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DEFINABLE EQUIVARIANT RETRACTIONS IN NON-ARCHIMEDEAN GEOMETRY

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ABSTRACT. For G an algebraic group definable over a model of ACVF, or more generally a definable subgroup of an algebraic group, we study the stable completion $\widehat{G}$ of G, as introduced by Loeser and the second author. For G connected and stably dominated, assuming G commutative or that the valued field is of equicharacteristic 0, we construct a pro-definable G-equivariant strong deformation retraction of $\widehat{G}$ onto the generic type of G.

For $G = S$ a semiabelian variety, we construct a pro-definable S-equivariant strong deformation retraction of $\widehat{S}$ onto a definable group which is internal to the value group. We show that, in case S is defined over a complete valued field K with value group a subgroup of $\mathbb{R}$, this map descends to an $S(K)$-equivariant strong deformation retraction of the Berkovich analytification S^{an} of S onto a piecewise linear group, namely onto the skeleton of S^{an}. This yields a construction of such a retraction without resorting to an analytic (non-algebraic) uniformization of S.

Furthermore, we prove a general result on abelian groups definable in an NIP theory: any such group G is a directed union of ∞-definable subgroups which all stabilize a generically stable Keisler measure on G.

CONTENTS

1. Introduction

In [9], Loeser and the second author have developed a novel model-theoretic approach to non-Archimedean (algebraic) geometry. For an algebraic variety V defined over some valued field K, they construct the so-called *stable completion* of V, a pro-definable space, more precisely a functor $\widehat{V}$ which is pro-definable over K and which associates to any valued field extension L of K a topological space $\widehat{V}(L)$ naturally containing $V(L)$, endowed with the valuation topology, as a subspace. In case L is non-trivially valued and algebraically closed, $V(L)$ is dense in $\widehat{V}(L)$.

The construction of $\widehat{V}$ is similar to that of the Berkovich analytification V^{an} of V ([2], see also [4]), which is only defined when K is complete with value group a subgroup of $\mathbb{R}$. As is the case for V^{an}, the stable completion $\widehat{V}$ has good topological properties, e.g., analogues of local (pathwise) connectedness and local compactness hold for $\widehat{V}$ in the definable category. Assuming that V is quasi-projective, strong topological tameness properties are established for $\widehat{V}$ in [9]. Most notably, it is shown that $\widehat{V}$ admits a pro-definable strong deformation retraction onto a Γ-internal space, i.e., onto a piecewise linear space in the definable category. Here, Γ denotes the value group. This parallels topological tameness results in the case of V^{an}, established by Berkovich. Actually, assuming K is complete with $\Gamma_K \leq \mathbb{R}$, it is shown in [9, Chapter 14] that, for some suitable $K^{max} \supseteq K$, V^{an} may be obtained as a topological quotient of $\widehat{V}(K^{max})$ and the pro-definable strong deformation retraction descends to a strong deformation retraction of V^{an} onto a finite simplicial complex, thus reproving and generalizing results by Berkovich, as this holds for every quasi-projective variety V, without any smoothness assumption on V. Moreover, in this context, local contractibility of V^{an} is established in [9], as well as the finiteness of the number of homotopy types in $\{\mathcal{V}_b^{\mathrm{an}} \mid b \in S(K)\}$, where $\mathcal{V} \to S$ is a quasi-projective family of algebraic varieties defined over K. For a survey of the results in [9] and in particular the consequences in the realm of Berkovich spaces, we refer to [5].

The main tools used in [9] concern the geometric model theory of the (first order) theory ACVF of algebraically closed non-trivially valued fields. The study of ACVF goes back to Abraham Robinson who established its model-completeness. His arguments actually yield quantifier elimination for ACVF in various natural languages. Haskell, Macpherson and the second author initiated the study of ACVF from a point of view of geometric model theory. In [6], they show that the imaginary sorts in ACVF are classified by certain natural group quotients, the so-called *geometric sorts*, and they identify the stable part of ACVF as the collection of definable sets which are internal in the residue field k. In [7] the authors systematically develop stable domination as a means to lift phenomena known in stable theories to the unstable context. Stably dominated types are definable types whose generic extension is entirely controlled by its stable part. In ACVF, these correspond to the definable types orthogonal to Γ.

Let us briefly describe the construction of the Berkovich analytification of an algebraic variety. Suppose that K is a field which is complete with respect to a non-Archimedean norm $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ and that V is an algebraic variety defined over K. The construction of the anaytification V^{an} is done by gluing affine pieces, so let us assume that V is affine. Then V^{an} is the set of all multiplicative semi-norms $|\cdot|_v\colon K[V] \to \mathbb{R}_{\geq 0}$ extending $|\cdot|$ on K, and the topology on V^{an} is defined as

the weakest topology such that for any $f \in K[V]$, the map $|\cdot|_v \mapsto |f|_v$ is continuous. In this way, the natural embedding of $V(K)$ into V^{an} becomes a homeomorphism.

Identifying $\mathbb{R}_{\geq 0}$ with $\mathbb{R} \cup \{\infty\}$ via $-\log$, the field K becomes a (complete) valued field with $\Gamma_K \leq \mathbb{R}$, and, by quantifier elimination in ACVF, as a set, V^{an} equals

$$\{p = \mathrm{tp}(\overline{a}/K \cup \mathbb{R}) \in S_V(K \cup \mathbb{R}) \mid \Gamma_{K(\overline{a})} \leq \mathbb{R}\},$$

where $S_V(K \cup \mathbb{R})$ denotes the set of complete types concentrating on V, in the theory ACVF, over the (2-sorted) parameter set $K \cup \mathbb{R}$.

The key insight of [9] is that one may obtain a model-theoretic analogue of V^{an} by considering those (global definable) types concentrating on V whose realizations do not increase the value group, relative to the base model over which one works. Consequently, the stable completion $\widehat{V}$ is defined as the set of stably dominated types concentrating on V, endowed with a topology which is similarly defined as in the case of V^{an}. It is an important feature that, when one identifies a definable type with its canonical base, $\widehat{V}$ naturally gets the structure of a (strict) pro-K-definable space in ACVF. Thus model-theoretic methods may be applied to $\widehat{V}$, in particular powerful tools both from stability theory and from o-minimality, two well developed and completely different strands of geometric model theory. Any type $\mathrm{tp}(\overline{a}/K)$ with $\mathrm{tr}(K(\overline{a})/K) = \mathrm{tr}(k_{K(\overline{a})}/k_K)$ is stably dominated, and the collection of these types, which corresponds to the sub-collection of all strongly stably dominated types, forms an ind-definable subspace $V^{\#}$ of $\widehat{V}$.

Now let us move to the equivariant situation. If K is a complete field with respect to an non-Archimedean norm and G is an algebraic group defined over K, then $G(K)$ naturally acts on G^{an}, and one may ask whether there is a $G(K)$-equivariant retraction of G^{an} onto a skeleton which is then naturally a piecewise linear group. For E a Tate elliptic curve defined over K, i.e., an elliptic curve with bad (split) multiplicative reduction, E^{an} does indeed admit an $E(K)$-equivariant strong deformation retraction onto the circle group $\mathbb{R}/\mathbb{Z}$, as one may for example show using the Tate uniformization; in case E is an elliptic curve with good reduction, E^{an} equivariantly retracts to the trivial group. Actually, whenever S is an abelian or even semi-abelian variety defined over K, passing to a finite separable extension of K if necessary, S^{an} admits an analytic uniformization by an analytic group which is equivariantly contractible, from which one obtains an $S(K)$-equivariant strong deformation retraction of S^{an} onto its skeleton (see [2, Section 6.5], where this is explained for abelian varieties).

In the present paper, we prove analogous results for the stable completion $\widehat{S}$ of a semiabelian variety S defined over some valued field K, confirming that the model-theoretic approach to non-Archimedean geometry developed in [9] is very adequate for the study of semiabelian varieties as well, as the topological tameness properties one expects do also hold in this context. We would like to highlight that, contrarily to the classical approach via analytic uniformizations (see [3]), our construction may be performed over an arbitrary valued field (with value group not necessarily Archimedean) and does not require the passage to a finite separable extension in the process. Moreover, we reprove the results for S^{an} mentioned in the previous paragraph in the same way topological tameness results for V^{an} could be deduced in [9] from the corresponding results for the stable completion $\widehat{V}$ in the definable category.

Before we may state the main contributions of our paper, we need to mention a result from [12], which is a crucial ingredient in our construction.

In [7], an important and very useful structural way to decompose types in ACVF is obtained: if M is a maximally complete model of ACVF and $\overline{a}$ is a (possibly imaginary) tuple from the monster model $\mathcal{U}$, then $\mathrm{tp}(\overline{a}/M \cup \Gamma(M\overline{a}))$ is stably dominated. So types may be understood, to some extent, by types in the value group and types in the residue field.[1] Moreover, ACVF has the invariant extension property, i.e., any type over an algebraically closed set admits a global automorphism invariant extension. It follows that ACVF is a *metastable* theory.

Metastability is an axiomatic framework capturing the phenomena we described in the previous paragraph. It has been introduced in [7] and further developed in work of Rideau-Kikuchi and the second author [12], who undertake a thorough study of groups definable in metastable theories. In their work, stably dominated groups, i.e., (∞-)definable groups admitting a stably dominated generic type, play a key role, an important reason being that many features of stable groups lift to them. As one of the main results, they establish a group version of the metastability property in the commutative case: for any definable abelian group G in ACVF, there is a definable homomorphism $\lambda : G \twoheadrightarrow \Lambda$ such that Λ is Γ-internal and $N := \ker(\lambda)$ is a connected group which is an increasing union (indexed essentially by the value group) of definable stably dominated subgroups.

We show that whenever $G = S$ is a semiabelian variety, N is in fact itself stably dominated. More precisely, if S is defined over a valued field $F \subseteq K \models$ ACVF, there is an F-definable decomposition

$$(1.1) \qquad\qquad 0 \to N \to S \to \Lambda \to 0$$

such that Λ is Γ-internal and N stably dominated, definable and connected. In particular, it follows that $p_N \in \widehat{S}$, where p_N denotes the generic type of N.

For abelian varieties, stable domination of N is shown in [12]. It is easy to see that this also holds for an algebraic torus, e.g., if $G = \mathbb{G}_m^n$, then $N = (\mathcal{O}^*)^n$ is stably dominated, with $G/N \cong \Gamma^n$. We show that stable domination of the corresponding N is preserved in short exact sequences of algebraic groups. As a semiabelian variety is an extension of an abelian variety by an algebraic torus, we may thus lift the result to the semiabelian case.

For G any definable group, we denote by $p_e \in \widehat{G}$ the type of the identity element in G. We obtain the following result.

Theorem A (Theorem 3.10)**.** *Let S be a semi-abelian variety defined over $F \subseteq K \models$ ACVF, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from* (1.1)*.*

Then there is an F-definable special[2] deformation retraction $\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$ with final image $\Sigma \subseteq \widehat{S}$ such that ρ is equivariant under the action of S by multiplication and for each $t < \infty$, $q_t = \rho(t, p_e)$ is the generic type of a connected strongly stably dominated definable subgroup N_t of N, with N_t Zariski dense in S.

Moreover, the morphism $\pi : S \to \Lambda$ induces definable bijection between Σ and Λ.

Modulo some continuity issues concerning the map given by the tensor product $\otimes$ (dealt with in Subsection 2.6), Theorem A follows in a rather straight forward

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[1]This may be seen as a very powerful incarnation of the Ax-Kochen-Ershov philosophy which postulates that model-theoretic questions about henselian valued fields may be reduced to questions about the residue field and the value group.

[2]See Definition 3.1 for the notion of a special deformation retraction.

way from the existence of a definable continuous path in $\widehat{N}$ between p_e and p_N (the generic type of N), along generic types of (Zariski-dense strongly stably dominated) definable subgroups N_t of N. Let us illustrate this first with two easy examples. Denote by $\eta_{c,\gamma}$ the generic type of the closed ball $B_{\geq\gamma}(c)$, for $\gamma \in \Gamma \cup \{\infty\}$.

- The standard deformation retraction $\rho : [0,\infty] \times \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$, sending $(\gamma, \eta_{c,\delta})$ to $\eta_{c,\min(\delta,\gamma)}$ is definable and $(\mathcal{O}, +)$-equivariant with final image $\{\eta_{0,0}\}$, and as an equivariant continuous map it is entirely determined by the path

$$q : [0,\infty] \to \widehat{\mathcal{O}}, \ q(\gamma) := \rho(\gamma, p_e) = \eta_{0,\gamma},$$

  where $p_e = \eta_{0,\infty}$. Thus, $N_\gamma = \gamma\mathcal{O}$ in this example.

- The map $\rho' : [0,\infty] \times \mathbb{G}_m \to \widehat{\mathbb{G}_m}$, $(\gamma, c) \mapsto \eta_{c,v(c)+\gamma}$ extends uniquely to a $\mathbb{G}_m$-equivariant deformation retraction $\rho : [0,\infty] \times \widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_m}$, via

$$\rho(\gamma, \eta_{c,v(c)+\delta}) = \eta_{c,v(c)+\min(\gamma,\delta)} \ \text{(for } c \neq 0 \text{ and } \delta \geq 0\text{)}.$$

  Its final image is $\{\eta_{c,v(c)} \mid c \neq 0\} = \{\eta_{0,\gamma} \mid \gamma \in \Gamma\} \cong \Gamma$, and one has $p_e = \eta_{1,\infty}$, $N = N_0 = \mathcal{O}^*$ and $N_\gamma = 1 + \gamma\mathcal{O}$ for all $\gamma \in (0,\infty)$. Setting $q_\gamma = \rho(\gamma, p_e) = \eta_{1,\gamma}$, one may check that

$$\rho(\gamma, p) = \widehat{m}(q_\gamma \otimes p),$$

  the convolution of q_γ and p. Here, m denotes the multiplication in $\mathbb{G}_m$.

In our paper, we give two proofs of the existence of a group path as above, one in arbitrary characteristic which is valid in all stably dominated abelian definable subgroups of an algebraic group, one in equicharacteristic 0, valid for all stably dominated subgroups of an algebraic group:

Theorem B (Theorem 3.6 & Theorem 4.17). *Let G be an algebraic group defined over a valued field F, and let N be an F-definable stably dominated connected subgroup of G. Assume that N is commutative or that F is of equicharacteristic 0. Then there is an F-definable path $q : [0,\infty] \to N^\#$ such that*

(i) $q_\infty = p_e$ and $q_0 = p_N$;

(ii) for any $t < \infty$, q_t is the generic type of a connected stably dominated definable subgroup N_t of N which is Zariski dense in N.

The construction in arbitrary characteristic relies on the existence of a definable path between p_e and p_N, which is then turned into a group path, using an averaging process (described in [12] and requiring commutativity of the group, for technical reasons) reminiscent of Zilber's Indecomposability Theorem. The second construction, only valid in equicharacteristic 0, is more explicit and does not require the group N to be commutative. It utilizes an intrinsic scale provided by the value group, and the subgroup N_t in the group path is found as the kernel of the homomorphism to the maximal $\mathcal{O}/t\mathcal{O}$-internal quotient of N. The proof may be understood as a linearization procedure at the level of generic types, which is available even if, in ACVF, there exists no exponential map in the definable category.

As we have already mentioned, we deduce Theorem A from Theorem B. Moreover, Theorem B yields the following strong contractibility result as a corollary.

Theorem C (Theorem 3.7 & Corollary 4.18). *Let N be a group which satisfies the assumptions of Theorem B. Then $\widehat{N}$ admits an F-definable N-equivariant special deformation retraction with final image $\{p_N\}$.*

It is worth mentioning the following connection to group schemes over $\mathcal{O}$. Assume the base field F is algebraically closed. If H is a connected Zariski-dense stably dominated (∞-definable) subgroup of the affine algebraic group G, there is a group scheme $\mathbb{H}$ over $\mathcal{O}$ such that H is definably isomorphic to the group of $\mathcal{O}$-valued points of $\mathbb{H}$. (See [12, Theorem 6.11].) Under additional assumptions on $\mathbb{H}$, a chain of stably dominated subgroups becomes visible geometrically, namely via the map $\mathrm{val}(c) \mapsto \ker(\mathbb{H}(\mathcal{O}) \to \mathbb{H}(\mathcal{O}/c\mathcal{O}))$.

Assume now that F is a valued field with value group $\Gamma_F \leq \mathbb{R}$ and that V is an algebraic variety defined over F. In [9, Chapter 14], strong topological links are established between the stable completion $\widehat{V}$ of V (the *definable* world) and the Berkovich analytification V^{an} (the *analytic* world). If F^{max} denotes an algebraically closed valued field extension of F with value group $\Gamma_{F^{\mathrm{max}}} = \mathbb{R}$ and residue field $k_{F^{\mathrm{max}}} = (k_F)^{\mathrm{alg}}$, then there is a natural continuous surjective and closed map $\pi_V : \widehat{V}(F^{\mathrm{max}}) \to V^{\mathrm{an}}$. Moreover, considering the parameter set $\mathbf{F} := (F, \mathbb{R})$, any pro-$\mathbf{F}$-definable (special) deformation retraction

$$H : [0, \infty] \times \widehat{V} \to \widehat{V}$$

descends to a (special) deformation retraction

$$\widetilde{H} : \mathbb{R}_\infty^+ \times V^{\mathrm{an}} \to V^{\mathrm{an}},$$

where $\mathbb{R}_\infty^+ = [0, \infty](\mathbf{F}) = \mathbb{R}_{\geq 0} \cup \{\infty\}$. If the final image Z of H is Γ-internal, then the final image of $\widetilde{H}$ equals $\mathbf{Z} = \pi_V(Z(F^{\mathrm{max}}))$, a set which is naturally homeomorphic to a piecewise linear subset of $\mathbb{R}_\infty^n$ (where $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ and $n \in \mathbb{N}$), carrying a $\mathbb{Q}$-tropical structure[3].

We call such deformations $\widetilde{H}$ *definably induced*. In case $V = G$ is an algebraic group and H is G-equivariant, the map $\widetilde{H}$ will automatically be $G(F)$-equivariant. Theorem A thus yields the following topological tameness result about analytifications of semi-abelian varieties.

Theorem D (Theorem 5.4). *Let S be a semi-abelian variety defined over a valued F with $\Gamma_F \leq \mathbb{R}$. Consider the F-definable decomposition $0 \to N \to S \to \Lambda \to 0$ from (1.1). Then there is a definably induced $S(F)$-equivariant special deformation retraction*

$$\tilde{\rho} : \mathbb{R}_\infty^+ \times S^{\mathrm{an}} \to S^{\mathrm{an}}$$

with final image a skeleton $\mathbf{\Sigma}$. The map $\pi : S \to \Lambda$ induces a bijection between Σ and $\Lambda(\mathbf{F})$, where $\Lambda(\mathbf{F})$ is a subset of $\mathbb{R}^n$ carrying the structure of a piecewise linear abelian group.

Let us mention that in case A is an abelian variety over a valued field F with $\Gamma_F \cong \mathbb{Z}$ and if $N \leq A$ is as in (1.1), then $N(F)$ corresponds to the $\mathcal{O}_F$-points of the identity component of the Néron model $\mathcal{A}$ of A.[4] Observe, though, that the (maximal) stably dominated subgroup N of A always exists, even when A is definable over a valued field with non-Archimedean value group.

In the present article, even if we recall the relevant key concepts and results, we assume the reader is familiar with the model theory of ACVF as developed in [6, 7] and more specifically with the model-theoretic approach to non-Archimedean

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[3]This means that all the coefficients in the inequalities describing $\mathbf{Z}$ may be taken from $\mathbb{Q}$.

[4]We thank Antoine Ducros and Peter Schneider for pointing this out.

geometry from [9]. Moreover, we will use results from [12] on stably dominated groups at many places, and so some acquaintance with these is certainly helpful.

Here is an overview of the article. In Section 2, we gather a number of facts which we will use in our paper, in particular on stable domination, the stable completion of an algebraic variety and on stably dominated groups in ACVF. The material we present is mostly from [7, 9, 12], but some facts are new, in particular one on the maximal internal quotient of a generically stable group (Proposition 2.9) and a continuity result concerning $\otimes$ in the stable completion (Proposition 2.23), which are both key ingredients in our construction.

In Section 3, we give our first construction of a definable group path, valid in all characteristics, and in Section 4, we present the explicit construction in equicharacteristic 0. The results about equivariant retractions are then transferred to the setting of Berkovich analytifications in Section 5, where we prove in particular Theorem D. Finally, in Section 6, we prove a general result on commutative groups definable in an arbitrary NIP theory (Proposition 6.3), which is a rather weak analogue of our main result, namely that any such group G may be written as a directed union of ∞-definable subgroups which all stabilize a generically stable Keisler measure on G.

2. Tying up some loose ends

We work in a complete theory T, and $\mathcal{U}$ denotes a monster model of T. We assume throughout that T eliminates quantifiers and imaginaries. Our notation and terminology is mostly standard. If D is a C-definable set, for C a subset of $\mathcal{U}$, we denote by $D(C)$ the set $D(\mathcal{U}) \cap \mathrm{dcl}(C)$. We will use various generalizations of definable sets in our paper, in particular (strict) pro-definable and ∞-definable sets, relatively definable subsets of such sets, and occasionally ind-definable and iso-definable sets. See [9, Section 2.2.] or [12] for definitions and properties of these notions. For basic facts about the model theory of ACVF, we refer to [6, 7].

2.1. **Stably embedded sets.** Let C be a small subset of $\mathcal{U}$. Recall that a (small) family $(X_i)_{i \in I}$ of C-definable sets in $\mathcal{U}$ is *stably embedded* if for any finite sequence $(i_1, \dots, i_m)$ of elements of I, any $\mathcal{U}$-definable subset of $\prod_{j=1}^{m} X_{i_j}$ is definable with parameters from $C \cup \bigcup_{i \in I} X_i$. If X and D are definable sets, X is said to be *D-internal* if $(X = \emptyset$ or) there are $m \in \mathbb{N}$ and a surjective $\mathcal{U}$-definable function $g : D^m \to X$.

Generically stable types are definable types that behave in some ways like types in a stable theory. For a definition of generic stability in an arbitrary theory T, see, e.g., [1]. We will only be concerned with generically stable types in NIP theories, where there are various useful characterizations of this notion (see [15, Theorem 2.29]).

Lemma 2.1. *Let p be a C-definable generically stable type, and D a C-definable set. Let f_- be a definable family of functions to D. Then the set of p-germs of instances of f_- is D-internal.*

Proof. Let X be the set of p-germs of instances of f_-, so X is a definable set since the type p is definable. Let $a_1, \dots, a_n$ be a sufficiently long Morley sequence of p over C. For a parameter c, let $g(c) = (f_c(a_1), \dots, f_c(a_n)) \in D^n$. Taking n large enough, if $g(c) = g(c')$ the germs of f_c and $f_{c'}$ are equal. Hence the map π from the image of g to X sending $g(c)$ to the p-germ of f_c shows that X is D-internal. $\quad \square$

Lemma 2.2. *(T is any theory.) Let X be an $\emptyset$-definable set. Assume that X is stably embedded, then so is $\mathrm{Int}(X)$: the union of the $\emptyset$-definable X-internal sets.*

Proof. First, let Y be $\emptyset$-definable and $f_a : X \to Y$ a definable bijection. Consider the $\emptyset$-definable set A of codes of bijections $f_{a'} : X \to Y$. For $f \in A$, define $g = f^{-1} \circ f_a$. Then $g : X \to X$ is a bijection. As X is stably embedded, $g = h_b$ for some definable function h_b, with b in some X^n. The function mapping b to f such that $h_b = f^{-1} \circ f_a$ is a-definable and defines a surjection from some definable subset of X^n to A. Hence A is X-internal. Now any definable subset of Y is definable using a parameter from A and parameters from X (since X is stably embedded), therefore any definable subset of Y is definable with parameters from $\mathrm{Int}(X)$.

Now if Y is an $\emptyset$-definable X-internal set, there is a definable surjection $f_a : X^m \to Y$. Let A be the set of codes of surjections $f_{a'} : X^m \to Y$. Then A is $\emptyset$-definable and any definable subset of Y is definable with parameters in X and A. It remains to show that A is X-internal. The argument is the same as above: let $f, f' \in A$, then we can define the correspondence $f'^{-1} \circ f := \{(x, x') \in X^m \times X^m : f(x) = f'(x')\}$. By stable embeddedness of X, any such correspondence can be written as h_b for some $b \in X^k$. Now fixing an element $f \in A$, we see that any element of A is definable from such a $b \in X^k$ along with f, hence A is X-internal as required. $\square$

2.2. **Stably dominated and strongly stably dominated types.** We briefly recall some facts around (strongly) stably dominated types. Given a set of parameters C, we denote by St_C the collection of all C-definable stable stably embedded sets. For a tuple a, we denote by $\mathrm{St}_C(a)$ the set $\mathrm{dcl}(Ca) \cap \mathrm{St}_C(\mathcal{U})$.

In the following definition, non-forking $\mathop{\smile}\limits_{}$ is meant with respect to the stable multi-sorted structure $\mathrm{St}_C(\mathcal{U})$.

Definition 2.3 ([7, 12]). A type $p = \mathrm{tp}(a/C)$ is called *stably dominated* if for any tuple b with $\mathrm{St}_C(a) \mathop{\smile}\limits_{\mathrm{St}_C(C)} \mathrm{St}_C(b)$, one has $\mathrm{tp}(a/\mathrm{St}_C(a)) \vdash \mathrm{tp}(a/\mathrm{St}_C(a)b)$ (or equivalently, $\mathrm{tp}(b/\mathrm{St}_C(a)) \vdash \mathrm{tp}(b/Ca)$). It is called *strongly stably dominated* if there exists a formula $\phi(x) \in \mathrm{tp}(a/\mathrm{St}_C(a))$ such that for any tuple b with $\mathrm{St}_C(a) \mathop{\smile}\limits_{\mathrm{St}_C(C)} \mathrm{St}_C(b)$, one has $\phi(x) \vdash \mathrm{tp}(a/\mathrm{St}_C(a)b)$.

Clearly, every strongly stably dominated type is stably dominated.

By [9, Proposition 2.6.12], if p is a definable type based on some $C = \mathrm{acl}(C)$ and $p|A$ is strongly stably dominated for some $A = \mathrm{acl}(A)$, then $p|C$ is strongly stably dominated.

Fact 2.4 ([7]). *Let p be a global definable type in some theory T.*

- *(1) If p is stably dominated, then it is generically stable.*
- *(2) In $T = \mathrm{ACVF}$, p is stably dominated if and only if it is generically stable if and only if it is orthogonal to Γ.[5]*

The following is a special case of [9, Proposition 8.1.2].

Fact 2.5. *We work in ACVF. Let q be an A-definable type on a variety V. Then q is strongly stably dominated if and only if $\dim(q) = \dim(h_*q)$ for some A-definable map h into a stable sort; where $\dim(h_*q)$ refers to Morley dimension.*

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[5]Recall that a global definable type p is called *orthogonal to* Γ if $\Gamma(\mathcal{U}a) = \Gamma(\mathcal{U})$ for $a \models p$.

2.3. **The stable completion of an algebraic variety.** We work in ACVF, and we will now briefly recall the notion of the *stable completion* of an algebraic variety from [9].[6] Given an algebraic variety V defined over a valued field F, its stable completion $\widehat{V}$ is a functor which to every set B over which V is defined associates the set $\widehat{V}(B)$ of all global B-definable stably dominated types concentrating on V. It is naturally given by a strict pro-F-definable set, i.e., by a projective limit of F-definable sets with surjective transition functions.

One endows $\widehat{V}$ with the following ind-definable topology: for any $F \subseteq K \models$ ACVF, a basis of open subsets for $\widehat{V}(K)$ is given by sets of the form $\{p \in \widehat{U} \mid g_*(p) \in \Omega\}$, for U an open affine subvariety of V defined over K, $\Omega \subseteq \Gamma_\infty$ an open subset and $g = \mathrm{val} \circ G$ for some regular function G on U. Here, $g_*(p) \in \Gamma_\infty(K)$ is defined as $g(a)$, where $a \models p|K$. As p is orthogonal to Γ, this is well-defined.

If $X \subseteq V$ is a definable subset, $\widehat{X} := \{p(x) \in \widehat{V} \mid p(x) \models x \in X\}$ is a relatively definable subset of $\widehat{V}$ which is endowed with the subspace topology. One denotes by $X^{\#}(B)$ the set of all strongly stably dominated types in $\widehat{X}(B)$. Then $X^{\#}$ is naturally an ind-definable set, and we endow it with the subspace topology.

We refer to [9] for the definition and study of definable analogues of various topological properties, in particular definable compactness and definable (path) connectedness.

2.4. **Pro-definable groups and generic types.** We now recall the concept of genericity in the context of pro-definable groups, following [12, Section 3]. Let G be a pro-C-definable group,[7] where C is some small parameter set. A global C-definable type p concentrating on G is called *right generic* in G over C if for all $g \in G(\mathcal{U})$, the type $g \cdot p$ is C-definable. Left genericity is defined in a similar way, as is right genericity in a pro-definable principal homogeneous space under G. A right or left generic type p of G is called *symmetric* if for any other global definable type q concentrating on G one has $p \otimes q = q \otimes p$. Note that in an NIP theory, this is equivalent to generic stability of p.

If G admits a smallest pro-$\mathcal{U}$-definable subgroup H of bounded index, H is called the *strong connected component* of G and denoted by G^{00}. Similarly, if the intersection H of all pro-$\mathcal{U}$-definable subgroups of finite index in G has bounded index in G, then H is called the *connected component* of G and denoted by G^0. The group G is called *connected* if $G = G^0$, and *strongly connected* if $G = G^{00}$.

The following fact combines [12, 3.4, 3.9, 3.11 & 3.12].

Fact 2.6. *(T is any theory.) Let G be a pro-definable group.*
Assume that p is a right (resp. left) generic type of G. Then the following holds:

(1) *G is pro-definably isomorphic to a pro-limit of definable groups. In particular, if G is ∞-definable, it is an intersection of definable groups.*
(2) *G^{00} exists and one has $G^{00} = G^0 = \mathrm{Stab}(p)$, where $\mathrm{Stab}(p)$ denotes the left (resp. right) stabilizer of p.*

Assume in addition that p is symmetric. Then the following holds:

(3) *Right and left generic types coincide in G, they are all symmetric, and G acts transitively on the set of generic types, which is bounded.*

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[6]We refer to [9] and also to [5] for details.

[7]See [12] for the definition of pro-definable and ∞-definable groups.

> *In particular, p is left and right generic, and G is connected if and only if p is the unique (right) generic type of G.*

In our work, mostly symmetric (right) generic types will play a role. We will call them *generic* types in what follows.

Definition 2.7. A pro-definable group is called *generically stable* (*stably dominated* or *strongly stably dominated*, respectively) if it admits a generic type which is generically stable (stably dominated or strongly stably dominated, respectively).

Note that all parts of Fact 2.6 apply to all generically stable and thus in particular to all stably dominated groups.

Lemma 2.8. *(T is any theory.) Let G and $H \leq G$ be pro-definable groups. Assume that G is stably dominated and that H admits a right generic type. Let a be generic in G. Then a is right generic in the coset $a \cdot H$ over $\pi(a)$, where $\pi : G \to G/H$ is the canonical projection.*

Proof. By Fact 2.6, we may assume that $a \in G^0$. Let b be right generic in H over a, such that $b \in H^0$, and so $b \in G^0$ as well. By symmetry of the generic type of G, a is generic in G over b, as is ab^{-1}. By symmetry again, b is right generic in H over ab^{-1}. Being a translate of b over ab^{-1}, $a = (ab^{-1}) \cdot b$ is right generic in $a \cdot H$ over ab^{-1}. (Note that $a \cdot H$ is definable over ab^{-1}.) $\qquad\square$

The following proposition is a key ingredient for our second construction of definable equivariant retractions in Section 4.

Proposition 2.9. *Let G be a pro-C-definable generically stable connected group, and let D be a stably embedded C-definable set. There exists a pro-C-definable group $\mathfrak{g}_D$ internal to D and a pro-C-definable homomorphism $g : G \to \mathfrak{g}_D$ which is maximal in the sense that for any other pro-C-definable group homomorphism $g' : G \to \mathfrak{g}'_D$ with $\mathfrak{g}'_D$ internal to D, g' factors through g.*

The generic of $\mathfrak{g}_D$ is interdefinable over C with $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$, where a is a generic of G over C and $\mathrm{Int}_C(D)$ is the union of C-definable, D-internal sets.

Proof. We follow the proof of [12, Proposition 4.6] very closely. By Lemma 2.2, $\mathrm{Int}_C(D)$ is stably embedded. Let p be the (by Fact 2.6 unique) generic type of G. For any a, let $\theta(a)$ enumerate $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$. Fixing $a \in G$ generic over C, consider the map $f_a : b \mapsto \theta(ab)$ which is a pro-C-definable map on G. As the generic p of G is generically stable, the p-germ $\tilde{a}$ of f_a is strong by [1, Theorem 2.2], which means that there is a function $f'_{\tilde{a}}$ such that $c := f'_{\tilde{a}}(b) = f_a(b)$ for $b \models p|Ca$. On the other hand, the germ $\tilde{a}$ lives in $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$ by Lemma 2.1 hence we can write $f'_{\theta(a)}$. Then by stable embeddedness, $\mathrm{tp}(c, \theta(a)/C\theta(b)) \vdash \mathrm{tp}(c, \theta(a)/Cb)$, so $c \in \mathrm{dcl}(\theta(a), \theta(b))$. We can now write $\theta(ab) = c = F(\theta(a), \theta(b))$ for $a \models p$ and $b \models p|Ca$.

By the group chunk theorem ([12, Proposition 3.15]), there is a pro-C-definable group $\mathfrak{g}_D$ for which F coincides generically with the multiplication map and θ extends to a pro-C-definable homomorphism $g : G \to \mathfrak{g}_D$.

Universality follows from the fact that we have taken all of $\mathrm{dcl}(Ca) \cap \mathrm{Int}_C(D)$ in the image of θ. $\qquad\square$

Remark 2.10. In the situation of Proposition 2.9, if $C \subseteq C'$, denoting the groups computed using C and C' by $\mathfrak{g}_D^C$ and $\mathfrak{g}_D^{C'}$, respectively, there is by construction a canonical surjective pro-C'-definable homomorphism $\mathfrak{g}_D^{C'} \to \mathfrak{g}_D^C$.

2.5. **Stably dominated groups in** ACVF. The following result is a consequence of [12, Proposition 4.6 & Corollary 4.11].

Fact 2.11. *Let N be a definable stably dominated group in* ACVF. *Then $N^0 = N^{00}$ has finite index in N.*

Lemma 2.12. *Let N be a definable stably dominated group in* ACVF. *Then the generic types of N are strongly stably dominated.*

Proof. By Fact 2.11, we may assume that N is connected. Let C be a model over which N is defined, and we work over C. By [12, Proposition 4.6], there is a definable homomorphism $h : N \to \mathfrak{g}$, where $\mathfrak{g}$ is a definable group in St_C such that the generics of N are stably dominated via h, i.e., if p is a generic type of N and $a \models p|C$, then for any tuple b with $h(a) \underset{\mathrm{St}_C(C)}{\bigcup} \mathrm{St}_C(b)$, one has $\mathrm{tp}(b/Ch(a)) \vdash \mathrm{tp}(b/Ca)$.

We show that $\mathrm{tp}(a/Ch(a))$ is isolated by the formula $h(x) = h(a)$, from which the result follows. By [12, Lemma 4.9], we have that $\mathrm{tp}(a/C)$ is generic in G if and only if $\mathrm{tp}(h(a)/C)$ is generic in $\mathfrak{g}$. Hence if $h(a') = h(a)$, then a' is generic in N over C, hence $\mathrm{tp}(a'/C) = \mathrm{tp}(a/C)$, hence $\mathrm{tp}(a'h(a')/C) = \mathrm{tp}(ah(a)/C)$ and finally $\mathrm{tp}(a'/Ch(a)) = \mathrm{tp}(a/Ch(a))$ as required. □

Here is a partial converse to the preceding lemma.

Lemma 2.13. *Let N be an ∞-definable strongly stably dominated connected subgroup of an algebraic group G in* ACVF. *Then N is definable.*

Proof. We may replace G by the Zariski closure of N in G and thus assume without loss of generality that N is Zariski dense in G. As the generic type of N is strongly stably dominated, definability of N follows from [12, Proposition 4.6 in combination with Corollary 4.16]. □

Lemma 2.14. *Let N be a definable stably dominated group in* ACVF$_{0,0}$. *Assume that N is abelian and connected. Then N is divisible.*

Proof. Let p be a prime number and assume that $pN \neq N$. Then pN has infinite index in N and the group N/pN is an infinite stably dominated abelian p-group. We know, e.g., by [12, Proposition 4.6], that this group maps to an infinite stable group $\mathfrak{g}$, where $\mathfrak{g}$ is then also an abelian p-group. As $\mathfrak{g}$ is internal to k, it is definably isomorphic to a group definable in ACF_0, whence definably isomorphic to an algebraic group. It is well known that there are no infinite commutative algebraic groups in characteristic 0 which are p-groups.[8] □

If V is an algebraic variety defined over a valued field K, there is a natural notion of a *bounded* subset of $V(K)$. In [13, pp. 81–83], this is presented for complete valued fields with archimedean value group, but it readily generalizes to arbitrary valued fields, and the results we will use from [13] hold in the general case as well. If V is an affine variety, a subset $B \subseteq V(K)$ is said to be bounded in $V(K)$ if for any $f \in K[V]$ there is $\gamma \in \Gamma_K$ such that $\mathrm{val}(f(B)) \subseteq [\gamma, \infty]$. If V is an arbitrary variety and $U_1, \ldots, U_m$ an open affine covering of V, a subset $B \subseteq V(K)$ is called bounded in $V(K)$ if for $i = 1, \ldots, m$ there is a set $B_i \subseteq U_i(K)$ which is bounded in $U_i(K)$ such that $B = \bigcup_i B_i$. We will use the following properties of bounded sets:

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[8]One may first reduce the statement to linear algebraic groups. For these, by the Jordan decomposition, the result follows from the unipotent and the semisimple cases, which are clear.

Fact 2.15 ([13, pp. 81–82]).

(1) *If $f : V \to W$ is a morphism of algebraic varieties and $B \subseteq V(K)$ is bounded in $V(K)$, then $f(B)$ is bounded in $W(K)$.*

(2) *If V is a closed subvariety of W and $B \subseteq V(K)$, then B is bounded in $V(K)$ if and only if B is bounded in $W(K)$.*

(3) *If $f : V \to W$ is a proper morphism of quasi-projective algebraic varieties and $B \subseteq W(K)$ is bounded in $W(K)$, then $f^{-1}(B)$ is bounded in $V(K)$.* □

In what follows, as in [9], boundedness of a definable subset D of V in an algebraic variety V just means boundedness of $D(K)$ in $V(K)$, where $K \models$ ACVF and D and V are defined over K. This does not depend on the model K.

Lemma 2.16. *Let G be an algebraic group, G_1 a normal algebraic subgroup and $G_2 := G/G_1$, with projection map $\pi : G \to G_2$. For a definable subgroup H of G, the following are equivalent:*

(1) *H is bounded in G;*

(2) *$H \cap G_1$ is bounded in G_1 and $\pi(H)$ is bounded in G_2.*

Proof. (1)⇒(2) follows from Fact 2.15(1+2). To prove the converse, we may assume that H is Zariski dense in G and that G is a connected algebraic group.

Claim. *There is a definable subset W of H which is bounded in G and such that $\pi(W) = \pi(H)$.*

To settle the claim, we first note that by the proof of [12, Corollary 4.5] there is a definable subset $Y \subseteq H$ such that $\pi(Y) = \pi(H)$ and $\pi{\restriction}_Y$ has finite fibers. As algebraic closure (in the field sort) in ACVF is given by field theoretic algebraic closure, we may assume that $Y = H \cap \widetilde{Y}$ for some constructible subset $\widetilde{Y}$ of G such that $\pi{\restriction}_{\widetilde{Y}}$ has finite fibers. Since $\pi(\widetilde{Y})$ is a Zariski dense constructible subset of G_2, there is an open affine subvariety $U \subseteq G_2$ contained in $\pi(\widetilde{Y})$ such that, setting $Z_U := \pi^{-1}(U) \cap \widetilde{Y}$, the map $\pi : Z_U \to U$ is a (surjective and) finite morphism, so is in particular proper. Moreover, note that $\pi(Z_U \cap H) = U \cap \pi(H)$.

As H is Zariski dense in G, we may choose finitely many elements $h_1, \dots, h_n \in H$ such that $(U_i)_{1 \leq i \leq n}$ is an (affine) open cover of G_2, where $U_i := \pi(h_i)U$. Thus $\pi(h_i Z_U \cap H) = U_i \cap \pi(H)$ for all i. Since $\pi(H)$ is bounded in G_2 by assumption, there are bounded definable subsets B_i of U_i such that $\pi(H) = \bigcup_{i=1}^n B_i$. As $\pi : h_i Z_U \to U_i$ is proper, $W_i := \pi^{-1}(B_i) \cap h_i Z_U$ is bounded in $h_i Z_U$ by Fact 2.15(3), so bounded in G as well, by Fact 2.15(2). Thus $W := \bigcup_{i=1}^n W_i$ is as required, proving the claim.

The set $(H \cap G_1) \times W$ is bounded in $G \times G$, and so $(H \cap G_1) \cdot W$ is bounded in G, as it is the image of a bounded set under the multiplication map $m : G^2 \to G$. Since $H \subseteq (H \cap G_1) \cdot W$, boundedness of H in G follows. □

Proposition 2.17. *Let N be a definable stably dominated subgroup of an algebraic group G. Then $\widehat{N}$ is definably compact.*

Proof. We may assume that G is a connected algebraic group. We will first show that N is a bounded subset of G. By Chevalley's structure theorem, there is a surjective homomorphism of algebraic groups $\pi : G \twoheadrightarrow A$, with A an abelian variety and $L := \ker(\pi)$ a linear algebraic group. By [12, Corollary 4.5], $N \cap L$ is a stably dominated group, and so it follows from [12, Proposition 6.9] that $(N \cap L)^0$ is a

bounded subset of its Zariski closure, which is an algebraic subgroup of L. Thus, $(N \cap L)^0$ and so also $N \cap L$ is a bounded subset of L. Moreover, $\pi(N)$ is bounded in A, as A is a projective variety and so bounded in itself. Thus N is bounded in G by Lemma 2.16, and N^2 is bounded in G^2.

By Fact 2.11, to prove that $\widehat{N}$ is definably compact, we may assume that $N = N^0$. Let p_N be the unique generic type of N. Then the pro-definable set

$$S = \{r(x,y) \in \widehat{G^2} \mid \widehat{\pi}_i(r) = p_N, i = 1, 2\}$$

is closed in $\widehat{G^2}$. Moreover, $S \subseteq \widehat{N^2}$, so S is a bounded subset of $\widehat{G^2}$, in the terminology of [9]. Thus, S is definably compact by [9, Theorem 4.2.19]. Let m denote the multiplication in G. Then $Z = \widehat{m}(S) \subseteq \widehat{G}$ is a definably compact subset of $\widehat{N}$ by [9, Proposition 4.2.9]. The set Z contains all realized types in N (i.e., simple points of $\widehat{N}$), since every element in N is a product of two generics. As the simple points are dense in $\widehat{N}$, we conclude that $\widehat{N} = Z$ is definably compact.   □

2.6. Continuity of the tensor product.
In this subsection, we continue to work in ACVF.

Lemma 2.18. *For any definable sets V, W, the map $\otimes : \widehat{V} \times \widehat{W} \to \widehat{V \times W}$ is pro-definable.*

Proof. Let $p(x) \in \widehat{V}$, $q(y) \in \widehat{W}$ and $r(x,y) = p(x) \otimes q(y)$. Fix some formula $\phi(x, y; t)$. Write $d_p\phi(x; y, t) = \theta_b(y, t)$, where θ depends only on ϕ and b is the canonical parameter of θ_b, so that b is definable from p seen as a point in $\widehat{V}$. For any parameter c, we have $\phi(x, y; c) \in r \iff c \models d_q\theta_b(y; t)$. This is definable from $(p, q) \in \widehat{V} \times \widehat{W}$.   □

Assume H is a finite-dimensional vector space over $K \models$ ACVF. A *semi-lattice* in H is an $\mathcal{O}$-submodule $u \leq H$ for which there is a vector subspace $U \leq H$ such that u/U is a lattice in H/U. Clearly, the set of semi-lattices $L(H)$ is definable. The *linear topology* on $L(H)$ is the definable topology whose pre-basic open sets are of the form $\Omega_h = \{u \in L(H) \mid h \notin u\}$ or of the form $\Theta_h = \{u \in L(H) \mid h \in \mathfrak{m}u\}$.

We leave the easy proof of the following lemma to the reader. (See [9, Chapter 5] for details concerning semi-lattices.).

Lemma 2.19.   (1) *Any semi-lattice $u \in L(H)$ is isomorphic to $\mathcal{O}^l \times K^r$, where $l + r = n := \dim(H)$. In particular, there is a basis $\overline{b}$ of H such that u is diagonal for $\overline{b}$, i.e., $u = \bigoplus_{i=1}^n (u \cap Kb_i)$.*
 (2) *If $H = \bigoplus_{i=1}^s H_i$, then $\prod_{i=1}^s L(H_i)$ embeds naturally into $L(H)$. This embedding is a definable homeomorphism onto a closed subset of $L(H)$.*
 (3) *Let $\overline{b} = (b_1, \ldots, b_n)$ be a basis of H. Then the map $d : \Gamma_\infty^n \to L(H)$,*

$$d(\gamma_1, \ldots, \gamma_n) := \left\{ \sum_{i=1}^n c_i b_i \mid \mathrm{val}(c_i) + \gamma_i \geq 0 \text{ for } i = 1, \ldots, n \right\},$$

is a definable homeomorphism onto the closed subset $\Delta_{\overline{b}}$ of $L(H)$ consisting of all semi-lattices in H which are diagonal for $\overline{b}$. In particular, $\Delta_{\overline{b}}$ is a Γ-internal definable subset of $L(H)$.

The following strong result about simultaneous diagonalisation is a converse to part (3) of the preceding lemma.

Fact 2.20 ([9, Lemma 6.2.2]). *Let $X \subseteq L(H)$ be a definable Γ-internal subset. Then there are finitely many K-bases $\overline{b}^1, \ldots, \overline{b}^N$ of H such that any $u \in X$ is diagonal for some $\overline{b}^i$.*

Lemma 2.21. *Let H and H' be finite-dimensional vector spaces.*

    (1) *If $u \in L(H)$ and $u' \in L(H')$, then $u \otimes u' \in L(H \otimes H')$, where the tensor product is taken over $\mathcal{O}$. Furthermore, the map $\tau : L(H) \times L(H') \to L(H \otimes H')$, $(u, u') \mapsto u \otimes u'$, is definable.*

    (2) *Let $\Sigma \subseteq L(H)$ be a Γ-internal definable subset. Then the restriction of τ to $\Sigma \times L(H')$ is continuous in the linear topology.*

Proof. (1) Clear.

(2) Let $\overline{b} = (b_1, \ldots, b_n)$ be a basis of H. We first show the continuity result in the special case where $\Sigma = \Delta_{\overline{b}} \subseteq L(H)$. Let $H_i = Kb_i$ and $E_i = H_i \otimes H' \leq H \otimes H'$. It follows from Lemma 2.19 that $\Delta_{\overline{b}} = \prod_{i=1}^n L(H_i) \cong \Gamma_\infty^n$ canonically, and that the map $\tau \restriction_{\Delta_{\overline{b}} \times L(H')}$ decomposes topologically into the product of maps $\tau_i : L(H_i) \times L(H') \to L(E_i)$. The continuity of τ_i may be easily checked directly. We leave the verification to the reader. This shows the result for $\Sigma = \Delta_{\overline{b}}$.

To prove the general case, using Fact 2.20, we may find bases $\overline{b}^1, \ldots, \overline{b}^N$ of H such that $\Sigma \subseteq \bigcup_{i=1}^N \Delta_{\overline{b}^i}$. Let $(u, u') \in \Sigma \times L(H')$, and let U be an open neighborhood of $u \otimes u'$ in $L(H \otimes H')$. Let I be the set of indices $i \in \{1, \ldots, N\}$ such that $u \in \Delta_{\overline{b}^i}$. For $i \in I$, choose open neighborhoods Ω_i of u and Ω'_i of u' such that $\tau((\Omega_i \cap \Delta_{\overline{b}^i}) \times \Omega'_i) \subseteq U$. For $i \notin I$, we set $\Omega_i = L(H) \setminus \Delta_{\overline{b}^i}$ (which is an open neighborhood of u by Lemma 2.19) and $\Omega'_i = L(H')$. Let $\Omega = \bigcap_{i=1}^N \Omega_i$ and $\Omega' = \bigcap_{i=1}^N \Omega'_i$. By construction, we get $\tau((\Omega \cap \Sigma) \times \Omega') \subseteq U$. $\hfill\square$

For $n, d \geq 0$ denote by $H_{n,d}$ the vector space of all polynomials of degree $\leq d$ in n variables over the valued field. In what follows, $L(H_{n,d})$ will be endowed with the linear topology. For $p \in \widehat{\mathbb{A}^n}$ let $J_{n,d}(p) := \{h \in H_{n,d} \mid \mathrm{val}(h(p)) \geq 0\}$. Then $J_{n,d}(p)$ is a definable $\mathcal{O}$-submodule of $H_{n,d}$, and it is easy to see that it is a semi-lattice.

Fact 2.22 ([9, Theorem 5.1.4]). *The maps $J_{n,d} : \widehat{\mathbb{A}^n} \to L(H_{n,d})$ are pro-definable and continuous, and the induced map $J_n := (J_{n,d})_d : \widehat{\mathbb{A}^n} \to \varprojlim_d L(H_{n,d})$ is a pro-definable homeomorphism onto its image.*

Proposition 2.23. *Let V, W be algebraic varieties and $\Sigma \subseteq \widehat{V}$ an iso-definable Γ-internal subset. Then the map $\otimes : \Sigma \times \widehat{W} \to \widehat{V \times W}$ is a (pro-definable) homeomorphism onto its image.*

Proof. Clearly, the map $\otimes : \Sigma \times \widehat{W} \to \widehat{V \times W}$ is injective and has a continuous inverse, since the inverse is given by the restriction of the canonical map $\widehat{V \times W} \to \widehat{V} \times \widehat{W}$. It thus suffices to show that $\otimes : \Sigma \times \widehat{W} \to \widehat{V \times W}$ is continuous. Passing to affine charts and some ambient affine spaces, we may assume that $V = \mathbb{A}^n$ and $W = \mathbb{A}^m$.

By Fact 2.22, it suffices to show that the map $f_d = J_{n+m,d} \circ \otimes : \widehat{\mathbb{A}^n} \times \widehat{\mathbb{A}^m} \to L(H_{n+m,d})$ restricted to $\Sigma \times \widehat{\mathbb{A}^m}$ is continuous for every d. It follows from the definitions of the tensor product of generically stable types and of the maps $J_{l,d}$ that f_d factors through $L(H_{n,d}) \times L(H_{m,d})$. More precisely, f_d decomposes as

$$\widehat{\mathbb{A}^n} \times \widehat{\mathbb{A}^m} \xrightarrow{J_{n,d} \times J_{m,d}} L(H_{n,d}) \times L(H_{m,d}) \xrightarrow{\tau} L(H_{n,d} \otimes H_{m,d}) \xrightarrow{\rho} L(H_{n+m,d}),$$

where τ is as in Lemma 2.21 and ρ is the (continuous definable) map induced by the inclusion $H_{n+m,d} \subseteq H_{n,d} \otimes H_{m,d}$.

As $J_{n,d}(\Sigma)$ is a Γ-internal definable subset of $L(H_{n,d})$, the continuity of the map $f_d \restriction_{\Sigma \times \widehat{\mathbb{A}^m}}$ follows from Lemma 2.21. $\qquad\square$

3. Existence of a definable equivariant retraction

For our construction of an equivariant deformation retraction, we will only need a special case of the main result of [9], namely Fact 3.2 below, which is much easier to prove than the general non-smooth case. Moreover, all deformation retractions which will appear in our paper are of a very special form. In order to simplify the exposition, we give them a name.

Definition 3.1. Let V be an algebraic variety defined over the valued field F, and let $X \subseteq V$ be an F-definable subset. A pro-F-definable continuous map

$$H : [0, \infty] \times \widehat{X} \to \widehat{X}$$

is called an *F-definable special deformation retraction of $\widehat{X}$ with final image Σ_0* if the following properties hold:

  (i) $H_\infty = \mathrm{id}_{\widehat{X}}$
 (ii) $H_0(\widehat{X}) \subseteq \Sigma_0$
(iii) $H_t\restriction_{\Sigma_0} = \mathrm{id}_{\Sigma_0}$ for all $t \in [0, \infty]$.
 (iv) For every open subvariety U of V, the set $\widehat{U} \cap \widehat{X}$ is invariant under H.
  (v) $H(0, x) = H(0, H(t, x))$ for any $x \in \widehat{X}$ and any $t \in [0, \infty]$.
 (vi) $X^{\#}$ is invariant under H.
(vii) $H_0(X) = H_0(\widehat{X}) = \Sigma_0$
(viii) Σ_0 is F-definably homeomorphic to a definable subset of Γ^w, for some finite F-definable set w.
 (ix) For any $x \in \widehat{X}$ and any $t < \infty$, $H(t, x)$ is Zariski generic in X.

Here, we denoted by H_t the map sending x to $H(t, x)$.

Remark. Let H be an F-definable special deformation retraction of $\widehat{X}$ with final image Σ_0. Then (vi) combined with (vii) and (ix) yield the following:

  (x) $\Sigma_0 \subseteq X^{\#}$, and every element of Σ_0 is Zariski generic in X.

Fact 3.2. *Let V be a smooth irreducible quasi-projective variety over a valued field F and let $X \subseteq V$ be a v-clopen F-definable subset such that $\widehat{X}$ is definably compact and definably connected.*

Then there is an F-definable special deformation retraction $H : [0, \infty] \times \widehat{X} \to \widehat{X}$ with final image Σ_0.

Proof. That such a map exists with properties (i-vii) follows from [9, Theorem 12.1.1 & Remark 12.2.3]. Moreover, as $\widehat{X}$ is definably connected, one may achieve in addition property (viii), by the discussion in [9, top of p. 178].

Finally, let us argue why one may achieve property (ix). It follows from the proof of [9, Theorem 12.1.1] that for any $x \in \widehat{X}$ and any $t < \infty$, the type $H(t, x)$ is Zariski generic in V, as this is true for the inflation homotopy H_{inf} defined in [9, p. 180] and H is constructed as the composition of H_{inf} with finitely many additional homotopies which are all Zariski generalizing. $\qquad\square$

Lemma 3.3. *Let V be an irreducible variety defined over a model of* ACVF, *and let $\eta : [0, \infty] \to \widehat{V}$ be a continuous definable path with $\eta(\infty)$ a Zariski generic type in V. Then η is constant in a neighborhood of ∞.*

Proof. Let $p = \eta(\infty)$. For any open subvariety U of V, there exists $\gamma_U < \infty$ such that $\eta([\gamma_U, \infty]) \subseteq \widehat{U}$. By the construction of $\widehat{V}$, it is enough to show that for any affine open subvariety U and any regular function f on U, the continuous definable function $T : [\gamma_U, \infty] \to \Gamma_\infty$, $T(t) := \mathrm{val}(f_*(\eta(t)))$ is constant in a neighborhood of ∞. If f is not identically 0 on U, then $\mathrm{val}(f_*(p)) \neq \infty$, since p is Zariski generic in U by assumption. The result now follows from quantifier elimination in Γ_∞. $\qquad\square$

Corollary 3.4. *Let V be a smooth irreducible quasi-projective variety defined over a valued field F and let $X \subseteq V$ be a v-clopen F-definable subset such that $\widehat{X}$ is definably compact and definably connected. Let $q_0, q_\infty \in \widehat{X}(F)$ with q_0 Zariski generic in V. Then there is an F-definable continuous path $\eta : [0, \infty] \to \widehat{X}$ with $\eta(0) = q_0$, $\eta(\infty) = q_\infty$ and $\eta(t)$ Zariski generic in V for any $t < \infty$.*
   Moreover, if $q_0, q_\infty \in X^\#$, then η may be required to have its image in $X^\#$.

Proof. Choose $H : [0, \infty] \times \widehat{X} \to \widehat{X}$ as in Fact 3.2. Let $r_t = H(t, q_0)$ and $u_t = H(t, q_\infty)$. By Lemma 3.3, there is $\gamma < \infty$ such that $r_t = r_\gamma$ for all $t \geq \gamma$. Moreover, since $\widehat{X}$ is definably connected and definably compact, so is $\Sigma_0 = H(0, \widehat{X})$. As Σ_0 is definably homeomorphic to a subspace of Γ^w for some finite set w, Σ_0 is definably path-connected and in particular there is a continuous definable path $s : [\alpha, \beta] \to \Sigma_0$ between r_0 and u_0, for some $\alpha < \beta < \infty$, i.e., parametrized by an interval of finite length. We may glue these three paths, which yields a definable path $\eta : [0, \infty] \to \widehat{X}$ between q_0 and q_∞ as required. (Zariski genericity of $\eta(t)$ for $t < \infty$ is a consequence of the fact that H satisfies property (ix) in Definition 3.1.)
   The moreover part follows, as H preserves $X^\#$ and $\Sigma_0 \subseteq X^\#$. $\qquad\square$

Lemma 3.5. *Let G be an algebraic group defined over a model of* ACVF, *and let $H = H^0$ be a connected definable subgroup of G. Then $\widehat{H}$ is definably connected.*

Proof. The group H acts transitively and definably on the (finite) set of definable connected components of $\widehat{H}$. The result follows, looking at stabilizers. $\qquad\square$

Theorem 3.6. *Let G be an algebraic group defined over a valued field F, and let N be a commutative F-definable stably dominated connected subgroup of G. Let p_e be the type of the identity in N, and let p_N be the generic type of N. Then there is an F-definable path $q : [0, \infty] \to N^\#$ such that*

   (i) $q_\infty = p_e$ and $q_0 = p_N$;
   (ii) for any $t < \infty$, q_t is the generic type of a connected stably dominated definable subgroup N_t of N which is Zariski dense in N.

Proof. First note that p_e is strongly stably dominated, as is p_N, by Lemma 2.12, so $p_e, p_N \in N^\#(F)$. Furthermore, we may assume that G equals the Zariski closure of N, so in particular G is connected and N is Zariski dense in G, hence N is v-clopen in G, as it is a subgroup with non-empty interior for the valuation topology. Moreover, $\widehat{N}$ is definably compact by Proposition 2.17 and definably connected by Lemma 3.5. We may thus infer from Corollary 3.4 that there is an F-definable path $\eta : [0, \infty] \to N^\#$, such that $\eta_\infty = p_e$ and $\eta_0 = p_N$, with η_t strongly stably dominated and Zariski generic in N for any $t < \infty$.

For $p, q \in \widehat{G}$ we denote by $p^{-1} \in \widehat{G}$ the image of p under the inversion map in G, and we set $p * q := \widehat{m}(p \otimes q) \in \widehat{G}$, where m denotes the multiplication map in G. Now fix n large enough and define $q_t = \eta_t * \eta_t^{-1} * \cdots$ where the product contains n instances of $\eta_t * \eta_t^{-1}$. Then the map $t \mapsto q_t$ is continuous by Proposition 2.23. Moreover, it follows from [9, Proposition 2.6.12] that q_t is strongly stably dominated. By [12, Lemma 5.1], q_t is the generic of an ∞-definable connected subgroup N_t of N. By Lemma 2.13, N_t is in fact definable. Clearly, $q_\infty = p_e$ and $q_0 = p_N$, since p_e and p_N are generic types of connected stably dominated definable subgroups. Moreover, the type q_t, obtained as an alternating sum of η_t, is strongly stably dominated and Zariski generic in N for any $t < \infty$. $\qquad\square$

Theorem 3.7. *Let G be an algebraic group defined over a valued field F, and let N be a commutative F-definable stably dominated connected subgroup of G.*

Then there is an F-definable special deformation retraction $\rho : [0, \infty] \times \widehat{N} \to \widehat{N}$ with final image $\{p_N\}$ such that ρ is equivariant under the action of N by multiplication and for each $t < \infty$, $q_t = \rho(t, p_e)$ is the generic type of a connected strongly stably dominated definable subgroup N_t of N which is Zariski dense in N.

Proof. Let $q : [0, \infty] \to N^{\#}$ be a definable path as in the conclusion of Theorem 3.6. Define $r : [0, \infty] \times \widehat{N} \to \widehat{N \times N}$, $(t, a) \mapsto q_t \otimes a$. Then r is continuous by Proposition 2.23, and so is $\rho := \widehat{m} \circ r : [0, \infty] \times \widehat{N} \to \widehat{N}$. Clearly, ρ is N-equivariant with final image $\{p_N\}$ and satisfies all the required properties from Definition 3.1. $\qquad\square$

For the definition of the *limit stably dominated subgroup* of a pro-definable group we refer to [12, Definition 5.6].

Lemma 3.8. *Let G be an algebraic group, G_1 a normal algebraic subgroup and $G_2 := G/G_1$, with projection map $\pi : G \to G_2$. Suppose that in G the limit stably dominated subgroup exists. Let $N \le G$ be this group, and suppose that G/N is Γ-internal. Set $N_1 := N \cap G$ and $N_2 := \pi(N)$. Then the following holds:*

(1) For $i = 1, 2$, N_i is the limit stably dominated subgroup of G_i, and G_i/N_i is Γ-internal.

(2) If N_1 and N_2 are stably dominated, so is N.

Proof. (1) Let $N = \bigcup_{t \models q} S_t$, where $(S_t)_{t \models q}$ is a limit stably dominated family for G in the sense of [12, Definition 5.6]. Then $S_t \cap G_1$ is stably dominated for any $t \models q$, by [12, Corollary 4.5]. Moreover, any connected stably dominated subgroup of G_1 is a subgroup of N. Thus, N_1 is the limit stably dominated subgroup of G_1. The group G_1/N_1 is Γ-internal, as it embeds into G/N.

Clearly, any $\pi(S_t)$ is stably dominated and connected. Moreover, since $G_2/N_2 = \pi(G/N)$ is Γ-internal, any stably dominated connected subgroup H of G_2 is necessarily a subgroup of N_2, as HN_2/N_2 is stably dominated and Γ-internal. It follows that $N_2 = \bigcup_{t \models q} \pi(S_t)$ is the limit stably dominated subgroup of G_2.

(2) follows from [12, Lemma 4.3]. $\qquad\square$

Fact 3.9. *Let S be a semi-abelian variety defined over a valued field $F \subseteq K \models$ ACVF. Then there is an F-definable decomposition*

$$0 \to N \to S \to \Lambda \to 0,$$

with Λ Γ-internal and N stably dominated, definable and connected. The group N is the unique maximal definable stably dominated and connected subgroup of S. Moreover, if $S = A$ is an abelian variety, Λ is definably compact.

Proof. If $S = A$ is an abelian variety, the statement is proven in [12, Corollary 6.19]. If $S = \mathbb{G}_m^n$, then $\mathbb{G}_m^n(\mathcal{O})$ is stably dominated and connected with quotient group $\Lambda \cong (\Gamma, +)^n$, yielding the case of a torus.

Now let S be an arbitrary semi-abelian variety. As S is commutative, by [12, Theorem 5.16], S admits a (unique) limit stably dominated definable subgroup N, with quotient S/N Γ-internal. The result then follows from Lemma 3.8, as S is the extension of an abelian variety by a torus.   □

Theorem 3.10. *Let S be a semi-abelian variety defined over $F \subseteq K \models \mathrm{ACVF}$, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from Fact 3.9.*

Then there is an F-definable special deformation retraction $\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$ with final image $\Sigma \subseteq \widehat{S}$ such that ρ is equivariant under the action of S by multiplication and for each $t < \infty$, $q_t = \rho(t, p_e)$ is the generic type of a connected strongly stably dominated definable subgroup N_t of N, with N_t Zariski dense in S.

Moreover, the morphism $\pi : S \to \Lambda$ induces definable bijection between Σ and Λ.

Proof. We proceed as in the proof of Theorem 3.7. We first define a map $r : [0, \infty] \times \widehat{S} \to \widehat{S \times S}$, $(t, a) \mapsto q_t \otimes a$, where q is as in Theorem 3.6. The map $\rho := \widehat{m} \circ r : [0, \infty] \times \widehat{S} \to \widehat{S}$ is continuous and satisfies all the required properties from Definition 3.1, and it is clearly S-equivariant. The restriction $\rho_S = \rho \upharpoonright_{[0,\infty] \times S}$ has final image equal to $\{a + p_N \mid a \in S\}$, a set which may be identified with $S/N \cong \Lambda$ and which is thus Γ-internal.

It follows from the definitions that ρ is the canonical extension (in the sense of [9, Section 3.8]) of ρ_S. Thus, the final image of ρ is equal to Λ as well, as $\widehat{\Lambda} = \Lambda$.   □

4. An explicit definable equivariant retraction in equicharacteristic 0

In this section, we will give an alternative, more explicit, construction of an equivariant definable special deformation retraction in the case of equicharacteristic 0. This construction does not require knowing in advance that a (non-equivariant) retraction exists. More importantly, it does not require that the stably dominated connected group N be commutative, in order to show that its stable completion $\widehat{N}$ allows for an N-equivariant definable special deformation retraction to the generic type of N. We believe that even in the commutative case, it might be useful in its own right.

4.1. **Internality of quotients in $\mathrm{ACVF}_{0,0}$.** The following result will be used to define an intrinsic scale, given by subgroups, in any stably dominated definable subgroup of an algebraic group in a model of $\mathrm{ACVF}_{0,0}$.

Lemma 4.1. *Work in $\mathrm{ACVF}_{0,0}$. Let D, D' be definable subgroups of $\mathcal{O}$. Then the following are equivalent:*

   (1) $D' \subseteq D$.
   (2) $\mathcal{O}/D$ is $\mathcal{O}/D'$-internal.
   (3) $\mathcal{O}/D$ is almost $\mathcal{O}/D'$-internal.

Proof. $1. \Rightarrow 2. \Rightarrow 3.$ is clear.

We now prove $3. \Rightarrow 1$. Note that every definable subgroup of $\mathcal{O}$ is of the form $\gamma\mathfrak{m}$ or $\gamma\mathcal{O}$ for some $0 \leq \gamma \leq \infty$, so in particular the set of definable subgroups of $\mathcal{O}$ is totally ordered by inclusion. Whenever $D \subsetneq D'$ are definable subgroups, there is $0 < \gamma \in \Gamma$ such that $D \subseteq \gamma\mathfrak{m} \subsetneq \gamma\mathcal{O} \subseteq D'$. It is thus enough to show that for any $0 < \gamma \in \Gamma$, the set $\mathcal{O}/\gamma\mathfrak{m}$ is not almost $\mathcal{O}/\gamma\mathcal{O}$-internal.

The idea of the proof is similar to [8, Lemma 5.1]. Consider the field of generalized power series $K = k((X^\Gamma))$, where k is an algebraically closed field of characteristic 0 and Γ a non-trivial divisible ordered abelian group. Let $l = k((T^\mathbb{Q}))$ (with the trivial valuation) and consider $L = l((X^\Gamma))$ with the X-adic valuation, an elementary extension of $K \models \mathrm{ACVF}_{0,0}$.

Given $0 < \gamma \in \Gamma(K) = \Gamma$ and $\rho \in L$ of valuation 0 (e.g., $\rho \in k^\times$) we now define an automorphism σ_ρ of L fixing $K \cup (\mathcal{O}/\gamma\mathcal{O})(L)$ pointwise such that for distinct $\rho, \rho' \in k^\times$, we have $\sigma_\rho(T + \gamma\mathfrak{m}) \neq \sigma_{\rho'}(T + \gamma\mathfrak{m})$. This will show the result, as then $(\mathcal{O}/\gamma\mathfrak{m})(L) \subsetneq \mathrm{acl}(K \cup (\mathcal{O}/\gamma\mathcal{O})(L))$.

Define $\sigma = \sigma_\rho$ on monomials as follows:

$$\sigma(aT^q X^g) := aT^q X^g \exp(q\rho X^\gamma).$$

Then extend the map by linearity to generalized series. We check that σ is an automorphism having the required properties:

(1) σ is a continuous automorphism: linearity and continuity follow from the definition. Multiplicativity can be checked on monomials: $\sigma(a_1 T^{q_1} X^{g_1}) \cdot \sigma(a_2 T^{q_2} X^{g_2}) = a_1 a_2 T^{q_1+q_2} X^{g_1+g_2} \exp(q_1 \rho X^\gamma) \exp(q_2 \rho X^\gamma) = \sigma(a_1 a_2 T^{q_1+q_2} X^{g_1+g_2})$.

For any $x \in \mathcal{O}$, we have

$$(4.1) \qquad\qquad v(x - \sigma(x)) \geq v(x) + \gamma.$$

This, along with continuity, implies that σ is a bijection: if we have y such that $v(\sigma(y) - x) =: \eta$, we let $y' = y + x - \sigma(y)$ and obtain a better candidate for a preimage of x, that is $v(\sigma(y') - x) > v(\sigma(y) - x)$. By transfinite induction, we find an actual preimage of x.

(2) σ fixes $K \cup (\mathcal{O}/\gamma\mathcal{O})(L)$ pointwise: For $x \in K$, $\sigma(x) = x$ is immediate from the definition and (4.1) gives the rest of the statement.

(3) $\sigma_\rho(T + \gamma\mathfrak{m}) \neq \sigma_{\rho'}(T + \gamma\mathfrak{m})$ for $\rho \neq \rho'$ follows from the computation $v(\sigma_\rho(T) - \sigma_{\rho'}(T)) = v((\rho - \rho')TX^\gamma) = \gamma$. $\qquad\square$

Remark 4.2. In positive and in mixed characteristic, the statement of Lemma 4.1 does not hold: in $\mathrm{ACVF}_{p,p}$, given any $\gamma \in \Gamma_{>0}$ and $c \in \mathcal{O}$ with $v(c) = \gamma$, the Frobenius automorphism induces a group isomorphism $\mathcal{O}/\gamma\mathcal{O} = \mathcal{O}/c\mathcal{O} \cong \mathcal{O}/c^p\mathcal{O} = \mathcal{O}/(p\gamma)\mathcal{O}$; in $\mathrm{ACVF}_{0,p}$, setting $\gamma_0 := \mathrm{val}(p) \in \Gamma_{>0}$, the map $x \mapsto x^p$ induces a definable surjection $\mathcal{O}/\gamma_0\mathcal{O} \twoheadrightarrow \mathcal{O}/(2\gamma_0)\mathcal{O}$.

Corollary 4.3. *Work in* $\mathrm{ACVF}_{0,0}$. *Let D be a definable subgroup of $\mathcal{O}$, and let $d \in \mathbb{N}$. Then $(\mathcal{O}/D)^d$ is the maximal (almost) $\mathcal{O}/D$-internal quotient of $\mathcal{O}^d$.*

Proof. It is clear that $(\mathcal{O}/D)^d$ is $\mathcal{O}/D$-internal. Conversely, let $\mathcal{O}^d/N$ be (almost) $\mathcal{O}/D$-internal for some $(\infty\text{-})$definable subgroup $N \leq \mathcal{O}^d$. For $i \in \{1, \ldots d\}$, let $B_i = \{0\}^{i-1} \times \mathcal{O} \times \{0\}^{d-i-1} \cong \mathcal{O}$. It follows that for any i the group $B_i/(N \cap B_i)$ is (almost) $\mathcal{O}/D$-internal, and so $\{0\}^{i-1} \times D \times \{0\}^{d-i-1} \leq N \cap B_i$ by Lemma 4.1. Thus $D^d \leq N$. $\qquad\square$

Corollary 4.4. *Work in* $\mathrm{ACVF}_{0,0}$. *Let C be a model (or more generally a base structure consisting of a field). Let $\gamma \in \Gamma$, and set $C_\gamma := \mathrm{acl}(C\gamma)$. Let D be a C_γ-definable subgroup of $\mathcal{O}$. If $a = (a_1, \ldots, a_d)$ is generic in $\mathcal{O}^d$ over C, the tuple $(a_1/D, \ldots, a_d/D)$ dcl-generates $\mathrm{dcl}(C_\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/D)$ over C_γ.*

More generally, if $a \in K^m$ realizes a strongly stably dominated type over C, there is a tuple $b = (b_1, \ldots, b_d)$ from $C(a)$ such that b is generic in $\mathcal{O}^d$ over C and $\mathrm{acl}(C_\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/D)$ is finitely acl-generated by $(b_1/D, \ldots, b_d/D)$ over C_γ.

Proof. First suppose that a is generic in $\mathcal{O}^d$ over C. By the previous corollary, $(\mathcal{O}/D)^d$ is the maximal (almost) $\mathcal{O}/D$-internal quotient of $\mathcal{O}^d$, over any set of parameters. As $\mathrm{tp}(a/C)$ is stably dominated and thus orthogonal to Γ, the tuple a is generic in $\mathcal{O}^d$ over C_γ. By the last sentence of Proposition 2.9, $\mathrm{dcl}(aC_\gamma) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/D)$ is then interdefinable with $(a_1/D, \ldots, a_d/D)$ over C_γ, proving the result.

Now suppose $a \in K^m$ with $\mathrm{tp}(a/C)$ strongly stably dominated, i.e., $\mathrm{tr}(a/C) = \mathrm{tr}(k_{C(a)}/k_C) = d$ by Fact 2.5. We find $b \in C(a)$ generic in $\mathcal{O}^d$ over C. As b and a are interalgebraic over C, we have $\mathrm{acl}(C\gamma b) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/D) = \mathrm{acl}(C\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/D)$, so the result follows from the special case. $\square$

4.2. **Linearization.** In this subsection, we work in $\mathrm{ACVF}_{0,0}$. Let C be a base structure consisting of a field. We fix a C-definable stably dominated connected subgroup N of an algebraic group. For $\gamma \in \Gamma$, let $C_\gamma = \mathrm{acl}(C\gamma)$. Set $d := \dim(N)$.

Fact 4.5 ([8, Section 7]). *For $\gamma \in \Gamma$, $\mathcal{O}/\gamma\mathcal{O}$ and $\mathcal{O}/\gamma\mathfrak{m}$ are stably embedded.*

By Proposition 2.9, we let N_γ be the kernel of the maximal $\mathcal{O}/\gamma\mathcal{O}$-internal quotient of N and similarly N_γ^+ is defined as the kernel of the maximal $\mathcal{O}/\gamma\mathfrak{m}$-internal quotient of N, both computed over C_γ.

The groups N_γ and N_γ^+ are ∞-definable. Note that if $\gamma < \delta$, then $N_\gamma \supseteq N_\gamma^+ \supseteq N_\delta \supseteq N_\delta^+$.

Lemma 4.6. *The groups N_γ and N_γ^+ have bounded index inside some definable group and are intersections of definable groups.*

Proof. By Fact 2.6(1), the quotient N/N_γ can be written as an inverse limit of definable groups $\pi_i(N)$. By Lemma 2.12, the generic type p_N of N is strongly stably dominated and by Corollary 4.4, over C_γ, the set $\mathrm{acl}(C_\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/\gamma\mathcal{O})$ is in the algebraic closure of a finite set in $\mathcal{O}/\gamma\mathcal{O}$. By construction of N_γ, if $a \models p_N|C$, then $\mathrm{acl}(C_\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/\gamma\mathcal{O})$ is interalgebraic over C_γ with the sequence $\pi_i(a)$. Hence we can find i_0 such that $\mathrm{acl}(C_\gamma a) \cap \mathrm{Int}_{C_\gamma}(\mathcal{O}/\gamma\mathcal{O})$ is already interalgebraic with $\pi_{i_0}(a)$ over C_γ. Then N_γ has bounded index inside the kernel of π_i. Also N_γ is the intersection of the kernels of π_i which are definable subgroups of N.

The same arguments work for N_γ^+. $\square$

Lemma 4.7. *The group N_γ is strongly stably dominated.*

Proof. We first show that the quotient $\mathfrak{g}_\gamma := N_\gamma/N_\gamma^+$ is stable of Morley rank d. Let $p = p_N$ be the generic type of N, and let $\mathrm{Lin}(p)$ be the generic type of $\mathcal{O}^d$. Then any $a \models p|C$ is interalgebraic over C with some $b = (b_1, \ldots, b_d) \models \mathrm{Lin}(p)|C$. Denote by r_γ (by r_γ^+, respectively) the generic type of the maximal $\mathcal{O}/\gamma\mathcal{O}$-internal (maximal $\mathcal{O}/\gamma\mathfrak{m}$-internal, respectively) quotient of N over C, with realizations a_γ and a_γ^+, images of a via the canonical projection. Let $b_\gamma = (b_1/\gamma\mathcal{O}, \ldots, b_d/\gamma\mathcal{O})$, and define similarly b_γ^+. We have the following:

- $\mathrm{acl}(C_\gamma b_\gamma^+) = \mathrm{acl}(C_\gamma a_\gamma^+)$;
- $\mathrm{acl}(C_\gamma b_\gamma) = \mathrm{acl}(C_\gamma a_\gamma)$ and as $\mathrm{tp}(b_\gamma^+ / \mathrm{acl}(C_\gamma b_\gamma))$ is interdefinable with the generic of a k^d-torsor, then $\mathrm{tp}(a_\gamma^+ / \mathrm{acl}(C_\gamma a_\gamma))$ is interalgebraic with the generic of a k^d-torsor. In particular, that type is stable of Morley rank d.
- The generic of $\ker(N/N_\gamma^+ \to N/N_\gamma) \cong \mathfrak{g}_\gamma$ is interdefinable with the non-forking extension of $\mathrm{tp}(a_\gamma^+ / C_\gamma a_\gamma)$ by Lemma 2.8, hence $\mathfrak{g}_\gamma$ is a stable group of Morley rank d.

Any type p_γ of N_γ projecting on the generic of $\mathfrak{g}_\gamma$ is strongly stably dominated. In particular, it is definable over C_γ. The same is true for any translate of p_γ, since the generic of $\mathfrak{g}_\gamma$ is invariant under translation. Hence p_γ is a generic type of N_γ and N_γ is strongly stably dominated. $\qquad\square$

Corollary 4.8. *The groups N_γ and N_γ^+ are definable.*

Proof. By Lemma 4.7, N_γ is strongly stably dominated, so it is definable by Lemma 2.13.

Now by the proof of Lemma 4.7, the quotient N_γ/N_γ^+ is ω-stable. By Lemma 4.6, write $N_\gamma^+ = \bigcap_i H_i$, where the H_i's are definable subgroups of N_γ. Since there is no infinite descending chain of definable subgroups of N_γ/N_γ^+, the intersection $\bigcap_i H_i$ is equal to a finite subintersection, hence N_γ^+ is definable. $\qquad\square$

Corollary 4.9. *The group $N_\gamma^{00} = N_\gamma^0$ is of finite index in N_γ.*

Proof. This is just Fact 2.11, combined with Fact 2.6(2). $\qquad\square$

Lemma 4.10. *Let $\gamma \in [0, \infty]$. Set $V_\gamma = \bigcup_{\delta > \gamma} N_\delta$ and $W_\gamma = \bigcap_{\delta < \gamma} N_\delta$. Then the following holds:*

(1) $V_\gamma = \bigcup_{\delta > \gamma} N_\delta^+ \leq N_\gamma^+$ and $(N_\gamma^+ : V_\gamma) < \infty$, so in particular $\left(N_\gamma^+\right)^0 = V_\gamma^0$.
(2) $N_\gamma \leq W_\gamma = \bigcap_{\delta < \gamma} N_\delta^+$ and $(W_\gamma : N_\gamma) < \infty$, so in particular $N_\gamma^0 = W_\gamma^0$.

Proof. We know that $V_\gamma = \bigcup_{\delta > \gamma} N_\delta^+ \subseteq N_\gamma^+$ and $N_\gamma \subseteq W_\gamma = \bigcap_{\delta < \gamma} N_\delta = \bigcap_{\delta < \gamma} N_\delta^+$.

Let us show that $N_\gamma^0 = W_\gamma^0$. We fix some saturated model M_0 and we work over C_γ. Suppose that the index of N_γ in W_γ is infinite. Let $a \in M_0$ realize the generic of N over C_γ and let a_γ be the image of a in N/N_γ under the projection map, as in Lemma 4.7, and let $\tilde{a}_\gamma$ be the image of a in N/W_γ. We have that $\tilde{a}_\gamma$ is algebraic over a_γ and that a_δ, $\delta < \gamma$ is algebraic over $\tilde{a}_\gamma$. However, by assumption, a_γ is not algebraic over $\tilde{a}_\gamma$ and there exists an automorphism fixing $C_\gamma \tilde{a}_\gamma$ along with a_δ, $\delta < \gamma$ and for which a_γ has an infinite orbit. We now define b_δ, $\delta \leq \gamma$ as in the proof of Lemma 4.7. Then each b_δ is interalgebraic with a_δ. Therefore b_γ has an infinite orbit under σ whereas each b_δ, $\delta < \gamma$ has a finite orbit. Since $b_\delta \in \mathrm{dcl}(C_\gamma b_{\delta'})$ for $\delta < \delta'$, taking a large enough power of σ, we may assume that σ actually fixes each b_δ, $\delta < \gamma$. This is impossible since any element d of $\mathcal{O}/\gamma\mathcal{O}$ is determined by the sequence $(d_\delta : \delta < \gamma)$ where d_δ is the image of d in $\mathcal{O}/\delta\mathcal{O}$.

The fact that $(N_\gamma^+ : V_\gamma) < \infty$ is proved in a similar way. Assume that V_γ has infinite index in N_γ^+. Let $\tilde{a}_\gamma^+$ be the image of a in N/V_γ, and a_γ^+ that in N/N_γ^+. Take as above an automorphism σ fixing $C_\gamma a_\gamma^+$ and under which $\tilde{a}_\gamma^+$ has infinite orbit. This implies that each a_δ, $\delta > \gamma$ has infinite orbit. The same is true for each b_δ, $\delta > \gamma$, however b_γ^+ has finite orbit under σ. Taking a power of σ, we may assume that b_γ^+ is fixed. As $b_\gamma^+ = b + \gamma\mathfrak{m}$, using $\sigma(\gamma) = \gamma$, we get $\sigma(b) + \gamma\mathfrak{m} = \sigma(b_\gamma^+)$

and so $\delta_0 = v(b - \sigma(b)) > \gamma$. This implies that $\sigma(b_{\delta_0}) = b_{\delta_0}$, which is the desired contradiction. □

Corollary 4.11. *For any $\gamma \in [0, \infty]$, one has $\left(N_\gamma^+\right)^0 = \left(N_\gamma^+\right)^{00}$.*

Proof. This follows from $\bigcup_{\delta > \gamma} N_\delta^0 = \left(N_\gamma^+\right)^0$ and Corollary 4.9 since an increasing union of strongly connected groups is strongly connected. □

Lemma 4.12. *Let H be a definable stably dominated group such that $N_\gamma \supseteq H \supseteq \left(N_\gamma^+\right)^0$. Then $(N_\gamma : H) < \infty$.*

Proof. We work over a model M containing C_γ. Suppose $\mathrm{acl}(a_\gamma) \subsetneq \mathrm{acl}(a') \subseteq \mathrm{acl}(a_\gamma^+)$, where, a_γ, a' and a_γ^+ are the images of $a \models p_N | M$ under the canonical projection of N onto N/N_γ, N/H and N/N_γ^+, respectively.

As non-forking extensions/restrictions and translates of (strongly) stably dominated types are (strongly) stably dominated, it follows from Lemma 2.8 that $\mathrm{tp}(a/a_\gamma)$ is generic in $a + N_\gamma$ and strongly stably dominated. Similarly, $\mathrm{tp}(a/a')$ is generic in $a + H$ and stably dominated.

As $a_\gamma^+ + N_\gamma^+ \in \mathrm{St}_{a_\gamma}(a)$ is of Morley rank d over a_γ, it follows that $\mathrm{St}_{a_\gamma}(a)$ is interalgebraic over a_γ with a_γ^+. Thus there is some non-algebraic element from $\mathrm{St}_{a_\gamma}(a)$ which is in $\mathrm{acl}(a')$. Note that $\mathrm{St}_{a_\gamma}(a)$ and $\mathrm{St}_{a'}(a)$ are interdefinable over a', since $a' \in \mathrm{St}_{a_\gamma}(a)$ and $a_\gamma \in \mathrm{dcl}(a') \subseteq \mathrm{dcl}(a)$ (cf. [7, Lemma 3.5 & Remark 7.9]). It follows that $\dim(\mathrm{St}_{a'}(a)) < d$, and so $\mathrm{tp}(a/\mathrm{acl}(a'))$ is not strongly stably dominated by Fact 2.5. Thus the generic type of H is not strongly stably dominated either, contradicting Lemma 2.12. □

Remark 4.13. Any ∞-definable group H such that $N_\gamma \supseteq H \supseteq N_\gamma^+$ is definable, since H/N_γ^+ is definable, as it is an ∞-definable subgroup of an ω-stable group.

4.3. **Definability and continuity.** We keep the notation and conventions from Subsection 4.2.

Lemma 4.14. *There are C-definable families $\left(\widetilde{N}_\gamma\right)_{\gamma \in [0, \infty]}$ and $\left(\widetilde{N}_\gamma^+\right)_{\gamma \in [0, \infty]}$ of definable sugroups of N such that for any γ, $N_\gamma \le \widetilde{N}_\gamma$ with $(\widetilde{N}_\gamma : N_\gamma) < \infty$ and similarly $N_\gamma^+ \le \widetilde{N}_\gamma^+$ with $(\widetilde{N}_\gamma^+ : N_\gamma^+) < \infty$.*

In addition, we may choose these families so that $\widetilde{N}_\gamma^+ \le \widetilde{N}_\gamma \le \widetilde{N}_\delta^+ \le \widetilde{N}_\delta$ whenever $\gamma < \delta$.

Proof. For $\gamma \in [0, \infty]$, there is a formula $\psi(x, y)$ with parameters from C such that $\psi(x, \gamma)$ defines N_γ. Corollary 4.4 together with Proposition 2.9 implies that there are $\overline{a}$ generic in N/N_γ over C_γ and $\overline{b}$ generic in $(\mathcal{O}/\gamma\mathcal{O})^d$ over C_γ such that $\overline{b} \in \mathrm{dcl}(C_\gamma \overline{a})$ and $\overline{a} \in \mathrm{acl}(C_\gamma \overline{b})$. By compactness, there is $\theta(y) \in \mathrm{tp}(\gamma/C)$ such that whenever $\models \theta(\delta)$, then $\psi(x, \delta)$ defines a subgroup N_δ' of N such that there are $\overline{a}$ generic in N/N_δ' over C_γ and $\overline{b}$ generic in $(\mathcal{O}/\gamma\mathcal{O})^d$ over C_γ with $\overline{b} \in \mathrm{dcl}(C_\gamma \overline{a})$ and $\overline{a} \in \mathrm{acl}(C_\gamma \overline{b})$. It follows from Proposition 2.9 that $N_\delta \le N_\delta'$ and $(N_\delta' : N_\delta) < \infty$ for any such δ. By compactness, we obtain a C-definable family $\left(\widetilde{N}_\gamma\right)_{\gamma \in [0, \infty]}$ with the required properties.

In exactly the same way, one proves the existence of the family $\left(\widetilde{N}_\gamma^+\right)_{\gamma \in [0, \infty]}$.

In order to achieve the additional requirement, it is enough to replace the group $\widetilde{N}_\gamma$ by $\bigcap_{\gamma \le \delta} \widetilde{N}_\delta \cap \bigcap_{\gamma < \delta} \widetilde{N}_\delta^+$, and similarly $\widetilde{N}_\gamma^+$ by $\bigcap_{\gamma \le \delta} \widetilde{N}_\delta \cap \bigcap_{\gamma \le \delta} \widetilde{N}_\delta^+$. □

Lemma 4.15. *The theory* ACVF *admits elimination of* $\exists^\infty$ *in imaginary sorts.*

Proof. By [6, Lemma 2.6.2], if D is a definable (imaginary) set, then either it is k-internal, or, for some m, there is a definable surjective map from D^m to an infinite interval of Γ. The second case is an open condition and implies that D is infinite. In the first case, since we know that k eliminates $\exists^\infty$, we also see that D being infinite can be expressed as a definable condition on the parameters defining D. $\qquad\square$

Lemma 4.16. *Let* $q_\gamma \in \widehat{N}$ *be the generic type of* N_γ^0. *Then the map* $q : [0,\infty] \to \widehat{N}$, $\gamma \mapsto q_\gamma$ *is pro-definable. Its image is iso-definable and* Γ-*internal.*

Proof. Let $\left(\widetilde{N}_\gamma\right)_{\gamma \in [0,\infty]}$ be a family as given by Lemma 4.14. Then $N_\gamma^0 = \widetilde{N}_\gamma^0$ for all γ. For every formula $\phi(x,y)$, the set

$$\mathrm{Gen}(\phi) := \{(\gamma, p) \in [0,\infty] \times \widehat{N} \mid p \text{ is } \phi\text{-generic in } \widetilde{N}_\gamma\}$$

is relatively definable in $[0,\infty] \times \widehat{N}$. This can be seen as follows. By generic stability, the ϕ-definitions of elements of $\widehat{N}$ are uniform. Now $\widetilde{N}_\gamma$ acts on the ϕ-definitions of elements of $\widehat{\widetilde{N}_\gamma}$. By Lemma 4.15, there is n_ϕ such that whenever the orbit of a ϕ-definition under this action is finite, it is of cardinality at most n_ϕ. It follows that the set $\mathrm{Gen} := \bigcap_\phi \mathrm{Gen}(\phi) \subseteq [0,\infty] \times \widehat{N}$ is pro-definable.

The set $\mathrm{PrGen} := \{(\gamma, p) \in \mathrm{Gen} \mid p * p = p\}$ is then pro-definable as well, and it is equal to $\mathrm{Graph}(q)$, since for a generic type p of a stably dominated group H, one has $p * p = p$ precisely if p is the principal generic of H. This proves pro-definability of q.

As q is injective, iso-definability of its image follows from compactness and then this image is Γ-internal by definition. $\qquad\square$

Theorem 4.17. *The map* q *is continuous. Thus,* $q : [0,\infty] \to N^\#$ *is a definable path between* $q_\infty = p_e$ *and* $q_0 = p_N$ *along generic types of strictly increasing strongly stably dominated connected definable subgroups of* N, *with* q_γ *Zariski generic in* N *for every* $\gamma < \infty$.

Proof. By definable compactness of $\widehat{N_\gamma}$ (Proposition 2.17), $l_\gamma^+ := \lim_{\delta \models \gamma^+} q_\delta$ exists and is in $\widehat{N_\gamma}$. We claim that $l_\gamma^+ = q_\gamma$. To see this, first note that, for any $\delta > \gamma$ and $a \in N_\delta$, we have $a \cdot q_\delta = q_\delta$, hence also $a \cdot l_\gamma^+ = l_\gamma^+$, as the map $q \mapsto a \cdot q$ is continuous (e.g., by Proposition 2.23). Moreover, as $q_\delta * q_\delta = q_\delta$ for all δ, we have $l_\gamma^+ = l_\gamma^+ * l_\gamma^+$ by continuity of $\otimes$ (Proposition 2.23). Hence l_γ^+ is the generic type of a subgroup of N_γ containing $\bigcup_{\delta > \gamma} N_\delta = N_\gamma^+$ (Lemma 4.10). By Lemma 4.12, l_γ^+ must be the generic of N_γ, namely $l_\gamma^+ = q_\gamma$ as required.

We next show continuity at γ^- (including $\gamma = \infty$): As before, by continuity of $\otimes$, we know that $l_\gamma^- := \lim_{\delta \models \gamma^-} q_\delta$ is an idempotent. Moreover, since $q_\gamma * q_\delta = q_\delta$, for every $\delta < \gamma$, this holds in the limit as well, i.e., $q_\gamma * l_\gamma^- = l_\gamma^-$. As $l_\gamma^- \in \bigcap_{\delta < \gamma} \widehat{N_\delta} = \widehat{W_\gamma}$, it follows from Lemma 4.10 that $l_\gamma^- = q_\gamma$, since the only idempotent generic type of W_γ is the principal generic. $\qquad\square$

The proofs of Theorem 3.7 and Theorem 3.10, respectively, show that Theorem 4.17 yields the following two corollaries.

Corollary 4.18. *Let* G *be an algebraic group defined over a model of* $\mathrm{ACVF}_{0,0}$, *and let* N *be a* C-*definable stably dominated connected subgroup of* G.

Then there is a C-definable special deformation retraction $\rho : [0, \infty] \times \widehat{N} \to \widehat{N}$ with final image $\rho(0, \widehat{N}) = \{p_N\}$ such that ρ is equivariant under the action of N by multiplication and for each $t < \infty$, $q_t = \rho(t, p_e)$ is the generic type of a definable strongly stably dominated connected Zariski dense subgroup N_t of N, with $N_s \supsetneq N_t$ if $s < t$.

Corollary 4.19. *Let S be a semi-abelian variety defined over $F \subseteq K \models \mathrm{ACVF}_{0,0}$, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from Fact 3.9.*

Then there is an F-definable special deformation retraction $\rho : [0, \infty] \times \widehat{S} \to \widehat{S}$ with final image $\Sigma := \rho(0, \widehat{S}) \subseteq \widehat{S}$ such that ρ is equivariant under the action of S by multiplication and for each $t < \infty$, $q_t = \rho(t, p_e)$ is the generic type of a definable strongly stably dominated connected subgroup N_t of N which is Zariski dense in S and such that $N_s \supsetneq N_t$ whenever $s < t$. Moreover, Σ is in definable bijection with Λ, canonically.

5. Application to the topology of S^{an}

Let S be a semi-abelian variety defined over a valued field F with $\Gamma_F \leq \mathbb{R}$. In this section, using the methods and results from [9, Chapter 14], we will deduce from our results on $\widehat{S}$ the existence of an equivariant strong deformation retraction of (the underlying topological space of) the Berkovich analytification S^{an} onto its skeleton, which is a simplicial complex carrying the structure of a piecewise linear group.

Recall that if B is a parameter set in some model of ACVF, a type $p \in S(B)$ is *almost orthogonal to* Γ, denoted by $p \perp^a \Gamma$, if $\Gamma(Ba) = \Gamma(B)$ for $a \models p$. If this is the case then for a B-definable function $g : X \to \Gamma_\infty$, with p concentrating on the definable set X, we set $g(p) := g(a)$, where $a \models p$. This is well-defined, as the value $g(a)$ does not depend on the realization a.

Until the end of this section, we fix a valued field F with $\Gamma_F \leq \mathbb{R}$. We will now introduce some objects and notation (mostly) from [9, Chapter 14]:

- $\mathbb{R}_\infty$ denotes $\mathbb{R} \cup \{\infty\}$, equipped with the order topology. Moreover, $\mathbb{R}_\infty^+ := \mathbb{R}^+ \cup \{\infty\}$ denotes the sub-interval of its non-negative elements.
- We set $\mathbf{F} := (F, \mathbb{R})$. In particular, $\mathbb{R}_\infty = \Gamma_\infty(\mathbf{F})$ and $\mathbb{R}_\infty^+ = [0, \infty](\mathbf{F})$.
- Let V be an algebraic variety defined over F and $X \subseteq V \times \Gamma_\infty^n$ an $\mathbf{F}$-definable subset. As a set, let $B_{\mathbf{F}}(X) := \{p \in S_X(\mathbf{F}) \mid p \perp^a \Gamma\}$.
- We endow $B_{\mathbf{F}}(V)$ with the topology whose basic open sets are given by finite intersections of sets of the form $\{p \in B_{\mathbf{F}}(U) \mid g(p) \in \Omega\}$, where U is an open affine subvariety of V defined over F, $\Omega \subseteq \mathbb{R}_\infty$ is an open interval and $g = \mathrm{val} \circ G$ for some $G \in F[U]$.
- We endow $B_{\mathbf{F}}(V \times \Gamma_\infty^n) = B_{\mathbf{F}}(V) \times \mathbb{R}_\infty^n$ with the product topology, and finally $B_{\mathbf{F}}(X) \subseteq B_{\mathbf{F}}(V) \times \mathbb{R}_\infty^n$ with the subspace topology.
- Any $\mathbf{F}$-definable map $f : X \to Y$ induces a map $B_{\mathbf{F}}(f) : B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(Y)$, which is continuous in case f is the restriction of a regular map between the ambient algebraic varieties.
- We fix a maximally complete algebraically closed extension F^{max} of F, with $k_{F^{\mathrm{max}}} = (k_F)^{\mathrm{alg}}$ and $\Gamma_{F^{\mathrm{max}}} = \mathbb{R}$. (Note that such an F^{max} is uniquely determined up to $\mathbf{F}$-isomorphism.)

The following is an adaptation of Definition 3.1 to the Berkovich setting.

Definition 5.1. Let V be an algebraic variety defined over the valued field F, and let $X \subseteq V$ be an **F**-definable subset. A continuous map

$$\widetilde{H} : \mathbb{R}_\infty^+ \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$$

is called a *special deformation retraction of $B_{\mathbf{F}}(X)$ with final image* **Z** if the following properties hold:

- $\widetilde{H}_\infty = \mathrm{id}_{B_{\mathbf{F}}(X)}$
- $\widetilde{H}_0(B_{\mathbf{F}}(X)) \subseteq \mathbf{Z}$
- $\widetilde{H}_t{\restriction}_{\mathbf{Z}} = \mathrm{id}_{\mathbf{Z}}$ for all $t \in \mathbb{R}_\infty^+$.
- For every open subvariety U of V defined over F, the set $B_{\mathbf{F}}(U) \cap B_{\mathbf{F}}(X)$ is invariant under $\widetilde{H}$.
- $\widetilde{H}(0, x) = \widetilde{H}(0, \widetilde{H}(t, x))$ for any $x \in B_{\mathbf{F}}(X)$ and any $t \in \mathbb{R}_\infty^+$.
- **Z** is homeomorphic to a piecewise linear subset of $\mathbb{R}^n$, for some $n \in \mathbb{N}$.
- For any $x \in B_{\mathbf{F}}(X)$ and any $t < \infty$, $\widetilde{H}(t, x)$ is Zariski generic in X.

Fact 5.2. *Let V, W be algebraic varieties defined over F, and let $X \subseteq V \times \Gamma_\infty^n$ and $Y \subseteq W \times \Gamma_\infty^m$ be **F**-definable subsets.*

(1) *The space $B_{\mathbf{F}}(V)$ is canonically homeomorphic to the underlying topological space of the Berkovich analytification V^{an} of V.*

(2) *The restriction map $\pi_X : \widehat{X}(F^{\mathrm{max}}) \to B_{\mathbf{F}}(X)$ is surjective, continuous and closed. If $F = F^{\mathrm{max}}$, it is a homeomorphism.*

(3) *Any continuous pro-**F**-definable function $h : \widehat{X} \to \widehat{Y}$ induces a continuous function $\tilde{h} : B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(Y)$ such that $\pi_Y \circ h = \tilde{h} \circ \pi_X$.*

(4) *Let $H : [0, \infty] \times \widehat{X} \to \widehat{X}$ be an **F**-definable special deformation retraction, with final image $H(0, \widehat{X}) = Z$. Let $\mathbf{Z} = \pi_X(Z(F^{\mathrm{max}}))$. Then H induces a special deformation retraction $\widetilde{H} : \mathbb{R}_\infty^+ \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$ with final image* **Z**.

Proof. This is a combination of 14.1.1, 14.1.2, 14.1.3 and 14.1.6 in [9], except for the fact in (4) that $\widetilde{H}$ is a special deformation retraction when H is assumed to be an **F**-definable special deformation retraction. This is easily checked by hand.   □

We will call a map $\tilde{h} : B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(Y)$ as in part (3) of Fact 5.2 *definably induced*.

Theorem 5.3. *Let G be an algebraic group defined over F, and let N be an F-definable stably dominated connected subgroup of G. Assume that N is commutative or that F is a valued field of equicharacteristic 0. Then there is a definably induced $N(F)$-equivariant special deformation retraction $\tilde{\rho} : \mathbb{R}_\infty^+ \times B_{\mathbf{F}}(N) \to B_{\mathbf{F}}(N)$ with final image $\tilde{\rho}(0, B_{\mathbf{F}}(N)) = \{p_N | \mathbf{F}\}$.*

Proof. It suffices to apply Fact 5.2(4) to Theorem 3.7 or to Corollary 4.18, respectively.   □

Similarly, Fact 5.2(4) applied to Theorem 3.10 or to Corollary 4.19, respectively, yields the following result, where we identify (the underlying topological space of) S^{an} with $B_{\mathbf{F}}(S)$.

Theorem 5.4. *Let S be a semi-abelian variety defined over F. Consider the F-definable decomposition $0 \to N \to S \to \Lambda \to 0$ from Fact 3.9. Then there is a definably induced $S(F)$-equivariant special deformation retraction $\tilde{\rho} : \mathbb{R}_\infty^+ \times S^{\mathrm{an}} \to$*

S^{an} *with final image a skeleton* $\mathbf{\Sigma}$. *The map* $\pi : S \to \Lambda$ *induces a bijection between* Σ *and* $\Lambda(\mathbf{F})$, *where* $\Lambda(\mathbf{F})$ *is a subset of* $\mathbb{R}^n$ *carrying the structure of a piecewise linear abelian group.*   □

Remark 5.5. Recall that in any NIP theory, in particular in ACVF, every ∞-definable group G admits a strong connected component G^{00}, by a result of Shelah (see, e.g., [15, Theorem 8.4]). Assume now that S is a semi-abelian variety defined over a model of ACVF, and let $0 \to N \to S \to \Lambda \to 0$ be the decomposition from Fact 3.9. As $N = N^{00}$, we have $S/S^{00} \cong \Lambda/\Lambda^{00}$. Thus the failure of strong connectedness of S may be entirely pushed to the piecewise linear world.

Moreover, if $S = A$ is an abelian variety, in the expansion of Γ to a real closed field $\mathcal{R}$, we get $\Lambda \cong S^1(\mathcal{R})^e$ for some integer e with $0 \le e \le \dim(A)$, and A^{an} is homotopy equivalent to $S^1(\mathbb{R})^e \cong \Lambda/\Lambda^{00}$. In this sense, the homotopy type of A^{an} is encoded in the failure of strong connectedness of A.

6. NIP ABELIAN GROUPS

In this section we prove an analogue of our main theorem, but starting with an arbitrary abelian group definable in an NIP theory. The result is of course much weaker, in particular there is no space of generics that could play the role of $\widehat{V}$. We obtain a directed system of ∞-definable subgroups $C(t)$ and instead of those groups having a stably dominated (or generically stable) generic type—which need not exist in general—we ask that they admit a generically stable invariant *measure*. Groups with such a generically stable invariant measure are called *fsg* and can be thought of as the *definably compact* groups in a general NIP theory.

We first recall some definitions, all of which can be found in more details in [15]. A (Keisler) measure μ on a definable set X over a model M is a finitely additive probability measure on M-definable subsets of X. If $X = G$ is a group, then we say that μ is invariant if $\mu(g \cdot Y) = \mu(Y)$ for every M-definable $Y \subseteq X$ and every $g \in G(M)$. A group admitting an invariant measure (over some, or equivalently any, model) is said to be definably amenable. In particular, if $G(M)$ is amenable as a pure group, then it is definably amenable and it follows that any solvable definable group is definably amenable.

Since a type is a special case of a measure (with values in $\{0, 1\}$ instead of $[0, 1]$), groups with a generically stable invariant type are definably amenable. Many conditions on types translate to measures, and this is the case in particular for generically stability. There are several equivalent definitions of a generically stable measure. We give one which is easy to state. In the following, if $a_1, \ldots, a_n$ are tuples from a model M and $\phi(x)$ is a formula over M, we let $Av(\phi(x); a_1, \ldots, a_n)$ denote $|\{i \le n : M \models \phi(a_i)\}|$.

Definition 6.1. Let T be an NIP theory and $\mu(x)$ a measure over a model M. We say that μ is generically stable if it can be uniformly approximated by finite averages of points in the following sense:

For any formula $\phi(x; y)$ and $\epsilon > 0$, there are $a_1, \ldots, a_n \in M^{|x|}$ such that for any $b \in M^{|y|}$,

$$|\mu(\phi(x; b)) - Av(\phi(x; b); a_1, \ldots, a_n)| \le \epsilon.$$

Note that a generically stable measure $\mu(x)$ is definable in the following sense: given $\phi(x; y)$ and $\epsilon > 0$, there are formulas $\theta_i(y, \bar{d})$, $i = 1, \ldots, n$, that cover y-space and such that for any b, b' if for some $i \le n$, $b, b \models \theta_i(y, \bar{d})$, then $|\mu(\phi(x; b)) -$

$\mu(\phi(x; b'))| \leq \epsilon$. Indeed, one can take $\bar{d}$ to consist of the points $a_1, \ldots, a_n$ as in the definition above for $\epsilon/2$ and let the formulas $\theta_i(y, \bar{d})$ enumerate the ϕ-types over $\bar{d}$. Note in particular that as $\phi(x; y)$ and ϵ vary, we only need $|T|$-many parameters for $\bar{d}$, since it is enough to take ϵ rational.

An ∞-definable group G which admits a generically stable invariant measure $\mu(x)$ is said to have fsg (*finitely satisfiable generics*). This condition can be thought of as an abstract version of compactness. For instance, in an o-minimal structure, a definable group has fsg if and only if it is definably compact. Similarly we can state:

Lemma 6.2. *Let A be an abelian variety definable in a model of* ACVF*, then A has fsg.*

Proof. Let $0 \to N \to A \to \Lambda \to 0$ be the decomposition from Fact 3.9. The group N is stably dominated (hence generically stable), so it has fsg. By [12, Corollary 6.19], Λ is a definably compact, Γ-internal group, hence Λ has fsg by [10, Theorem 8.1]. By Proposition 4.5 of that same paper, an extension of a group with fsg by a group with fsg has fsg. Hence A has fsg. $\qquad\square$

If $\mu(x)$ and $\eta(x)$ are two generically stable (Keisler) measures on a definable group G, we can define the convolution $\mu * \eta(x)$ by

$$\mu * \eta(\phi(x)) = \int_y \mu(\phi(x \cdot y)) d\lambda(y).$$

We refer to [15, Section 7.4] for explanation of why this integral makes sense. Then $\mu * \eta$ is again a generically stable measure. Note that if G is abelian, then for any $g \in G$,

(A) $g \cdot (\mu * \eta) = (g \cdot \mu) * \eta = \mu * (g \cdot \eta)$.

Proposition 6.3. *Let T be NIP and let G be a definable abelian group. Then there is a pro-definable set S and ∞-definable subgroups $N(t)$, for $t \in S$ forming a directed system (any small family has an upper bound) and such that $\bigcup_{t \in S} N(t) = G$ and each $N(t)$ stabilizes a generically stable measure on G.*

Proof. As G is abelian, it is definably amenable. Let M be a $|T|^+$-saturated model and let μ_M be a G-invariant measure over M. We can extend μ_M to a global measure μ which is $G(M)$-invariant and generically stable (see [11, Lemma 7.6] or [14, Proposition 3.4]). Let $N_\mu = \{g \in G(\mathcal{U}) \mid g \cdot \mu = \mu\}$ be the stabilizer of μ. Then N_μ is a subgroup of G containing $G(M)$. By definability of μ, N_μ is ∞-definable over a set of size $|T|$. Let t_0 be an enumeration of the parameters needed to define μ and write $\mu = \mu_{t_0}$. Then also N_μ is defined over t_0 and we can write $N_\mu = N(t_0)$. Let $S = \text{tp}(t_0)$. Then S is pro-definable and for every tuple $t \in S$, μ_t is a well-defined generically stable measure (where μ_t is defined over t using the same definition scheme as μ over t_0). Then also $N_{\mu_t} = N(t)$ is defined over t the same way N_μ is defined over t_0. We have constructed a pro-definable family of ∞-definable subgroups of G.

Since $N(t_0)$ contains $G(M)$, by compactness, for any (small) model M', there is $t' \equiv t_0$ such that $N(t')$ contains $G(M')$. This proves that $\bigcup_{t \in S} N(t) = G$.

It remains to show that the family is directed. Let $(r_i)_{i \in I}$ be a small family of elements of S and write $\mu_i = \mu_{r_i}$. For any finite $I_0 \subseteq I$, let $\mu_{I_0} = *_{i \in I_0} \mu_i$. As G is abelian, this product is independent of the order of the factors. By (A), each $N(r_i)$,

$i \in I_0$ is in $Stab(\mu_{I_0})$. Fix a model M' over which all the μ_i's are defined. Let μ be a limit of the μ_{I_0} over M' along an ultrafilter containing $\{I_1 : I_1 \supseteq I_0\}$ for each finite $I_0 \subseteq I$. Then μ is M'-invariant and its stabilizer contains each $N(r_i)$, $i \in I$. As above, we can extend μ to a global measure λ which is generically stable and whose stabilizer still contains all the $N(r_i)$'s. Now λ is defined over some model of size $|T|$ and hence there is an automorphism σ such that $\sigma(\lambda) =: \lambda_0$ is generically stable over M. Let t_0 be as in the first paragraph of the proof. Then $N(t_0)$ contains $G(M)$ and as λ_0 is finitely satisfiable in M, we have $\lambda_0(N(t_0)) = 1$. This implies that $Stab(\lambda_0) \leq N(t_0)$. So $Stab(\lambda) \leq N(\sigma^{-1}(t_0))$, with $\sigma^{-1}(t_0) \in S$. Therefore each $N(t_i)$ is a subgroup of $N(\sigma^{-1}(t_0))$, which finishes the proof. $\square$

Corollary 6.4. *If G is an abelian group with no indiscernible linearly ordered family of ∞-definable groups, then G has fsg.*

Proof. By the previous proposition and since the family $C(t)$ is directed, it must be that $G = C(t)$ for some t. Then G stabilizes a generically stable measure and hence G has fsg. $\square$

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