Covering for the Dodd-Jensen core model below $0^\dagger$

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Preface

The covering lemma is one of the most central theorems of inner model theory. In fact it is not a single theorem, it is rather a family of theorems which apply to different core models under specific smallness assumptions. The first proof of a covering lemma is due to Jensen and proved for the constructible hierarchy L in [2]. The covering lemma for L is the following theorem:

Theorem (Jensen). *Exactly one of the following statements holds.*

- *L covers , i.e. for all sets X of ordinals there is some $Y \in L$ such, that $X \subseteq Y$ and $\overline{\overline{Y}} \leqslant \overline{\overline{X}} + \aleph_1$.*
- *$0^\sharp$ exists.*

Thus, in the latter case, higher core models can be constructed. The proof relies on fine structure theory, which was developped by Jensen in [9] to allow an in depth study of the constructible hierarchy. Later Dodd and Jensen constructed a new core model K in [3] and proved covering for it in [4] and [5]. We want to rework the proof of these papers, to be precise, we want to prove the following theorem:

Theorem (Dodd-Jensen). *Assume $\neg(0^\dagger)$ then one of the following statements holds true:*

- *K covers, i.e. for all sets X of ordinals there is some $Y \in K$ such, that $X \subseteq Y$ and $\overline{\overline{Y}} \leqslant \overline{\overline{X}} + \aleph_1$.*
- *There is a Prikry generic sequence C over K such that $K[C]$ covers, i.e. for all sets X of ordinals there is some $Y \in K[C]$ such, that $X \subseteq Y$ and $\overline{\overline{Y}} \leqslant \overline{\overline{X}} + \aleph_1$.*

Notice that the assumption $\neg(0^\dagger)$ implies that no inner model has two measurable cardinals. If one allows the core models to have a regular limit of measurables, then covering may fail, that is, one can no longer expect finding a maximal Prikry system such that $K[C]$ covers. If one allows such cardinals in the core model, the covering lemma will have to be be different, but one can still prove that for every set x one can find a Prikry system C_x such that $K[C_x]$ covers x, an in depth study of the behavior of such Prikry sequences can be found in [7].

We assume the reader is familiar with fine structure theory as in [11], with inner model theory as in [12] and with basic facts about Prikry forcing, mainly about Prikry sequences as in [8].

This paper has the following structure:

In the first section we will fix our notation and recall some definition and properties of premice and of K.

The second section introduce collapsing mice and presents in lemma 2.12 how to get an inner model theoritic grasp on the question. It's a crucial lemma which is needed in every proof of covering in a form or another. As a matter of fact this lemma restates the covering theorem as a pure inner model theoritic problem.

Then to prove covering we will split the proof in two parts, by considering a new hypothesis:

(H) If $\mu > \aleph_2$ is singular in V, then μ is singular in K too.

and proving the following: Suppose $\neg(0^\dagger)$

i. (H) implies that K covers.

ii. If (H) fails there is a Prikry generic C over K such that $K[C]$ covers.

section three is devoted to a more detailed analysis of elementary substructures of K, which leads to the main lemma of the proof of covering: the fact, that for carefully chosen –but nonetheless sufficiently many– elementary substructures of K, their transitive collapses are not moved in the coiteration with K. In the second part of section two, we will prove the first part of covering, i.e. under (H) the core model already covers.

In section four we will study the $\neg(H)$ case, we will show how the above mentioned coiterations give rise to Preprikry sequences and we will derive a unique sequence, which will behave well enough to be "truly" Prikry.

In the fifth section, we prove the measurability of the least counter example to (H), and finish the proof of the covering theorem by using all methods of section two and the insights given by section three.

I want to thank Professor Dr. Ralf Schindler for allowing me to work on this interesting subject and his advice while writing this paper, Philipp Doebler for his corrections of my grammar and spelling errors, Klaus Loerke for his LaTeX advice and last but not least, my parents for their unconditional support during my studies.

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1 Preliminaries

If not noted elsewhere, every notation is supposed to be found in [11] for fine structural notations or [12] for Σ^*-theoretic or inner model theoretic concepts.
We write $\mathrm{cf}(\mu)$ for the cofinality of μ, and $\mathrm{cp}(\pi)$ for the critical point of π.

- Let $\mathcal{M}$ be a premouse, $\rho_n(\mathcal{M})$ is the n^{th} projectum of $\mathcal{M}$, as in [11] definition 5.1,
- $p(\mathcal{M})$ the standard parameter of $\mathcal{M}$, as in [11] definition 6.3,
- we write $\mathcal{M}^n$ for $\mathcal{M}^{n,p(\mathcal{M})}$, as in [11] definition 5.1,
- $H^n_{\mathcal{M}} = H^{\mathcal{M}}_{\rho_n(\mathcal{M})}$ as in [12] p. 18.
- $h^{n+1}_{\mathcal{M}}(\xi, p)$ is the iterated composition of the Σ_1 skolem functions of the appropriated reducts as in [11] p. 33,
- $h_{\mathcal{M}}(X)$ denotes the Σ_1 skolem hull of $\mathcal{M}$ with parameter from X as in [11] convention, p. 14,
- $h^{n+1}_{\mathcal{M}}(X)$ is the hull formed by the iterated composition of the Σ_1 skolem functions of the appropriated reducts as in [11] p. 33, these are denoted $\tilde{h}^{n+1}_{\mathcal{M}}(X)$ in [12].
- $\Gamma(\kappa, \mathcal{M})$ is the set of all good $\mathbf{\Sigma}^{(n)}_1(\mathcal{M})$-functions $f : \kappa \to \mathcal{M}$ for n such that $\rho_{n+1}(\mathcal{M}) > \kappa$ as in [12] p. 73.
- Let $\sigma : J^E_\tau \to J^{E'}_{\tau'}$ cofinally, $\mathcal{M} \rhd J^E_\tau$ such that τ is a cardinal in $\mathcal{M}$, then $\Gamma(\sigma, \mathcal{M})$ is the set of all good $\mathbf{\Sigma}^{(n)}_1(\mathcal{M})$ functions $f : \gamma \to \mathcal{M}$ such that $\gamma < \tau$ and $\rho_{n+1}(\mathcal{M}) \geqslant \tau$ as in [12] p. 96.
- $\Gamma^k(\kappa, \mathcal{M})$ is the set of all good $\mathbf{\Sigma}^{(n)}_1(\mathcal{M})$-functions $f : \kappa \to \mathcal{M}$ for $n < k$ such that $\rho_{n+1}(\mathcal{M}) > \kappa$ as in [12] p. 93.
- $\mathrm{Ult}(\mathcal{M}, U)$ denotes the coarse ultrapower by a measure U.
- $\mathrm{Ult}^*(\mathcal{M}, E)$ denotes the Σ^* ultrapower as in [12] section 3.1; typical elements are of the form $[\alpha, f]$ with $\alpha < \mathrm{lh}(E)$ and $f \in \Gamma(\mathrm{cp}(E), \mathcal{M})$, one can show that they have the form $\pi(f)(\alpha)$ too ([12], lemma 3.1.5).
- $\mathrm{Ult}^{(n)}(\mathcal{M}, E)$ denotes the $\Sigma^{(n)}$ ultrapower as in [12] section 3.5; typical elements are of the form $[a, f]$ with $a \in E$ and $f \in \Gamma^k(\mathrm{cp}(E), \mathcal{M})$.
- Let $\sigma : Q \to Q'$, for $\mathcal{M} \rhd Q \| \tau$ such that τ is a cardinal in $\mathcal{M}$. We denote by $\mathrm{Ult}(\mathcal{M}, \sigma \upharpoonright \tau)$ the model $\mathcal{N}$ such that $\tilde{\sigma} : \mathcal{M} \to \mathcal{N}$ is the canonical extension of $\sigma \upharpoonright \kappa : Q|\tau \to Q'|\sup \sigma"\tau$ as in [12] p. 102.

We use the iterability concepts of [12]. $K[C]$ is the model given by Prikry forcing over the filter generated by C, for more on Prikry forcing see [8].
In diagrams representing iteration a curly arrow means that we allow drops to occur, a normal arrow means an iteration map, hence that no drop occurs.

Definition 1.1 ([12] p. 109). A premouse is an acceptable J-structure $\mathcal{M} = (J^E_\alpha, E_{\omega\alpha})$ satisfying:

i. $E \subseteq \{(\nu, x);\ \nu < \omega\alpha \wedge x \subseteq \nu\}$. Set $E_\nu = \{x; (\nu, x) \in E\}$.

ii. $\forall \nu < \omega\alpha$, either $E_\nu = \emptyset$ or ν is a limit, S^E_ν has a largest cardinal κ and E_ν is a normal measure over S^E_ν with critical point κ and the stucture $\mathcal{M}\|\nu = (J^E_\nu, E_{\omega\nu})$ is amenable.

iii. (Coherency) Let $\nu \leqslant \omega\alpha$ and

$$\pi : S^E_\nu \to_{E_\nu} \mathcal{N} \text{ weakly}$$

where $\mathcal{N} = (|\mathcal{N}|;\ E')$. Then $\nu + 1 \subseteq \text{wfcore}(\mathcal{N})$, $E' \upharpoonright \nu = E \upharpoonright \nu$ and $E'_\nu = \emptyset$.

iv. $\mathcal{M}\|\nu$ is sound for all $\nu < \alpha$.

We will call the mouse active (or having an active measure) if $E_{\omega\alpha} \neq \emptyset$ and passive otherwise.
As said before $\mathcal{M}\|\nu = (J^{E^\mathcal{M}}_\nu, E^\mathcal{M}_{\omega\nu})$, let further $\mathcal{M}|\nu = (J^{E^\mathcal{M}}_\nu, \emptyset)$.
We write $\mathcal{M} \trianglelefteq \mathcal{N}$ if $\mathcal{M} = \mathcal{N}\|\nu$ for some $\nu \leqslant OR \cap \mathcal{N}$ and $\mathcal{M} \triangleleft \mathcal{N}$ if $\mathcal{M} = \mathcal{N}\|\nu$ for some $\nu < OR \cap \mathcal{N}$

Definition 1.2.

- $(0^\sharp)$ is the statement: There is an iterable premouse with an active measure.
- $(0^\dagger)$ is the statement: There is an iterable premouse $\mathcal{M}$ and $\alpha, \beta \in OR$ such that $E^\mathcal{M}_\alpha$ and $E^\mathcal{M}_\beta$ are active measures and $\beta < \alpha$.
- $(0^\ddagger)$ is the statement: There is an iterable "premouse" $\mathcal{M} = (J^E_\nu, E_\nu, E_{\nu+1})$ such that both E_ν and $E_{\nu+1}$ have same critical point, further $E_{\nu+1}$ is a measure of order 1 in $\mathcal{M}$. For an exact definition of $0^\ddagger$ see [12] p. 200.

Notice that the mouse in the $(0^\ddagger)$ case doesn't match with our definition, one should weaken the conditions on premice to allow such a mouse, but since we work under $\neg(0^\ddagger)$ during all this work, we won't formulate it, the only thing to remember, is that no mouse can have two measures with the same critical point.

Definition 1.3. A weasel W is a class sized model of the form $L[E]$ such that $W\|\alpha$ is a mouse for all $\alpha \in OR$, see [12] p. 175.
A weasel W is universal if and only if the coiteration of W with any coiterable premouse terminates, i.e. its length is strictly less than ∞, see [12] p. 184.
Let W be a universal weasel. A measure U on W with critical point κ is W-correct if and only if

- setting $\nu = \kappa^{+W}$, (J^E_ν, U) is a premouse;
- $\text{Ult}(W, U)$ is well-founded.

We stress that we do not assume that $U \in W$. ([12], p. 218)

Definition 1.4 ([12] p. 212 ff.). $(\neg 0^\ddagger)$
A mouse $\mathcal{M}$ is strong if and only if there is a universal weasel W such that $\mathcal{M} = W\|\alpha$ for an $\alpha \in OR$
The measure sequence E^K is inductively define as follows:

$$E^K_{\omega\nu} = \begin{cases} F & \text{if } F \text{ is the only measure such that } (J^E_\nu, F) \text{ is strong;} \\ \emptyset & \text{if } J^E_\nu \text{ is strong and for no measure } F \text{ is } (J^E_\nu, F) \text{ strong.} \end{cases}$$

We set

$$K = L[E^K] = \bigcup_{\alpha \in OR} J_\alpha^{E^K}.$$

K is called the core model; if $\neg(0^\dagger)$ then K is called the Dodd-Jensen core model.

Theorem 1.5 ([12] p. 226 and 232). *Suppose* $\neg(0^\ddagger)$.

- *K is a universal weasel.*
- *Let U be K-correct, then there is a ν such that $U = E_\nu^K$.*
- *Let $\sigma : K \to W$ be an elementary map from K to a universal weasel, then W is a simple iterate of K and σ is the iteration map.*

2 Collapsing mice and embeddings

To prove covering, we must first give it more structure, i.e. reduce the problem to something on which we can "unleash" inner model theory and fine structure theory, this will be done in the next lemma in which we show, that "good" maps already covers V, such that we only have to check, that such maps have their range in K or $K[C]$.

Definition 2.1. Let $\mathcal{N}$ be a premouse with a largest cardinal η. A mouse $\mathcal{M} \trianglerighteq \mathcal{N}$ is called a collapsing mouse for $\mathcal{N}$ if and only if

- $\mathcal{N} \cap OR$ is a cardinal of $\mathcal{M}$, if $\mathcal{M} \triangleright \mathcal{N}$, and
- there is some n such that $\rho_{n+1}(\mathcal{M}) \leqslant \eta < \rho_n(\mathcal{M})$ and
$$\mathcal{M} = h_{\mathcal{M}}^{n+1}(\eta \cup \{p(\mathcal{M})\}),$$
hence $\mathcal{M}$ is η-sound.

Lemma 2.2. *For a given $\mathcal{N}$, there is at most one collapsing mouse for $\mathcal{N}$.*

Proof. Deny. Let $\mathcal{M}, \mathcal{M}'$ be two collapsing mice for $\mathcal{N}$ and $\nu = \mathcal{N} \cap OR$ and η the largest cardinal of $\mathcal{N}$. We coiterate $\mathcal{M}$ and $\mathcal{M}'$. Let Q be the last model on the $\mathcal{M}$-side and Q' the last model of the $\mathcal{M}'$-side. As both are mice only one side is non simple(c.f. [12] lemma 5.3.1), let us suppose without loss of generality that the $\mathcal{M}$-side is simple and let π be the associated iteration map:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\pi\quad} & Q \\ & & \trianglelefteq \\ \mathcal{M}' & \rightsquigarrow & Q' \end{array}$$

As the $\mathcal{M}'$-side is the non-simple side, $Q \trianglelefteq Q'$(c.f. [12] lemma 4.4.2). Further as $\mathcal{M}\|\nu = \mathcal{N} = \mathcal{M}\|\nu$ both iterations are above η as η is the largest cardinal in $\mathcal{M}\|\nu$. Let $\mathcal{M}'_i$ be the structure of the $\mathcal{M}'$-side of the coiteration, $\mathcal{M}_i$ the structure on the $\mathcal{M}$-side, (ν_i, α_i) the indices of the coiteration. Let n be such that $\rho_{n+1}(\mathcal{M}) \leqslant \eta < \rho_n(\mathcal{M})$.

1. Case $Q \triangleleft Q'$.

Then Q is sound hence $Q = \mathcal{M}$. Let
$$a \in \left(\mathbf{\Sigma}_1^{(n+1)}(\mathcal{M}) \cap \mathcal{P}(\eta)\right) \setminus \mathcal{M},$$
We have that $a \in Q'$, hence $a \in \mathcal{M}'$ as $\mathcal{P}(\eta) \cap Q' \subseteq \mathcal{P}(\eta) \cap \mathcal{M}'$ (c.f. [12] 4.2.2). But $\mathcal{P}(\eta) \cap \mathcal{M}' = \mathcal{P}(\eta) \cap \mathcal{M}$ as ν is a cardinal in both structures, hence $a \in \mathcal{M}$, a contradiction.

2. Case $Q = Q'$.

Suppose $Q \neq \mathcal{M}$ then Q is not η-sound, hence $Q = Q' \neq \mathcal{M}'$. Since Q and $\mathcal{M}$ are mice they are solid (c.f. [12] 5.2.1). As Q is solid, we have that $\pi(p(\mathcal{M})) = p(Q)$. Let X be the closure of $\eta \cup p(Q)$ under functions of $\Gamma(\eta, Q)$. The transitive collapse of X is $\mathcal{M}$, since $\mathcal{M}$ is transitive and $\pi(h_{\mathcal{M}}^{n+1}(\xi, p(\mathcal{M}))) = h_Q^{n+1}(\xi, p(Q))$ as $\xi < \eta$. Thus $\pi : \mathcal{M} \xrightarrow{\sim} X$. Let $\mathcal{M}^*$ be the last truncate on the $\mathcal{M}'$-side, hence:

$$\begin{array}{ccccc} \mathcal{M} & & \xrightarrow{\quad\pi\quad} & & Q \\ & & & & \| \\ \mathcal{M}' & \rightsquigarrow & \mathcal{M}^* & \xrightarrow{\sigma} & Q' \end{array}$$

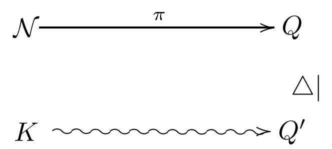
Since $\mathcal{M}^*$ is a truncate, it is sound, let σ be the associated iteration map from $\mathcal{N}^*$ to Q'. We also have that Q' and $\mathcal{M}^*$ are solid and that the standard parameter of $\mathcal{M}^*$ is mapped on the standard parameter of Q', thus

$$\mathcal{M}^* = h^{n+1}_{\mathcal{M}^*}(\eta \cup \{p(\mathcal{M}^*)\}) \cong h^{n+1}_{Q}(\eta \cup \{p(Q)\}),$$

thus $\mathcal{M} = \mathcal{M}^*$. Let i be such that $\mathcal{M}^* = \mathcal{M}'_i \| \alpha_i$. Let ν_j be the index of the first used measure on the $\mathcal{M}$-side. Suppose $i > j$. By the normality of the iteration $E^{\mathcal{M}_i}_{\nu_j} = E^{\mathcal{M}'_i}_{\nu_j} = \emptyset$, but on the other side $E^{\mathcal{M}'_i}_{\nu_j} = E^{\mathcal{M}}_{\nu_j} \neq \emptyset$ since that measure was used. Thus $i = j$ but then as $\nu_j = \nu_i$ we have that $E^{\mathcal{M}}_{\nu_j} = E^{\mathcal{M}^*}_{\nu_i}$, thus we wouldn't use that measure, a contradiction.
The only possibillity to get $\mathcal{M} = Q = Q'$, if $\mathcal{M}'$ is moved, Q' could not be η-sound, hence $\mathcal{M}'$ is not moved and $\mathcal{M} = \mathcal{M}'$ □(Lemma 2.2)

Remark 2.3. If $\mathcal{N} = K|\gamma$, such that $\mathcal{N}$ has a largest cardinal η with $K \models$ "η is a cardinal", and $\mathcal{M} \trianglerighteq \mathcal{N}$ is a collapsing mouse for $\mathcal{N}$, then $\mathcal{M} \lhd K$.

Proof. We coiterate $\mathcal{N}$ with K, since K is universal the $\mathcal{N}$ side is simple. We have the following diagram, where Q is the last structure on the $\mathcal{N}$-side and Q' the last structure on the Q' side, let π be the associated iteration map.



1. Case $\mathcal{N}$ is not moved.
Then $\mathcal{N} \lhd Q'$, this is clear if there is no drop on the K-side, if there is a drop it suffices to notice that Q' is not η-sound if a measure is used on the K side. Let ν be the index of the first measure used in the K-side, ν is a cardinal in all structure of the K-side with the possible exception of K and $\gamma \leqslant \nu$, but then $\mathcal{N} \trianglelefteq Q' \| \nu$ since $\mathcal{N}$ projects to $\eta < \nu$ and ν is a cardinal in Q'. Then as $K \| \nu = Q \| \nu$ we have in fact that $\mathcal{N} \lhd K$.
2. Case $\mathcal{N}$ is moved.
Hence Q is not sound, thus K must be moved, and there is a drop on the K-side. $Q = Q'$, since Q' is a mouse and Q is not sound. Recall that $\mathcal{N} \rhd K|\gamma$, hence the coiteration is above η. Hence we are in the same case as case 2 of the proof of 2.2 the same argument leads to $\mathcal{N} \lhd K$. □(Remark 2.3)

Definition 2.4. Let $\mathcal{N}$ be a premouse. A mouse $\mathcal{M} \trianglerighteq \mathcal{N}$ is called a generalized collapsing mouse for $\mathcal{N}$ if and only if

- $\mathcal{N} \cap OR = \tau$ is a cardinal of $\mathcal{M}$, if $\mathcal{M} \rhd \mathcal{N}$, and
- if $\mathcal{N}$ has a largest cardinal μ, then $\mathcal{M}$ is μ-sound or else
- there is a $n < \omega$ such that $\rho_{n+1}(\mathcal{M}) < \tau \leqslant \rho_n(\mathcal{M})$ and

$$\mathcal{M} = h^{n+1}_{\mathcal{M}}(\tau \cup \{p(\mathcal{M})\}).$$

In this situation, we say that $\mathcal{M}$ is τ-sound.

Remark 2.5. If $\mathcal{M}$ is a collapsing mouse for $\mathcal{N}$, then $\mathcal{M}$ is a generalized collapsing mouse for $\mathcal{N}$.

Remark 2.6. If $\mathcal{M}$ is a generalized collapsing mouse for $\mathcal{N}$, and if $\overline{\mathcal{M}} \trianglerighteq \mathcal{N}$ such that $\tau = \mathcal{N} \cap OR$ is a cardinal of $\overline{\mathcal{M}}$ (if $\mathcal{N} \triangleleft \overline{\mathcal{M}}$), $\rho_{n+1}(\overline{\mathcal{M}}) \leqslant \tau$ and $\overline{\mathcal{M}} = h^{n+1}_{\overline{\mathcal{M}}}(\tau \cup \{p(\overline{\mathcal{M}})\})$, then $\overline{\mathcal{M}} \trianglelefteq \mathcal{M}$.
In particular, there is at most one generalized collapsing mouse for $\mathcal{N}$.

Proof. If $\mathcal{N}$ has a largest cardinal, then we have already proved this result. If τ is a limit cardinal, then the coiteration of $\mathcal{M}$ with $\overline{\mathcal{M}}$ is above τ and the arguments of 2.2 shows that there are no step in the coiteration, hence $\overline{\mathcal{M}} \trianglelefteq \mathcal{M}$ or $\mathcal{M} \triangleleft \overline{\mathcal{M}}$.
Let us suppose that $\mathcal{M} \triangleleft \overline{\mathcal{M}}$, as $\rho_{n+1}(\mathcal{M}) < \tau$, τ can not be a cardinal in a extension of $\mathcal{M}$, but it is one in $\overline{\mathcal{M}}$, a contradiction! □(Remark 2.6)

Notation 2.7. Let μ be a regular cardinal, $\overline{K} \cong X \prec K \| \mu$ such that $\overline{K}$ is transitiv and $\pi : \overline{K} \to K\|\mu$ the uncollapsing map.
Let $(\kappa^\pi_i : i < \alpha^\pi)$ be an enumeration of the transfinite cardinals of $\overline{K}$, and $\kappa^\pi_{\alpha^\pi} = OR \cap \overline{K}$. Let $\kappa^{\pi -}_i$ be either the cardinal predecessor of κ^π_i in $\overline{K}$ if it exists or κ^π_i else. For $i \leqslant \alpha^\pi$ let $\mathcal{M}^\pi_i$ be either

i. K, if $\pi \restriction \kappa^{\pi -}_i = \mathrm{id}$ and $\mathcal{P}(\kappa^{\pi -}_i) \cap K \subseteq \overline{K}$, or else

ii. the generalized collapsing mouse for $\overline{K}|\kappa^\pi_i$ if it exists.

For each $i \leqslant \alpha^\pi$ let n^π_i be either

i. 0, if $\mathcal{M}^\pi_i = K$, or else

ii. the n such that $\rho_{n+1}(\mathcal{M}^\pi_i) < \kappa^\pi_i \leqslant \rho_n(\mathcal{M}^\pi_i)$.

We write $\tilde{\mathcal{M}}^\pi_i$ for $\mathrm{Ult}(\mathcal{M}^\pi_i, \pi \restriction \kappa^\pi_i)$ and call it the lift up of $\mathcal{M}^\pi_i$ through π. For $X \prec K\|\mu$, let $\pi_X : \overline{K}_X \cong K^X \to K\|\mu$ be the uncollapsing map.

Definition 2.8. Let μ be a regular cardinal and $\pi : \overline{H} \to H_\mu$ fully elementary such that $\overline{H}$ is transitive. We call π almost good if and only if for all $i \leqslant \alpha^\pi$ if $\mathcal{M}^\pi_i$ exists, then $\tilde{\mathcal{M}}^\pi_i$ is normally iterable above $\pi(\kappa^{\pi -}_i)$.

Remark 2.9. $\mathcal{M}^\pi_i$ is undefined if and only if $\pi \restriction \kappa^{\pi -}_i \neq \mathrm{id}$ and there is no generalized collapsing mouse for $\overline{K}|\kappa^\pi_i$.

Lemma 2.10. *Let π, $\overline{K}$ as above, then for all $i < \alpha^\pi$ such that $\mathcal{M}^\pi_i$ exists and is not K, $\tilde{\mathcal{M}}^\pi_i$ is $\pi(\kappa^{\pi -}_i)$-sound, further the following is equivalent:*

i. $\tilde{\mathcal{M}}^\pi_i$ is normally iterable above $\kappa^{\pi -}_i$,

ii. $\tilde{\mathcal{M}}^\pi_i$ is an initial segment of K,

iii. $\tilde{\mathcal{M}}^\pi_i$ is iterable.

Hence if one of these conditions holds, $\tilde{\mathcal{M}}^\pi_i$ is the generalized collapsing mouse for $K|\pi(\kappa^\pi_i)$.

Proof. We already know that $\tilde{\mathcal{M}}^\pi_i \triangleright K|\pi(\kappa^\pi_i)$ (c.f. [12] 3.6.3), by [12] lemma 3.6.9 $\tilde{\mathcal{M}}^\pi_i$ is $\pi(\kappa^{\pi -}_i)$-sound as $\mathcal{M}^\pi_i$ is $\kappa^{\pi -}_i$-sound.
The only non trivial part of the equivalence is (i)⇒(ii). Let $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^\pi_i$, we drop all π and i in the notation and write τ for κ^π_i, κ for $\kappa^{\pi -}_i$. Suppose $\tilde{\mathcal{M}}$ is normally iterable above κ. As $\tilde{\mathcal{M}}|\tau = K|\tau$ we can coiterate $\tilde{\mathcal{M}}$ and K since coiterations are normal iterations and the coiteration will be above κ. Thus we are strictly in the

same situation as in the proof of remark 2.3 if $\kappa < \tau$, and thus it leads to the same result. If $\kappa = \tau$, then Case 1 of the proof of 2.3 still holds, and the other part as well using the fact that the coiteration is above τ. $\square$(Lemma 2.10)

Lemma 2.11. *Let $X \prec K\|\mu$ and $i < \alpha^{\pi_X}$ such that $\mathcal{M}_i^{\pi_X} = K$ then the following is equivalent:*

i. $\tilde{\mathcal{M}}_i^{\pi_X}$ is normally iterable above $\kappa_i^{\pi_X -}$,

ii. $\tilde{\mathcal{M}}_i^{\pi_X}$ is iterable.

Proof. We only have to show (i)$\Rightarrow$(ii). Let $W = \mathrm{Ult}(K, \pi_X \restriction \kappa_i^{\pi_X})$, $\eta = \kappa_i^{\pi_X -}$ and $\tau = \kappa_i^{\pi_X}$. If $\pi_X \restriction \tau = \mathrm{id}$ then $W = K$ and we have nothing to show, thus we can suppose that η is the critical point of π_X. As $K|\tau = W|\tau$ the coiteration is above η, hence W and K are coiterable. Let Q be the last model on the W-side, K^* the last model on the K-side. As K is universal the W-side is simple, let π be the associated iteration map:

$$\begin{array}{ccc} W & \xrightarrow{\quad\pi\quad} & Q \\ & & \trianglelefteq \\ K & \rightsquigarrow & K^* \end{array}$$

Thus Q is iterable. Thus, as π is fully elementary, W is also iterable. $\square$(Lemma 2.11)

Lemma 2.12 (frequent extension of embeddings). *Let θ be a cardinal and $\kappa \geq \aleph_1$ be a regular cardinal. The set*

$$S = \{\mathrm{ran}(\pi) \cap \theta; \pi \textit{ is almost good and } \mathrm{card}(\pi) = \kappa\}$$

is stationary in $[\theta]^\kappa$.

Proof. Let $\mathfrak{A} = (\theta; (f_i : i < \kappa))$ be an algebra. Let μ be a regular cardinal which is large enough. We recursively define sequences $(Y_i : i \leq \kappa)$, $(\overline{K}_i : i \leq \kappa)$ and $(\pi_i : i \leq \kappa)$ such that:

i. $Y_i \prec K\|\mu$, for all $i \leq \kappa$,

ii. $\overline{\overline{Y_i}} < \kappa$, for all $i < \kappa$,

iii. $Y_\lambda = \bigcup_{i<\lambda} Y_i$, for all limit ordinals $\lambda \leq \kappa$,

iv. $Y_{i+1} \supseteq f_j{}''Y_i^{<\omega}$, for all $j < i < \kappa$,

v. $\pi_i : \overline{K_i} \cong Y_i$, where $\overline{K_i}$ is transitiv, and

vi. If $j \leqslant \alpha^{\pi_i}$ and $\tilde{\mathcal{M}}_j^{\pi_i}$ is not normally iterable above $\pi_i(\kappa_j^{\pi_i -})$, then let

$$\mathcal{N}_{i,j} \cong X_{i,j} = \left\{[\alpha_k^{i,j}, f_k^{i,j}]; k < \omega\right\} \prec \tilde{\mathcal{M}}_j^{\pi_i},$$

where $\alpha_k^{i,j} \in \pi_i(\mathrm{dom}(f_k^{i,j}))$ and $f_k^{i,j} \in \Gamma(\pi_i, \mathcal{M}_j^{\pi_i})$, be such that $\mathcal{N}_{i,j}$ is transitive and $X_{i,j}$ is a witness for $\tilde{\mathcal{M}}_j^{\pi_i}$ not being normally iterable above $\pi_i(\kappa_j^{\pi_i -})$. In this situation let $\left\{\alpha_k^{i,j}; k < \omega\right\} \subseteq Y_{i+1}$ for each such j.

Let $\sigma^i_j : \mathcal{N}_{i,j} \to \tilde{\mathcal{M}}^{\pi_i}_j$ be the uncollapsing map, $\overline{\pi}_{i,j} = \pi_j^{-1} \circ \pi_i$, for $i < j \leqslant \kappa$, and $\overline{Y}_i = \pi_\kappa^{-1}"Y_i = \mathrm{ran}(\overline{\pi}_{i,\kappa})$.

We claim that (Y_κ, π_κ) is as desired. Obviously Y_κ is closed under the f_i's. Let us assume that π_κ is not as desired and work toward contradiction. By assumption there is a $j < \alpha^{\pi_\kappa}$ such that $\tilde{\mathcal{M}}^{\pi_\kappa}_j$ is not normally iterable above $\pi_\kappa(\kappa_j^{\pi_\kappa -})$. Let

$$X = \left\{[\alpha^{\kappa,j}_k, f^{\kappa,j}_k]; k < \omega\right\} \prec \tilde{\mathcal{M}}^{\pi_\kappa}_j,$$

where $\alpha^{\kappa,j}_k \in \pi_\kappa(\mathrm{dom}(f^{\kappa,j}_k))$ and $f^{\kappa,j}_k \in \Gamma(\pi_\kappa, \mathcal{M}^{\pi_\kappa}_j)$, be a witness to the non normal iterability above $\pi_\kappa(\kappa_j^{\pi_\kappa -})$, $\mathcal{N}_{\kappa,j}$ its transitive collaps and $\sigma^\kappa_j : \mathcal{N}_{\kappa,j} \to \mathcal{M}^{\pi_\kappa}_j$ the uncollapsing map.

Let $\tilde{\mu} > \mu$ be a regular cardinal which is large enough and pick a $Z \prec H_{\tilde{\mu}}$ with the following property:

i. $\overline{\overline{Z}} < \kappa$

ii. $\left\{f^{\kappa,j}_k; k < \omega\right\} \cup \left\{\overline{K}_\kappa\right\} \subseteq Z$, and

iii. $Z \cap \overline{K}_\kappa = \overline{Y}_{i_0}$ for an $i_0 < \kappa$,

We construct Z in ω steps, first pick a $Z_0 \prec H_{\tilde{\mu}}$ such that (i) and (ii) holds, if Z_n is defined then pick a i_{n+1} such that $Z_n \cap \overline{K}_\kappa \subseteq \overline{Y}_{i_{n+1}}$, there is such a $\overline{Y}_{i_{n+1}}$ by the regularity of κ, and let $Z_{n+1} \prec H_{\tilde{\mu}}$ such that $\overline{Y}_{n+1} \cup \sup\left\{\alpha; \alpha \in \overline{Y}_{n+1} \cap OR\right\} \subseteq Z_{n+1}$ and (i) and (ii) holds for Z_{n+1}. Then

$$Z = \bigcup_{n<\omega} Z_n$$

is a structure with the properties (i),(ii) and (iii). The construction shows that $Z \cap \kappa \in \kappa$. Let $\sigma : \tilde{K} \to Z$ be the uncollapsing map, we have that $\sigma(\overline{K}_{i_0}) = \overline{K}_\kappa$ and $\sigma \restriction \overline{K}_{i_0} = \overline{\pi}_{i_0,\kappa}$. Further let $\overline{j}$ be such that $\sigma(\kappa^{\pi_{i_0}}_{\overline{j}}) = \kappa^{\pi_\kappa}_j$ und $n = n^{\pi_{i_0}}_{\overline{j}}$. We have that:

$$Z \vDash "\mathcal{M}^{\pi_\kappa}_j \text{ is the generalized collapsing mouse for } \overline{K}_\kappa | \kappa^{\pi_\kappa}_j",$$

hence

$$\tilde{K} \vDash "\sigma^{-1}(\mathcal{M}^{\pi_\kappa}_j) \text{ is the generalized collapsing mouse for } \sigma^{-1}(\overline{K}_\kappa | \kappa^{\pi_\kappa}_j)",$$

but as $\overline{K}_\kappa | \kappa^{\pi_\kappa}_j = \sigma(\overline{K}_{i_0} | \kappa^{\pi_{i_0}}_{\overline{j}})$, we have that $\sigma^{-1}(\mathcal{M}^{\pi_\kappa}_j) = \mathcal{M}^{\pi_{i_0}}_{\overline{j}}$ as generalized collapsing mice are unique. Hence, by the elementarity of σ, $n = n^{\pi_\kappa}_j$. We define:

$$\begin{aligned} \Phi : \mathcal{N}_{\kappa,j} &\to \tilde{\mathcal{M}}^{\pi_{i_0}}_{\overline{j}} \\ (\sigma^\kappa_j)^{-1}([\alpha, f]) &\mapsto [\alpha, \sigma^{-1}(f)], \end{aligned}$$

where $\alpha \in \pi_\kappa(\mathrm{dom}(f))$ and $f \in \Gamma(\pi_\kappa, \mathcal{M}^{\pi_\kappa}_j)$. For a $\Sigma^{(n)}_0$ formula φ we have following equivalences:

$$\begin{aligned}
&\tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}} \models \varphi(\Phi((\sigma_{\overline{j}}^{\kappa})^{-1}([\alpha, f])))\\
&\iff \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}} \models \varphi([\alpha, \sigma^{-1}(f)])\\
&\iff \alpha \in \pi_{i_0}(\Big\{u; u \in \mathrm{dom}(\sigma^{-1}(f)) \wedge \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}} \models \varphi(\sigma^{-1}(f)(u))\Big\})\\
&\iff \alpha \in \pi_\kappa \circ \overline{\pi}_{i_0,\kappa}(\Big\{u; u \in \mathrm{dom}(\sigma^{-1}(f)) \wedge \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}} \models \varphi(\sigma^{-1}(f)(u))\Big\})\\
&\iff \alpha \in \pi_\kappa \circ \sigma(\big\{u; u \in \mathrm{dom}(\sigma^{-1}(f)) \wedge \sigma^{-1}(\mathcal{M}_j^{\pi_\kappa}) \models \varphi(\sigma^{-1}(f)(u))\big\})\\
&\iff \alpha \in \pi_\kappa(\big\{u; u \in \mathrm{dom}(f) \wedge \mathcal{M}_j^{\pi_\kappa} \models \varphi(f(u))\big\})\\
&\iff \tilde{\mathcal{M}}_j^{\pi_\kappa} \models \varphi([\alpha, f])\\
&\iff X \models \varphi([\alpha, f])
\end{aligned}$$

Hence Φ is well defined and $\Phi : \mathcal{N}_{\kappa,j} \rightarrow_{\Sigma_0^{(n)}} \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}}$.

Suppose $\tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}}$ is normally iterable above $\pi_{i_0}(\kappa_{\overline{j}}^{\pi_{i_0}-})$, then by 2.10 it is already iterable. Thus it is normally iterable above $\Phi(\rho_{n+1}(\mathcal{N}_{\kappa,j}))$ and hence by [12] 4.3.7 $\mathcal{N}_{\kappa,j}$ is normaly iterable above $\rho_{n+1}(\mathcal{N}_{\kappa,j})$, but this is a contradiction to $\rho_{n+1}(\mathcal{N}_{\kappa,j}) \leqslant (\sigma_{\overline{j}}^{\kappa})^{-1}(\kappa_\kappa^{\pi_\kappa-})$. Thus $\tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}}$ is not iterable, hence not normally iterable above $\pi_{i_0}(\kappa_{\overline{j}}^{\pi_{i_0}-})$, and we already had choosen a witness $\mathcal{N}_{i,\overline{j}} \prec \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}}$ to the non normal iterability above $\pi_{i_0}(\kappa_{\overline{j}}^{\pi_{i_0}-})$. We define Ψ as follows:

$$\begin{aligned}
\Psi : \mathcal{N}_{i_0,\overline{j}} &\rightarrow \mathcal{M}_j^{\pi_\kappa}\\
(\sigma_{\overline{j}}^{i_0})^{-1}([\alpha_k^{i_0,j}, f_k^{i_0,j}]) &\mapsto \overline{\pi}_{i_0,\kappa}(f_k^{i_0,j})(\pi_\kappa^{-1}(\alpha_k^{i_0,j}))
\end{aligned}$$

Let $(\sigma_{\overline{j}}^{i_0})^{-1}([\alpha_k^{i_0,j}, f_k^{i_0,j}]) \in \mathcal{N}_{i_0,\overline{j}}$ and φ be a $\Sigma_0^{(n)}$-formula, we write α, f for $\alpha_k^{i_0,j}, f_k^{i_0,j}$.

$$\begin{aligned}
\mathcal{N}_{i_0,\overline{j}} \models \varphi((\sigma_{\overline{j}}^{i_0})^{-1}([\alpha, f])) &\iff \tilde{\mathcal{M}}_{\overline{j}}^{\pi_{i_0}} \models \varphi([\alpha, f])\\
&\iff \alpha \in \pi_{i_0}(\Big\{u; u \in \mathrm{dom}(f) \wedge \mathcal{M}_{\overline{j}}^{\pi_{i_0}} \models \varphi(f(u))\Big\})\\
&\iff \pi_\kappa^{-1}(\alpha) \in \overline{\pi}_{i_0,\kappa}(\Big\{u; u \in \mathrm{dom}(f) \wedge \mathcal{M}_{\overline{j}}^{\pi_{i_0}} \models \varphi(f(u))\Big\})\\
&\iff \pi_\kappa^{-1}(\alpha) \in \big\{u; u \in \mathrm{dom}(\overline{\pi}_{i_0,\kappa}(f)) \wedge \mathcal{M}_j^{\pi_\kappa} \models \varphi(\overline{\pi}_{i_0,\kappa}(f)(u))\big\}\\
&\iff \mathcal{M}_j^{\pi_\kappa} \models \varphi(\overline{\pi}_{i_0,\kappa}(f)(\pi_\kappa^{-1}(\alpha)))\\
&\iff \mathcal{M}_j^{\pi_\kappa} \models \varphi(\Psi((\sigma_{\overline{j}}^{i_0})^{-1}([\alpha, f])))
\end{aligned}$$

The fourth equivalence holds since $\sigma(\mathcal{M}_{\overline{j}}^{\pi_{i_0}}) = \mathcal{M}_j^{\pi_\kappa}$ and $\sigma \restriction \overline{K}_{i_0} = \overline{\pi}_{i_0,\kappa}$. Hence Ψ is $\Sigma_0^{(n)}$. $\mathcal{N}_{i_0,\overline{j}}$ is not iterable above $(\sigma_{\overline{j}}^{i_0})^{-1}(\kappa_{\overline{j}}^{\pi_{i_0}-})$, but $\mathcal{M}_j^{\pi_\kappa}$ is iterable, a contradiction to [12] 4.3.7! □(Lemma 2.12)

Corollary 2.13. *Let* $\overline{K} \cong X \prec K\|\mu$ *such that* π_X *is almost good,* $\mathrm{cp}(\pi_X) = \kappa$ *exists and* $\mathcal{P}(\kappa) \cap \overline{K} = \mathcal{P}(\kappa) \cap K$. *Then* $\mathrm{Ult}(K, U)$ *is iterable, where*

$$U = \{x \in \mathcal{P}(\kappa) \cap K;\ \kappa \in \pi_X(x)\}\,.$$

Proof. We define an embedding from $\mathrm{Ult}(K, U)$ in $\tilde{K}$, the lift up of K through $\pi \restriction \mathrm{cp}(\pi)^{+\overline{K}}$.

$$\begin{aligned}
k : \mathrm{Ult}(K, U) &\rightarrow \tilde{K}\\
[f] &\mapsto [\kappa, f]
\end{aligned}$$

Let $f:\kappa\to K$ and φ be a formula, then $\kappa\in\pi_X(\mathrm{dom}(f))$ and

$$\begin{aligned}\mathrm{Ult}(K,U)\vDash\varphi([f]) &\iff \{\alpha;K\vDash\varphi(f(\alpha))\}\in U\\ &\iff \kappa\in\pi_X(\{\alpha;K\vDash\varphi(f(\alpha))\})\\ &\iff \kappa\in\pi_X(\{\alpha;\alpha\in\mathrm{dom}(f)\wedge K\vDash\varphi(f(\alpha))\})\\ &\iff \tilde{K}\vDash\varphi([\kappa,f])\end{aligned}$$

Hence k is elementary and thus $\mathrm{Ult}(K,U)$ is iterable.

Remark that, if i_U is the cannonical ultrapowermap and $\tilde{\pi}$ the cannonical map from K to $\tilde{K}$, then $k\circ i_U=\tilde{\pi}$ by definition, moreover, as U is a normal ultrafilter, $k(\kappa)=k([\mathrm{id}])=[\kappa,\mathrm{id}]=\pi_X(\mathrm{id})(\kappa)=\kappa$, hence $\kappa<\mathrm{cp}(k)$. Further

$$\kappa^{+K}\leqslant\kappa^{+\,\mathrm{Ult}(K,U)}\leqslant\kappa^{+\tilde{K}}\leqslant\kappa^{+K}$$

hence all are equal and $\mathrm{cp}(k)>\kappa^{+\,\mathrm{Ult}(K,U)}$. □(Corollary 2.13)

3 When K covers

Definition 3.1. Let W be an inner model, that is $W \models ZFC$ and $W \subseteq V$ and $\mathrm{OR} \subseteq W$. We say W covers if and only if for all sets X of ordinals there is some $Y \in W$ such, that $X \subseteq Y$ and $\overline{\overline{Y}} \leqslant \overline{\overline{X}} + \aleph_1$.
We say W covers strongly if and only if the following holds:
if $\kappa \geqslant \aleph_1$ is a cardinal and $\theta \geqslant \kappa$, then $[\theta]^\kappa \cap W$ is stationary in $[\theta]^\kappa$.

Theorem 3.2 (Covering). *Assume $\neg(0^\dagger)$ then one of the following statements holds true:*

- *K covers.*
- *There is a Prikry generic sequence C over K such that $K[C]$ covers.*

We will prove a stronger version:

Theorem 3.3. *Assume $\neg(0^\dagger)$. Exactly one of the following statements holds true.*

i. K covers strongly.

ii. There is a $\mu > \aleph_2$ such that $K \models$"μ is measurable" and there is some $C \subseteq \mu$ Prikry generic over K such that $K[C]$ covers strongly.

We want to restrict the study of this problem to regular κs:

Lemma 3.4. *Let W be a inner model and $\kappa > \aleph_1$ a singular cardinal, such that $[\theta]^\lambda \cap W$ is stationary in $[\theta]^\lambda$ for all $\theta \geqslant \lambda$ and all $\lambda < \kappa$, then $[\theta]^\kappa \cap W$ is stationary in $[\theta]^\kappa$ for all $\theta \geqslant \kappa$.*

Proof. Let $\kappa \geqslant \aleph_1$ be as in the lemma $(\kappa_i; i < \mathrm{cf}(\kappa))$ a witness to the singularity of κ. Let further $\mathfrak{A} = (\theta, (f_j; j < \kappa))$ be an algebra with $\theta \geqslant \kappa$ a cardinal. By assumption for all i and all $n < \omega$ and for all $X_m \in [\theta]^{\leqslant \kappa_i}$ $m < n$, we can find a $\tilde{X}$ such that $\tilde{X}$ is closed under $(f_j; j < \kappa_i)$ and $\bigcup_{m<n} X_m \subseteq \tilde{X}$. Hence there is (in V) a function

$$\Phi_i^n : [[\theta]^{\leqslant \kappa_i}]^n \cap W \to [\theta]^{\leqslant \kappa_i} \cap W,$$

such that for all $(X_m; m < n) \in [[\theta]^{\leqslant \kappa_i}]^n$, $\Phi_i^n((X_m; m < n))$ is closed under $(f_j; j < \kappa_i)$.
Let $\mathfrak{B}_i = ([\theta]^{\leqslant \kappa}; (\Phi_i^n; n < \omega \wedge i < \mathrm{cf}(\kappa)))$; as we can choose in W a bijection between $[\theta]^{\leqslant \kappa}$ and some θ' and $\omega \cdot \mathrm{cf}(\kappa) < \kappa$ there is a $Z \subseteq [\theta]^{\leqslant \kappa}$ such that $Z \in W$ and Z is closed under all Φ_i^n. We want to show that $\bigcup Z$ is closed under all f_j. Let $\alpha_1, \dots, \alpha_n \in \bigcup Z$ and $j < \kappa$. There are $X_1, \dots, X_n \in Z$ and $i < \mathrm{cf}(\kappa)$ such that $j < \kappa_i$ and $\alpha_m \in X_m$ for all $m \leqslant n$. Then $\Phi_i^n((X_m; m \leqslant n)) \in Z$ and $f_j(\alpha_1, \dots, \alpha_n) \in \Phi_i^n((X_m; m \leqslant n))$ thus $f_j(\alpha_1, \dots, \alpha_n) \in \bigcup Z$. □(Lemma 3.4)
Let us now consider the following hypothesis:

(H) If $\mu > \aleph_2$ is singular in V, then μ is singular in K too.

We will split the proof of covering in two parts:

i. (H) implies that K covers strongly.

ii. If (H) fails and $0^\dagger$ doesn't exist, then there is a Prikry generic C over K such that $K[C]$ covers strongly.

We intend to prove the first part in this section, but as most of the lemmata don't use (H) and in fact are needed in later sections, it will be explicitely stated in the lemmata if the results are proven under (H).

Definition 3.5. Let $\pi : \overline{H} \to H_\mu$ be elementary and $\overline{H}$ transitive. The embedding π will be called a good map if π is almost good as defined in 2.8, is iterable and if π is continuous at points of cofinality ω.

Until the end of this section fix a π such that π is good and $\kappa = \mathrm{cp}(\pi)$ exists, let further $\overline{K} = K^{\mathrm{dom}(\pi)}$.

Lemma 3.6. $\mathcal{P}(\kappa) \cap K \not\subseteq \overline{K}$.

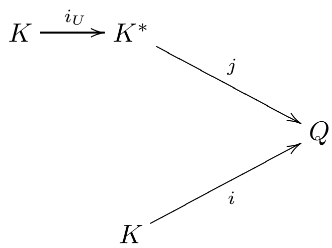
Proof. Deny. As $\mathcal{P}(\kappa) \cap \overline{K} = \mathcal{P}(\kappa) \cap K$, $\kappa^{+\overline{K}} = \kappa^{+K}$. Let

$$U = \{X \subseteq \kappa;\ X \in \mathcal{P}(\kappa) \cap K, \kappa \in \pi(X)\}.$$

By Corollary 2.13, we know that $K^* = \mathrm{Ult}(K, U)$ is iterable, even $k : K^* \to \tilde{K}$ is an elementary embedding such that $\mathrm{cp}(k) > \kappa^{+K} = \kappa^{+K^*} = \kappa^{+\tilde{K}}$, where $\tilde{K}$ was the lift up of K trough π. Hence

$$(*) \qquad K^* \| \kappa^{+K} = \tilde{K} \| \kappa^{+\tilde{K}} = K \| \kappa^{+K}.$$

We coiterate K^* with K, since both are universal weasels, we get the following diagram:



where i_U is the ultrapower map by U. Because of $(*)$, $i \restriction \kappa^{+K} + 1 = j \restriction \kappa^{+K} = \mathrm{id}$ and $\mathrm{cp}(j \circ i_U) = \kappa$ and $\mathrm{cp}(j) > \kappa^{+K} + 1$. This is an outright contradiction to the fact that there can be only one elementary embedding from K to a universal weasel, as we have noted in 1.5. $\square$(Lemma 3.6)

Lemma 3.7. *Let η be minimal such that $\mathcal{P}(\eta) \cap \overline{K} \neq \mathcal{P}(\eta) \cap K$, then*

$$\eta < \kappa \Rightarrow \kappa = \eta^{+\overline{K}}.$$

Proof.
As GCH holds in $\overline{K}$, if $\delta < \kappa$, then $\mathrm{card}^{\overline{K}}(\mathcal{P}(\delta)) = \delta^{+\overline{K}} \leqslant \kappa$.
1.Case $\kappa = \delta^{+\overline{K}}$.
Because of acceptability, we have a one to one function $f : \kappa \xrightarrow{\sim} \mathcal{P}^{\overline{K}}(\delta)$. Thus a one to one function $\pi(f) : \pi(\kappa) \xrightarrow{\sim} \mathcal{P}^K(\delta)$, since $\pi(\delta) = \delta$. As $\mathrm{card}^K(\kappa) < \mathrm{card}^K(\pi(\kappa))$, $\mathrm{card}(\mathcal{P}^{\overline{K}}(\delta)) < \mathrm{card}(\mathcal{P}^K(\delta))$. Hence $\mathcal{P}(\delta) \cap \overline{K} \subsetneq \mathcal{P}(\delta) \cap K$, since $\pi(f) \restriction \kappa = f$.
2. Case $\delta^{+\overline{K}} < \kappa$.
We have a one to one function $f : \delta^{+\overline{K}} \xrightarrow{\sim} \mathcal{P}^{\overline{K}}(\delta)$. Hence $\pi(f) : \delta^{+\overline{K}} \xrightarrow{\sim} \mathcal{P}^K(\delta)$, as $\pi(\delta^{+\overline{K}}) = \delta^{+\overline{K}}$ and $\pi(\delta) = \delta$. But for a $X \subseteq \delta$ $\pi(X) = X$, hence for all $\alpha < \delta^{+\overline{K}}$ $f(\alpha) = \pi(f)(\alpha)$ and therefore $\pi(f) = f$. Thus $\mathcal{P}(\delta) \cap \overline{K} = \mathcal{P}(\delta) \cap K$.

Hence if κ is a limit cardinal only case 2 occurs and $\mu = \kappa$ and if κ is a successor cardinal then case 1 shows that $\mu^{+\overline{K}} = \kappa$. $\square$(Lemma 3.7)
Therefore the coiteration of $\overline{K}$ with K is above η and as K is a universal weasel, the $\overline{K}$-side is simple. There must be a drop in the first step of the coiteration since $\eta^{+\overline{K}} < \eta^{+K}$. Let $\mathcal{M}_0 = K\|\zeta$ denote the truncated model.

Lemma 3.8. *$\overline{K}$ is not moved in the coiteration of $\overline{K}$ and K.*

Proof. Let ν_i be the iteration indices, κ_i the associated critical points and let $\overline{K}_i$ and $\mathcal{M}_i$ be the structures on the $\overline{K}$- side and $\mathcal{M}_0$-side of the coiteration. We write $\overline{E}_{\nu_i}$ for $E^{\overline{K}_i}_{\nu_i}$ and E_{ν_i} for $E^{\mathcal{M}_i}_{\nu_i}$. We treat the coiteration of K with $\overline{K}$ as if it were the coiteration of $\mathcal{M}_0$ with $\overline{K}$.

$$\begin{array}{ccc} \overline{K} & \longrightarrow & \overline{K}_\infty \\ & & \trianglelefteq| \\ \mathcal{M}_0 & \rightsquigarrow & M_\infty \end{array}$$

As we work towards contradiction, let us suppose that $\overline{K}$ is moved. Let ζ_i be the maximal ζ such that ν_i is the cardinal successor of κ_i in $\mathcal{M}_i\|\zeta$ or, in other words, the maximal ζ such that $\overline{E}_{\nu_i}$ is a total measure over the corresponding truncate.
Claim 1. Every mouse $\mathcal{M}_i^* = \mathcal{M}_i\|\zeta_i$ projects to κ_i and is κ_i-sound.
Proof.
Suppose first that $\zeta_i < \mathrm{ht}(\mathcal{M}_i)$.
$\mathcal{M}_i^*$ is a truncate of $\mathcal{M}_i$ hence it is sound, hence it is sound above κ_i. Thus

$$\mathcal{P}(\kappa_i) \cap \overline{K}_i \subsetneq \mathcal{P}(\kappa_i) \cap \mathcal{M}_i,$$

hence $\kappa_i^{+\overline{K}_i}$ is not a cardinal in $\mathcal{M}_i$, and it will be collapsed in $\mathcal{M}_i\|\zeta_i + \omega$, by the definition of ζ_i, thus $\mathcal{M}_i\|\zeta_i$ projects to κ_i. Suppose now that $\xi_i < \mathrm{ht}\,\mathcal{M}_i$, we prove this case by induction on i. For the successor step: let $\mathcal{M}_{i+1} = \mathrm{Ult}^*(\mathcal{M}_i^*, E_{\nu_i})$, where $\mathcal{M}_i^* = \mathcal{M}_i\|\zeta_i$ satisfies the claim. Let n be such that $\rho_{n+1}(\mathcal{M}_i^*) \leqslant \kappa_i < \rho_n(\mathcal{M}_i^*)$.
Each $x \in \mathcal{M}_{i+1}$ is of the form $\pi^{\mathcal{M}}_{i,i+1}(f)(\kappa_i)$ where $\pi^{\mathcal{M}}_{i,i+1}$ is the associated ultrapower map and $f \in \Gamma(\kappa_i, \mathcal{M}_i^*)$. There is a good $\Sigma_1^{(n-1)}(\mathcal{M}_i^*)$ function g such that $f \simeq g(\xi, p)$ with a parameter $p \in \mathcal{M}_i^*$.
As $\mathcal{M}_i^*$ is κ_i-sound by induction hypothesis for all $p \in \mathcal{M}_i^*$:

$$p = h^{n+1}_{\mathcal{M}_i^*}(\xi, p(\mathcal{M}_i^*)),$$

where $\xi < \kappa_i$. Hence

$$\pi_{i,i+1}(p) \in h^{n+1}_{\mathcal{M}_{i+1}}(\kappa_i \cup \{p(\mathcal{M}_{i+1})\}),$$

as iteration maps are Σ^*-preserving. Thus we even get:

$$x = g'(\kappa_i, h^{n+1}_{\mathcal{M}_{i+1}}(\xi, \pi_{i,i+1}(p(\mathcal{M}_{i+1})))),$$

where g' is a good $\Sigma_1^{(n-1)}(\mathcal{M}_{i+1})$ function with the same functionally absolute definition as g, as

$$\pi^{\mathcal{M}}_{i,i+1}(p(\mathcal{M}_i^*)) = p(\mathcal{M}_{i+1}).$$

We just have seen that x is $\Sigma_1^{(n)}(\mathcal{M})$-definable with parameters less than $\kappa_i + 1$. Hence:

$$\mathcal{M}_{i+1} = h^{n+1}_{\mathcal{M}_{i+1}}((\kappa_i + 1) \cup \{p(\mathcal{M}_{i+1})\}).$$

If λ is a limit ordinal, then every $x \in \mathcal{M}_\lambda$ is of the form $\pi^{\mathcal{M}}_{i,\lambda}(x_i)$ for an i large enough and an $x_i \in \mathcal{M}_i$. By induction hypothesis we know that for every x_i:

$$x_i \in h^{n+1}_{\mathcal{M}_i}(\kappa_i \cup \{p(\mathcal{M}_i)\}),$$

where n is such that $\rho_{n+1}(\mathcal{M}^*_i) \leqslant \kappa_i < \rho_n(\mathcal{M}^*_i)$. Without loss of generality one can choose i large enough such that there is no truncation between i and λ and that

$$\rho_{n+1}(\mathcal{M}^*_j) \leqslant \kappa_j < \rho_n(\mathcal{M}^*_j)$$

for all $i \leqslant j \leqslant \lambda$. Because of the elementarity of the iteration maps, the claim follows as in the successor step. $\square$(Claim 1)

Let i be the first index such that $E^{\overline{K}_i}_{\nu_i} \neq \emptyset$. Then we have that $\overline{K}_i = \overline{K}$. Let $E^{\overline{K}_i}_{\nu_i} = \overline{E}_{\nu_i}$.

Claim 2. The model $\text{Ult}^*(\mathcal{M}^*_i, \overline{E}_{\nu_i})$ is iterable above κ_i.

Proof. Let $\sigma : \mathcal{M}^*_i \to^*_{\overline{E}_{\nu_i}} \tilde{\mathcal{M}} = \text{Ult}^*(\mathcal{M}^*_i, \overline{E}_{\nu_i})$

We have that $\mathcal{M}^*_i \| \kappa_i = \overline{K} \| \kappa_i$ and that $\mathcal{M}^*_i$ is κ_i-sound and that it projects to κ_i. Hence $\mathcal{M}^*_i$ is the collapsing mouse to $\overline{K}|\nu_i$. Hence $\overline{\mathcal{M}}$ the liftup of $\mathcal{M}^*_i$ through π is iterable, as π is good. Thus $\overline{\mathcal{M}} \lhd K$ by 2.10. Let $\overline{\pi} : \mathcal{M}^*_i \to \overline{\mathcal{M}}$ be the associated liftup of π. Since $\overline{E}_{\nu_i}$ is a measure on $\mathcal{M}^*_i$, we have that $\pi(\overline{E}_{\nu_i})$ is a total measure on K too and $\text{Ult}(K, \pi(\overline{E}_{\nu_i}))$ is iterable. Let $\sigma' : K \to \text{Ult}(K, \pi(\overline{E}_{\nu_i}))$ be the associated ultrapower map. $K \vDash$ "$\overline{\mathcal{M}}$ is iterable", hence

$$\text{Ult}(K, \pi(\overline{E}_{\nu_i})) \vDash \text{"}\sigma'\left(\overline{\mathcal{M}}\right) \text{ is iterable"},$$

and thus $\sigma'(\overline{\mathcal{M}})$ is truly iterable. We want to embed $\tilde{\mathcal{M}}$ in $\sigma'\left(\overline{\mathcal{M}}\right)$:

$$\begin{array}{ccc} \mathcal{M}^*_i & \xrightarrow{\overline{\pi}} & \overline{\mathcal{M}} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\ \tilde{\mathcal{M}} & \overset{\Phi}{\dashrightarrow} & \sigma'\left(\overline{\mathcal{M}}\right) \end{array}$$

Therefore we define:

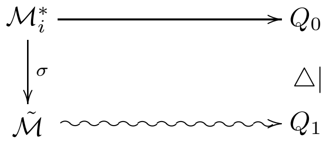
$$\begin{aligned} \Phi : \tilde{\mathcal{M}} &\to \sigma'\left(\overline{\mathcal{M}}\right) \\ [f] &\mapsto \sigma'(\overline{\pi}(f))(\pi(\kappa_i)). \end{aligned}$$

Let us compute the level of elementarity of Φ. For a function $f \in \Gamma(\kappa_i, \mathcal{M}^*_i)$ and a $\Sigma^{(m)}_0$ fomula φ, with m such that $\rho_{m+1}(\mathcal{M}^*_i) \leqslant \kappa_i < \rho_m(\mathcal{M}^*_i)$ we have that:

$$\begin{aligned} \tilde{\mathcal{M}} \vDash \varphi([f]) &\iff \{\xi < \kappa_i; \mathcal{M}^*_i \vDash \varphi(f(\xi))\} \in \overline{E}_{\nu_i} \\ &\iff \overline{\pi}(\{\xi < \kappa_i; \mathcal{M}^*_i \vDash \varphi(f(\xi))\}) \in \pi(\overline{E}_{\nu_i}) \\ &\iff \left\{\xi < \pi(\kappa_i); \overline{\mathcal{M}} \vDash \varphi(\overline{\pi}(f)(\xi))\right\} \in \pi(\overline{E}_{\nu_i}) \\ &\iff \left\{\xi < \pi(\kappa_i); K \vDash \text{"}\overline{\mathcal{M}} \vDash \varphi(\overline{\pi}(f)(\xi))\text{"}\right\} \in \pi(\overline{E}_{\nu_i}) \\ &\iff \text{Ult}(K, \pi(\overline{E}_{\nu_i})) \vDash \text{"}\sigma'\left(\overline{\mathcal{M}}\right) \vDash \varphi\left(\sigma'\left(\overline{\pi}(f)(\pi(\kappa_i))\right)\right)\text{"} \\ &\iff \overline{\mathcal{M}} \vDash \varphi(\overline{\pi}(f)(\xi)). \end{aligned}$$

Notice for the second equivalence that as $\overline{\pi}$ is the lift up of π, $\pi \restriction \overline{K}|\nu_i = \overline{\pi} \restriction \overline{K}|\nu_i$. Hence Φ is $\Sigma^{(m)}_0$-elementary (notice that we used that $\overline{\pi}$ was $\Sigma^{(m)}_0$-elementary too). As $\sigma'\left(\overline{\mathcal{M}}\right)$ is iterable, $\tilde{\mathcal{M}}$ is normally iterable above $\rho_{m+1}(\tilde{\mathcal{M}})$ by [12] 4.3.7. Hence it is iterable above κ_i if $\rho_{m+1}(\tilde{\mathcal{M}}) \leqslant \kappa_i$, but since $\mathcal{M}^*_i$ is m-sound, $\rho_{m+1}(\tilde{\mathcal{M}}) \leqslant \rho_{m+1}(\mathcal{M}^*_i)$ (c.f. [12] 3.2.3),thus we have proved the claim. $\square$(Claim 2)

Since $\mathcal{M}_i^*\|\kappa_i = \tilde{\mathcal{M}}\|\kappa_i$ and κ_i is a cardinal in both structures, the coiteration is above κ_i and exists. Let Q_0 be the last sturcture on the $\mathcal{M}_i^*$-side and Q_1 the last structure on the $\tilde{\mathcal{M}}$-side. By [12] 5.1.6. the $\mathcal{M}_i^*$-side of the coiteration is simple, and we get the following diagram:



Claim 3.

i. $\mathcal{M}_i^*$ coiterate simply with $\tilde{\mathcal{M}}$ above κ_i to a common mouse Q.

ii. $\tilde{\pi} \circ \sigma = \pi^*$, where $\tilde{\pi}$ and π^* are the respective iteration maps.

iii. $E_{\nu_i}^{\mathcal{M}_i^*} = \overline{E}_{\nu_i}$

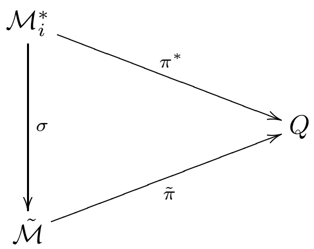
Proof.

i. We know that the $\mathcal{M}_i^*$-side must be simple, hence $Q_0 \lhd Q_1$ or $Q_0 = Q_1$.

   Let us suppose that $Q_0 \lhd Q_1$.

   Let
$$a \in \left(\Sigma_1^{(m)}(\mathcal{M}_i^*) \cap \mathcal{P}(\kappa_i)\right) \setminus \mathcal{M}_i^*.$$

   Then $a \in \Sigma_1^{(m)}(Q_0)$. We can see Q_0 as an iterate of $\tilde{\mathcal{M}}$, if we extend the iteration of $\tilde{\mathcal{M}}$ to Q_1 with a truncation to Q_0. But then a has to be in Q_1 and therefore in $\tilde{\mathcal{M}}$ too, as the coiteration is above κ_i. If we lengthen the iteration at the beginning, by regarding $\tilde{\mathcal{M}}$ as an iterate of $\mathcal{M}_i^*$, the iteration is still above κ_i and we have that $a \in \mathcal{M}_i^*$, a contradiction! Now we have the following diagramm:



ii. As the coiteration is above κ_i, we have that:
$$\tilde{\pi} \circ \sigma \restriction \kappa_i = \pi^* \restriction \kappa_i$$

   Because of the soundness of $\mathcal{M}_i^*$, every $x \in \mathcal{M}_i^*$ is of the form $h_{\mathcal{M}_i^*}^{m+1}(\xi, p(\mathcal{M}_i^*))$ for a $\xi < \kappa_i$. From the elementarity of the maps we now get that:
$$\pi^*(x) = h_Q^{m+1}(\xi, \pi^*(p(\mathcal{M}_i^*))) = h_Q^{m+1}(\xi, \tilde{\pi} \circ \sigma(p(\mathcal{M}_i^*))) = \tilde{\pi} \circ \sigma(x).$$

iii. Now $\mathrm{cp}(\pi^*) = \kappa_i$, hence, the first measure of the $\mathcal{M}_i^*$-side had to be $E_{\nu_i}^{\mathcal{M}_i^*}$. Hence $E_{\nu_i}^{\mathcal{M}_i^*} \neq \emptyset$ and for $a \in \mathcal{P}(\kappa_i) \cap \mathcal{M}_i^*$, we have that:
$$a \in E_{\nu_i}^{\mathcal{M}_i^*} \iff \kappa_i \in \pi^*(a) = \tilde{\pi} \circ \sigma(a) \iff \kappa_i \in \sigma(a) \iff a \in \overline{E}_{\nu_i}.$$

For the second equivalence we have to check that $\mathrm{cp}(\tilde{\pi}) > \kappa_i$. We already know that $\mathcal{M}_i^* \rhd \overline{K}|\nu_i$, hence $\tilde{\mathcal{M}} = \mathrm{Ult}(\mathcal{M}_i^*, \overline{E}_{\nu_i}) \rhd \mathrm{Ult}(\overline{K}|\nu_i, \overline{E}_{\nu_i}) = \mathcal{N}$. By coherency $E_{\nu_i}^{\mathcal{N}} = \emptyset$, thus $E^{\tilde{\mathcal{M}}_{\nu_i}} = \emptyset$ and the measure used in the first step had a critical point greater than κ_i, but as the iteration is normal, κ_i can not be a critical point later.

□(Claim 3)

But as $\overline{E}_{\nu_i}$ was used in the coiteration with K, they can't be equal, a contradiction!

□(Lemma 3.8)

Now we know that for a good π, if $\mathcal{I}$ is the K-side of the coiteration of K with $\overline{K}$, then for all $i < \mathrm{lh}(\mathcal{I})$

$$\overline{K}\|\kappa_i^{+\overline{K}} \lhd \mathcal{M}_i^{\mathcal{I}}$$

and $\mathcal{M}_i^{\mathcal{I}}$ is the collapsing mouse for $\overline{K}|\kappa_i^{+\overline{K}}$, where in a slight abuse of notation we allow $\kappa_i^{+\overline{K}}$ to be $\overline{K} \cap OR$ if κ_i is the largest cardinal of $\overline{K}$.

Theorem 3.9 (weak covering). *Suppose $0^\ddagger$ does not exist, then for all $\beta \geqslant \aleph_2$, $\mathrm{cf}^V(\beta^{+K}) \geqslant \overline{\overline{\beta}}^V$.*

Proof. Deny. Let β be minimal with $\theta = \mathrm{cf}^V(\beta^{+K}) < \overline{\overline{\beta}}^V$. Remark that all we have proved thus far was only under $\neg(0^\ddagger)$. Let π be good such that $\{\beta_i;\ i \leqslant \theta\} \subseteq \mathrm{ran}(\pi)$, where $(\beta_i, i < \theta)$ is a strictly monotonous sequence witnessing the cofinality of β^{+K}, $\beta_\theta = \beta^{+K}$ and $\mathrm{card}(\mathrm{dom}(\pi)) = \aleph_1 \cdot \theta$. Let $\overline{K} = K^{\mathrm{dom}(\pi)}$. As we have seen there is a collapsing mouse for $\overline{K}\|\beta^{+\overline{K}}$ and its lift up $\tilde{\mathcal{M}}$ is an initial segment of K, moreover (c.f.[12] 3.6.5.)

$$\beta^{+\tilde{\mathcal{M}}} = \sup \pi\text{"} \left\{\alpha; \alpha < (\pi^{-1}(\beta))^{+\overline{K}}\right\} = \beta^{+K}.$$

Further $\tilde{\mathcal{M}}$ is a collapsing mouse for $K|\beta^{+\tilde{\mathcal{M}}}$, hence $\tilde{\mathcal{M}}$ projects to β, hence there is a subset of β that is not in $\tilde{\mathcal{M}}$ thus β^{+K} will be collapsed to β in K, a contradiction to the fact that β^{+K} is a cardinal in K! □(Theorem 3.9)

Lemma 3.10. (H)
Let π be good and $\overline{K} = K^{\mathrm{dom}(\pi)}$. In the coiteration of $\overline{K}$ with K, there are only finitely many measures that are used on the K-side.

Proof. Deny. Let $\mathcal{I}$ be the iteration tree of the K-side. Then there is a i such that $i + \omega < \mathrm{lh}(\mathcal{I})$ and $\pi_{i,i+n}^{\mathcal{I}}(\kappa_i) = \kappa_{i+n}$, where $\kappa_n = \mathrm{cp}(\pi_{i+n,i+n+1}^{\mathcal{I}})$. Let $\overline{\kappa} = \sup\{\kappa_j; i \leqslant j < i + \omega\}$.
As $\overline{\kappa}$ is measurable in $\mathcal{M}_{i+\omega}^{\mathcal{I}}$, it has to be inaccessible in $\overline{K}$, hence $\pi(\overline{\kappa})$ is inaccessible in K. But $\pi\text{"}\{\kappa_j; i \leqslant j < i + \omega\}$ is cofinal in $\pi_{\overline{\kappa}}$, hence with (H) it must be less than $\aleph_2$. A contradiction if we suppose that $\mathrm{ran}(\pi) \cap \aleph_2 \in \omega_2 + 1$.

□(Lemma 3.10)

Lemma 3.11. (H)
K covers strongly.

Proof. We prove this by induction on θ, for regular θ. We know that if $\kappa \geqslant \aleph_1$ and $\theta \geqslant \kappa$, then $\{\mathrm{ran}(\pi) \cap\ \theta; \pi \text{ is good and } \mathrm{card}(\pi) = \kappa\}$ is stationary in $[\theta]^\kappa$. We now have to check that for such a π:

$$\mathrm{ran}(\pi) \cap\ \theta \in K.$$

Let $\overline{K} = K^{\mathrm{dom}(\pi)}$, $\mathcal{I}$ the iteration tree of the K side of the coiteration of $\overline{K}$ with K, k such that $\mathcal{M}_k^{\mathcal{I}}$ is the collaps mouse for $\overline{K}|\theta^{+\overline{K}}$, n such that $\rho_{n+1}(\mathcal{M}) \leqslant \pi^{-1}(\theta) < \rho_n(\mathcal{M})$, $\tilde{\mathcal{M}}$ the lift up of $\mathcal{M}$ through π. Then:

$$\mathrm{ran}(\pi) \cap \ \theta = h_{\tilde{\mathcal{M}}}^{n+1}\left(\pi\text{"}\rho_{n+1}(\mathcal{M}_0^{\mathcal{I}}) \cup \left\{\pi(p(\mathcal{M}_k^{\mathcal{I}})), \pi(\kappa_0^{\mathcal{I}}), \ldots, \pi(\kappa_k^{\mathcal{I}})\right\}\right) \cap \theta,$$

By induction hypothesis $\pi\text{"}\rho_{n+1}(\mathcal{M}_0^{\mathcal{I}}) \in K$ and thus $\mathrm{ran}(\pi) \cap \theta \in K$. □(Lemma 3.11)

4 Towards a unique Prikry generic sequence

Lemma 4.1. *Let $\mathcal{M}$ be a sound mouse and $\mathcal{I}$ a simple iteration of $\mathcal{M}$ above $\rho_\omega(\mathcal{M})$. Then $\kappa \in \mathcal{M}^{\mathcal{I}}_\infty$ is a critical point of the iteration if and only if $\kappa \in h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\})$, with n is such that $\rho_{n+1}(\mathcal{M}) \leqslant \kappa < \rho_n(\mathcal{M})$.*

Proof. Let κ_j be the critical points of the iteration. Suppose κ is a critical point and $\kappa \in h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\})$. Let $\pi^{\mathcal{I}}_{i,i+1} : \mathcal{M}^{\mathcal{I}}_i \to \mathcal{M}^{\mathcal{I}}_{i+1}$ be the iteration map such that $\mathrm{cp}(\pi^{\mathcal{I}}_{i,i+1}) = \kappa$. Then $\kappa \notin h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\})$ implies that

$$\kappa \in h^{n+1}_{\mathcal{M}^{\mathcal{I}}_{i+1}}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_{i+1})\}),$$

but

$$h^{n+1}_{\mathcal{M}^{\mathcal{I}}_{i+1}}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_{i+1})\}) \subseteq \mathrm{ran}(\pi^{\mathcal{I}}_{i,i+1}),$$

a contradiction!
Suppose conversly that $\kappa \notin h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\}$. Suppose κ is not a critical point. Let i be minimal such that either:

- $\mathrm{cp}(\pi^{\mathcal{I}}_{i,i+1}) > \kappa$, or
- if there are no j sich that $\mathrm{cp}(\pi^{\mathcal{I}}_{j,j+1}) > \kappa$, then $i = \mathrm{lh}(\mathcal{I}) = \theta$.

Then $\pi^{\mathcal{I}}_{i,\theta} \restriction \kappa + 1 = \mathrm{id}$, as κ is not a critical point. By [12] 4.2.4.

$$\mathcal{M}^{\mathcal{I}}_i = h^{n+1}_{\mathcal{M}^{\mathcal{I}}_i}(\rho_{n+1}(\mathcal{M}) \cup \{\kappa_j ; j < i\} \cup \{p(\mathcal{M}^{\mathcal{I}}_i)\}).$$

But we have chosen i such that $\kappa \geqslant \sup \{\kappa_j ; j < i\}$, hence

$$\mathcal{M}^{\mathcal{I}}_i = h^{n+1}_{\mathcal{M}^{\mathcal{I}}_i}(\rho_{n+1}(\mathcal{M}) \cup \sup \{\kappa_j ; j < i\} \cup \{p(\mathcal{M}^{\mathcal{I}}_i)\}) = h^{n+1}_{\mathcal{M}^{\mathcal{I}}_i}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_i)\}).$$

Hence $\mathrm{ran}\,\pi^{\mathcal{I}}_{i,\theta} = h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\})$, since $\pi^{\mathcal{I}}_{i,\theta} \restriction \kappa + 1 = \mathrm{id}$, but therefore $\kappa \in \mathrm{ran}\,\pi_{i,\theta}$ too, thus $\kappa \in h^{n+1}_{\mathcal{M}^{\mathcal{I}}_\infty}(\kappa \cup \{p(\mathcal{M}^{\mathcal{I}}_\infty)\})$, a contradiction! □(Lemma 4.1)
We now work under $(\neg H)$, let $\mu > \aleph_2$ be minimal with $\mathrm{cf}(\mu) < \overline{\overline{\mu}}$ and μ regular in K.

Definition 4.2. A map $\pi : \overline{H} \to H_\theta$, with $\theta > \mu^{+V}$ a large enough regular cardinal, will be said to be very good if:

- every $\tilde{\mathcal{M}}^{\pi}_i$, as defined in 2.7, is iterable,
- π is continuous at points of cofinality ω,
- $\mu \in \mathrm{ran}(\pi)$,
- $\mathrm{ran}(\pi) \cap \mu$ is cofinal in μ,
- $\overline{\overline{\pi}} = \mathrm{cf}(\mu) \cdot \aleph_1$.

Such πs exist, since $\mathrm{cf}(\mu) \cdot \aleph_1 < \overline{\overline{\mu}}$.
Let $\mathcal{I} = \mathcal{I}^\pi$ be the K-side of the coiteration with $\overline{K} \| \pi^{-1}(\mu)$, where $\overline{K} = K^{\mathrm{dom}(\pi)}$. $\mathcal{M}^{\mathcal{I}}_\infty$ cannot be a finite iterate of K, else μ would be singular in K. On the other side there can be no $\overline{\kappa} < \mu$, which would be the supremum of the first ω many critical points of the iteration, because of the minimality of μ, therefore $\mathrm{cf}(\mu) = \omega$ and $\mathcal{M}^{\mathcal{I}}_\infty = \mathcal{M}^{\mathcal{I}}_\omega$.

Notation 4.3. Let $\{\kappa_n^\pi; n < \omega\}$ be the first ω many critical points of the iteration $\mathcal{I}$, we write $\overline{C}_\pi$ for $\{\kappa_n^\pi; n < \omega\}$ and C_π for $\{\pi(\kappa_n^\pi); n < \omega\}$.
Further $X \subseteq^{\text{fin}} Y$, if $X \setminus Y$ is finite and $X =^{\text{fin}} Y$, if $X \subseteq^{\text{fin}} Y$ and $Y \subseteq^{\text{fin}} X$.

Lemma 4.4. *Let π, π' be very good and $\operatorname{ran}(\pi') \subseteq \operatorname{ran}(\pi)$, then $C_\pi \cap \operatorname{ran}(\pi') \subseteq^{\text{fin}} C_{\pi'}$.*

Proof. Let $\tilde{\mathcal{M}} = \operatorname{Ult}(\mathcal{M}_\infty^{\mathcal{I}^{\pi'}}; \pi^{-1} \circ \pi' \restriction \pi'^{-1}(\mu))$, write $\overline{\mu} = \pi^{-1}(\mu)$.
$\tilde{\mathcal{M}}$ is $\overline{\mu}$-sound, and $\rho_\omega(\tilde{\mathcal{M}}) \leqslant \overline{\mu}$. But since $\mathcal{M}_\infty^{\mathcal{I}^\pi}$ $\overline{\mu}$-sound and $\rho_\omega(\mathcal{M}_\infty^{\mathcal{I}^\pi}) < \overline{\mu}$, we have $\tilde{\mathcal{M}} \trianglelefteq \mathcal{M}_\infty^{\mathcal{I}^{\pi'}}$, by 2.5.
1. Case $\tilde{\mathcal{M}} = \mathcal{M}_\infty^{\mathcal{I}^\pi}$, in this case we want to show even more:

$$C_\pi \cap \operatorname{ran}(\pi') =^{\text{fin}} C_{\pi'}.$$

Let

$$\xi \in \overline{C}_\pi \cap \operatorname{ran}(\pi^{-1} \circ \pi').$$

If

$$(\pi^{-1} \circ \pi')^{-1}(\xi) = \overline{\xi} \notin \overline{C}_{\pi'}$$

then it is generated by $\vec{\eta'} < \overline{\xi}$, that is there is a term τ such that:

$$\overline{\xi} = \tau^{\mathcal{M}_\infty^{\mathcal{I}^{\pi'}}}(\vec{\eta'}, p(\mathcal{M}_\infty^{\mathcal{I}^{\pi'}})),$$

but then

$$\xi = \pi^{-1} \circ \pi'(\overline{\xi}) = \tau^{\tilde{\mathcal{M}}}(\vec{\eta}, p(\tilde{\mathcal{M}}))$$

with $\vec{\eta} < \xi$. Thus it could not have been in $\overline{C}_\pi$.
On the other side, if $\overline{\xi} \in \overline{C}_{\pi'}$ and $\xi = \pi^{-1} \circ \pi'(\overline{\xi}) \notin \overline{C}_\pi$ then there is a term τ such that

$$\xi = \tau^{\tilde{\mathcal{M}}}(\vec{\eta}, p(\tilde{\mathcal{M}}))$$

with $\vec{\eta} < \xi$ and ξ is uniquely determined by:

$$\tilde{\mathcal{M}} \models \exists \vec{\eta} < \xi \ \xi = \tau^{\tilde{\mathcal{M}}}(\vec{\eta}, p(\tilde{\mathcal{M}})).$$

Thus it holds for $\overline{\xi}$ in $\mathcal{M}_\infty^{\mathcal{I}^{\pi'}}$ and $\overline{\xi} \notin \overline{C}_{\pi'}$. Hence $C_\pi \cap \operatorname{ran}(\pi) =^{\text{fin}} C_{\pi'}$.
2. Case $\tilde{\mathcal{M}} \triangleleft \mathcal{M}_\infty^{\mathcal{I}^\pi}$
There is a term τ such that

$$\left\{\tilde{\mathcal{M}}, p(\tilde{\mathcal{M}})\right\} = \tau^{\mathcal{M}_\infty^{\mathcal{I}^\pi}}(\vec{\xi}, \vec{\eta}, p(\mathcal{M}_\infty^{\mathcal{I}^{\pi'}}))$$

where $\vec{\xi} < \rho_\omega(\mathcal{M}_\omega^{\mathcal{I}^\pi})$ and $\vec{\eta} \in \overline{C}_\pi$. Let

$$\kappa \in \operatorname{ran}(\pi^{-1} \circ \pi') \setminus \ (\pi^{-1} \circ \pi')\text{"}\overline{C}_{\pi'}$$

large enough, i.e. $\vec{\eta} < \kappa$, such that $\kappa = \sigma^{\tilde{\mathcal{M}}}(\vec{\xi'}, p(\tilde{\mathcal{M}}))$, with $\vec{\xi'} < \kappa$ and σ some term. Since $\tilde{\mathcal{M}}$ and $p(\tilde{\mathcal{M}})$ are terms in $\mathcal{M}_\infty^{\mathcal{I}^\pi}$, there is a term τ^* such that

$$\kappa = \tau^{*\mathcal{M}_\infty^{\mathcal{I}^\pi}}(\vec{\xi^*}, \vec{\eta^*}, p(\mathcal{M}_\infty^{\mathcal{I}^{\pi'}}))$$

with $\vec{\xi^*} < \rho_\omega(\mathcal{M}_\infty^{\mathcal{I}^\pi})$ and $\vec{\eta^*} \in \overline{C}_\pi \cap \kappa$. Thus $\kappa \notin \overline{C}_\pi$. □(Lemma 4.4)

Lemma 4.5. *There are cofinally many very good π such that for all very good π_1 with $\operatorname{ran}(\pi) \subseteq \operatorname{ran}(\pi_1)$ there is a very good π_2 with $\operatorname{ran}(\pi_1) \subseteq \operatorname{ran}(\pi_2)$ and $C_{\pi_2} \subseteq \operatorname{ran}(\pi)$.*

Proof. Deny. Let $(\pi_i; i \leqslant \omega_1)$ be continuous at ω_1 such that for all i there is no $\tilde{\pi}$ with $\operatorname{ran}(\pi_{i+1}) \subseteq \operatorname{ran}(\tilde{\pi})$ and $C_{\tilde{\pi}} \subseteq \operatorname{ran}(\pi_i)$. But since $\operatorname{otp}(\overline{C}_{\pi_{\omega_1}}) = \omega$ there must be a i with $\overline{C}_{\pi_{\omega_1}} \subseteq \operatorname{ran}(\pi_i)$, and $\operatorname{ran}(\pi_i) \subseteq \operatorname{ran}(\pi_{\omega_1})$. Contradiction!

□(Lemma 4.5)

Lemma 4.6. *There is a very good π such that for all very good π_1 with $\operatorname{ran}(\pi) \subseteq \operatorname{ran}(\pi_1)$ there is some very good π_2 with $\operatorname{ran}(\pi_1) \subseteq \operatorname{ran}(\pi_2)$ and $C_{\pi_2} =^{\text{fin}} C_\pi$.*

Proof. Deny. Let $(\pi_i; i \leqslant \omega_1)$ be continuous at ω_1 such that for all $i < \omega_1$ $\operatorname{ran}(\pi_i) \subseteq \operatorname{ran}(\pi_{i+1})$, and whenever $\tilde{\pi}$ is such that $\operatorname{ran}(\pi_{i+1}) \subseteq \operatorname{ran}(\tilde{\pi})$ then $C_{\tilde{\pi}} \neq^{\text{fin}} C_{\pi_i}$. We write $\mathcal{I}^i$ for $\mathcal{I}^{\pi_i}$. We want to thin out this sequence until we get a contradiction. We already know by Lemma 4.4 that $C_{\pi_{\omega_1}} \cap \operatorname{ran}(\pi_i) \subseteq^{\text{fin}} C_{\pi_i}$ for all $i < \omega_1$. Hence $C_{\pi_{\omega_1}} \subseteq^{\text{fin}} C_{\pi_i}$ for all but boundedly many $i < \omega_1$. Let us assume without loss of generality that $\gamma_0 < \mu$ is such that $C_{\pi_{\omega_1}} \setminus \gamma_0 \subseteq C_{\pi_i}$ for all $i < \omega_1$. Further, with the help of lemma 4.5 we may choose the π_i such that $C_{\pi_i} \subseteq \operatorname{ran}(\pi_0)$, thus by lemma 4.4 we can find a $\gamma_0 < \gamma_1 < \mu$ such that for all i $C_{\pi_i} \setminus \gamma_1 \subseteq C_{\pi_0}$.

Let $\tilde{\mathcal{M}}_i = \operatorname{Ult}(\mathcal{M}_\infty^{\mathcal{I}^i}, \pi_{\omega_1}^{-1} \circ \pi_i \restriction \pi_i^{-1}(\mu))$ and $\widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}$ the associated lift up of $\pi_{\omega_1}^{-1} \circ \pi_i$. We may assume that $\tilde{\mathcal{M}}_i \lhd \mathcal{M}_\infty^{\mathcal{I}^{\omega_1}} = \mathcal{N}$, because otherwise $C_{\pi_i} =^{\text{fin}} C_{\pi_{\omega_1}}$ as we have seen in the first case of the proof of lemma 4.4.

Claim 1. $\mathcal{N} \| \mu'^{+\mathcal{N}} = \bigcup_{i<\omega_1} \tilde{\mathcal{M}}_i$, where $\mu' = \pi_{\omega_1}^{-1}(\mu)$.

Proof. For $i \leqslant \omega_1$ let $\tilde{\pi}_i$ be the associated map of the liftup of $\mathcal{M}_\infty^{\mathcal{I}^{\pi_i}}$ to K, to prove the claim it suffices to show that

$$\mathcal{P}(\mu) \cap \operatorname{ran}(\tilde{\pi}_{\omega_1}) = \bigcup_{i<\omega_1} \mathcal{P}(\mu) \cap \operatorname{ran}(\tilde{\pi}_i).$$

For $i \leqslant \omega_1$ let $\mathcal{M}_k^{\mathcal{I}^i}$ be the k^{th}-structure of the iteration $\mathcal{I}^i$, $\pi_{k,l}^{\mathcal{I}^i}$ the compositions of the associated iteration maps and $\kappa_k^{\pi^i}$ the associated critical points. Let $\tilde{a} \in \mathcal{P}(\mu) \cap \operatorname{ran}(\tilde{\pi}_{\omega_1})$ and

$$a = \tilde{\pi}_{\omega_1}^{-1}(\tilde{a}) \in \mathcal{P}(\mu') \cap \mathcal{N}.$$

As a is in a direct limit there is some n and some $\overline{a}$ such that for the iteration map $\pi_{n,\omega}^{\mathcal{I}^{\omega_1}}$, $a = \pi_{n,\omega}^{\mathcal{I}^{\omega_1}}(\overline{a})$, with

$$\overline{a} \in \mathcal{P}(\kappa_n^{\pi_{\omega_1}}) \cap \mathcal{M}_n^{\mathcal{I}^{\omega_1}}.$$

There is an i such that for some m:

$$\pi_{\omega_1}(\kappa_n^{\pi_{\omega_1}}) = \pi_i(\kappa_m^{\pi_i}),$$

since $C_{\pi_{\omega_1}} \setminus \gamma_0 \subseteq C_i$. Let $b, \overline{b}$ such that

$$\overline{a} = \pi_{\omega_1}^{-1} \circ \pi_i(\overline{b}) \text{ and } b = \pi_{m,\omega}^{\mathcal{I}^i}(\overline{b}).$$

It suffices to prove that $\tilde{\pi}_i(b) = \tilde{a}$, since $\operatorname{ran}(\tilde{\pi}_i) \cap \mathcal{P}(\mu) \subseteq \operatorname{ran}(\pi_{i+1})$.

$$\overline{b} = \tau^{\mathcal{M}_m^{\mathcal{I}^i}}(\vec{\gamma}, p(\mathcal{M}_m^{\mathcal{I}^i})) \text{ with } \vec{\gamma} < \kappa_m^{\pi_i},$$

hence

$$b = \tau^{\mathcal{M}_\infty^{\mathcal{I}^i}}(\vec{\gamma}, p(\mathcal{M}_\infty^{\mathcal{I}^i})).$$

Without loss of generality:

$$p(\tilde{\mathcal{M}}_i) = \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(p(\mathcal{M}_\infty^{\mathcal{I}^i})) \in \operatorname{ran}(\pi_{n,\omega}^{\mathcal{I}^{\omega_1}})$$

and $\pi_{\omega_1}^{-1} \circ \pi_i(\vec{\gamma}) \in \operatorname{ran}(\pi_{n,\omega}^{\mathcal{I}^{\omega_1}})$.
We have then:

$$\begin{aligned} \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(b) &= \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(\pi_{m,\omega}^{\mathcal{I}^i}(\overline{b})) \\ &= \tau^{\tilde{\mathcal{M}}_i}(\pi_{\omega_1}^{-1} \circ \pi_i(\vec{\gamma}), p(\tilde{\mathcal{M}}_i)) \\ &\in \operatorname{ran}(\pi_{n,\omega}^{\mathcal{I}^i}). \end{aligned}$$

Say $\widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(b) = \pi_{n,\omega}^{\mathcal{I}^i}(a')$. We want to show $a' = \overline{a}$. For $\delta < \kappa_n^{\mathcal{I}^{\omega_1}}$:

$$\begin{aligned} \delta \in a' &\iff \delta \in \pi_{n,\omega}^{\mathcal{I}^{\omega_1}}(a') \\ &\iff \delta \in \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(b) \\ &\iff \delta \in \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(\pi_{m,\omega}^{\mathcal{I}^i}(\overline{b})) \\ &\iff \delta \in \widetilde{\pi_{\omega_1}^{-1} \circ \pi_i}(\overline{b}) \text{ as } \overline{b} \subseteq \kappa_m^{\mathcal{I}^i} \\ &\iff \delta \in \pi_{\omega_1}^{-1} \circ \pi_i(\overline{b}) = \overline{a}. \end{aligned}$$

□(Claim 1)

In particular $\operatorname{cf}^V(\mu'^{+\mathcal{N}}) = \omega_1$. As $\mathcal{N}$ is the generalized collapsing mouse for μ', there is a n such that $\rho_{n+1}(\mathcal{N}) \leqslant \mu' < \rho_n(\mathcal{N})$.

Claim 2. $\operatorname{cf}(\mathcal{N}^n \cap \mathrm{OR}) = \omega_1$

Proof. Let us suppose the contrary and work toward contradiction.

1. Case $\operatorname{cf}(\mathcal{N}^n \cap \mathrm{OR}) = \omega$.

Let $(\eta_k, k < \omega)$ be a monotonous sequence, cofinal in $\mathcal{N}^n \cap \mathrm{OR}$. We have that

$$h^1_{\mathcal{N}^n \| \eta_k}(\mu' \cup \{p(\mathcal{N}^n)\}) \prec_{\Sigma_1} \mathcal{N}^n \| \eta_k$$

But as $\mathcal{N}$ is μ'-sound, it follows that

$$\begin{aligned} \mathcal{N} &= h^1_{\mathcal{N}^n}(\mu' \cup \{p(\mathcal{N})\}) \\ &= \bigcup_{k<\omega} h^1_{\mathcal{N}^n \| \eta_k}(\mu' \cup \{p(\mathcal{N}^n)\}) \end{aligned}$$

Let $\mu_k = \sup\left(h^1_{\mathcal{N} \| \eta_k}(\mu' \cup \{p(\mathcal{N}^n)\}) \cap \mu'^{+\mathcal{N}^n}\right)$, $\mu_k < \mu'^{+\mathcal{N}}$ as the skolemhulls all have cardinality μ. but then $(\mu_k, k < \omega)$ is a monotonous sequence, cofinal in $\mu^{+\mathcal{N}}$, a contradiction to $\operatorname{cf}(\mu'^{+\mathcal{N}}) = \omega_1$!

2. Case $\operatorname{cf}(\mathcal{N}^n \cap \mathrm{OR}) > \omega_1$.

We use the same argument, pick a monotonous, cofinal sequence $(\eta_k; k < \theta)$ with $\theta = \operatorname{cf}(\mathcal{N}^n \cap \mathrm{OR}) > \omega_1$. Define $(\mu_k, k < \theta)$ as above, then the μ_k are a sequence which is cofinal in $\mu'^{+\mathcal{N}}$ and of order type $\theta > \omega_1$, a contradiction! □(Claim 2)

Let $(\eta_i; i < \omega_1)$ be a monotonous cofinal sequence in $\mathcal{N}^n \cap \mathrm{OR}$, let

$$\sigma_i : \overline{\mathcal{N}}_i \xrightarrow{\sim} h_{\mathcal{N}^n \| \eta_i}(\mu' \cup \{p(\mathcal{N})\}) \prec_{\Sigma_1} \mathcal{N}^n \| \eta_i$$

where σ_i is the uncollapsing map and let $\overline{p}_i$ be such that $\sigma_i(\overline{p}_i) = p(\mathcal{N}^n)$. We know that $\overline{\mathcal{N}}_i \in \mathcal{N}$. Hence $\overline{\mathcal{N}}_i \in \mathcal{N} \| \mu^{+\mathcal{N}}$, thus we can fix a $j(i)$ for all $i < \omega_1$ such that $\overline{\mathcal{N}}_i \in \tilde{\mathcal{M}}_{j(i)}$. Thus $\{\overline{\mathcal{N}}_i, \overline{p}_i\} \in \tilde{\mathcal{M}}_{j(i)}$, further

$$\{\overline{\mathcal{N}}_i, \overline{p}_i\} \in h^{n_{j(i)}+1}_{\tilde{\mathcal{M}}_{j(i)}}(\mu' \cup \left\{p(\tilde{\mathcal{M}}_{j(i)})\right\}),$$

for $n_{j(i)}$ such that $\rho_{n_{j(i)}+1}(\tilde{\mathcal{M}}_{j(i)}) \leqslant \mu' < \rho_{n_{j(i)}}(\tilde{\mathcal{M}}_{j(i)})$. Pick a monotonous sequence $(\varepsilon_k, k < \omega)$ cofinal in μ'. For all i choose a $\tilde{\gamma}_i$ such that:

- $\{\overline{\mathcal{N}}_i, \overline{p}_i\} \in h^{n_{j(i)}+1}_{\tilde{\mathcal{M}}_{j(i)}}(\tilde{\gamma}_i \cup \left\{p(\tilde{\mathcal{M}}_{j(i)})\right\})$,
- $\tilde{\gamma}_i = \varepsilon_k$ for some k.

By the pigeonhole principle, there must be a $k < \omega$ such that for ω_1 many i's $\gamma_i = \varepsilon_k$. Hence without loss of generality $\tilde{\gamma}_i = \tilde{\gamma}_j = \gamma_2 < \mu'$ for all $i, j < \omega_1$. We have now that $C_{\pi_i} \setminus \gamma_1 \subseteq C_{\pi_0}$ for all $i \leqslant \omega_1$, $C_{\pi_{\omega_1}} \setminus \gamma_0 \subseteq C_{\pi_i}$ for all $i \leqslant \omega_1$ and $C_{\pi_{\omega_1}} \neq^{\text{fin}} C_{\pi_i}$ for all $i < \omega_1$, since we are working towards contradiction. Hence we can choose a $\xi \in C_{\pi_0} \setminus \gamma_1$ such that $\xi \in C_{\pi_i} \setminus C_{\pi_{\omega_1}}$ and $\overline{\xi} = \pi^{-1}_{\omega_1}(\xi) > \gamma_2$.
We are now ready to produce the contradiction:
$\overline{\xi} \in h_{\mathcal{N}^n}(\overline{\xi} \cup \{p(\mathcal{N}^n)\})$, hence $\overline{\xi} \in h_{\mathcal{N}^n \| \eta_i}(\overline{\xi} \cup \{p(\mathcal{N}^n)\})$ for almost all i. Thus $\overline{\xi} \in h_{\overline{\mathcal{N}}_i}(\overline{\xi} \cup \overline{p}_i)$. Let $j = j(i)$, then $\overline{\xi} \in h^{n_j+1}_{\tilde{\mathcal{M}}_j}(\overline{\xi} \cup \tilde{\gamma}_i \cup \left\{p(\tilde{\mathcal{M}}_j)\right\})$, but $\tilde{\gamma}_i = \gamma_2 < \overline{\xi}$. Hence

$$\overline{\xi} \in h^{n_j+1}_{\tilde{\mathcal{M}}_j}(\overline{\xi} \cup \left\{p(\tilde{\mathcal{M}}_j)\right\}),$$

and thus ξ couldn't have been in a C_{π_i}, a contradiction! □(Lemma 4.6)

5 When $K[C]$ covers

We still work under $(\neg H)$. As all results are based on 5.2, all the following only holds below $\neg(0^\dagger)$.

Notation 5.1. Let π be a very good function (in the sense of definition 4.2).
Let $\tilde{\mathcal{M}}^\pi = \mathrm{Ult}\left(\mathcal{M}_\infty^{\mathcal{I}^\pi}, \pi \restriction \pi^{-1}(\mu)\right)$ be the liftup of $\mathcal{M}_\infty^{\mathcal{I}^\pi}$ to K, $\tilde{\pi}$ its associated ultrapower map.
Then let $\tilde{U}^\pi$ denote the active measure of $\tilde{\mathcal{M}}^\pi$ and U^π that of $\mathcal{M}_\infty^{\mathcal{I}^\pi}$, further let $U = \bigcup\left\{\tilde{U}^\pi; \pi \text{ very good}\right\}$.

Remark 5.2. For all $X \in \mathcal{P}(\mu) \cap \mathrm{ran}(\pi) \cap K$ either $C_\pi \subseteq^{\mathrm{fin}} X$ or $C_\pi \subseteq^{\mathrm{fin}} \mu \setminus X$.

Proof. Let $X \in \mathcal{P}(\mu) \cap \mathrm{ran}(\pi) \cap K$, $\overline{X} = \pi^{-1}(X)$ and $\{\kappa_k^\pi; k < \omega\}$ be the critical points of the iteration $\mathcal{I}^\pi$, $\mathcal{M}_k^{\mathcal{I}^\pi}$ the structures of the iteration and $\pi_{k,l}^{\mathcal{I}^\pi}$ the compositions of the associated iteration maps, then

$$\overline{X} \in \mathcal{M}_\infty^{\mathcal{I}^\pi} = h^{n+1}_{\mathcal{M}_\infty^{\mathcal{I}^\pi}}\left(\mathrm{ran}(\pi_{0,\infty}^{\mathcal{I}^\pi}) \cup \{\kappa_k^\pi; k < \omega\}\right),$$

with n such that $\rho_{n+1}(\mathcal{M}_\infty^{\mathcal{I}^\pi}) \leqslant \pi^{-1}(\mu) < \rho_n(\mathcal{M}_\infty^{\mathcal{I}^\pi})$. Therefore there is a term τ such that

$$\overline{X} = \tau^{\mathcal{M}_\infty^{\mathcal{I}^\pi}}\left(\pi_{0,\infty}^{\mathcal{I}^\pi}(x), \kappa_0^\pi, \dots, \kappa_{l-1}^\pi\right).$$

But then for all $k \geqslant l$

$$\overline{X} = \pi_{k,\infty}^{\mathcal{I}^\pi}\left(\tau^{\mathcal{M}_k^{\mathcal{I}^\pi}}(\pi_{0,k}^{\mathcal{I}^\pi}(x), \kappa_0^\pi, \dots, \kappa_{l-1}^\pi)\right).$$

Let $\overline{X}_k = \tau^{\mathcal{M}_k^{\mathcal{I}^\pi}}\left(\pi_{0,k}^{\mathcal{I}^\pi}(x), \kappa_0^\pi, \dots, \kappa_{l-1}^\pi\right)$, we have:

$$\begin{aligned}
\kappa_l^\pi \in \overline{X}_l = \pi_{l+1,\infty}^{\mathcal{I}} \circ \pi_{l,l+1}^{\mathcal{I}}(\overline{X}_l) &\iff \kappa_l^\pi \in \overline{X}_{l+1} = \pi_{l,l+1}^{\mathcal{I}}(\overline{X}_l) \\
&\iff \overline{X}_l \in E_l^{\mathcal{I}^\pi} \\
&\iff \pi_{l,k}^{\mathcal{I}}(\overline{X}_l) = \overline{X}_k \in \pi_{l,k}^{\mathcal{I}}(E_l^{\mathcal{I}^\pi}) = E_k^{\mathcal{I}^\pi} \quad (*) \\
&\iff \kappa_k^\pi \in \overline{X}_{k+1} \\
&\iff \kappa_k^\pi \in \overline{X}
\end{aligned}$$

$(*)$: That equality holds because we work below $0^\dagger$, and therefore there can be only one active measure on a mouse. Notice that we only need $0^\ddagger$ to prove this, since under $0^\ddagger$ there can be no mouse with two measure having the same critical point.

□(Remark 5.2)

We want to prove that U is a $< \mu$-complete ultrafilter over K. In order to prove it, we will heavily use that it is in fact generated by the C_π:

Lemma 5.3. *For all very good π and for all $X \in \tilde{\mathcal{M}}^\pi$*

$$X \in \tilde{U}^\pi \iff C_\pi \subseteq^{\mathrm{fin}} X.$$

Proof. It suffices to show that for all $X \in \tilde{\mathcal{M}}^\pi = \tilde{\mathcal{M}}$:

$$X \in \tilde{U}^\pi \Rightarrow C_\pi \subseteq^{\mathrm{fin}} X.$$

Let $X \in \mathcal{P}(\mu) \cap \tilde{\mathcal{M}}$ then there is a $\delta < \mu^{+\tilde{\mathcal{M}}}$ such that $X \in \tilde{\mathcal{M}}\|\delta$. Since $\tilde{\mathcal{M}}$ is acceptable there is an

$$f : \mu \to \mathcal{P}(\mu) \cap \tilde{\mathcal{M}}\|\delta$$

with f surjective and

$$f \in h^{n+1}_{\tilde{\mathcal{M}}}(\delta' \cup \{p\}),$$

where p is the standard parameter of $\tilde{\mathcal{M}}$, n is such that $\rho_{n+1}(\tilde{\mathcal{M}}) \leqslant \mu < \rho_n(\tilde{\mathcal{M}})$ and $i < \delta' < \mu$ such that $\delta' \in \operatorname{ran}(\pi)$. Hence $f = \vec{X} = (X_i; i < \mu) \in \operatorname{ran}(\tilde{\pi})$ and there is an i such that $X_i = X$. Then

$$\tilde{\pi}^{-1}(\vec{X}) \restriction \pi^{-1}(\delta') \in h^{n+1}_{\mathcal{M}^{\mathcal{I}^\pi}_\infty}\left(\pi^{-1}(\delta') \cup \{\overline{p}\}\right),$$

where $\overline{p}$ is the standard parameter of $\mathcal{M}^{\mathcal{I}^\pi}_\infty$ and n is such that $\rho_{n+1}(\mathcal{M}^{\mathcal{I}^\pi}_\infty) \leqslant \pi^{-1}(\mu) < \rho_n(\mathcal{M}^{\mathcal{I}^\pi}_\infty)$.

If $\xi \in \overline{C}_\pi$, ξ large enough, then

$$\mathcal{M}^{\mathcal{I}^\pi}_\infty \vDash \forall j < \pi^{-1}(\delta') \left[\left(\pi^{-1}(\vec{X})\right)_j \in U^\pi \leftrightarrow \xi \in \left(\pi^{-1}(\vec{X})\right)_j\right].$$

This is proved as in 5.2. Hence

$$\tilde{\mathcal{M}} \vDash \forall j < \delta' \left((\vec{X})_j = X_j \in \tilde{U}^\pi \leftrightarrow \pi(\xi) \in (\vec{X})_j = X_j\right).$$

Thus $X \in \tilde{U}^\pi \iff \xi^* \in X$ for all sufficiently large $\xi^* \in C_\pi$; and this is what we wanted to prove. □(Lemma 5.3)

Now we want to show that U is an ultrafilter, therefore we need to show that the $\tilde{U}^\pi$ are compatible enough, i.e. if an X has measure one somewhere, then for every other measure, that measures X, the set X is still measure one.

Lemma 5.4. *For all very good π and π', for all $X \in \operatorname{ran}(\pi) \cap \operatorname{ran}(\pi')$ we have that*

$$X \in \tilde{U}^\pi \iff X \in \tilde{U}^{\pi'}.$$

Proof. Pick a $\tilde{\pi}$ such that $\operatorname{ran}(\pi) \cup \operatorname{ran}(\pi') \subseteq \operatorname{ran}(\tilde{\pi})$, then by lemma 4.4 $C_{\tilde{\pi}} \subseteq^{\text{fin}} C_\pi$ and $C_{\tilde{\pi}} \subseteq^{\text{fin}} C_{\pi'}$. If $X \in \tilde{U}^\pi$ then $C_\pi \subseteq^{\text{fin}} X$, hence $C_{\tilde{\pi}} \subseteq^{\text{fin}} X$. On the other hand if $X \in \tilde{U}^{\tilde{\pi}}$, then $C_{\tilde{\pi}} \subseteq^{\text{fin}} X$. As $C_{\tilde{\pi}}$ is cofinal in C_π (as each C_π has order type ω), C_π cannot be in $\mu \setminus X$ modulo finitely many elements, therefore $C_\pi \subseteq^{\text{fin}} X$, hence $X \in \tilde{U}^\pi$.

Analogously, one can prove $X \in \tilde{U}^{\pi'} \iff X \in \tilde{U}^{\tilde{\pi}}$. □(Lemma 5.4)

Lemma 5.5. *U is a $< \mu$-complete ultrafilter on K.*

Proof. Deny. Let $\vec{X} \in K$ be such that $\bigcap_{i<\lambda}(\vec{X})_i$ is a counter-example. Then there is a very good π such that $\vec{X}, \lambda \in \operatorname{ran}(\pi)$. Thus for all $i \in \lambda \cap \operatorname{ran}(\pi)$, $(\vec{X})_i \in \tilde{U}^\pi$. Hence $\vec{X} = \tau^{\mathcal{M}^{\mathcal{I}^\pi}_\infty}(\kappa^\pi_0, \ldots, \kappa^\pi_n, \{p(\mathcal{M}^{\mathcal{I}^\pi}_\infty)\})$ with κ^π_i being the critical points of the iteration $\mathcal{I}^\pi$, and as we have seen before:

for all $\kappa^\pi_k > \kappa^\pi_n, \pi^{-1}(\lambda)$ and for all $i < \pi^{-1}(\lambda)$

$$\kappa^\pi_k \in (\pi^{-1}(\vec{X}))_i.$$

This shows that

$$\overline{C}_\pi \subseteq^{\text{fin}} \bigcap \left\{(\pi^{-1}(\vec{X}))_i; i < \pi^{-1}(\lambda)\right\},$$

hence $C_\pi \subseteq^{\text{fin}} \bigcap \left\{(\vec{X})_i; i < \lambda\right\}$; a contradiction! □(Lemma 5.5)

The same argument proves that U is a $< \mu$-complete normal ultrafilter on K. In order to prove that $U \in K$ it suffices to verify (c.f. Theorem 1.5):

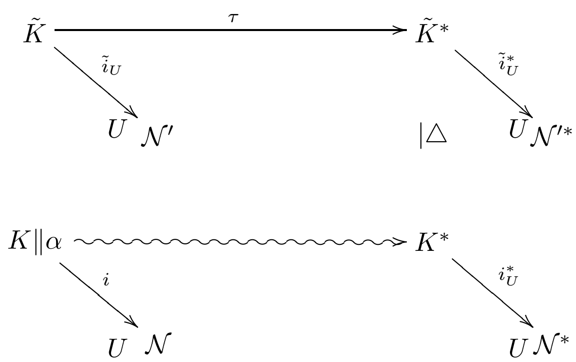
Lemma 5.6. Ult$(K; U)$ *is iterable.*

Proof. We already know that for cofinally many α, the ultrapower Ult$(K\|\alpha, U)$ is iterable, as cofinally many are liftups from very good maps. Suppose that Ult$(K; U)$ is not iterable. Let $\sigma : \overline{V} \to V_\theta$ be an elementary embedding with θ large enough, such that

- $\sigma(\overline{K}) = K\|\theta$,
- $\sigma(\overline{U}) = U$,
- $\sigma(\overline{\mu}) = \mu$ and
- Ult$(\overline{K}, \overline{U})$ is not iterable.

Let $\tilde{K} = \mathrm{Ult}(\overline{K}|\overline{\mu}^{+\overline{K}}; \sigma \upharpoonright \overline{\mu}^{+\overline{K}})$, then Ult$(\tilde{K}, U)$ is not iterable. We coiterate K and $\tilde{K}$. The same argument as in the proof of 3.8 shows that there is a drop in the first step, say $\alpha \geqslant \mu^{+\tilde{K}}$, such that α is minimal with $\rho_\omega(K\|\alpha) \leqslant \mu$ moreover $\mu^{+\tilde{K}} = \sup \sigma"\overline{\mu}^{+\overline{K}} < \mu^{+K}$ (c.f. 3.6.4) since $\mathrm{cf}(\mu^{+K}) \geqslant \omega_1$ by weak covering. Therefore the coiteration of $\tilde{K}$ with K is in fact a coiteration of $\tilde{K}$ with $K\|\alpha$ for some $\alpha < \mu^{+K}$. Let $\tilde{K}^*$ the last model on the $\tilde{K}$-side and K^* that on the K-side, the $\tilde{K}$-side is simple, further $\tilde{K}^* \trianglelefteq K^*$.

Let $\mathcal{N}$, $\mathcal{N}^*$, $\mathcal{N}'$ and $\mathcal{N}'^*$ be respectively the U-ultrapower of $K\|\alpha$, K^*, $\tilde{K}$ and $\tilde{K}^*$ and i, i^*_U, $\tilde{i}_U$ and $\tilde{i}^*_U$ the respective ultrapower map.



We know that $\mathcal{N}$ is iterable and $\mathcal{N}'$ is not. We want to show that $\mathcal{N}'^*$ is neither. We define:

$$\Phi : \mathcal{N}' \to \mathcal{N}'^*$$
$$\tilde{i}_U(f)(\mu) \mapsto \tilde{i}^*_U(\tau(f))(\mu),$$

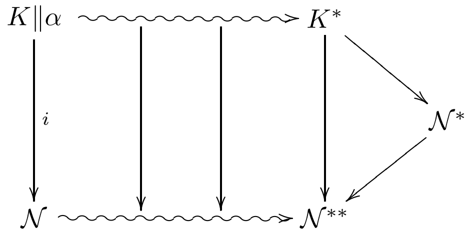
where τ is the coiteration map from $\tilde{K}$ to $\tilde{K}^*$ and i_Us beeing the right ultrapower map (using U) as in the diagramm. Now we have:

$$\begin{aligned} \mathcal{N}' \models \varphi(\tilde{i}_U(f)(\mu)) &\iff \Big\{\xi < \mu; \tilde{K} \models \varphi(f(\xi))\Big\} \in U \\ &\iff \Big\{\xi < \mu; \tilde{K}^* \models \varphi(\tau(f)(\xi))\Big\} \in U \\ &\iff \mathcal{N}'^* \models \varphi(\tilde{i}^*_U(\tau(f))(\mu)). \end{aligned}$$

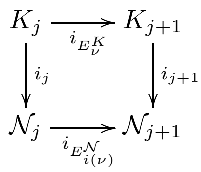
Notice that the second equivalence holds since

$$\left\{\xi < \mu; \tilde{K}^* \models \varphi(\tau(f)(\xi))\right\} = \tau\text{"} \left\{\xi < \mu; \tilde{K} \models \varphi(f(\xi))\right\} = \left\{\xi < \mu; \tilde{K} \models \varphi(f(\xi))\right\},$$

as $\tau \restriction \mu = \mathrm{id}$. Now if we prove that $\mathcal{N}^*$ is iterable we get an outright contradiction. We want to copy the iteration of $K\|\alpha$ to K^* to an iteration of $\mathcal{N}$ to $\mathcal{N}^{**}$.



There can be no drop at the beginning, else we just had chosen α a little smaller. As i is at least Σ^* preserving, we can copy the iteration (c.f. [12] 4.3.1). We want to analyze the copy maps further. Let $\mathcal{N}_j$ and K_j be the structures of the iterations. Let $\nu = \nu_{i+1}$ be an index and its critical point κ. Let $i_{E_\nu^K} : K_j \to_{E_\nu^K} K_{j+1}$, and $i_{E_{i(\nu)}^{\mathcal{N}}} : \mathcal{N}_j \to_{E_{i(\nu)}^{\mathcal{N}}} \mathcal{N}_{j+1}$, where $i : K_j \to \mathcal{N}_j$ is the ultrapower map by U. We know that we can complete the diagram such that it commutes,



by defining:

$$\begin{aligned} i_{j+1} &: K_{j+1} \to \mathcal{N}_{j+1} \\ i_{E_\nu^K}(f)(\kappa) &\mapsto i_{E_{i_j(\nu)}^{\mathcal{N}}}(i_j(f))(i_j(\kappa)). \end{aligned}$$

Then we want to show :

$$\begin{aligned} \mu \in i_{j+1}(X) &\iff \mu \in i_j(X) \\ &\iff X \in U. \end{aligned}$$

If $\kappa > \mu$ then

$$\begin{aligned} \mu \in i_j(X) &\iff \mu \in i_{E_{i_j(\nu)}^{\mathcal{N}}} \circ i_j(X) \\ &\iff \mu \in i_{j+1} \circ i_{E_\nu^K}(X) \\ &\iff \mu \in i_{j+1}(X). \end{aligned}$$

If $\mu = \kappa$ (this can only happen in the first step of the iteration), then $X = i_{E_\nu^K}(X) \cap \kappa$ thus:

$$\begin{aligned} \mu = \kappa \in i_{j+1}(X) &\iff \mu \in i_{j+1} \circ i_{E_\nu^K}(X) \cap i_{j+1}(\kappa) \\ &\iff \mu \in i_{E_{i_j(\nu)}^{\mathcal{N}}} \circ i_j(X) \\ &\iff \mu \in i_j(X). \end{aligned}$$

Remark that this holds since there is no drop in the first step. We want to show that we can embed $\mathrm{Ult}(K_{j+1}, U)$ in $\mathcal{N}_{j+1}$. Let $k : \mathrm{Ult}(K_{j+1}, U) \to \mathcal{N}_{j+1}$ such that

$$k(i_U^{K_{j+1}}(f)(\mu)) = i_{j+1}(f)(\mu),$$

where $i_U^{K_{j+1}}$ is the associated ultrapower map. Let $f : \mu \to K^*$ be a map in K^* and φ a formula.

$$\begin{aligned}
\mathrm{Ult}(K_{j+1}, U) \models \varphi(i_U^{K_{j+1}}(f)(\mu) &\iff \{\alpha < \mu; K_{j+1} \models \varphi(f(\alpha))\} \in U \\
&\iff \mu \in i_j(\{\alpha < \mu; K_{j+1} \models \varphi(f(\alpha))\}) \\
&\iff \mu \in i_{j+1}(\{\alpha < \mu; K_{j+1} \models \varphi(f(\alpha))\}) \\
&\iff \mu \in \{\alpha < i_j(\mu); \mathcal{N}_{j+1} \models \varphi(f(\alpha))\} \\
&\iff \mathcal{N}_{j+1} \models \varphi(f(\mu))
\end{aligned}$$

Thus we can embedd $\mathcal{N}^*$ in $\mathcal{N}^{**}$, and we arrive at the announced contradiction.

□(Lemma 5.6)

We now let $C = C_\pi$ with a π as in lemma 3.6. All that remain to be shown is that if $\kappa \geqslant \aleph_1$ is a cardinal and $\theta > \kappa$ then $[\theta]^\kappa \cap K[C]$ is stationary in $[\theta]^\kappa$. Therefore it suffices to proof the following:

Lemma 5.7. *Suppose $0^\dagger$ does not exists. Let $\tilde{\pi} : \overline{H} \to H_\eta$ be elemantary such that $\overline{H}$ is transitive. Let $\overline{K} = K^{\overline{H}}$. Let us assume that $\tilde{\pi}$ is very good and that $C \in \mathrm{ran}(\tilde{\pi})$. Then $\tilde{\pi}``(\overline{H} \cap OR) \in K[C]$.*

Proof.

First, as we work below $0^\dagger$, there can be no $\tilde{\mu}$ above μ which is regular in K and singular in V, as we could repeat the arguments of the last two sections, and this would lead to mice having two active measures. Thus (H) holds above μ and by repeating the arguments of 3.10 we get: if $\mathcal{I}$ denotes the shortest normal tree on K such that $\overline{K} \trianglelefteq \mathcal{M}_\infty^{\mathcal{I}}$ and if $C_{\tilde{\pi}}$ denotes $\left\{\tilde{\pi}(\mathrm{cp}(\pi_{i,i+1}^{\mathcal{I}})); i+1 < \mathrm{lh}(\mathcal{I})\right\}$, then

i. for all $\varepsilon < \mu$, $C_{\tilde{\pi}} \cap \varepsilon$ is finite, and

ii. $C_{\tilde{\pi}} \setminus \mu$ is finite.

There is a π as given by 4.6 with $\pi \in \mathrm{ran}(\tilde{\pi})$ and $C_\pi =^{\mathrm{fin}} C$, therefore $C_{\tilde{\pi}} \subseteq^{\mathrm{fin}} C_\pi$. Let $\tilde{\mathcal{M}}$ be the liftup of $\mathcal{M}_i^{\mathcal{I}}$ via $\tilde{\pi}$, where $i \leqslant \infty$ is the least such that $\overline{K} \| \tilde{\pi}^{-1}(\mu) \trianglelefteq \mathcal{M}_i^{\mathcal{I}}$. As C is Prikry-generic over K, for all but finitely many $\xi < \xi' \in C$:

$$\xi \in h_{\tilde{\mathcal{M}}}^{n+1}(\xi \cup \left\{p(\tilde{\mathcal{M}})\right\}) \iff \xi' \in h_{\tilde{\mathcal{M}}}^{n+1}(\xi' \cup \left\{p(\tilde{\mathcal{M}})\right\}),$$

where n is such that $\rho_{n+1}(\tilde{\mathcal{M}}) \leqslant \tilde{\pi}^{-1}(\mu) < \rho_n(\tilde{\mathcal{M}})$. Therefore if $C_{\tilde{\pi}} \cap \mu$ is unbounded in μ then $C \setminus C_{\tilde{\pi}}$ cannot be infinite. Thus if $\mathrm{ran}(\tilde{\pi}) \cap OR \notin K$ then $C_{\tilde{\pi}} =^{\mathrm{fin}} C$. And for all $\overline{K}$ cardinals κ,

$$\tilde{\pi}``(\kappa) = h_{\tilde{\mathcal{M}}}^{n+1}(\tilde{\pi}``(\rho_{n+1}(\mathcal{M}_\infty^{\mathcal{I}})) \cup C \cup \{p\}) \cap \tilde{\pi}(\kappa) \in K[C].$$

□(Lemma 5.7)

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