}(\bar{x}, \bar{x}', \bar{x}_r)$ of $\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b})$ (over M) restricts to the unique generic types p and $p' = p^\sigma$ of $\varphi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}', \sigma(\bar{b}))$, respectively ($p' = \sigma(p)$ being a consequence of invariance). Choose $\tilde{a} = (\bar{a}, \bar{a}', \bar{a}_r) \models \tilde{p}$ (with $\tilde{a}$ from some $M^* \succcurlyeq_{\mathcal{L}} M$). So $\bar{a} \models p$ and $\bar{a}' \models p^\sigma$. Going to some extension of M^* if necessary, we may thus assume that there is $\sigma^* \in \mathrm{Aut}(M^*)$ extending σ, such that $\sigma^*(\bar{a}) = \bar{a}'$. In particular, $(M^*, \sigma^*) \models \exists \bar{x}\, \bar{x}_r \psi(\bar{x}, \sigma(\bar{x}), \bar{x}_r, \tilde{b})$. So the same is true in (M, σ), as this is an existentially closed model, and (M, σ) satisfies the geometric axiom corresponding to ψ and φ.

Conversely, let (M, σ) be a model of T_σ together with all geometric axioms. Let $(M, \sigma) \subseteq (N, \sigma) \models T_\sigma$ and $\bar{a}_0 \in N$, satisfying some quantifier-free $\mathcal{L}_\sigma$-formula with parameters from M. A standard reduction shows that we may assume that this formula is of the form $\chi(\bar{x}_0, \sigma(\bar{x}_0), \bar{b}_0)$, where $\chi(\bar{x}_0, \bar{x}'_0, \bar{z}_0)$ is an $\mathcal{L}$-formula (without quantifiers). Put $p_0 := \mathrm{tp}_{\mathcal{L}}(\bar{a}_0/M)$ and $p'_0 = p_0^\sigma := \mathrm{tp}_{\mathcal{L}}(\sigma(\bar{a}_0)/M) \in S_n(M)$. By condition (3) in Definition 2.4, there is a nice type $p \in S_{n+m}(M)$ restricting to p_0. Replacing (N, σ) by some extension $(\tilde{N}, \tilde{\sigma})$ if necessary, we may assume that there exists a tuple $\bar{a} \in N$ containing $\bar{a}_0$, such that $\models p(\bar{a})$. So $\sigma(\bar{a}) \supseteq \sigma(\bar{a}_0)$. Again by (3), applied to $\mathrm{tp}_{\mathcal{L}}(\bar{a}, \sigma(\bar{a})/M)$, we may choose a nice type $\tilde{p}(\bar{x}, \bar{x}', \bar{x}_r) \in S(M)$ restricting to $p(\bar{x})$ and $p' = p^\sigma$, respectively.

Now we choose some arbitrary $\varphi(\bar{x}, \bar{b}) \in N(p)$, then we choose $\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b}) \in N(\tilde{p})$, such that $\psi(\bar{x}, \bar{x}', \bar{x}_r, \tilde{b})$ implies $\chi(\bar{x}_0, \bar{x}'_0, \bar{b}_0) \wedge \varphi(\bar{x}, \bar{b}) \wedge \varphi(\bar{x}', \sigma(\bar{b}))$ (this is possible since $N(\tilde{p}) \vdash \tilde{p}$).

Since $(M, \sigma) \models \Theta_{\psi,\varphi}(\tilde{b}, \bar{b})$, the corresponding axiom ensures that there are tuples $\bar{\alpha}, \bar{\alpha}_r \in M$, such that $M \models \psi(\bar{\alpha}, \sigma(\bar{\alpha}), \bar{\alpha}_r, \tilde{b})$. In particular, $(M, \sigma) \models \chi(\bar{\alpha}_0, \sigma(\bar{\alpha}_0), \bar{b}_0)$, where $\bar{\alpha}_0$ denotes the appropriate subtuple of $\bar{\alpha}$. This shows that (M, σ) is an existentially closed model. $\square$

Example 2.6. The following known proofs of existence of TA are instances of Proposition 2.5. The first two examples are from [**9**], the third example from [**4**].

(1) Let T be a theory of finite additive Morley rank with DMP (see Example 2.3).

Genericity with respect to MR gives rise to a geometric notion of genericity. Nice formulas correspond to formulas with all instances of degree 1. Properties (1) and (2) from Definition 2.4 are easily verified, (4) follows from the additivity of MR combined with the DMP as indicated in Example 2.3, and (3) is a consequence of the DMP (this is a degenerate case since all types over models are nice).

(2) Let T be the theory of a totally transcendental module, or more generally a complete theory of a one-based group G that is totally transcendental. Any definable subset of G^n is given by a boolean combination of cosets of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable (connected) subgroups of G^n (see, for example, [**22**, Corollary 4.4.6]); any strong type p is the (unique) generic type of a coset of its stabilizer $\mathrm{Stab}(p)$.

It is straightforward to check that genericity with respect to Morley rank (or with respect to stable group theory; this amounts to the same in this context) gives rise to a geometric notion of genericity. Nice formulas are formulas with all instances of Morley degree 1.

(3) Let DCF_0 be the theory of differentially closed fields in characteristic 0. It is shown in [**4**, Corollary 2.15] (rephrased in our terminology) that *D-genericity* with respect to the Kolchin topology is a geometric notion of genericity.

3. *Green fields*

We present in this section a sketch of Poizat's construction of a *green field* in characteristic 0 (see [**24**]) as well as the construction of a *bad field* [**3**]. The green field is obtained using Hrushovski's amalgamation method (without collapse), whereas the bad field is constructed by collapsing the former to a field of finite Morley rank. (We refer to Section 6 for a more systematic treatment of this amalgamation method.)

In both constructions, uniformity results for intersections of tori with algebraic varieties in characteristic 0 have to be used in order to establish the necessary definability properties which make Hrushovski's amalgamation method work. We recall these uniformity results (called *weak CIT*) since they will be used in our construction of a geometric notion of genericity in the green field and also in the proof that the bad fields have the DMP (see Section 5).

3.1. *Dimension, codimension and predimension*

In the following, we gather the results that will be needed to get a definable control on the (pre-)dimension in the green fields.

Let us fix some notation (mainly following [**3**]): $\mathbb{C}$ denotes an algebraically closed field of characteristic 0. A variety V will always be a closed subvariety of some $\mathbb{G}_m^n$ (which may be identified with the set $(\mathbb{C}^*)^n$ of its $\mathbb{C}$-rational points). A *torus* is a connected algebraic subgroup of $\mathbb{G}_m^n$. It is described by finitely many equations of the form $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 1$, where $r_i \in \mathbb{Z}$. If T is a torus and $\bar{a}$ is generic in T over $\mathbb{C}$, then the $\mathbb{Q}$-linear dimension of $\bar{a}$ over $\mathbb{C}^*$ (modulo torsion) equals the algebraic dimension of T (as a variety) and will be denoted by $\mathrm{l.dim}_{\mathbb{Q}}(T)$ or $\dim(T)$. Given a closed and irreducible subvariety W in $\mathbb{G}_m^n$, its *minimal torus* is the smallest torus T, such that W lies in some coset $\bar{a} \cdot T$. In this case, we define its *codimension* $\mathrm{cd}(W) := \dim(T) - \dim(W) = \mathrm{l.dim}_{\mathbb{Q}}(W) - \dim(W)$, where $\mathrm{l.dim}_{\mathbb{Q}}(W) := \dim(T)$. The *predimension* of W is given by $\delta(W) := 2\dim(W) - \dim(T) = \dim(W) - \mathrm{cd}(W)$.

An irreducible subvariety $W \subseteq V$ is cd-*maximal in* V if $\mathrm{cd}(W') > \mathrm{cd}(W)$ for every irreducible subvariety $W \subsetneq W' \subseteq V$. Clearly, irreducible components of V and cosets of tori maximally contained in V are examples of cd-maximal subvarieties.

We now present a result that was stated by Poizat [**24**, Corollaire 3.7]. It is a reformulation of a result proved by Zilber [**30**] (and later generalized by Kirby [**20**] to the context of semiabelian varieties).

FACT 3.1. Let $\mathcal{V} = \{V_{\bar{b}} \mid \bar{b} \models \theta(\bar{z})\}$ be a uniformly definable family of closed subvarieties of $\mathbb{G}_m^n$. There exists a finite collection of tori $\mathcal{T}(\mathcal{V}) = \{T_0, \ldots, T_r\}$, such that, for any member $V_{\bar{b}}$ of the family and any cd-maximal subvariety W of $V_{\bar{b}}$, the minimal torus of W belongs to $\mathcal{T}(\mathcal{V})$.

This property, which Zilber called *weak CIT*, is at the origin of a series of definability results, as we shall see in what follows.

A matrix $M = (m_{i,j}) \in \mathrm{Mat}(n \times n, \mathbb{Z})$ acts on $\mathbb{G}_m^n$. For $\bar{a} \in \mathbb{G}_m^n$ we put

$$\bar{a}^M := \left(\prod_{j=1}^n a_j^{m_{1,j}}, \ldots, \prod_{j=1}^n a_j^{m_{n,j}} \right).$$

DEFINITION 3.2. Let $V \subseteq \mathbb{G}_m^n$ be an irreducible variety defined over the algebraically closed field K. The variety V is called *free* if its minimal torus is equal to $\mathbb{G}_m^n$. It is called *rotund* if it is free and if, for any K-generic tuple $\bar{a}$ in V and any $M \in \mathrm{Mat}(n \times n, \mathbb{Z})$, putting $W := \mathrm{locus}(\bar{a}^M/K)$, one has $\delta(W) \geqslant 0$ (In Zilber's terminology [**31**], our notion of rotund corresponds to 'G-normal' and 'G-free'; the term 'rotund' is taken from [**20**]. Since we only use rotund varieties that are free as well, we include the freeness condition in our definition.).

A property $\mathcal{P}$ of algebraic varieties is called *definable* if, for any uniformly definable family of algebraic varieties $\mathcal{V} = \{V_{\bar{b}} \mid \bar{b} \models \theta(\bar{z})\}$, the set of parameters $\bar{b}$, such that $V_{\bar{b}}$ has the property $\mathcal{P}$ is definable.

FACT 3.3 [**20**]. (1) Freeness is a definable property.
(2) Rotundity is a definable property.

Proof. For convenience, we include the argument. Let $V_{\bar{b}}$ be an instance of a uniformly definable family $\mathcal{V}$ of irreducible varieties in $\mathbb{G}_m^n$. Since the minimal torus of $V_{\bar{b}}$ lies in the finite collection of tori $\{T_0, \ldots, T_r\}$ attached to $\mathcal{V}$, it is sufficient to avoid all $T_i \neq \mathbb{G}_m^n$ from this collection to force the minimal torus of $V_{\bar{b}}$ to be equal to $\mathbb{G}_m^n$. This can be done definably and shows (1).

To prove (2), we may assume that the dimension of $V_{\bar{b}}$ is equal to k throughout the family, and that all instances are free. Thus, $2k - n = d \geqslant 0$, and it suffices to impose the following crucial condition:

(∗) *For generic* $\bar{g} \in V_{\bar{b}}$ *and* $T \in \mathcal{T}(\mathcal{V})$ *and any irreducible component* W *of* $V_{\bar{b}} \cap \bar{g} \cdot T$ *of maximal dimension,* $\dim(W) - \mathrm{cd}(W) \leqslant d$ *holds.*

The finiteness of $\mathcal{T}(\mathcal{V})$ implies that cd is definable. It is well known that dim is definable as well. Using definability of types in ACF, it follows that (∗) is a definable condition. It is not hard to see that (∗) is enough to guarantee the rotundity of $V_{\bar{b}}$ (cf. the proof of [**3**, Lemma 4.3]). □

Let us mention that freeness is also a definable property in positive characteristic. One may prove this using Zilber's Indecomposibility Theorem. (We thank Martin Bays for pointing this out to us.)

Lemma 3.4. *Let V be an irreducible subvariety of $\mathbb{G}_m^n$ and let $\mathcal{T}(V)$ be the finite family of tori given in Fact* 3.1. *Assume that V is free. Let $W \subsetneq V$ be a proper irreducible subvariety, such that $\delta(W) \geqslant \delta(V)$.*

Then the minimal torus of W is contained in some $T \in \mathcal{T}(V)$ with $T \subsetneq \mathbb{G}_m^n$.

Proof. From $\dim(W) < \dim(V)$ and $\delta(W) \geqslant \delta(V)$, we infer $\mathrm{cd}(W) < \mathrm{cd}(V)$. Let W' be cd-maximal, such that $W \subseteq W' \subseteq V$ and $\mathrm{cd}(W') \leqslant \mathrm{cd}(W)$. The minimal torus of W is contained in the minimal torus T of W'. Clearly, T is a proper subtorus of $\mathbb{G}_m^n$; moreover, $T \in \mathcal{T}(V)$ by Fact 3.1. □

3.2. *Green colour, green fields of Poizat and bad fields*

We now recall the construction of the green field of Poizat [**24**] and of the bad field [**3**]. We expand the language of rings by a new unary predicate $\ddot{U}$ and thus work in $\mathcal{L} = \mathcal{L}_{\text{rings}} \cup \{\ddot{U}\}$. Elements in $\ddot{U}$ will be called *green*, those not in $\ddot{U}$ are *white*. We consider $\mathcal{L}$-structures of the form $K = (K, +, -, \times, 0, 1, \ddot{U}(K))$, such that K is an algebraically closed field of characteristic 0 and $\ddot{U}(K)$ a subgroup of the multiplicative group $K^\times$ which is divisible and torsion-free. So $\ddot{U}$ is a vector space over $\mathbb{Q}$. If we write $K \subseteq L$, then we mean that K is an $\mathcal{L}$-substructure of L, in particular $\ddot{U}(L) \cap K = \ddot{U}(K)$.

We call K a *green field* if $\delta(k) = 2\,\mathrm{tr.deg}(k) - \mathrm{l.dim}_{\mathbb{Q}}(\ddot{U}(k)) \geqslant 0$ for every $k \subseteq K$ of finite transcendence degree. Here $\delta(k)$ is called the *predimension* of $(k, \ddot{U}(k))$. More generally, for $K \subseteq L$, such that $\mathrm{tr.deg}(L/K)$ and $\mathrm{l.dim}_{\mathbb{Q}}(\ddot{U}(L)/\ddot{U}(K))$ are finite, we put

$$\delta(L/K) = 2\,\mathrm{tr.deg}(L/K) - \mathrm{l.dim}_{\mathbb{Q}}(\ddot{U}(L)/\ddot{U}(K)).$$

An extension $K \subseteq L$ of green fields is called *self-sufficient* if $\delta(L'/K) \geqslant 0$ for every green field L', such that $K \subseteq L' \subseteq L$ and $\mathrm{tr.deg}(L'/K)$ is finite; we write $K \leqslant L$ if this holds. If $A \subseteq L$ is any subset, then there is a minimal (with respect to inclusion) green field K', such that $A \subseteq K' \leqslant L$; this is called the *self-sufficient closure of A in L* and denoted by $\mathrm{cl}_\omega(A) = \mathrm{cl}_\omega^L(A)$. Note that this notion depends on L, but often we omit the superscript if L is clear from the context. If A contains a $\mathbb{Q}$-basis of the green points of its self-sufficient closure in L, then we also write $A \leqslant L$ (by a slight abuse of notation), and A is called a *self-sufficient subset* of L. Note that in this case $\mathrm{cl}_\omega^L(A)$ is given by A^{alg}, the algebraic closure of A in the field sense.

If $\bar{a}$ is a finite tuple from L and $B \subseteq L$, then the *dimension of $\bar{a}$ over B* is given by $\mathrm{d}(\bar{a}/B) = \mathrm{d}^L(\bar{a}/B) = \delta(\mathrm{cl}_\omega(B\bar{a})/\mathrm{cl}_\omega(B))$. Note that if $K \subseteq A \subseteq L$ for some $K \leqslant L$ with $\mathrm{tr.deg}(A/K) < \infty$, then $\mathrm{tr.deg}(\mathrm{cl}_\omega(A)/K) < \infty$ as well.

The following lemma is a direct consequence of the definitions.

Lemma 3.5. *Let $K \subseteq L$ be an extension of green fields. Assume that $(g_1, \ldots, g_n)$ is a basis of $\ddot{U}(L)$ over $\ddot{U}(K)$. Then $K \leqslant L$ if and only if* $\mathrm{locus}(\bar{g}/K)$ *is rotund.*

Let $(\mathcal{C}_0, \leqslant)$ be the class of green fields with self-sufficient embeddings.

Fact 3.6 [**24**]. (a) The class $\mathcal{C}_0$ is elementary, and $(\mathcal{C}_0, \leqslant)$ has the amalgamation property (AP) and the joint embedding property (JEP). Moreover, up to $\mathcal{L}$-isomorphism, the subclass $\mathcal{C}_0^{\mathrm{fin}}$ of green fields of finite transcendence degree is countable.

(b) Let K_ω be the Fraïssé–Hrushovski limit of $(\mathcal{C}_0^{\mathrm{fin}}, \leqslant)$. Then K_ω is a saturated model of its $\mathcal{L}$-theory T_ω.

(c) The algebraic closure in T_ω equals the self-sufficient closure.

(d) Let $A, A' \subseteq K \models T_\omega$. Then $\mathrm{tp}_{T_\omega}(A) = \mathrm{tp}_{T_\omega}(A') \Leftrightarrow \mathrm{cl}_\omega(A) \simeq_{\mathcal{L}} \mathrm{cl}_\omega(A')$ (over the map sending A to A').

(e) The theory T_ω is ω-stable of Morley rank $\omega \cdot 2$, with $\mathrm{MR}(\ddot{U}) = \omega$. Moreover, $\mathrm{MR}(\bar{a}/A) < \omega \Leftrightarrow \mathrm{d}(\bar{a}/A) = 0$ for all sets A and finite tuples $\bar{a}$.

(f) Let $K \models T_\omega$. Then any non-zero element of K may be written in the form $(a+b) \times (c+d)$ for some green elements a, b, c, d. In particular, K is in the definable closure of $\ddot{U}(K)$.

Remark 3.7. Assuming SC, Zilber shows in [**31**] that there is a natural model of T_ω, namely, the structure $(\mathbb{C}, +, \times, \ddot{U})$, where the set of green points is given by $\ddot{U} := \{\exp(t(1+i)+q) \mid t \in \mathbb{R},\ q \in \mathbb{Q}\}$.

The construction of Poizat provides a 'bad field of infinite rank' in characteristic 0. It is possible to collapse the theory T_ω to obtain a *bad field*, that is, a field of finite Morley rank with a definable infinite proper subgroup of the multiplicative group of the field. In [**3**], Baudisch, Martin-Pizarro, Wagner and the author construct an elementary subclass $\mathcal{C}_0^\mu \subseteq \mathcal{C}_0$, such that $(\mathcal{C}_0^\mu, \leqslant)$ has (AP) and (JEP). Let K_μ be the corresponding Fraïssé–Hrushovski limit, and T_μ be its $\mathcal{L}$-theory.

Fact 3.8 [**3**]. (a) The Fraïssé–Hrushovski limit K_μ is saturated.

(b) Let $A, A' \subseteq K \models T_\mu$. Then $\mathrm{tp}_{T_\mu}(A) = \mathrm{tp}_{T_\mu}(A') \Leftrightarrow \mathrm{cl}_\omega(A) \simeq_{\mathcal{L}} \mathrm{cl}_\omega(A')$ (over the map sending A to A').

(c) The theory T_μ is of Morley rank 2, and $\ddot{U}$ is strongly minimal.

(d) For all $A \subseteq K \models T_\mu$ and $\bar{a} \in K$, one has $\mathrm{d}(\bar{a}/A) = \mathrm{MR}(\bar{a}/A)$. In particular, $\mathrm{acl}_\mu(A) = \{a \in K \mid \mathrm{d}(a/A) = 0\}$, where acl_μ denotes the algebraic closure in T_μ.

(e) Let $K \models T_\mu$. Then any non-zero element of K may be written in the form $(a+b) \times (c+d)$ for some green elements a, b, c, d. In particular, K is in the definable closure of $\ddot{U}(K)$, so T_μ is almost strongly minimal.

(f) The $\mathcal{L}$-theory T_μ is model-complete.

The following result will thus apply to the theory T_μ.

Fact 3.9 [**27**]. Let K be a field of finite Morley rank (in some expansion $\mathcal{L}$ of the language of rings).

(a) Any algebraically closed subset of K is an elementary substructure.

(b) The theory $\mathrm{Th}_{\mathcal{L}}(K)$ eliminates imaginaries.

We mention another fact which will be needed later on. It is a direct consequence of Baudisch, Hils, Martin Pizarro and Wagner [**3**, Lemma 10.3(2)].

Fact 3.10. We work in T_ω or in T_μ. For any $d \geqslant 0$ and any variable tuples $\bar{x}$ and $\bar{z}$, there is a partial type $\pi_d(\bar{x}, \bar{z})$, such that, for any $\bar{a}$ and $\bar{b}$, one has $\models \pi_d(\bar{a}, \bar{b})$ if and only if $\mathrm{d}(\bar{a}/\bar{b}) \geqslant d$.

We finish this section with an example showing that in both T_ω and T_μ, we cannot infer from the characterizations of types in Facts 3.6(d) and 3.8(b), respectively, that two self-sufficient green tuples having the same field type (over an algebraically closed and self-sufficient base) must have the same type. The problem is that one has to *choose green roots*.

Example 3.11. Let L be a model of T_ω or T_μ, $K = \mathbb{Q}^{\mathrm{alg}} \subseteq L$, and let $\bar{a} = (a_1, a_2, a_3), \bar{a}' = (a_1', a_2', a_3') \in L$ be green tuples. Put $A = K(\bar{a})^{\mathrm{alg}}$, $A' = K(\bar{a}')^{\mathrm{alg}}$. Suppose that A and A' are self-sufficient in L, and that $\bar{a}$ is a $\mathbb{Q}$-basis of $\ddot{U}(A)$ over K, and similarly for $\bar{a}'$ and $\ddot{U}(A')$.

Suppose that both $\bar{a}$ and $\bar{a}'$ are generic in the variety V given by the equation $X = (Y + Z)^2$. Note that exactly one of the two square roots of a_1 (and of a'_1) is green. Suppose that $a_2 + a_3$ and $-a'_2 - a'_3$ are green. Then $\bar{a}$ and $\bar{a}'$ do not have the same type over K (not even over $\emptyset$).

4. *A definability result for algebraic varieties*

In this section, we prove a definability result for varieties in characteristic 0, which will allow us to deal with uniformity issues around multiplicity in green fields: it is the major ingredient in the proof to show that the bad fields constructed in [**3**] have the DMP and that the green fields of Poizat admit a geometric notion of genericity.

Definition 4.1. (1) Let L/K be a field extension with $K \models \mathrm{ACF}_0$, and let $l \geqslant 2$ be an integer. A tuple $\bar{a} = (a_1, \ldots, a_n)$ from $L^\times$ is called *l-Kummer generic over* K if $\mathrm{Gal}(K(\sqrt[l]{a_1}, \ldots, \sqrt[l]{a_n})/K(\bar{a})) \simeq (\mathbb{Z}/l\mathbb{Z})^n$.

The tuple $\bar{a}$ is called *Kummer generic over* K if it is l-Kummer generic over K for every $l \geqslant 2$.

(2) Let $V \subseteq \mathbb{G}_m^n$ be an irreducible closed subvariety of the standard torus $\mathbb{G}_m^n$, V defined over $K \models \mathrm{ACF}_0$. The variety V is called *l-Kummer generic* or *Kummer generic* if every tuple $\bar{a} = (a_1, \ldots, a_n)$ which is generic in V over K is l-Kummer generic or Kummer generic over K, respectively.

The notion of a Kummer generic tuple is taken from [**32**], although Zilber calls such a tuple *simple*. Note that the definition of a Kummer generic variety does not depend on the choice of the algebraically closed field K.

Let A be an abelian group, B be a subgroup of A and $l \geqslant 2$ be a natural number. Recall that B is an *l-pure* subgroup of A if whenever the equation $lx = b$ has a solution in A, where $b \in B$, then it has already a solution in B. If B is l-pure in A for every l, then it is called a pure subgroup. Note that if $\mathrm{Tor}(A) \subseteq B$, then B is l-pure in A if and only if A/B has trivial l-torsion.

For a field extension L/K and $X \subseteq L^\times$, we denote $K^\times\langle X\rangle$ the subgroup of $L^\times$ generated by $K^\times \cup X$. Let M be an algebraically closed field and Γ be a subgroup of the multiplicative group $M^\times$ of M. Then the divisible hull of Γ (that is, the set of elements $m \in M^\times$, such that $m^n \in \Gamma$ for some $n \geqslant 1$) is denoted by $\mathrm{div}(\Gamma)$.

Fact 4.2. Let K be an algebraically closed field of characteristic 0 and $V \subseteq \mathbb{G}_m^n$ be a closed irreducible subvariety defined over K.

(1) Let L/K be a field extension, $\bar{a}$ be an n-tuple from $L^\times$ and $l \geqslant 2$ be an integer. The following are equivalent:
 - (a) $\bar{a}$ is l-Kummer generic over K;
 - (b) $\bar{a}$ is multiplicatively independent over $K^\times$ and $(L_0^\times)^l \cap \langle a_1, \ldots, a_n\rangle = \langle a_1^l, \ldots, a_n^l\rangle$, where $L_0 = K(\bar{a})$ and $(L_0^\times)^l = \{b^l \mid b \in L_0^\times\}$;
 - (c) the elements $a_1/K^\times, \ldots, a_n/K^\times$ generate an l-pure subgroup of rank n inside the group $K(\bar{a})^\times/K^\times$; and
 - (d) if α_i is an lth root of a_i, then $\mathrm{tp}_{\mathrm{ACF}_0}(\alpha_1, \ldots, \alpha_n/K\bar{a})$ is determined by the formulas $\{x_i^l = a_i\}_{1 \leqslant i \leqslant n}$.

(2) The following are equivalent:
 - (a) V is l-Kummer generic; and
 - (b) the variety $\sqrt[l]{V} \subseteq \mathbb{G}_m^n$ given by '$(X_1^l, \ldots, X_n^l) \in V$' is irreducible.

(3) Let $\bar{a}$ be generic in V over K. Then the following are equivalent:
   (a) V is Kummer generic;
   (b) any group automorphism of $\mathrm{div}(K^{\times}\langle\bar{a}\rangle)$ fixing $K^{\times}\langle\bar{a}\rangle$ pointwise lifts to a field automorphism of $K(\bar{a})^{\mathrm{alg}}$, that is, the natural map (given by restriction) $\mathrm{Gal}(K(\bar{a})) \to \mathrm{Aut}_{gp}(\mathrm{div}(K^{\times}\langle\bar{a}\rangle)/K^{\times}\langle\bar{a}\rangle)$ is surjective;
   (c) V is p-Kummer generic for every prime number p; and
   (d) the elements $a_1/K^{\times}, \ldots, a_n/K^{\times}$ generate a pure subgroup of $K(\bar{a})^{\times}/K^{\times}$ of rank n.

Proof. Note that, with the notation from (1.b), letting $A := (L_0^{\times})^l\langle a_1, \ldots, a_n\rangle$, one has $A/(L_0^{\times})^l \cong \langle a_1, \ldots, a_n\rangle/\langle a_1, \ldots, a_n\rangle \cap (L_0^{\times})^l$, so (a) $\Longleftrightarrow$ (b) in (1) follows from Kummer theory (see, for example, [**21**, VI. Theorem 8.1]). The other equivalences are easily verified.

In (2), note that if W is an irreducible component of maximal dimension of $\sqrt[l]{V}$, then all the other irreducible components are multiplicative translates of W by some l-torsion element $\zeta \in \mathbb{G}_m^n$. In particular, $\sqrt[l]{V}$ is equidimensional. Now (2) follows, using (a)$\Leftrightarrow$(d) in (1). Part (3) is left to the reader. □

The pathology we encountered in Example 3.11 does not exist in case the tuples are Kummer generic, as is shown by the following corollary.

Corollary 4.3. *Let $K = K^{\mathrm{alg}} \leqslant L \models T$, where T equals T_ω or T_μ. Let $\bar{a}, \bar{a}' \in L$ be such that $K\bar{a} \leqslant L$ and $K\bar{a}' \leqslant L$. Suppose that $\bar{a}$ and $\bar{a}'$ are coloured in the same way and that $\bar{a}$ and $\bar{a}'$ have the same field type over K. Moreover, suppose that $\bar{a}$ is Kummer generic over K. Then $\mathrm{tp}_T(\bar{a}/K) = \mathrm{tp}_T(\bar{a}'/K)$.*

Proof. Note that since $K\bar{a} \leqslant L$, we have $\mathrm{cl}_\omega(K\bar{a}) = K(\bar{a})^{\mathrm{alg}}$ (similarly for $\bar{a}'$). Choose a field isomorphism $\alpha : K(\bar{a})^{\mathrm{alg}} \simeq K(\bar{a}')^{\mathrm{alg}}$ extending the map $K\bar{a} \mapsto K\bar{a}'$. We have $\alpha(\mathrm{div}(K^{\times}\langle\bar{a}\rangle)) = \mathrm{div}(K^{\times}\langle\bar{a}'\rangle)$, and it is easy to see that there exists $\sigma_0 \in \mathrm{Aut}_{gp}(\mathrm{div}(K^{\times}\langle\bar{a}\rangle)/K^{\times}\langle\bar{a}\rangle)$, such that

$$\alpha_0 \circ \sigma_0 : (\mathrm{div}(K^{\times}\langle\bar{a}\rangle), \ddot{U} \cap \mathrm{div}(K^{\times}\langle\bar{a}\rangle)) \simeq (\mathrm{div}(K^{\times}\langle\bar{a}'\rangle), \ddot{U} \cap \mathrm{div}(K^{\times}\langle\bar{a}'\rangle))$$

is an isomorphism of groups respecting the green points (here α_0 denotes the map $\alpha \restriction_{\mathrm{div}(K^{\times}\langle\bar{a}\rangle)}$). Since $\bar{a}$ contains a basis of $\ddot{U} \cap K(\bar{a})^{\mathrm{alg}}$ over $\ddot{U}(K)$, necessarily $\ddot{U} \cap K(\bar{a})^{\mathrm{alg}} = \ddot{U} \cap \mathrm{div}(K^{\times}\langle\bar{a}\rangle)$ (similarly for $\bar{a}'$). By Fact 4.2 there exists $\sigma \in \mathrm{Gal}(K(\bar{a}))$ restricting to σ_0, and we obtain an $\mathcal{L}$-isomorphism

$$\alpha \circ \sigma : (K(\bar{a})^{\mathrm{alg}}, \ddot{U} \cap K(\bar{a})^{\mathrm{alg}}) \simeq (K(\bar{a}')^{\mathrm{alg}}, \ddot{U} \cap K(\bar{a}')^{\mathrm{alg}}).$$

The result follows, using Facts 3.6(d) or 3.8(b), respectively. □

Fact 4.4 [**32**, Lemma 2.1]. Let K be an algebraically closed field and L/K be a finitely generated field extension. Then $L^{\times}/K^{\times}$ is a free abelian group.

This fact is proved by embedding $L^{\times}/K^{\times}$ into the group of Weil divisors of a suitably chosen variety V, such that $K(V) = L$. In the proof of the following key definability result, we give an effective version of this argument.

Proposition 4.5. *Being a Kummer generic (irreducible) variety is a definable property in characteristic 0.*

Proof. Let $\mathcal{V} = \{V_{\bar{b}} \mid \models \theta(\bar{b})\}$ be a uniformly definable family of closed subvarieties of $\mathbb{G}_m^n$. If $V_{\bar{b}}$ is not Kummer generic, then it is easy to construct a formula $\theta'(\bar{z}) \in \mathrm{tp}(\bar{b})$, such that whenever $\models \theta'(\bar{b}')$, then $V_{\bar{b}'}$ is not Kummer generic. By compactness, it thus suffices to construct, for every tuple $\bar{b}_0$, such that $V_{\bar{b}_0}$ is Kummer generic, some $\theta_0(\bar{z}) \in \mathrm{tp}(\bar{b}_0)$, such that $V_{\bar{b}'}$ is Kummer generic whenever $\models \theta_0(\bar{b}')$.

So assume that V_{b_0} is Kummer generic. Suppose that $V_{\bar{b}_0}$ is irreducible of dimension d. Choose $I \subseteq \{1, \ldots, n\}$, $|I| = d$, such that, for generic $\bar{a}$ in $V_{\bar{b}_0}$ (over $K \models \mathrm{ACF}_0$ containing $\bar{b}_0$), one has $\bar{a} \in K(\bar{a}_I)^{\mathrm{alg}}$, with $\bar{a}_I = (a_i)_{i \in I}$ (that is, $\bar{a}_I$ is a transcendence basis of $K(V_{\bar{b}_0}) = K(\bar{a})$ over K). Strengthening θ and choosing appropriate natural numbers m and k, we may assume that every variety $V = V_{\bar{b}}$ from the family $\mathcal{V}$ satisfies the following conditions (below, we shall always work over an algebraically closed field K over which V is defined).

(a) The variety V is irreducible of dimension d.

(b) Let $\bar{a}$ be generic in V. Then $\bar{a}_I$ is a transcendence basis of $K(V) = K(\bar{a})$ over K. Moreover, $[K(\bar{a}) : K(\bar{a}_I)] \leqslant m$.

(c) Let $\bar{a}$ be generic in V, and $\varepsilon : I \to \{-1, 1\}$ be any function. Denote the tuple $(a_i^{\varepsilon(i)})_{i \in I}$ by $\bar{a}_I^{\varepsilon}$. Then any a_j satisfies a polynomial equation over $K(\bar{a}_I^{\varepsilon})$ of the form

$$Y^m + \frac{f_{m-1}(\bar{a}_I^{\varepsilon})}{g_{m-1}(\bar{a}_I^{\varepsilon})} Y^{m-1} + \ldots + \frac{f_0(\bar{a}_I^{\varepsilon})}{g_0(\bar{a}_I^{\varepsilon})} = 0,$$

where $f_l(\bar{a}_I^{\varepsilon})$ and $g_l(\bar{a}_I^{\varepsilon})$ are polynomials in $\bar{a}_I^{\varepsilon}$ of total degree at most k (for $0 \leqslant l < m$).

(d) Let $\bar{a}$ be generic in V. Then $\bar{a}$ is multiplicatively independent over $K^\times$.

By standard arguments we may achieve (a)–(c). The property (d) is definable by Fact 3.3(1).

Claim. For a given $l \geqslant 2$ there is $\theta_l(\bar{z}) \in \mathrm{tp}(\bar{b}_0/K)$, such that, for any $\bar{b}$ with $\models \theta_l(\bar{b})$, the variety $V_{\bar{b}}$ is l-Kummer generic.

By Fact 4.2, $V_{\bar{b}}$ is l-Kummer generic if and only if the variety defined by the condition '$(X_1^l, \ldots, X_n^l) \in V_{\bar{b}}$' is irreducible. This proves the claim, for the latter condition is definable in $\bar{b}$.

Since a variety is Kummer generic if it is p-Kummer generic for every prime number p, using the previous claim, the proof of the proposition is thus completed once the following lemma is established. □

Lemma 4.6. *Let $V \subseteq \mathbb{G}_m^n$ be as above, satisfying* (a)–(d) (*with m and k as in* (b) *and* (c), *respectively*). *Let $p > n! m^n k^n$ be a prime number. Then V is p-Kummer generic.*

Proof. (i) We consider the *group of* (*Weil*) *divisors* of the function field $K(V)/K$, given by

$$\mathrm{Div}(K(V)/K) := \bigoplus_{v \in \mathfrak{Reg}} \mathbb{Z} \cdot v,$$

where $\mathfrak{Reg} = \mathfrak{Reg}(K(V)/K)$ denotes the set of all discrete valuations of $K(V)$ that are trivial on K and such that the residue field is of transcendence degree $d-1$ over K. (Alternatively, the classical and more geometric way would be to work with the group of Weil divisors of a certain projective variety V', namely, the normalization of $\mathbb{P}^d$ in the field $K(V) \supseteq K(\mathbb{P}^d) = K(\bar{a}_I)$.).

For any $f \in K(V)$ there is only a finite number of $v \in \mathfrak{Reg}(K(V)/K)$, such that $v(f) \neq 0$, and one has $f \in K$ if and only if $v(f) = 0$ for all $v \in \mathfrak{Reg}(K(V)/K)$. This follows from standard arguments in valuation theory. (We refer to [**29**, VI.§ 14 and VII.§ 4bis].) The following map is thus a group homomorphism that induces an embedding of $K(V)^\times / K^\times$ into $\mathrm{Div}(K(V)/K)$:

$$K(V)^\times \longrightarrow \mathrm{Div}(K(V)/K), \quad f \longmapsto (f) := \sum_v v(f) \cdot v.$$

(ii) Let v' be in $\mathfrak{Reg}(K(\bar{a}_I)/K)$. Suppose $v'(a_i) \geqslant 0$ for all $i \in I$. (Replacing $\bar{a}_I$ by a suitable $\bar{a}_I^{\varepsilon}$, we may always achieve this.) It follows from the assumption on v' that the ideal of $K[\bar{a}_I]$ given by the elements of positive valuation is a prime ideal of height 1, and so equal to (P) for some irreducible polynomial $P = P(\bar{a}_I)$. This means that $v'(P^z f(\bar{a}_I)/g(\bar{a}_I)) = z$ for all f and g that are not divisible by P. Denote this valuation by v'_P.

(iii) Let $v \in \mathfrak{Reg}(K(\bar{a})/K)$, let v' be its restriction to $K(\bar{a}_I)/K$ (which is an element of $\mathfrak{Reg}(K(\bar{a}_I)/K)$ by standard properties of algebraic extensions of valuations). By (ii), we may assume that $v' = v(P)v'_P$ for some irreducible polynomial $P = P(\bar{a}_I)$.

We will show that $|v(a_j)| \leqslant mk$ for all $j \leqslant n$. By (c),

$$a_j^m + \frac{f_{m-1}(\bar{a}_I)}{g_{m-1}(\bar{a}_I)} a_j^{m-1} + \ldots + \frac{f_0(\bar{a}_I)}{g_0(\bar{a}_I)} = 0,$$

so there exists $r < m$, such that $v(a_j^m) = v((f_r(\bar{a}_I)/g_r(\bar{a}_I))a_j^r)$ and thus

$$(m-r)v(a_j) = v\left(\frac{f_r(\bar{a}_I)}{g_r(\bar{a}_I)}\right).$$

By (b) and the fundamental inequality, $|v(f)| \leqslant m|v'_P(f)|$ for any $f \in K(\bar{a}_I)$.

Moreover, since the total degrees of f_r and g_r are bounded by k (by (c)), it follows that $|v'_P(f_r(\bar{a}_I)/g_r(\bar{a}_I))| \leqslant k$. Thus, $|v(a_j)|$ is bounded by mk.

(iv) Consider the elements $(a_1), \ldots, (a_n)$ of $\mathrm{Div}(K(V)/K)$. By (d) and (i), they are linearly independent over $\mathbb{Z}$, so there are valuations $v_1, \ldots, v_n \in \mathfrak{Reg}(K(V)/K)$, such that the square matrix $M = (v_i(a_j))_{i,j}$ has non-zero determinant. Now $|\det(M)| \leqslant n!m^n k^n$ by (iii) and the Leibniz formula, so $\det(M) \not\equiv 0 \mod p$ (as $p > n!m^n k^n$ by assumption). This means that no element of the form $\sum_{i=1}^n r_i(a_i)$, with $0 \leqslant r_i < p$ not all 0, is divisible by p in $\mathrm{Div}(K(V)/K)$. It follows that $\prod_{i=1}^n a_i^{r_i}$ does not have a pth root in $K(\bar{a})$. By Fact 4.2, this shows that V is p-Kummer generic. □

Remark 4.7. Gabber suggested a completely different proof for definability of Kummer genericity, a proof which generalizes to semiabelian varieties in arbitrary characteristic.

In joint work with Bays and Gavrilovich (see the forthcoming paper 'Some definability results in abstract Kummer theory'), we extract the 'Galois theoretic' essence of Gabber's argument and give a model-theoretic proof that applies to any definable abelian group of finite Morley rank with the DMP.

Before we finish this section, let us mention an important corollary of Proposition 4.5. It was observed by Roche that there is a gap in the construction of the bad field as given in [**3**]. The reason for this is intimately related to the problem raised in Example 3.11. In fact, [**3**, Bemerkung 6.7] is not true in general, and so the proof of the economic amalgamation lemma [**3**, Satz 9.2] is not correct. Fortunately, we may provide the necessary technical improvement, the existence of strongly minimal codes, in Corollary 4.8, so that the proof of the economic amalgamation lemma goes through without any changes.

In his thesis (see the forthcoming dissertation 'Fusion d'un corps algébriquement clos avec un sous-groupe non-algébrique d'une variété abélienne'), Roche considers so-called *octarine fields*, certain expansions of abelian varieties by a predicate for a non-algebraic subgroup, a context that is similar to bad fields. It is explained in detail there how strongly minimal codes are used to prove the economic amalgamation lemma. The same arguments apply in the context of bad fields.

Corollary 4.8. *There is a collection of codes satisfying all the requirements of Baudisch, Hils, Martin Pizarro and Wagner [**3**, Definition 4.7] and, moreover, that the instances of any code are strongly minimal definable sets.*

Proof. We adopt the terminology and notation from [**3**, Definition 4.7], restricting our attention to minimal prealgebraic formulas $\varphi(\bar{x})$, such that the corresponding variety is Kummer generic (equivalently, any generic solution of φ over an algebraically closed field is Kummer generic). We strengthen the definition of a code $\varphi_\alpha(\bar{x}, \bar{z})$ by adding that, for any non-empty instance $\varphi_\alpha(\bar{x}, \bar{b})$, its Zariski closure $V_\alpha(\bar{x}, \bar{b})$ is a Kummer generic variety (this is a definable property by Proposition 4.5).

It follows from Corollary 4.3 that $\varphi_\alpha(\bar{x}, \bar{z}) \wedge \bigwedge_i \ddot{U}(x_i)$ is a family of strongly minimal sets. □

5. *Generic automorphisms of green and bad fields*

In this section, we will establish the axiomatizability of the generic automorphism in the green and bad fields. We use the notation from Section 3.

5.1. *Generic automorphisms of the green field of Poizat*

Lemma 5.1. *The theory T_ω admits a geometric notion of genericity.*

Proof. Consider a type $\mathrm{tp}(\tilde{a}/K)$ where $\tilde{a}$ is a finite tuple from $\mathfrak{C} \succcurlyeq K \models T_\omega$ of the form $\bar{g}\overline{g'}\bar{w}\overline{w'}$ (maybe after reordering), satisfying the following conditions:

(i) the elements from $\bar{g}\overline{g'}$ are green, and those from $\bar{w}\overline{w'}$ are white;
(ii) $K\tilde{a} \leqslant \mathfrak{C}$, and $\bar{g} = (g_1, \ldots, g_n)$ is a basis of $\ddot{U}(K(\tilde{a})^{\mathrm{alg}})$ over $\ddot{U}(K)$, such that $\bar{g}' \in \langle \ddot{U}(K)\bar{g}\rangle$;
(iii) $\bar{w}$ is multiplicatively independent over $K\bar{g}$, and $\bar{w}' \in K^\times\langle \bar{g}\bar{w}\rangle$;
(iv) $\bar{w} \in K[g_1, \ldots, g_n, 1/g_1, \ldots, 1/g_n]$; and
(v) $\bar{g}$ is Kummer generic over K.

Call a type *special* if it satisfies (i)–(v) above.

Below, we define a geometric notion of genericity where the nice types are given by the special types. Let us first show that there are ‘enough’ special types. Let $K \models T_\omega$ and $\bar{a}$ be an arbitrary finite tuple from $\mathfrak{C} \succcurlyeq K$. Choose some finite green tuple $\bar{u}$, such that $\bar{a} \in K[\bar{u}]$. Such a tuple exists by Fact 3.6(f).

Combining the fact that $\mathrm{tr.deg}(\mathrm{cl}_\omega(K\bar{u})/K)$ is finite with Fact 4.4, we may find some finite tuple $\tilde{a}$ containing $\bar{a}$, $\tilde{a} = \bar{g}\overline{g'}\bar{w}\overline{w'}$ (where all the elements outside $\bar{a}$ may be taken to be green), such that $\mathrm{tp}(\tilde{a}/K)$ is special.

Now suppose that $\mathrm{tp}(\tilde{a}/K)$ is special, with $\tilde{a} = \bar{g}\overline{g'}\bar{w}\overline{w'}$ as above. Choose a finite $\bar{b} \in K$ such that the following properties are satisfied:

(1) $\mathrm{locus}(\bar{g}/K) = U(\bar{x}, \bar{b})$ is defined over $\bar{b}$, similarly $\mathrm{locus}(\bar{g}, \bar{w}/K) = V(\bar{x}, \bar{y}, \bar{b})$ and $\mathrm{locus}(\tilde{a}/K) = W(\bar{x}, \bar{x}', \bar{y}, \bar{y}', \bar{b})$ (this is equivalent to $\mathrm{Cb}_{\mathrm{ACF}}(\tilde{a}/K) \subseteq \bar{b}$);
(2) for any g' from $\overline{g'}$ there exists a green $b_{g'}$ from $\bar{b}$ and $m_{1,g'}, \ldots, m_{n,g'} \in \mathbb{Z}$, such that
$$g' = b_{g'} \prod_{i=1}^{n} g_i^{m_{i,g'}}; \tag{5.1}$$
(3) for any w' from $\overline{w'}$ there exists some $b_{w'}$ from $\bar{b}$, integers $m_{i,w'}$, $1 \leqslant i \leqslant n$ and $n_{i,w'}$, $1 \leqslant i \leqslant l$, such that
$$w' = b_{w'} \prod_{i=1}^{n} g_i^{m_{i,g'}} \prod_{i=1}^{l} w_i^{n_{i,w'}}. \tag{5.2}$$

Moreover, if $n_{i,w'} = 0$ for all i, then $b_{w'}$ is a white element; and

(4) for any w from $\bar{w}$ there is a polynomial $f_w \in K[\bar{u}, \bar{v}]$ with coefficients from $\bar{b}$ (so we may write i as $f_w = F_w(\bar{u}, \bar{v}, \bar{b})$), such that

$$w = F_w\left(g_1, \ldots, g_n, \frac{1}{g_1}, \ldots, \frac{1}{g_n}, \bar{b}\right). \tag{5.3}$$

Let $k := \dim(U) = \dim(W) = \operatorname{tr.deg}(\bar{g}/K)$ and $d := \mathrm{d}(\tilde{a}/K) = 2k - n$. The following conditions (a)–(g) hold for $\bar{b}' = \bar{b}$, and they are definable in $\bar{b}'$ (definability follows from Fact 3.3 and Proposition 4.5):

(a) $U(\bar{x}, \bar{b}')$ is irreducible of dimension k;
(b) $U(\bar{x}, \bar{b}')$ is rotund;
(c) $U(\bar{x}, \bar{b}')$ is Kummer generic;
(d) $V(\bar{x}, \bar{y}, \bar{b}')$ is equal to the variety given by $U(\bar{x}, \bar{b}')$ together with the equations from (5.3), that is, $y_w = F_w(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n, \bar{b}')$, where y_w is the variable corresponding to w;
(e) $V(\bar{x}, \bar{y}, \bar{b}') \subseteq \mathbb{G}_m^{n+l}$ is free (where $l = \lg(\bar{y})$); and
(f) $W(\bar{x}, \bar{x}', \bar{y}, \bar{y}', \bar{b}')$ is equal to the variety given by $V(\bar{x}, \bar{y}, \bar{b}')$ together with the corresponding equations from (5.1) and (5.2); and
(g) the quantifier-free types of $\bar{b}$ and $\bar{b}'$ in the language $\{\ddot{U}, =\}$ coincide.

Let $\theta(\bar{z})$ be an $\mathcal{L}$-formula, such that $\models \theta(\bar{b}')$ if and only if the conditions (a)–(g) are satisfied.

Claim. Suppose that $\bar{b}' \in K \models T_\omega$, such that $\models \theta(\bar{b}')$. Let $\tilde{a} = \bar{g}\overline{g'}\bar{w}\overline{w'}$ be a K-generic solution of $W_{\bar{b}'}$, and let $L := K(\tilde{a})^{\mathrm{alg}} = K(\bar{g})^{\mathrm{alg}}$, $\ddot{U}(L) := \operatorname{div}(\langle \ddot{U}(K)\bar{g}\rangle)$. Then $(L, \ddot{U}(L))$ is a self-sufficient extension of $(K, \ddot{U}(K))$, with $\delta(L/K) = d$. The tuple $\bar{g}\overline{g'}$ consists of green elements, whereas the elements from $\bar{w}\overline{w'}$ are white. Assume in addition that L is self-sufficient in $\mathfrak{C}$. Then $\mathrm{tp}_{\mathcal{L}}(\tilde{a}/K)$ is uniquely determined by: $\tilde{a}$ is (field) generic in $W_{\bar{b}'}$ over K, $\bar{g}\overline{g'}$ is green, $\bar{w}\overline{w'}$ is white and $K\tilde{a} \leqslant \mathfrak{C}$.

By construction, $K\tilde{a} \leqslant L$ (and so also $K \leqslant L$) follows from (b). The fact that $\bar{g}\overline{g'}$ is a green tuple is true by construction, together with (f) and (g). The colour assigned to each element w of $\bar{w}$ is white, since this is a multiplicatively independent tuple over $K\bar{g}$ by (e) and (d). Combining (f) and (g), we see that $\bar{w}'$ consists of white elements only. Note that the irreducibility of $W_{\bar{b}'}$ as well as $\delta(L/K) = d$ is an easy consequence of (a) together with the other conditions.

Finally, if $L \leqslant \mathfrak{C}$ (from which we deduce $K\tilde{a} \leqslant \mathfrak{C}$), then the type of $\tilde{a}$ over K is determined in the described way, since $U_{\bar{b}'}$ is Kummer generic (and $W_{\bar{b}'}$ irreducible). This follows from Corollary 4.3 and proves the claim.

Assume that U, V, W, θ, d are given as before, in the variables $\tilde{x}, \bar{z}$, where $\tilde{x} = \bar{x}\bar{x}'\bar{y}\bar{y}'$.

(1) Let $\varphi_{\ddot{U}}(\tilde{x})$ be a formula expressing that the elements from $\bar{x}, \bar{x}'$ are green, and those from $\bar{y}, \bar{y}'$ white.

(2) Let $\varphi_d(\tilde{x}, \bar{z})$ be an arbitrary formula from the partial type $\pi_d(\tilde{x}, \bar{z})$ (introduced in Fact 3.10).

(3) Let $Z(\tilde{x}, \bar{z})$ be a uniform family of varieties.

(4) Let $\chi_1(\bar{z})$ be a formula, such that $\models \chi_1(\bar{b}')$ if and only if $Z_{\bar{b}'}$ is a proper subvariety of $W_{\bar{b}'}$.

(5) Let $\chi(\bar{z}) = \chi_1(\bar{z}) \wedge \theta(\bar{z})$.

A *special* formula is a formula of the form

$$\varphi(\tilde{x}, \bar{z}) = W(\tilde{x}, \bar{z}) \wedge \neg Z(\tilde{x}, \bar{z}) \wedge \varphi_{\ddot{U}}(\tilde{x}) \wedge \varphi_d(\tilde{x}, \bar{z}) \wedge \chi(\bar{z}). \tag{5.4}$$

A non-empty instance of a special formula will also be called special.

By the claim, for any $\bar{b}' \in K \models T_\omega$, such that $\models \chi(\bar{b}')$ there is a unique special type $p(\tilde{x}) \in S(K)$ containing $\varphi(\tilde{x}, \bar{b}')$, such that any realization $\tilde{a}$ of p is generic (in the field sense) in $W_{\bar{b}'}$ over K. Moreover, using Fact 3.10, it is easy to see that the set of (instances of) special formulas in a given special type is dense in it.

We now define a notion of genericity, only using special formulas and special types. Let $\varphi(\tilde{x}, \bar{b}')$ and $p \in S(K)$ be special, and assume that p contains $\varphi(\tilde{x}, \bar{b}')$. We say that p is *generic* in $\varphi(\tilde{x}, \bar{b}')$ if any $\tilde{a} \models p$ is field generic in $W_{\bar{b}'}$, where $W(\tilde{x}, \bar{b}')$ is as in (5.4).

By what we have seen, special formulas correspond to nice formulas, and special types correspond to nice types. Since there are enough special types, in order to show that the notion of genericity we defined is a geometric notion of genericity, it is sufficient to show that it satisfies property (4) from Definition 2.4 (the remaining properties are clear).

To prove property (4), assume that p is generic in $\varphi(\tilde{x}, \bar{b})$, p_0 is generic in $\varphi_0(\tilde{x}_0, \bar{b}_0)$, $\tilde{x}_0$ is a subtuple of $\tilde{x}$ and that p restricts to p_0. We have to find $\delta(\bar{z}, \bar{z}_0) \in \mathrm{tp}(\bar{b}\bar{b}_0)$, such that whenever $\models \delta(\bar{b}', \bar{b}'_0)$, the generic type of $\varphi(\tilde{x}, \bar{b}')$ restricts to the generic type of $\varphi_0(\tilde{x}_0, \bar{b}_0)$. Choose $\tilde{a} \models p$. Then $\tilde{a}_0 = \bar{g}_0\bar{g}'_0\bar{w}_0\bar{w}'_0 \models p_0$ and we observe

$$K\tilde{a}_0 \leqslant K(\tilde{a})^{\mathrm{alg}} =: L \text{ or, equivalently, } K\bar{g}_0 \leqslant L, \tag{5.5}$$

$$W(\tilde{x}, \bar{b}) \text{ projects onto a generic subset of } W_0(\tilde{x}_0, \bar{b}_0). \tag{5.6}$$

Extend $\bar{g}_0$ to a (green) basis $\bar{g}_0\bar{g}_1 \subseteq \bar{g}\bar{g}'$ of $\ddot{U}(L)$ over $\ddot{U}(K)$, and let $\overline{x}_1$ be the variable tuple corresponding to $\bar{g}_1$. Choose a formula $\delta(\bar{z}, \bar{z}_0) \in \mathrm{tp}(\bar{b}, \bar{b}_0)$, such that, for any pair $(\bar{b}', \bar{b}'_0)$ with $\models \delta(\bar{b}', \bar{b}'_0)$, the following three conditions hold:

(1) $\models \chi(\bar{b}') \wedge \chi_0(\bar{b}'_0))$;
(2) $W(\tilde{x}, \bar{b}')$ projects onto a generic subset of $W_0(\tilde{x}_0, \bar{b}'_0)$; and
(3) for generic $\tilde{a} = \bar{g}\bar{g}'\bar{w}\bar{w}'$ in $W_{\bar{b}'}$, the variety $\mathrm{locus}(\bar{g}_1/\mathbb{Q}(\bar{b}'\bar{b}'_0\bar{g}_0)^{\mathrm{alg}})$ is rotund.

Note that the last property can be guaranteed using the definability of types in algebraically closed fields, combined with Fact 3.3.

Let $\bar{b}', \bar{b}'_0 \in K' \models T_\omega$, such that $K' \models \delta(\bar{b}', \bar{b}'_0)$. By the above conditions on δ, the generic type $p'(\tilde{x}) \in S(K')$ of the special formula $\varphi(\tilde{x}, \bar{b}')$ restricts to the generic type $p'_0(\tilde{x}_0)$ of $\varphi_0(\tilde{x}_0, \bar{b}'_0)$. This is clear for the algebraic part of the type as for the colouring. Moreover, if $\tilde{a} \models p'$, then $K\tilde{a}_0 \leqslant L = K(\tilde{a})^{\mathrm{alg}}$ follows from (3) and Lemma 3.5. Since $L \leqslant \mathfrak{C}$, we deduce that $K\tilde{a}_0 \leqslant \mathfrak{C}$, and so $\tilde{a}_0 \models p'_0$. □

THEOREM 5.2. *Let T_ω be the theory of the green field of Poizat (considered in an expansion by definition so that it eliminates quantifiers). Then $T_\omega A$ exists. Its reduct to the language of difference fields is equal to $ACFA_0$.*

Proof. Lemma 5.1 shows that T_ω admits a geometric notion of genericity. Thus, $T_\omega A$ exists by Proposition 2.5.

Now consider $(K, \ddot{U}(K), \sigma) \models T_\omega A$. Suppose that $(K, \sigma) \subseteq (L, \sigma) \models ACFA_0$. Putting $\ddot{U}(L) := \ddot{U}(K)$, we may expand the difference field (L, σ) to a green field with automorphism $(L, \ddot{U}(L), \sigma)$. Then $K \leqslant L$, and there is $(L, \ddot{U}(L), \sigma) \subseteq (M, \ddot{U}(M), \sigma) \models T_\omega A$, such that $L \leqslant M$. Since $(K, \ddot{U}(K), \sigma) \preccurlyeq (M, \ddot{U}(M), \sigma)$, it follows in particular that (K, σ) is existentially closed in (L, σ). Thus, (K, σ) is an existentially closed difference field, that is, a model of $ACFA_0$.

Let us now show that every completion of $ACFA_0$ is attained in this manner. Note that, for any $\sigma \in \mathrm{Gal}(\mathbb{Q})$, the green field with automorphism $(\mathbb{Q}^{\mathrm{alg}}, \{1\}, \sigma)$ embeds (in a self-sufficient way) into a model of $T_\omega A$. By Fact 2.1(2), any completion of $ACFA_0$ is determined by the action of σ on $\mathbb{Q}^{\mathrm{alg}}$. This shows the result. □

We already mentioned that, for stable T, the existence of TA implies that T does not have the finite cover property. Thus, Theorem 5.2 implies the following result of Evans [10].

Corollary 5.3. *The green field of Poizat does not have the finite cover property.*

5.2. *Generic automorphisms of bad fields*

Theorem 5.4. *The theory T_μ has the DMP.*

Proof. Since Morley rank is finite, definable and additive in T_μ, to show the DMP, it is sufficient to find, for any type $p \in S_m(M)$ over a model M, a formula $\varphi(\bar{x}, \bar{b}) \in p$, such that $\mathrm{MRD}(p) = \mathrm{MRD}(\varphi(\bar{x}, \bar{b})) = (d, 1)$ and $\mathrm{MRD}(\varphi(\bar{x}, \bar{b}') = (d, 1)$ whenever $\varphi(\bar{x}, \bar{b}')$ is consistent. We call such a type p *good.*

Claim. Suppose that $q(\tilde{x}) \in S_m(M)$ is a good type which is a finite cover of $p(\bar{x}) \in S_n(M)$, that is, there is a partial M-definable function f with finite fibres, such that $f_*(q) = p$. Then p is good. (The proof is left to the reader.)

Now let $p = \mathrm{tp}(\bar{b}/M) \in S_m(M)$ for $M \models T_\mu$. Note that there is a finite green tuple $\bar{a}'$ that is algebraic (in the sense of T_μ) over $M\bar{b}$ and such that $\bar{b} \in \mathrm{dcl}_\mu(M\bar{a}')$ (since $M' := \mathrm{acl}_\mu(M\bar{b}) \succcurlyeq M$ and $\mathrm{dcl}_\mu(\ddot{U}(M')) = M'$ by Facts 3.9 and 3.8). We may even assume that $\bar{a}'$ is a green basis of a self-sufficient extension of M. By Fact 4.4 there are elements $a_1, \ldots, a_n \in M(\bar{a}')$, the field generated by $\bar{a}'$ over M, such that $M^\times\langle\bar{a}\rangle = \mathrm{div}(M^\times\langle\bar{a}'\rangle) \cap M(\bar{a}')$, that is, $\bar{a}$ is Kummer generic over K. Replacing a_i by $\zeta(i)a_i$ for some root of unity $\zeta(i)$, we may arrange that a_i is green for all i.

We still have $\bar{a} \in \mathrm{acl}_\mu(M\bar{b})$ and $\bar{b} \in \mathrm{dcl}_\mu(M\bar{a})$. So by the claim, it is sufficient to show that $q = \mathrm{tp}(\bar{a}/M)$ is good. Note that $V = \mathrm{locus}(\bar{a}/M)$ is a Kummer generic variety. Thus, by Corollary 4.3, one has $\bar{a}_1 \models q$ if and only if the following conditions hold:

(1) $\bar{a}_1$ is generic (in the field sense) in V over M, that is, $\mathrm{locus}(\bar{a}_1/M) = V$;
(2) the tuple $\bar{a}_1$ consists of green elements; and
(3) $\mathrm{d}(\bar{a}_1/M) = \delta(\bar{a}_1/M) = 2\dim(V) - n = \delta(V) = d$.

Suppose $\mathrm{MRD}(V(\bar{x}) \wedge \bigwedge_{i=1}^n \ddot{U}(x_i)) = (d', m') > (d, 1)$. Then there is a proper subvariety W of V containing a type of maximal Morley rank d'. Since $\mathrm{MR} = \mathrm{d} \leqslant \delta$, W is contained in a coset of some $T \in \mathcal{T}(V)$, $T \neq \mathbb{G}_m^n$, by Lemma 3.4. It will suffice to remove from V (performing an induction) a finite number of such cosets to get a definable set of MRD equal to $(d, 1)$, for at each such step, either the Morley rank or the Morley degree will drop. After a finite number of steps we thus arrive at a formula of the form

$$\varphi(\bar{x}, \bar{b}, \bar{c}) = \bigwedge_{i=1}^n \ddot{U}(x_i) \wedge \bar{x} \in V_{\bar{b}} \setminus \left(\bigcup_{\mathbb{G}_m^n \neq T \in \mathcal{T}} \bigcup_{i=1}^{n_T} \bar{c}_{T,i} \cdot T \right),$$

such that $\mathrm{MR}(\varphi(\bar{x}, \bar{b}, \bar{c})) = (d, 1)$ with q as its unique generic type.

The following are definable conditions in the parameters $\bar{b}', \bar{c}'$ (by Proposition 4.5 and Fact 3.8, since Morley rank is definable in an almost strongly minimal theory):

$(*)$ $V_{\bar{b}'}$ is Kummer generic and $\delta(V_{b'}) = d$;
$(**)$ $\mathrm{MR}(\varphi(\bar{x}, \bar{b}', \bar{c}')) = d$; and
$(***)$ for any $T \in \mathcal{T}(V)$, such that $T \neq \mathbb{G}_m^n$, the intersection of $\varphi(\bar{x}, \bar{b}', \bar{c}')$ with any coset of T is of Morley rank $< d$.

If $(*), (**)$ and $(***)$ are satisfied, then $\mathrm{MRD}(\varphi(\bar{x}, \bar{b}', \bar{c}')) = (d, 1)$, showing that q is a good type. □

Note that in the previous proof, the conditions $(*), (**)$ and $(***)$ guarantee that when assigning the green colour to a generic point $\bar{a}'$ of $V_{\bar{b}'}$ (over $K' \models T_\mu$), we obtain a self-sufficient extension of K' that stays in the class $\mathcal{C}_0^\mu$. A priori, it is not clear that this is a definable condition in the parameters.

THEOREM 5.5. *The theory $T_\mu A$ exists. The $\mathcal{L}_{\text{rings}} \cup \{\sigma\}$-reduct of $T_\mu A$ equals $ACFA_0$.*

Proof. The existence of $T_\mu A$ follows from Theorem 5.4, using Example 2.3.

Note that if $(K, \ddot{U}(K))$ is a green field from the class $\mathcal{C}_0^\mu$ and L is an algebraically closed field containing K, then $(L, \ddot{U}(K)) \in \mathcal{C}_0^\mu$ (see [**3**, Folgerung 8.3]). Using this, the argument concerning the reduct to the language of difference fields is the same as in the proof of Theorem 5.2. □

5.3. *Bad pseudofinite fields*

We now give an application to pseudofinite fields, showing that the fixed field of a model of $T_\mu A$ is what might be called a 'bad pseudofinite field' of characteristic 0.

Recall that every pseudofinite field is supersimple of SU-rank 1, with $\mathrm{SU}(\bar{a}/K) = \mathrm{tr.deg}(\bar{a}/K)$ (see [**16**, **26**] for pseudofinite fields and simple theories).

COROLLARY 5.6. *Let F' be a pseudofinite field of characteristic 0. Then there is $F \succcurlyeq F'$ and an infinite divisible torsion-free subgroup $\ddot{U}$ of the multiplicative group of F, such that $(F, +, \times, \ddot{U})$ is supersimple of SU-rank 2, with $\ddot{U}$ of SU-rank 1.*

Proof. Choose $K \models T_\mu A$ (sufficiently saturated), such that the fixed field F is an elementary extension of F'. (It is easy to see that there is $(K, \sigma) \models ACFA_0$ with this property [**6**]; by Theorem 5.5, this is sufficient.) Note that F is stably embedded, by Facts 2.1(4) and 3.9. We first show that the full induced structure on F is supersimple of SU-rank 2, with $\mathrm{SU}(\ddot{U}) = 1$. We denote this theory by $\mathrm{Th}(F)$. Choose an element $g \in \ddot{U}(F)$. Note that $\mathrm{acl}_{\mathrm{Th}(F)}(B) = \mathrm{acl}_{T_\mu A}(B) \cap F = \mathrm{acl}_\mu(B) \cap F$ for any $B \subseteq F$ (by Fact 2.1(1)). Now assume $g \notin \mathrm{acl}_\mu(\emptyset)$. Then $\mathrm{d}(g) = \mathrm{MR}_{T_\mu}(g/\emptyset) = 1$. For every $B \subseteq F$ the following holds:

$$g \mathop{\smile\!\!\!\!\!\!\;|}^{T_\mu} B \Leftrightarrow g \notin \mathrm{acl}_\mu(B) \Leftrightarrow g \notin \mathrm{acl}_{\mathrm{Th}(F)}(B) \Leftrightarrow g \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{Th}(F)} B.$$

So the only forking extensions of $\mathrm{tp}_{\mathrm{Th}(F)}(g)$ are algebraic, from which we deduce $\mathrm{SU}_{\mathrm{Th}(F)}(g) = 1$, thus $\mathrm{SU}_{\mathrm{Th}(F)}(\ddot{U}) = 1$. Next we show that there is a 1-type in $\mathrm{Th}(F)$ of rank 2. Choose generic independent green elements g_1, g_2 in F, and put $w = g_1 + g_2$. Then $\delta(g_1, g_2/w) = 0$, so w and (g_1, g_2) are interalgebraic (in T_μ, hence also in $\mathrm{Th}(F)$). By the Lascar inequalities, we compute $\mathrm{SU}_{\mathrm{Th}(F)}(w) = \mathrm{SU}_{\mathrm{Th}(F)}(g_1, g_2) = 2$.

On the other hand, using the characterization of non-forking in Fact 2.1(3), by an easy induction on Morley rank we show that, for any $\bar{a} \in F$ and $B \subseteq F$, one has $\mathrm{MR}_{T_\mu}(\bar{a}/B) \geqslant \mathrm{SU}_{\mathrm{Th}(F)}(\bar{a}/B)$.

The structure $F_{\ddot{U}} = (F, +, \times, \ddot{U})$ is a reduct of the full induced structure on F. Moreover, as $\ddot{U}$ is an infinite definable subgroup of infinite index in the multiplicative group of F, it follows that $\mathrm{SU}(F_{\ddot{U}}) \geqslant 2$. We complete the proof using the following general lemma (it is folklore; for convenience, we include a proof). □

LEMMA 5.7. *Let T' be a reduct of the simple theory T, and π' be a partial T'-type, such that $\mathrm{SU}_T(\pi') < \omega$. Then $\mathrm{SU}_{T'}(\pi') \leqslant \mathrm{SU}_T(\pi')$.*

Proof. We argue by induction on $\mathrm{SU}_{T'}(\pi') = n \in \mathbb{N}$, the case $n = 0$ being trivial. Suppose $\mathrm{SU}_{T'}(\pi') = n + 1$, where π' is a partial type over A. Taking a T'-non-forking extension if necessary, we may suppose $A = M \models T$. Let $B \supset M$ and $a \models \pi'$, such that $\mathrm{SU}_{T'}(a/B) = n$. By induction, we know that $\mathrm{SU}_T(a/B) \geqslant n$. Moreover, since $a \not\perp^{T'}_M B$ is witnessed by any T'-Morley sequence in $\mathrm{tp}_{T'}(B/M)$ (see, for example, [**26**, Theorem 2.4.7]), in particular by any T-coheir sequence in $\mathrm{tp}_T(B/M)$, we may deduce $a \not\perp^{T}_M B$ from $a \not\perp^{T'}_M B$. $\square$

QUESTION 5.8. Is it possible to obtain the green pseudofinite field $(F, \ddot{U})$ of rank 2 (or the one of infinite rank) as an ultraproduct of coloured finite fields $(\mathbb{F}_q, N_q)$?

6. *Other Hrushovski amalgams*

We briefly review Hrushovski's amalgamation method (see [**11**] for a detailed account of this method). It is a variation of Fraïssé's original method, and a powerful tool to construct stable structures with prescribed pregeometry.

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures, $\mathcal{C}^{\mathrm{fin}} \subseteq \mathcal{C}$ be the class of finite (or 'finitely generated' in some sense) structures in $\mathcal{C}$ and $\delta : \mathcal{C}^{\mathrm{fin}} \to \mathbb{Z}$ be a *predimension function* satisfying some natural conditions. For $A \subseteq B$ in $\mathcal{C}^{\mathrm{fin}}$ put $\delta(B/A) := \delta(B) - \delta(A)$ (this definition may be extended to infinite A, as long as B is finitely generated over A). The structure A is said to be *self-sufficient* in B (denoted by $A \leqslant B$) if $\delta(B'/A) \geqslant 0$ for any $A \subseteq B' \subseteq B$ with B' finitely generated over A. Let $\mathcal{C}_0 = \{M \in \mathcal{C}_0 \mid \emptyset \leqslant M\}$ and consider the class $(\mathcal{C}_0, \leqslant)$. In all the examples that we consider, $\mathcal{C}_0$ is an elementary class, $\mathcal{C}_0^{\mathrm{fin}}$ is countable up to isomorphism, and $(\mathcal{C}_0, \leqslant)$ has AP and JEP. So there is a unique countable structure M_ω in $\mathcal{C}_0$ which is homogeneous with respect to $(\mathcal{C}_0, \leqslant)$, the *Fraïssé–Hrushovski limit* of $(\mathcal{C}_0^{\mathrm{fin}}, \leqslant)$. In order to establish the desired properties for $T_\omega = \mathrm{Th}(M_\omega)$, one has to show that M_ω is saturated.

The theory T_ω obtained in this way is usually of infinite (Morley) rank, and a more intricate second step, the so-called *collapse*, is needed to obtain a theory of finite Morley rank, where the rank is given by the *dimension* (that is, the 'eventual predimension') that comes out of the construction, $\mathrm{d}(A) := \min\{\delta(A') \mid A \subseteq A' \subseteq K\}$. The rough idea is to bound uniformly the number of realizations of types in T_ω of dimension 0. Technically, this is done by choosing families of strongly minimal sets in T_ω that coordinatize all such types of dimension 0 and to associate to any such family $\mathcal{F}$ a natural number $\mu(\mathcal{F})$. One obtains an elementary subclass $\mathcal{C}_0^\mu \subseteq \mathcal{C}_0$. The most delicate parts of the construction are to establish that $(\mathcal{C}_0^\mu, \leqslant)$ has AP, and the fact that the Fraïssé–Hrushovski limit M_μ of the finite structures in $(\mathcal{C}_0^\mu, \leqslant)$ is saturated. All this is analogous to the construction of green and bad fields, which was outlined in Section 3.

A famous instance of the aforementioned amalgamation method is Hrushovski's *fusion construction*, where two arbitrary strongly minimal theories (with DMP) are fused into a single strongly minimal theory ([**13**]; see also [**11**] for a detailed exposition of the uncollapsed fusion). For $i = 1, 2$, let T_i be strongly minimal $\mathcal{L}_i$-theories with DMP. We may assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjoint relational languages and that T_i has quantifier elimination. For $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$ consider the class $\mathcal{C}$ of models of the $\mathcal{L}$-theory $T_1^\forall \cup T_2^\forall$ and, for finite $A \in \mathcal{C}$, put $\delta(A) = \mathrm{d}_1(A) + \mathrm{d}_2(A) - |A|$, where $\mathrm{d}_i(A)$ is Morley rank in the sense of T_i. The above techniques apply. The theory T_ω is called the *free fusion* of T_1 and T_2 (over equality); the desired strongly minimal fusion is given by T_μ.

FACT 6.1. Let T_ω be the free fusion of the strongly minimal theories T_1 and T_2.

(1) The free fusion T_ω is ω-stable.

(2) Let A and A' be self-sufficient subsets of $\mathfrak{C} \models T_\omega$. Then $\mathrm{tp}_\omega(A) = \mathrm{tp}_\omega(A')$ if and only if $\mathrm{tp}_{T_i}(A) = \mathrm{tp}_{T_i}(A')$ for $i = 1, 2$.

(3) Let $K \preccurlyeq \mathfrak{C}$ and $\bar{a} \in \mathfrak{C}$ be a finite tuple. There is some finite $\hat{a} \supseteq \bar{a}$, such that $K\hat{a} \leqslant \mathfrak{C}$.

(4) Let $\bar{x} = (x_0, \ldots, x_{n-1})$, $0 \leqslant d \leqslant n$ and $\bar{z}$ be an arbitrary tuple of variables. Then there is a partial type $\pi_d(\bar{x}, \bar{z})$, such that, for any $\bar{a}, \bar{b}$, one has $\models \pi_d(\bar{a}, \bar{b})$ if and only if $\mathrm{d}(\bar{a}/\bar{b}) \geqslant d$.

Proof. The first three items are proved in [11], and the last part is an easy consequence of definability of Morley rank in strongly minimal theories. □

THEOREM 6.2. *For the following theories obtained by Hrushovski's amalgamation method without collapse, all ω-stable of infinite rank, the generic automorphism is axiomatizable.*

(1) *The ab initio construction* [14].

(2) *The free fusion of two strongly minimal theories T_1 and T_2, where both T_1 and T_2 have DMP* [13] (*see also* [11]).

(3) *The free fusion of two strongly minimal theories T_1 and T_2 over a common subtheory T_0, where both T_1 and T_2 have DMP and T_0 is ω-categorical, modular and satisfies $\mathrm{acl}_{T_0} = \mathrm{dcl}_{T_0}$* (*for example, for T_0 the theory of an infinite vector space over some finite field*) [11].

(4) *The black fields of Poizat in all characteristics* [23].

(5) *The red fields of Poizat in positive characteristic* [24].

(6) *The theory of the generic plane curve over an algebraically closed field constructed in* [5].

Proof. We give the argument for (2), the other cases being similar. So let T_ω be the free fusion of two strongly minimal theories T_1 and T_2 having DMP. We shall exhibit a geometric notion of genericity and apply Proposition 2.5. The construction is parallel to the one given in Lemma 5.1, although the definability problems we encountered in the case of Poizat's green fields do not arise in the context of the free fusion.

Let $K \preccurlyeq \mathfrak{C}$ and let $\bar{a} \in \mathfrak{C}$ be a finite tuple. Then $\mathrm{tp}_\omega(\bar{a}/K)$ is called *special* if $K\bar{a} \leqslant \mathfrak{C}$. Now let $p(\bar{x}) = \mathrm{tp}_\omega(\bar{a}/K)$ be special. For convenience we assume that $\bar{a} = (a_0, \ldots, a_{n-1})$ enumerates $K\bar{a} \setminus K$ (without repetitions). By Fact 6.1, p is determined by $p_1 = \mathrm{tp}_1(\bar{a}/K)$ and $p_2 = \mathrm{tp}_2(\bar{a}/K)$. For $I \subseteq \{0, \ldots, n-1\} = \mathbf{n}$, let $k_i^I := \mathrm{MR}_{T_i}(a_I/K)$. Then (by the assumptions) the following constraints are satisfied:

$$k_1^I > 0 \text{ and } k_2^I > 0 \text{ whenever } I \neq \emptyset, \tag{6.1}$$

$$k_1^I + k_2^I - |I| \geqslant 0 \text{ for all } I \subseteq \mathbf{n}, \tag{6.2}$$

$$d(\bar{a}/K) = \delta(\bar{a}/K) = k_1^{\mathbf{n}} + k_2^{\mathbf{n}} - n. \tag{6.3}$$

We choose $\mathcal{L}_i$-formulas $\varphi_i(\bar{x}, \bar{z}_i)$ and $\bar{b}_i \in K$, such that the following conditions are satisfied:

(i) p_i is the unique $\mathcal{L}_i$-generic type in $\varphi_i(\bar{x}, \bar{b}_i)$ over K (for $i = 1, 2$);
(ii) the formulas $\varphi_i(\bar{x}, \bar{z}_i)$ avoid all diagonals;
(iii) if $\varphi_i(\bar{x}, \bar{b}') \neq \emptyset$, then this is a formula of Morley degree 1; and
(iv) if $\bar{a}'$ is $\mathcal{L}_i$-generic in $\varphi_i(\bar{x}, \bar{b}_i')$ over K' (where $\bar{b}_i' \in K'$), then $\mathrm{MR}_{T_i}(a_I'/K') = k_i^I$ for all $I \subseteq \mathbf{n}$.

Using DMP in T_i, it is easy to see that formulas of the form $\varphi_i(\bar{x}, \bar{b}_i)$ exist and are dense in p_i.

Put $d = d(\bar{a}/K) = k_1^{\mathbf{n}} + k_2^{\mathbf{n}} - n$. A *special* formula is a formula of the form

$$\varphi(\bar{x}, \bar{z}) = \varphi_1(\bar{x}, \bar{z}_1) \wedge \varphi_2(\bar{x}, \bar{z}_2) \wedge \varphi_d(\bar{x}, \bar{z}),$$

where $\bar{z} \supseteq \bar{z}_1, \bar{z}_2$ is some tuple of variables, $\varphi_d(\bar{x}, \bar{z})$ is a formula from $\pi_d(\bar{x}, \bar{z})$ (see Fact 6.1) and φ_1, φ_2 are as above, satisfying (i–iv).

If $p(\bar{x}) = \mathrm{tp}_\omega(\bar{a}/K)$ is special with $\mathrm{d}(\bar{a}/K) = d$, then $p_1(\bar{x}) \cup p_2(\bar{x}) \cup \pi_d(\bar{x}, K) \vdash p(\bar{x})$, where $p_i = p{\restriction}_{\mathcal{L}_i}$. It follows that (instances of) special formulas are dense in any special type.

Claim. Let $\varphi(\bar{x},\bar{z}) = \varphi_1(\bar{x},\bar{z}_1) \wedge \varphi_2(\bar{x},\bar{z}_2) \wedge \varphi_d(\bar{x},\bar{z})$ be a special formula and $\bar{b}' \in K' \models T_\omega$, such that $\exists\bar{x}\varphi_1(\bar{x},\bar{b}'_1) \wedge \exists\bar{x}\varphi_2(\bar{x},\bar{b}'_2)$. Then there is a unique special type $p \in S_n(K')$, such that $p_i = p\restriction_{\mathcal{L}_i}$ is generic in $\varphi_i(\bar{x},\bar{b}'_i)$ for $i = 1, 2$.

Let p_i be the (by (iii) unique) generic $\mathcal{L}_i$-type in $\varphi_i(\bar{x},\bar{b}'_i)$. Note that by (ii) the types p_1 and p_2 agree on their reduct to mere equality. By Hasson and Hils [**11**, Lemma 3.10] and (6.2) there is $\bar{a}' \models p_1 \cup p_2$, such that $K'\bar{a}' \leqslant \mathfrak{C}$. Then $\mathrm{tp}(\bar{a}'/K')$ is special and uniquely determined by these data by Fact 6.1(2). From $K\bar{a}' \leqslant \mathfrak{C}$ we deduce that $\mathrm{d}(\bar{a}'/K') = \delta(\bar{a}'/K') = k_1^{\mathbf{n}} + k_2^{\mathbf{n}} - n = d$, so $\models \pi_d(\bar{a}',\bar{b}')$ and in particular $\models \varphi(\bar{a}',\bar{b}')$. This proves the claim.

We now define a notion of genericity $R_{\mathfrak{g}}$ for special types and formulas. We say that the special type $p(\bar{x}) \in S(K)$ is *generic* in the special formula $\varphi(\bar{x},\bar{b}) = \varphi_1(\bar{x},\bar{b}_1) \wedge \varphi_2(\bar{x},\bar{b}_2) \wedge \varphi_d(\bar{x},\bar{b})$ if $p_i = p\restriction_{\mathcal{L}_i}$ is generic in $\varphi_i(\bar{x},\bar{b}_i)$ for $i = 1, 2$.

It follows from the claim that the special types/formulas are precisely the nice types/formulas with respect to $R_{\mathfrak{g}}$. Clearly, $R_{\mathfrak{g}}$ is invariant and coherent. Moreover, by Fact 6.1(3), there are enough nice types. In order to prove that $R_{\mathfrak{g}}$ is geometric, it remains to show property (4) from Definition 2.4, that is, the definability of generic projections.

Let $\bar{x} = \bar{x}'\cup\bar{x}''$, $\bar{x}' = x_{I'}$ for $I' \subseteq \mathbf{n}$, and let $\varphi'(\bar{x}',\bar{z}') = \varphi'_1(\bar{x}',\bar{z}'_1) \wedge \varphi'_2(\bar{x}',\bar{z}'_2) \wedge \varphi'_{d'}(\bar{x}',\bar{z}')$ and $\varphi(\bar{x},\bar{z}) = \varphi_1(\bar{x},\bar{z}_1) \wedge \varphi_2(\bar{x},\bar{z}_2) \wedge \varphi_d(\bar{x},\bar{z})$ be special formulas, where the integers k_i^I are associated with $\varphi(\bar{x},\bar{z})$. Let $\bar{b},\bar{b}' \in K \models T_\omega$ be such that both $\varphi(\bar{x},\bar{b})$ and $\varphi'(\bar{x}',\bar{b}')$ are non-empty. Clearly, the projection of $\varphi(\bar{x},\bar{b})$ onto the $\bar{x}'$-variables is generic in $\varphi'(\bar{x}',\bar{b}')$ if and only if, for any generic (over K) $\bar{a} \models \varphi(\bar{x},\bar{b})$, the tuple $\bar{a}' = a_{I'}$ is generic in $\varphi'(\bar{x}',\bar{b}')$ over K, that is, if $\bar{a}'$ is $\mathcal{L}_i$-generic in $\varphi'_i(\bar{x}',\bar{b}'_i)$ over K for $i = 1, 2$ and $K\bar{a}' \leqslant \mathfrak{C}$, or equivalently $K\bar{a}' \leqslant K\bar{a}$. This is the case if and only if the following two definable properties hold (note that the second one is either always or never satisfied for a given pair of special formulas):

(1) $\exists\bar{x}''\varphi_i(\bar{x}'\bar{x}'',\bar{b}_i)$ is $\mathcal{L}_i$-generic in $\varphi'_i(\bar{x}',\bar{b}'_i)$ for $i = 1, 2$; and
(2) $k_1^J + k_2^J - |J| \geqslant k_1^{I'} + k_2^{I'} - |I'|$ for all $I' \subseteq J \subseteq \mathbf{n}$.

□

It is known that all the theories from Theorem 6.2 may be collapsed onto theories of finite rank. Using arguments that are similar (albeit much simpler) to the proof of Theorem 5.4, one obtains the following result.

Theorem 6.3. *The collapsed versions of all the theories from Theorem* 6.2 *have finite and additive Morley rank with DMP. In particular, the generic automorphism is axiomatizable in these collapsed theories using geometric axioms as in Example* 2.3.

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