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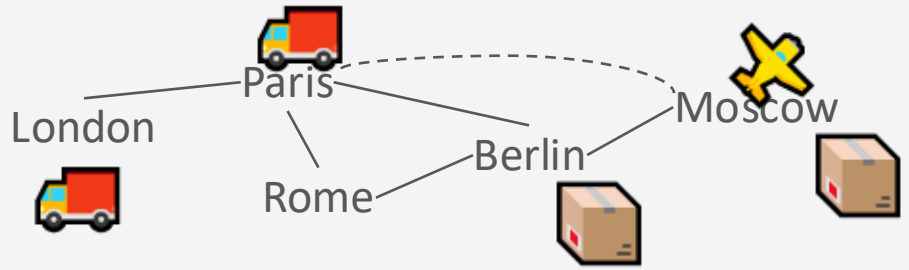

Automated Planning and Acting

Decision Making: Structure

living.knowledge

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Content: Planning and Acting

1. With **Deterministic** Models
2. With **Temporal** Models
3. With **Nondeterministic** Models
4. With **Probabilistic** Models
5. By **Decision Making**
   - *A. Foundations*
   - *B. Extensions*
   - *C. Structure*
     - Symmetries
     - Relations
6. **Human-aware** Planning

Outline: Decision Making – Structure

Structure by Groups in the Agent Set

- Agent types
- Partitioned decPOMDPs

Structure by Relations in the State Space

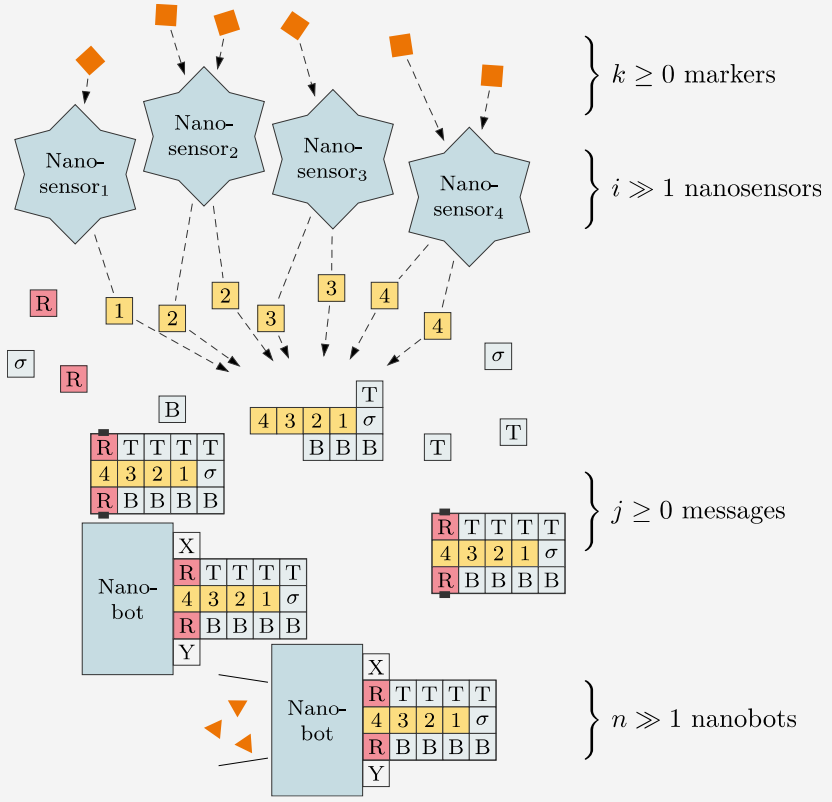
- Situation calculus
- First-order MDPs

Structure by Features in the State Space

- Dynamic Bayesian networks
- Factored MDPs

Example: Medical Nanoscale Systems

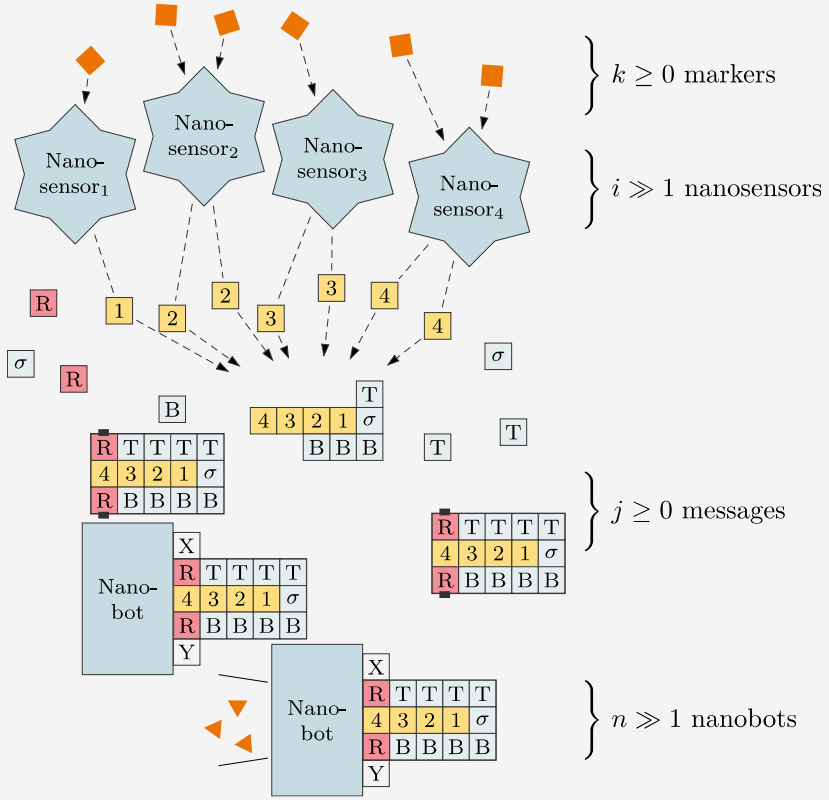
- Nanoscale systems regularly consist of $> 10{,}000$ nanoagents
  - Different types of agents: nanosensors, nanobots
- Application: DNA-based medical system
  - E.g., for diagnosis (modelled as an AND gate)
    - Nanosensors receptive to individual markers for a specific disease
      - Release individual tiles in presence of their individual markers
    - Tiles assemble themselves to form messages
    - Nanobots receptive to completely formed messages
      - Release markers of their own that signify presense of the disease
- Formal model necessary to argue about
  - Success rates
  - Sizes of agent sets



Florian Lau, Tanya Braun, Ralf Möller, and Stefan Fischer: An Extended Framework to Holistically Model Nanostructures and Nanonetworks as decPOMDPcoms. In ACM-*NanoCom-22 Proceedings of the 9th ACM International Conference on Nanoscale Computing and Communication*, 2022

Example: Medical Nanoscale Systems as a DecPOMDP

- Set of agents I consisting of nanosensors, nanobots
- Observations O_i: markers / messages present (or not)
  - Noisy process → probabilistic behaviour
- Actions A_i: release of tiles / markers (or not)
  - Noisy process → probabilistic behaviour
- Environment → probabilistic behaviour
  - Presence in general of agents, markers, tiles, messages, or position more specifically → movement over time
- Reward: Qualitative measure
  - Positive diagnosis only in presence of disease



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Reprise: Worst-case Complexity of DecPOMDP

- Space complexity
  - Transition model: $\mathcal{O}(s \cdot s \cdot a^N)$
  - Sensor model: $\mathcal{O}(s \cdot o^N)$ or $\mathcal{O}(s \cdot o^N \cdot a^N)$
  - Reward function: $\mathcal{O}(s)$ or $\mathcal{O}(s \cdot a^N)$
- Runtime complexity of brute-force search
  - Evaluation cost of a joint policy: $\mathcal{O}\left(s \cdot o^{Nh}\right)$
  - Policy space: $\mathcal{O}\left(a^{\frac{N\left(o^h-1\right)}{o-1}}\right)$
- Notations
  - $s = |S|$
    - State space size
  - $a = \max_{i \in I} |A_i|$
    - Largest individual action space size
  - $o = \max_{i \in I} |O_i|$
    - Largest individual action space size
  - h
    - Horizon

Agent Types & Partitioned DecPOMDPs

- Types: Agents with the same sets of actions and observations
  - E.g., two nanosensors $1,2$ receptive to the same marker and releasing the same tile
    - $A_1 = A_2 = \{0,1\}$; 0: do nothing, 1: release tile
    - $O_1 = O_2 = \{0,1\}$; 0: marker not present, 1: marker present

→ Partitions the set of agents regarding actions, observations

  - Agent set $I = \{I_1, \dots, I_K\}$ with $I_1, \dots, I_K$ a partitioning of I ($I = \bigcup_k I_k$, $I_k \cap I_{k'} = \emptyset, I_k \neq \emptyset$)
  - For each partition I_k: one set of actions A_k, one set of observations O_k for all agents in I_k
  - Expectation that $K \ll N$
- Additional constraints / assumptions on same behaviour in T, R, Ω

→ Partitions the set of agents completely, enabling more compact encodings

TB, Marcel Gehrke, Florian Lau, and Ralf Möller: Lifting in Multi-agent Systems under Uncertainty. In *UAI-22 International Conference on Uncertainty in Artificial Intelligence*, 2022

Additional Assumption 1

- Symmetric behaviour in T, R, Ω
  - Idea: It does not matter which agent of a partition does something, only how many
  - Formally, for two agents $i, j \in I_k$, exchanging their actions / observations does not have an effect

$$T(s, s', a_i, a_j, a_{-i,-j}) = T(s, s', a_j, a_i, a_{-i,-j})$$
$$R(s, a_i, a_j, a_{-i,-j}) = R(s, a_j, a_i, a_{-i,-j})$$
$$\Omega(s, o_i, o_j, o_{-i,-j}) = \Omega(s, s', o_j, o_i, o_{-i,-j})$$

    - Can be generlised to whole paritions, over all possible permutations θ of actions $\vec{a}_k$ / observations o_k, e.g., for T:

$$T(s, s', \vec{a}_k, \vec{a}_{-k}) = T(s, s', \theta(\vec{a}_k), \vec{a}_{-k})$$

  - Enables counting how many agents do something and not which in particular did something
    - Encode in a histogram $[\#(a_1), \dots, \#(a_l)]$ how many agents did actions $A_k = \{a_1, \dots, a_l\}$
    - Number of histograms $\binom{|I_k|+l-1}{l-1} \leq |I_k|^l$

Counting in DecPOMDPs

- Also called counting constraint / assumption in T, R, Ω
  - All permutations $\theta(\vec{a}_k)$ of a partition action $\vec{a}_k$ map to the same probability
  - Allows for replacing joint actions / observations over N agents in T, R, Ω with histograms for each partition
  - Example with two possible actions and a single partition

S	S'	$A_1^{\#}$	$\bar{T}(s, s', a_1') = P(s'\|s, a_1')$
0	0	[0,2]	0.01
0	0	[1,1]	0.02
0	0	[2,0]	0.03
0	1	[0,2]	0.015
0	1	[1,1]	0.012
0	1	[2,0]	0.01
1	0	[0,2]	0.01
⋮			

S	S'	A_1	A_2	$T(s, s', a_1, a_2) = P(s'\|s, a_1, a_2)$
0	0	0	0	0.01
0	0	0	1	0.02
0	0	1	0	0.02
0	0	1	1	0.03
0	1	0	0	0.015
0	1	0	1	0.012
0	1	1	0	0.012
0	1	1	1	0.01
1	0	0	0	0.01
⋮				

Additional Assumption 2

- Conditional independence among agents in T, R, Ω
  - Idea: Given a state and the other partitions, two agents are conditionally independent
  - Formally, for two agents $i, j \in I_k$,
    $$T(s, s', a_i, a_j, a_{-i,-j}) = P(s' | s, a_i, a_j, a_{-i,-j}) = P_i(s' | s, a_i, a_{-i,-j}) \cdot P_j(s' | s, a_j, a_{-i,-j})$$
    - Analogous for R (with addition) and Ω
    - Can also be generalised to whole partitions, e.g., for T
      $$T(s, s', \vec{a}_k, \vec{a}_{-k}) = P(s' | s, \vec{a}_k, \vec{a}_{-k}) = \prod_{i \in I_k} P_i(s' | s, a_i, \vec{a}_{-k})$$
- Combining Assumption 1 + 2 yields that $P_i = P_j$ for all $i, j \in I_k$ for all $k \in \{1, \dots, K\}$
  - Instead of having one large function T over $S, S', A_1, \dots, A_N$
  - We have K smaller functions T_k over $S, S', A_1, \dots, A_K$

Isomorphism in DecPOMDPs

- Assumption 1 only used with Assumption 2 (not on its own)
- Also called isomorphic constraint / assumption in T, R, Ω: Conditional independence between agents of a partition given joint state
  → Enables factorisation of T, R, Ω
  - E.g., $T(s, s', a_1, a_2) = \underbrace{T_1(s, s', a_1) \cdot T_2(s, s', a_2)}_{T_1 = T_2 = T'} = \prod_{i \in I_k} T'(s, s', a_i)$
- Allows for replacing T, R, Ω with joint actions / observations over N agents with K functions each, using single representative actions / observations

S	S'	A_i	$T'(s, s', a_i) = P(s' \mid s, a_i)$
0	0	0	0.01
0	0	1	0.03
0	1	0	0.015
0	1	1	0.01
1	0	0	0.01
		⋮	

Complexities

- Complexity-wise, with $n = \max_k |I_k| \leq N$
  - Counting
    - Transition model: $\mathcal{O}(s \cdot s \cdot n^{Ka})$
    - Sensor model: $\mathcal{O}(s \cdot n^{Ko})$
    - Reward function: $\mathcal{O}(s \cdot n^{Ka})$
  - Isomorphic
    - Transition model: $\mathcal{O}(K \cdot s \cdot s \cdot a^{K})$
    - Sensor model: $\mathcal{O}(K \cdot s \cdot o^{K})$
    - Reward function: $\mathcal{O}(K \cdot s \cdot a^{K})$
  - Reduction if $K \ll N$
- How about solving partitioned DecPOMDPs?

S	S'	A_i	$T'(s,s',a_i) = P(s'\|s,a_i)$
0	0	0	0.01
0	0	1	0.03
0	1	0	0.015
0	1	1	0.01
1	0	0	0.01
⋮			

S	S'	$A_1^{\#}$	$\bar{T}(s,s',a_1') = P(s'\|s,a_1')$
0	0	[0,2]	0.01
0	0	[1,1]	0.02
0	0	[2,0]	0.03
0	1	[0,2]	0.015
0	1	[1,1]	0.012
0	1	[2,0]	0.01
1	0	[0,2]	0.01
⋮			

S	S'	A_1	A_2	$T(s,s',a_1,a_2) = P(s'\|s,a_1,a_2)$
				0.01
				0.02
				0.02
0	0	1	1	0.03
0	1	0	0	0.015
0	1	0	1	0.012
0	1	1	0	0.012
0	1	1	1	0.01
1	0	0	0	0.01
⋮				

Remember: Multi-agent Dynamic Programming

- Iteratively eliminate weakly-dominated policy trees using a dynamic programming
  - A policy $\pi_{i,j}^t$ of agent i is weakly dominated if there always exists another policy $\pi_{i,j'}^t$ with higher value over all possible states s and all possible policies of the other agents $\boldsymbol{\pi}_{-i}^t$:

$$\forall s \in S, \boldsymbol{\pi}_{-i}^t \in \boldsymbol{\Pi}_{-i}^t \; \exists \pi_{i,j'}^t \in \Pi_i^t : V_i\left(s, \pi_{i,j}^t, \boldsymbol{\pi}_{-i}^t\right) \leq V_i\left(s, \pi_{i,j'}^t, \boldsymbol{\pi}_{-i}^t\right)$$

    - Solvable by a linear programme

Dynamic Programming Operator:

Input: sets of policies and corresponding value vectors for each agent.

Process:

- Perform exhaustive backup using policies for each agent.
- Calculate new value vectors.
- Prune weakly dominated policies per agent.

Output: Updated sets of policies and value vectors for each agent.

Hansen, E. A., Bernstein, D. S., & Zilberstein, S. 2004

Multi-agent Dynamic Programming for Isomorphic DecPOMDPs

- Isomorphic DecPOMDP: tuple $\left(I = \{I_1, \dots, I_K\}, S, \{A_k\}_{k=1}^K, \{O_k\}_{k=1}^K, \bar{T}, \bar{R}, \bar{\Omega}\right)$
  - Transition function $\bar{T} = \{T_k(s', s, a_1, \dots, a_K)\}_{k=1}^K$
  - Reward function $\bar{R} = \{R_k(s, a_1, \dots, a_K)\}_{k=1}^K$
  - Sensor model (observation function) $\bar{\Omega} = \{\Omega_k(s, o_1, \dots, o_K)\}_{k=1}^K$
- Dynamic programming:
  - Exhaustive backup: once per partition using A_k, O_k
  - Value computation: once per partition (value function needs to be updated accordingly)
  - Pruning: Check for weakly dominated policies per partition
    - A weakly dominated policy of $i \in I_k$ is also a weakly dominated policy of $j \in I_k$
    - Only need to consider every possible state and representative policy of each partition
  - Complexity: polynomial in the number of agents (due to value function computation)

Counting DecPOMDPs

- Counting DecPOMDP: tuple $\left(I = \{I_1, \dots, I_K\}, S, \{\#A_k\}_{k=1}^{K}, \{\#O_k\}_{k=1}^{K}, \bar{T}, \bar{R}, \bar{\Omega}\right)$
  - Transition function $\bar{T} = T(s, s', h_1^a, \dots, h_K^a)$
  - Reward function $\bar{R} = R(s, h_1^a, \dots, h_K^a)$
  - Sensor model (observation function) $\bar{\Omega} = \Omega(s, h_1^o, \dots, h_K^o)$
- Problem:
  - Builds policies out of histograms
  - Actually leads to a blow-up in the policy space: $\mathcal{O}\left(n^{\frac{aK\left(n^{ho}-1\right)}{n^o-1}}\right)$
- Solution:
  - Use A_k, O_k in partitions
  - Count how many agents of each partitions follow which representative policies instead

Multi-agent Dynamic Programming for Policy-counted DecPOMDPs

- Policy-counted DecPOMDP: tuple $\left(I = \{I_1, \dots, I_K\}, S, \{A_k\}_{k=1}^K, \{O_k\}_{k=1}^K, \bar{T}, \bar{R}, \bar{\Omega}\right)$
  - Transition function $\bar{T} = T(s, s', h_1^a, \dots, h_K^a)$
  - Reward function $\bar{R} = R(s, h_1^a, \dots, h_K^a)$
  - Sensor model (observation function) $\bar{\Omega} = \Omega(s, h_1^o, \dots, h_K^o)$
- Dynamic programming: Difference to isomorphic case
  - As before: Backup, value computation per partition (accordingly updated value function)
  - But, cannot check pruning for a single representative policy for each partition
    - Cannot just consider the representative policies of other partitions
  - Thus, exploit structure in policy space by counting how many agents follow which policy
    - Consider all possible counted policies (all histograms of representative policies)
  - Complexity: polynomial in the number of agents

Partitioned POSGs

- Remember: Partially observable stochastic game (POSG)
  - Dec-POMDP $(I, S, \{A_i\}_{i \in I}, \{O_i\}_{i \in I}, T, \cancel{R}, \Omega)$ but with individual reward functions $\{R_i\}_{i \in I}$
  - Reward function R_i for each agent $i \in I$
- Analogous assumptions for POSGs
  - Additionally, $R_i = R_j$ for all $i, j \in I_k$ for each k
  - Same mathematical transformations possible
  - Allow for defining isomorphic and policy-counted (and counting) POSGs
- Multi-agent dynamic programming also a solution method for POSGs, therefore results carry over
  - Slightly different value computation, rest stays the same
- Complexities: polynomial in the number of agents

Nazlı Nur Karabulut and TB: Lifting Partially Observable Stochastic Games. *In* SUM-24 Proceedings of the 16th International Conference on Scalable Uncertainty Management, 2024
Nazlı Nur Karabulut and TB: Counting Agents in Partially Observable Stochastic Games. In ECSQARU-25 Proceedings of the 18th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, 2025

Interim Summary: Structure by Groups in the Agent Set

- Types of agents with identical action and observation space
- Assumptions: symmetric behaviour, conditional independence
  - Counting
    - Permutations of actions of agents of the same partition map to the same probability / reward
    - Count occurrences → encode in histograms
  - Model complexity polynomial in the number of agents
- Isomorphic DecPOMDP: Use single representative agent per partition for policies
- Counting DecPOMDPs: Use histograms for policies → blows up state space
- Policy-counted DecPOMDP: Count how many agents follow each representative policy
- Dynamic programming possible per partition in all cases
  - Dynamic programming for isomorphic and policy-counted DecPOMDPs polynomial in the number of agents

Outline: Decision Making – Structure

Structure by Groups in the Agent Set

- Agent types
- Partitioned decPOMDPs

Structure by Relations in the State Space

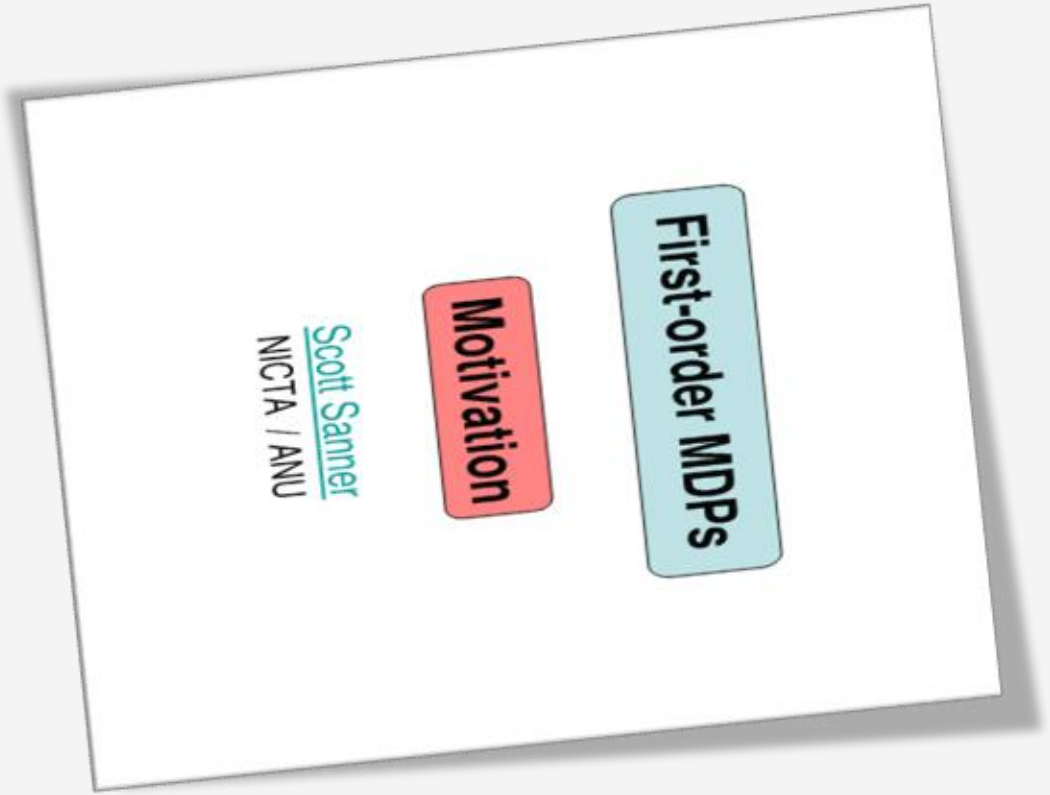
- Situation calculus
- First-order MDPs

Structure by Features in the State Space

- Dynamic Bayesian networks
- Factored MDPs

Acknowledgement

- Thanks to Scott Sanner!

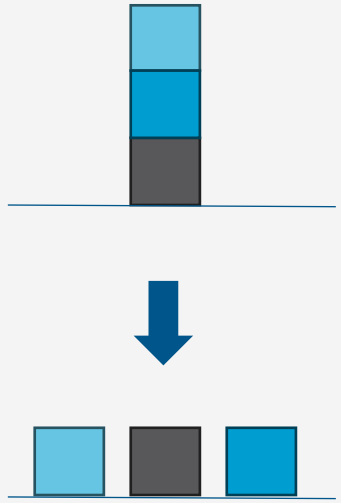


Motivation: Planning Languages

- Common languages:
  - STRIPS
  - PDDL
    - More expressive than STRIPS
    - For example, universal and conditional effects:

```
(:action put-all-blue-blocks-on-table
        :parameters ( )
        :precondition ( )
        :effect (forall (?b)
                (when (Blue ?b)
                (not (OnTable ?b)))))
```

  - General Game Playing (GGP)
    - One or more agents

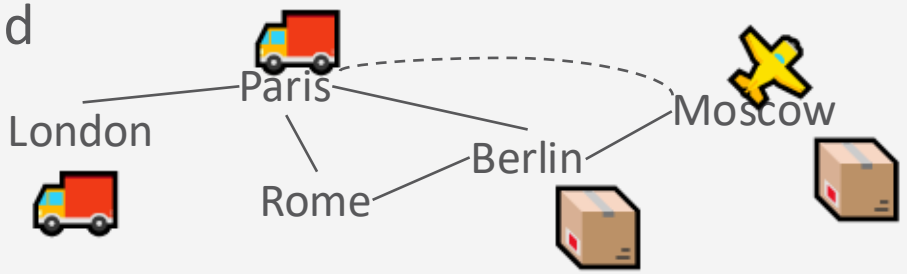


Motivation: Benefits of Relational Languages

- STRIPS, PDDL, GGP are relational languages…
  - Refer to relational fluents:
    - E.g., $BoxIn(?b, ?c)$, $OnTable(?b)$
  - Specify relations between objects
  - Change over time
- Use first-order logic to specify…
  - Action preconditions
  - Action effects
  - Goals / rewards
    - E.g., `(forall (?b ?c) ((Destination ?b ?c) ⇒ (BoxIn ?b ?c)))`
- Are domain-independent and often compact!

Motivation: How to Solve?

- Relational planning problem
  - E.g., box world




```
(:action load-box-on-truck-in-city
        :parameters (?b - box ?t - truck ?c – city)
        :precondition (and (BoxIn ?b ?c) (TruckIn ?t ?c))
        :effect (and (On ?b ?t) (not (BoxIn ?b ?c))))
```

- Solve *ground* problem for each domain instance?
  - E.g., instance with 3 trucks, 2 planes, 3 boxes
- Or solve lifted specification for *all* domains at once?

Box World: Full (Relational) Specification

- Relational fluents: $BoxIn(Box, City)$, $TruckIn(Truck, City)$, $BoxOn(Box, Truck)$
- Goal: $[\exists Box : b.\, BoxIn(b, paris)]$
- Actions:
  - $load(Box : b, Truck : t)$:
    - Effects:
      - when $[\exists City : c.\, BoxIn(b, c) \wedge TruckIn(t, c)]$ then $[BoxOn(b, t)]$
      - $\forall City : c.$ when $[BoxIn(b, c) \wedge TruckIn(t, c)]$ then $[\neg BoxIn(b, c)]$
  - $unload(Box : b, Truck : t)$:
    - Effects:
      - $\forall City : c.$ when $[BoxOn(b, t) \wedge TruckIn(t, c)]$ then $[BoxIn(b, c)]$
      - when $[\exists City : c.\, BoxOn(b, t) \wedge TruckIn(t, c)]$ then $[\neg BoxOn(b, t)]$
  - $drive(Truck : t, City : c)$:
    - Effects:
      - when $[\exists City : c_1.\, TruckIn(t, c_1)]$ then $[TruckIn(t, c)]$
      - $\forall City : c_1.\, when\ [TruckIn(t, c_1)]$ then $[\neg TruckIn(t, c_1)]$

Solving Ground Box World

- Apply planner to Box World grounded with respect to domain, e.g.,
  - Domain object instantiations:
    - $Box = \{box_1, box_2, box_3\}, Truck = \{truck_1, truck_2\}, City = \{paris, berlin, rome\}$
  - Ground fluents:
    - $BoxIn$: $\{BoxIn(box_1, paris), BoxIn(box_2, paris), BoxIn(box_3, paris), BoxIn(box_1, berlin), BoxIn(box_2, berlin), BoxIn(box_3, berlin), BoxIn(box_1, rome), BoxIn(box_2, rome), BoxIn(box_3, rome)\}$
    - $TruckIn$: $\{TruckIn(truck_1, paris), TruckIn(truck_2, paris), TruckIn(truck_1, berlin), TruckIn(truck_2, berlin), TruckIn(truck_1, rome), TruckIn(truck_2, rome)\}$
    - $BoxOn$: $\{BoxOn(box_1, truck_1), BoxOn(box_2, truck_1), BoxOn(box_3, truck_1), BoxOn(box_1, truck_2), BoxOn(box_2, truck_2), BoxOn(box_3, truck_2)\}$
  - Ground actions:
    - $load$: $\{load(box_1, truck_1), load(box_2, truck_1), load(box_3, truck_1), load(box_1, truck_2), load(box_2, truck_2), load(box_3, truck_2)\}$
    - $unload$: $\{unload(box_1, truck_1), unload(box_2, truck_1), unload(box_3, truck_1), unload(box_1, truck_2), unload(box_2, truck_2), unload(box_3, truck_2)\}$
    - $drive$: $\{drive(truck_1, paris), drive(truck_2, paris), drive(truck_1, berlin), drive(truck_2, berlin), drive(truck_1, rome), drive(truck_2, rome)\}$
  - Goal: $[BoxIn(box_1, paris) \vee BoxIn(box_2, paris) \vee BoxIn(box_3, paris)]$

Number of fluents exponential in arity

Number of actions exponential in arity

Goal description exponential in number of nested quantifiers

A First-order Solution to Box World

- Derive solution deductively at lifted PDDL level → Optimal for any domain instantiation!

  if $(\exists b.\, BoxIn(b, paris))$ **then**
  do $noop$
  else if $\left(\exists b^*, t^*.\, TruckIn(t^*, paris) \wedge BoxOn(b^*, t^*)\right)$ **then**
  do $unload(b^*, t^*)$
  else if $(\exists b, c, t^*.\, BoxOn(b, t^*) \wedge TruckIn(t, c))$ **then**
  do $drive(t^*, paris)$
  else if $(\exists b^*, c, t^*.\, BoxIn(b^*, c) \wedge TruckIn(t^*, c))$ **then**
  do $load(b^*, t^*)$
  else if $(\exists b, c_1^*, t^*, c_2.\, BoxIn(b, c_1^*) \wedge TruckIn(t^*, c_2))$ **then**
  do $drive(t^*, c_1^*)$
  else do $noop$

- Great, but how do I obtain this solution?

Situation Calculus

- Logic formalism designed for representing and reasoning about dynamic domains
  - First introduced by John McCarthy in 1963
- Basic elements
  - Actions that can be performed in the world
  - Situations
  - Fluents that describe the state of the world
- Domain
  - Action precondition axioms, one for each action
  - Successor state axioms, one for each fluent
  - Axioms describing the world in various situations
  - Foundational axioms of the situation calculus: situations are histories, induction on situations

Situation Calculus: Ingredients

- Actions
  - First-order terms with action parameters
  - E.g., $load(b,t)$, $unload(b,t)$, $drive(t,c)$
- Situations
  - Term that encodes action history
  - E.g., s, s_0, $do(load(b,t),s)$, $do(load(b,t),drive(t,c),s)$
- Fluents
  - Relation whose truth value varies between situations
  - E.g., $BoxOn(b,t,s)$, $TruckIn(t,c,s)$, $Box(t,c,s)$
- Effects?

Situation Calculus: PDDL to Effects

- Translate action effects into positive and negative effect axioms
  - $load(Box:b, Truck:t)$:
    - when $[\exists City:c.\, BoxIn(b,c) \wedge TruckIn(t,c)]$ then $[BoxOn(b,t)]$
    - $\forall City:c.$ when $[BoxIn(b,c) \wedge TruckIn(t,c)]$ then $[\neg BoxIn(b,c)]$
  - $unload(Box:b, Truck:t)$:
    - $\forall City:c.$ when $[BoxOn(b,t) \wedge TruckIn(t,c)]$ then $[BoxIn(b,c)]$
    - when $[\exists City:c.\, BoxOn(b,t) \wedge TruckIn(t,c)]$ then $[\neg BoxOn(b,t)]$
  - $drive(Truck:t, City:c)$:
    - when $[\exists City:c_1.\, TruckIn(t,c_1)]$ then $[TruckIn(t,c)]$
    - $\forall City:c_1.\, when\ [TruckIn(t,c_1)]$ then $[\neg TruckIn(t,c_1)]$

- $[\exists c.\, a = load(b,t) \wedge BoxIn(b,c,s) \wedge TruckIn(t,c,s)] \Rightarrow BoxOn(b,t,do(a,s))$
- $[\exists t.\, a = load(b,t) \wedge BoxIn(b,c,s) \wedge TruckIn(t,c,s)] \Rightarrow \neg BoxIn(b,c,do(a,s))$
- $[\exists t.\, a = unload(b,t) \wedge BoxOn(b,t,s) \wedge TruckIn(t,c,s)] \Rightarrow BoxIn(b,c,do(a,s))$
- $[\exists c.\, a = unload(b,t) \wedge BoxOn(b,t,s) \wedge TruckIn(t,c,s)] \Rightarrow \neg BoxOn(b,t,do(a,s))$
- $[\exists c_1.\, a = drive(t,c) \wedge TruckIn(t,c_1,s)] \Rightarrow TruckIn(t,c,do(a,s))$
- $[\exists c.\, a = drive(t,c) \wedge TruckIn(t,c_1,s)] \Rightarrow \neg TruckIn(t,c_1,do(a,s))$

Situation Calculus: PDDL to Effects

- Use rule to combine multiple effects $C_1 \Rightarrow F$, $C_2 \Rightarrow F$ over the same fluent F into effect axioms: $\gamma_F^+(\vec{x}, a, s) \Rightarrow F\big(\vec{x}, do(a, s)\big), \gamma_F^-(\vec{x}, a, s) \Rightarrow F\big(\vec{x}, do(a, s)\big)$
  - Rule: $[(C_1 \Rightarrow F) \wedge (C_2 \Rightarrow F)] \equiv [(C_1 \vee C_2) \Rightarrow F]$
  - As a sort of shorthand notation
    - E.g.,
      - $[\exists c.\, a = load(b,t) \wedge BIn(b,c,s) \wedge TIn(t,c,s)] \Rightarrow BOn\big(b,t,do(a,s)\big) \rightarrow \gamma_{BOn}^+(\vec{x},a,s) \Rightarrow BOn\big(\vec{x},do(a,s)\big)$
      - $[\exists c.\, a = unload(b,t) \wedge BOn(b,t,s) \wedge TIn(t,c,s)] \Rightarrow \neg BOn\big(b,t,do(a,s)\big) \rightarrow \gamma_{BOn}^-(\vec{x},a,s) \Rightarrow \neg BOn\big(\vec{x},do(a,s)\big)$
      - $[\exists t.\, a = unload(b,t) \wedge BOn(b,t,s) \wedge TIn(t,c,s)] \Rightarrow BIn\big(b,c,do(a,s)\big) \rightarrow \gamma_{BIn}^+(\vec{x},a,s) \Rightarrow BIn\big(\vec{x},do(a,s)\big)$
      - $[\exists \mathrm{t}.\, a = load(b,t) \wedge BIn(b,c,s) \wedge TIn(t,c,s)] \Rightarrow \neg BIn\big(b,c,do(a,s)\big) \rightarrow \gamma_{BIn}^-(\vec{x},a,s) \Rightarrow \neg BIn\big(\vec{x},do(a,s)\big)$
      - $[\exists c_1.\, a = drive(t,c) \wedge TIn(t,c_1,s)] \Rightarrow TIn\big(t,c,do(a,s)\big) \rightarrow \gamma_{TIn}^+(\vec{x},a,s) \Rightarrow TIn\big(\vec{x},do(a,s)\big)$
      - $[\exists c.\, a = drive(t,c) \wedge TIn(t,c_1,s)] \Rightarrow \neg TIn\big(t,c_1,do(a,s)\big) \rightarrow \gamma_{TIn}^-(\vec{x},a,s) \Rightarrow \neg TIn\big(\vec{x},do(a,s)\big)$

Frame Problem

- Positive and negative effect axioms specify what changes
  - $\gamma^{+}_{BOn}(\vec{x}, a, s) \Rightarrow BOn\big(\vec{x}, do(a, s)\big)$ $\quad\quad$ $\gamma^{-}_{BOn}(\vec{x}, a, s) \Rightarrow \neg BOn\big(\vec{x}, do(a, s)\big)$
  - $\gamma^{+}_{BIn}(\vec{x}, a, s) \Rightarrow BIn\big(\vec{x}, do(a, s)\big)$ $\quad\quad$ $\gamma^{-}_{BIn}(\vec{x}, a, s) \Rightarrow \neg BIn\big(\vec{x}, do(a, s)\big)$
  - $\gamma^{+}_{TIn}(\vec{x}, a, s) \Rightarrow TIn\big(\vec{x}, do(a, s)\big)$ $\quad\quad$ $\gamma^{-}_{TIn}(\vec{x}, a, s) \Rightarrow \neg TIn\big(\vec{x}, do(a, s)\big)$
- Assume completeness regarding these effect axioms:
  - That is, assume that $\gamma^{+}_{F}(\vec{x}, a, s) \Rightarrow F\big(\vec{x}, do(a, s)\big), \gamma^{-}_{F}(\vec{x}, a, s) \Rightarrow \neg F\big(\vec{x}, do(a, s)\big)$ characterise all the conditions under which an action a changes the value of fluent F
  - Formalise as explanation closure axioms
    - $\neg F(\vec{x}, s) \wedge F\big(\vec{x}, do(a, s)\big) \Rightarrow \gamma^{+}_{F}(\vec{x}, a\,, s)$ $\quad\quad$ $\equiv \neg F(\vec{x}, s) \wedge \neg\gamma^{+}_{F}(\vec{x}, a\,, s) \Rightarrow \neg F\big(\vec{x}, do(a, s)\big)$
      - If F was false and was made true by doing action a, then condition γ^{+}_{F} must have been true
    - $F(\vec{x}, s) \wedge \neg F\big(\vec{x}, do(a, s)\big) \Rightarrow \gamma^{-}_{F}(\vec{x}, a\,, s)$ $\quad\quad$ $\equiv F(\vec{x}, s) \wedge \neg\gamma^{-}_{F}(\vec{x}, a\,, s) \Rightarrow F\big(\vec{x}, do(a, s)\big)$
      - If F was true and was made false by doing action a then condition γ^{-}_{F} must have been true

Successor State Axioms (SSAs)

- Inputs / Requirements
  - Unique names for actions / arguments
  - Positive and negative effect axioms
    - $\gamma_F^+(\vec{x}, a, s) \Rightarrow F\left(\vec{x}, do(a, s)\right), \gamma_F^-(\vec{x}, a, s) \Rightarrow F\left(\vec{x}, do(a, s)\right)$
  - Explanation closure axioms
    - $\neg F(\vec{x}, s) \wedge F\left(\vec{x}, do(a, s)\right) \Rightarrow \gamma_F^+(\vec{x}, a\ , s), F(\vec{x}, s) \wedge \neg F\left(\vec{x}, do(a, s)\right) \Rightarrow \gamma_F^-(\vec{x}, a\ , s)$
  - Integrity: $\neg\exists \vec{x}, a, s. \gamma_F^+(\vec{x}, a, s) \wedge \gamma_F^-(\vec{x}, a, s)$
- SSA for each F:
  - $F\left(\vec{x}, do(a, s)\right) \equiv \gamma_F^+(\vec{x}, a, s) \vee \left(F(\vec{x}, s) \wedge \neg\gamma_F^-(\vec{x}, a, s)\right)$
  - Shorthand:
    - $F\left(\vec{x}, do(a, s)\right) \equiv \Phi_F(\vec{x}, a, s)$

Successor State Axioms (SSAs): Example

- SSA for each F: $F(\vec{x}, do(a,s)) \equiv \gamma_F^+(\vec{x},a,s) \vee (F(\vec{x},s) \wedge \neg\gamma_F^-(\vec{x},a,s))$
  - Shorthand: $F(\vec{x}, do(a,s)) \equiv \Phi_F(\vec{x},a,s)$
- $BoxOn(b,t,do(a,s)) \equiv \Phi_{BoxOn}(b,t,a,s)$
  $\equiv [\exists c.\, a = load(b,t) \wedge BoxIn(b,c,s) \wedge TruckIn(t,c,s)]$
  $\vee (BoxOn(b,t,s) \wedge \neg[\exists c.\, a = unload(b,t) \wedge BoxOn(b,t,s) \wedge TruckIn(t,c,s)])$
- $BoxIn(b,c,do(a,s)) \equiv \Phi_{BoxIn}(b,c,a,s)$
  $\equiv [\exists t.\, a = unload(b,t) \wedge BoxOn(b,t,s) \wedge TruckIn(t,c,s)]$
  $\vee (BoxIn(b,c,s) \wedge \neg[\exists t.\, a = load(b,t) \wedge BoxIn(b,c,s) \wedge TruckIn(t,c,s)])$
- $TruckIn(t,c,do(a,s)) \equiv \Phi_{TruckIn}(t,c,a,s)$
  $\equiv [\exists c_1.\, a = drive(t,c) \wedge TruckIn(t,c_1,s)]$
  $\vee (TruckIn(t,c,s) \wedge \neg[\exists c_1.\, a = drive(t,c) \wedge TruckIn(t,c_1,s)])$

Regression

- Idea: Use SSAs to regress from goal towards a (possibly only partially defined) intial state
  - A bit like lifted backward search

- Regression
  - If ϕ held after action a, then *regression* is the ϕ' that held before action a
  - Exploit following properties
    - $Regr(\neg\psi) = \neg Regr(\psi)$
    - $Regr(\psi_1 \wedge \psi_2) = Regr(\psi_1) \wedge Regr(\psi_2)$
    - $Regr\big((\exists x)\psi\big) = (\exists x) Regr(\psi)$
    - $Regr\left(F\big(\vec{x}, do(a,s)\big)\right) = \Phi_F(\vec{x}, a, s)$

Regression: Example

- If ϕ held after action a, then *regression* is the ϕ' that held before action a
- Exploit following properties
  - $Regr(\neg\psi) = \neg Regr(\psi)$
  - $Regr(\psi_1 \wedge \psi_2) = Regr(\psi_1) \wedge Regr(\psi_2)$
  - $Regr\big((\exists x)\psi\big) = (\exists x) Regr(\psi)$
  - $Regr\Big(F\big(\vec{x}, do(a,s)\big)\Big) = \Phi_F(\vec{x}, a, s)$

- Given: $\exists b.\, BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)$
- Regress through $unload(b^*, t^*)$
  - $Regr\Big(\exists b.\, BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)\Big)$

$= \exists b.\, Regr\Big(BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)\Big)$

$= \exists b.\, \Phi_{BoxIn}(b, paris, unload(b^*, t^*), s)$

$= \exists b.\, [\exists t.\, unload(b^*, t^*) = unload(b, t) \wedge BoxOn(b, t, s) \wedge TruckIn(t, paris, s)]$
$\vee\, (BoxIn(b, paris, s)$
$\wedge \neg[\exists t.\, unload(b^*, t^*) = load(b, t) \wedge BoxIn(b, paris, s) \wedge TruckIn(t, paris, s)])$

$= [\exists b, t.\, b = b^* \wedge t = t^* \wedge BoxOn(b, t, s) \wedge TruckIn(t, paris, s)] \vee \exists b.\, BoxIn(b, paris, s)$

$= [(\exists b.\, b = b^*) \wedge (\exists t.\, t = t^*) \wedge BoxOn(b^*, t^*, s) \wedge TruckIn(t^*, paris, s)]$
$\vee \exists b.\, BoxIn(b, paris, s)$

$= [BoxOn(b^*, t^*, s) \wedge TruckIn(t^*, paris, s)] \vee \exists b.\, BoxIn(b, paris, s)$

Cannot be made true
$\rightarrow \phi \wedge \neg[\bot] \equiv \phi \wedge \top \equiv \phi$

Make non-empty domain assumption for b, t

Regression: Example

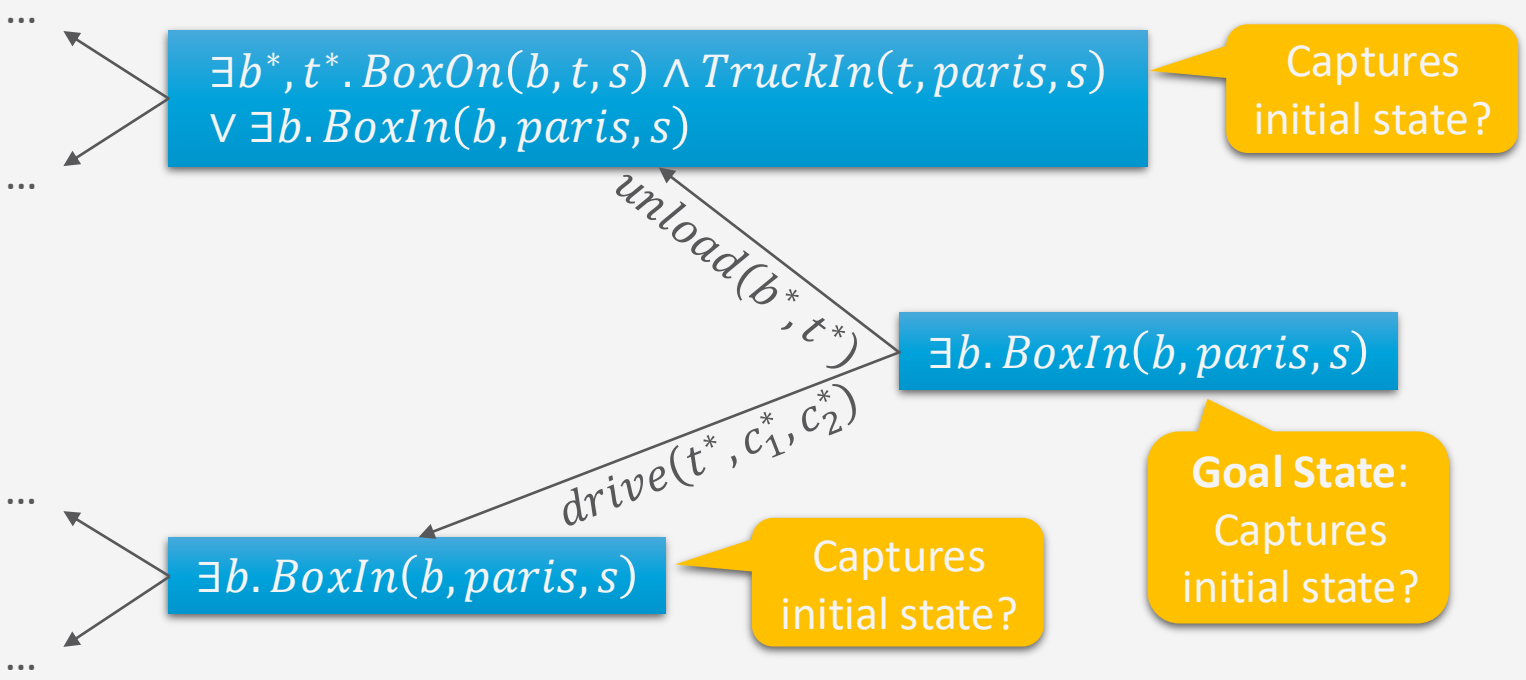
- Given: $\exists b.\, BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)$
- Regress through $unload(b^*, t^*)$
  - $Regr\left(\exists b.\, BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)\right)$
    $= [BoxOn(b^*, t^*, s) \wedge TruckIn(t^*, paris, s)] \vee \exists b.\, BoxIn(b, paris, s)$
- To get action instantiations of $unload(b^*, t^*)$, query knowledge base (KB, i.e., planning domain)
  - Existentially quantify b^*, t^* and obtain instances via query extraction w.r.t. KB
    - KB consists of first-order state and action abstraction → do not have to enumerate all states, b^*, t^*
    - $\exists b^*, t^*.\, Regr\left(\exists b.\, BoxIn\big(b, paris, do(unload(b^*, t^*), s)\big)\right)$
      $= \exists b^*, t^*.\, [BoxOn(b^*, t^*, s) \wedge TruckIn(t^*, paris, s)] \vee \exists b.\, BoxIn(b, paris, s)$
      $= [\exists b^*, t^*.\, BoxOn(b^*, t^*, s) \wedge TruckIn(t^*, paris, s)] \vee \exists b.\, BoxIn(b, paris, s)$

Regression Planning

- Define abstract goal state
  - E.g., $\exists b. BoxIn(b, paris, s)$
  - Check if regression through action sequence holds in initial state

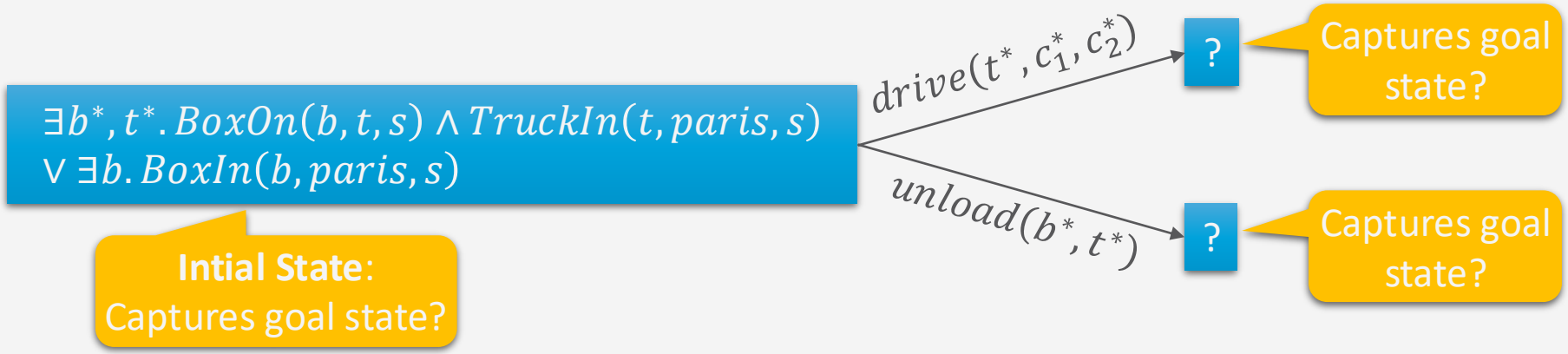
→ Goal regression planning

  - Provide initial state, actions
    - Initial state description can be partial
  - Use regression to tell whether goal will hold



Progression and Forward Search?

- Can we do lifted forward-search planning?



- Progression not first-order definable! (Reiter, 2001)
- Could progress ground state
  - But this does not exploit first-order structure

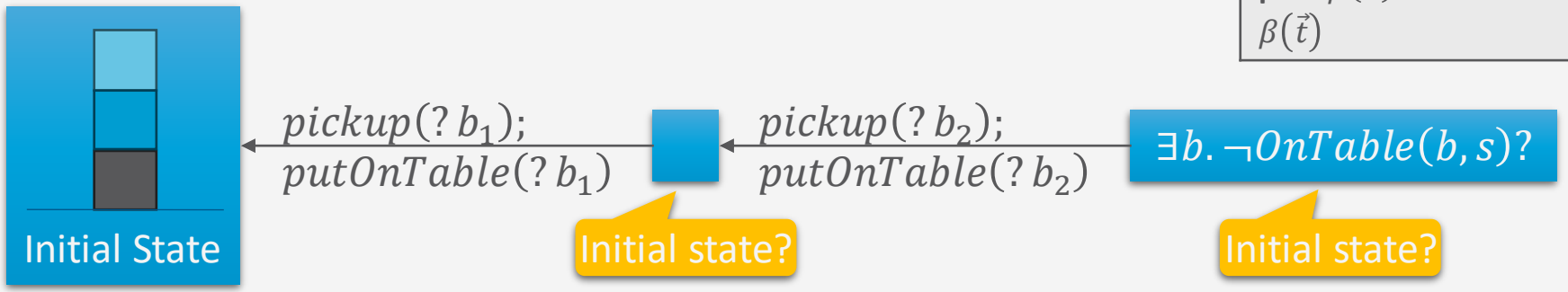
Golog: Restricted Plan Search

- Al**GO**l in **LOG**ic
  - Search the space of sequential action plans
  - Regress actions to initial state to test reachability
  - Restrict action space with program:

α	primitive action
$\phi?$	condition test
(δ_1, δ_2)	sequence
if ϕ **then** δ **endIf**	conditional
while ϕ **then** δ **endWhile**	loop
$(\delta_1 \mid \delta_2)$	nondeterministic choice of actions
$\pi\ \vec{x}\ [\delta]$	nondeterministic choice of arguments
δ^*	nondeterministic iteration
proc $\beta(\vec{x})\ \delta$ **endProc**	procedure call definition
$\beta(\vec{t})$	procedure call

Golog: Example

- Golog program
  - $(\pi b[\neg OnTable(b,s)?, pickup(b), putOnTable(b)])^*,$ $\forall b.\, OnTable(b,s)?$
- Diagram of Golog planning



  - Heavily restricted action sequences
  - Program exploits first-order action abstraction
  - Initial state need not be fully known

α	primitive action
$\phi?$	condition test
(δ_1, δ_2)	sequence
if ϕ **then** δ **endIf**	conditional
while ϕ **then** δ **endWhile**	loop
$(\delta_1 \mid \delta_2)$	nondeterministic choice of actions
$\pi\, \vec{x}\, [\delta]$	nondeterministic choice of arguments
δ^*	nondeterministic iteration
proc $\beta(\vec{x})$ δ **endProc**	procedure call definition
$\beta(\vec{t})$	procedure call

Frame Problem

- Frame problem: How to (*compactly*) specify what does not change?
  - Intuition: “What does not change, remains the same.”
    - Reiter’s so-called Default Solution
  - Not so easy to specify
    - Moving one thing might move another thing, even though the other thing is never directly touched
    - How to distinguish between relevant and irrelevant side effects? And use that efficiently?

- Default solution to frame problem given as successor state axioms (SSAs), which we construct next

Interim (Interim) Summary

- Situation calculus to describe a relational world
  - Can convert PDDL (and state-variable domains) into effect axioms
  - Derive SSAs from effect axioms
    - Using default solution to frame problem
- Regression operator
  - Extract action instantiation to achieve goal
- Regression planning
  - Initial state need not be fully specified
  - Exploit state and action abstraction
    - Avoid enumerating all state and action instances
- Frame problem

Next step: Extend this idea for decision-theoretic planning with uncertain action outcomes

$$rCase(s) = \begin{array}{|c|c|} \hline \forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s) & 1 \\ \hline \neg\big(\forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s)\big) & 0 \\ \hline \end{array}$$

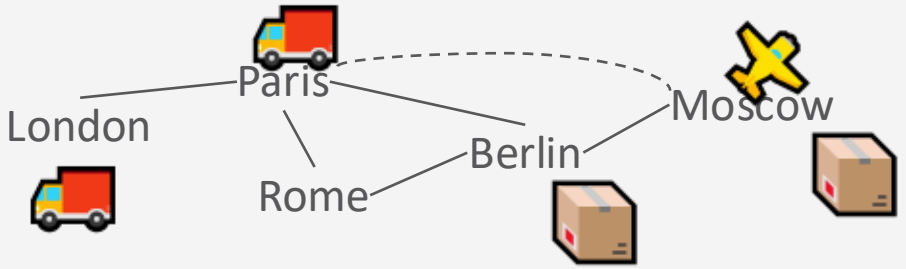
First-order MDPs

Structure by Relations in the State Space

First-order MDP (FOMDP)

- Components of MDP in an FOMDP specified as a collection of *case statements*
  - E.g., express reward in Box World FOMDP as

$$rCase(s) = \begin{array}{|c|c|} \hline \forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s) & 1 \\ \hline \neg\big(\forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s)\big) & 0 \\ \hline \end{array}$$



- Operators: define unary and binary case operations
  - E.g., cross-sum $\oplus$ (or $\ominus$, $\otimes$) of cases

$$\begin{array}{|c|c|} \hline \phi & 10 \\ \hline \neg\phi & 20 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \varphi & 3 \\ \hline \neg\varphi & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline \phi \wedge \varphi & 10+3 \\ \hline \phi \wedge \neg\varphi & 10+4 \\ \hline \neg\phi \wedge \varphi & 20+3 \\ \hline \neg\phi \wedge \neg\varphi & 20+4 \\ \hline \end{array}$$

Stochastic Actions and First-order Decision-theoretic Regression (FODTR)

- Stochastic actions using deterministic situation calculus
  - User's stochastic action, e.g., $A(x) = load(b,t)$
  - Nature's choice, e.g., $n(x) \in \{loadS(b,t), loadF(b,t)\}$
  - Probability distribution over nature's choice, e.g.,

Probability distribution → Adds up to 1 over success and failure choice

$0.1 + 0.9 = 1$

$0.6 + 0.4 = 1$

$P(loadS(b,t)|load(b,t)) =$

$snow(s)$	0.1
$\neg snow(s)$	0.6

$P(loadF(b,t)|load(b,t)) =$

$snow(s)$	0.9
$\neg snow(s)$	0.4

- First-order decision-theoretic regression (FODTR)
  - FODTR = *expectation* of regression:

$$FODTR[vCase(s), A(\vec{x})] = \boldsymbol{E}_{P(n(\vec{x})|A(\vec{x}))}\left[Regr\left(vCase(s), n(\vec{x})\right)\right]$$

FODTR & Q-Functions

- Result of FODTR is a case statement encoding a first-order Q-function

$$FODTR[vCase(s), A(\vec{x})] = R(s) \oplus \gamma \bigoplus_{j=1}^{k} P\left(n_j(\vec{x}), A(\vec{x}), s\right) \otimes Regr\left(V\left(do\left(n_j(\vec{x})\right), s\right)\right)$$

  - E.g.,

$$FODTR[vCase(s), unload(b^*, t^*)]$$

$$= rCase(s) \oplus \gamma \bigoplus_{j=1}^{k} pCase\left(n_j(\vec{x}), unload(b^*, t^*), s\right)$$

$$\otimes \begin{array}{|l|l|} \hline Regr\left(\exists b.\, BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right) & 10 \\ \hline Regr\left(\neg\exists b.\, BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right) & 0 \\ \hline \end{array}$$

$$rCase(s) = \begin{array}{|l|l|} \hline \exists b.\, BoxIn(b, paris, s) & 10 \\ \hline \neg\left(\exists b.\, BoxIn(b, paris, s)\right) & 0 \\ \hline \end{array}$$

$$pCase(loadS(b,t), load(b,t), s) = \begin{array}{|l|l|} \hline \top & 0.9 \\ \hline \end{array}$$

$$pCase(unloadS(b,t), unload(b,t), s) = \begin{array}{|l|l|} \hline \top & 0.9 \\ \hline \end{array}$$

$$pCase(driveS(c,t), drive(c,t), s) = \begin{array}{|l|l|} \hline \top & 1 \\ \hline \end{array}$$

FODTR & Q-Functions

$$FODTR[vCase(s), unload(b^*, t^*)]$$

$$= rCase(s) \oplus \gamma \bigoplus_{j=1}^{k} pCase\left(n_j(\vec{x}), unload(b^*, t^*), s\right) \otimes$$

$Regr\left(\exists b.BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right)$	10
$Regr\left(\neg\exists b.BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right)$	0.0

$$= rCase(s) \oplus \gamma \Bigg[$$

T	0.9	$Regr\left(\exists b.BoxIn\left(b, paris, do(unloadS(b^*, t^*), s)\right)\right)$	10
		$Regr\left(\neg\exists b.BoxIn\left(b, paris, do(unloadS(b^*, t^*), s)\right)\right)$	0.0

$$\oplus$$

T	0.1	$Regr\left(\exists b.BoxIn\left(b, paris, do\left(unloadF(b^*, t^*)\right)\right)\right)$	10
		$Regr\left(\neg\exists b.BoxIn\left(b, paris, do\left(unloadF(b^*, t^*)\right)\right)\right)$	0.0

$$\otimes \Bigg]$$

FODTR & Q-Functions

$$FODTR[vCase(s), unload(b^*, t^*)]$$

$$= rCase(s) \oplus \gamma \bigoplus_{j=1}^{k} pCase\left(n_j(\vec{x}), unload(b^*, t^*), s\right) \otimes$$

$Regr\left(\exists b.BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right)$	10
$Regr\left(\neg\exists b.BoxIn\left(b, paris, do\left(n_j(\vec{x}), s\right)\right)\right)$	0.0

$=$

$\exists b.BoxIn(b, paris, s)$	10
$\neg$“	0.0

$\oplus$

$\exists b.BoxIn(b, paris, s)$	9.0
$\neg$“	0.0

$\oplus$

$\exists b^*, t^*.BoxOn(b, t, s) \wedge TruckIn(t, paris, s) \vee \exists b.BoxIn(b, paris, s)$	8.1
$\neg$“	0.0

$=$

$\exists b.BoxIn(b, paris, s)$	19.0	$\rightarrow noop$
$\neg$“ $\wedge [\exists b, t.BoxOn(b, t, s) \wedge TruckIn(t, paris, s)]$	8.1	$\rightarrow unload(b^*, t^*)$
$\neg$“	0.0	$\rightarrow noop$

Symbolic Dynamic Programming (SDP)

- What value if 0-stages-to-go?
  - Immediate reward: $V^0(s) = rCase(s)$
- What value if 1-state-to-go?
  - We know value for each action → Take maximum for each state

$$V^1(s) = \max_s \begin{cases} \begin{array}{|c|c|}\hline \phi_1 & 9 \\ \hline \phi_2 & 0 \\ \hline \end{array} & = V^0(s, A_1) \\ \begin{array}{|c|c|}\hline \phi_3 & 3 \\ \hline \phi_4 & 1 \\ \hline \end{array} & = V^0(s, A_2) \end{cases}$$

$$V^1(s) = \begin{array}{|l|c|}\hline \phi_1 & 9 \\ \hline \textbf{else } \phi_3 & 3 \\ \hline \textbf{else } \phi_4 & 1 \\ \hline \textbf{else } \phi_2 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \phi_1 & 9 \\ \hline \phi_2 & 0 \\ \hline \end{array} = V^0(s, A_1) \qquad \begin{array}{|c|c|}\hline \phi_3 & 3 \\ \hline \phi_4 & 1 \\ \hline \end{array} \quad V^0(s, A_2)$$

- Value iteration
  - Obtain V^{n+1} from V^n until $(V^{n-1} \ominus V^n) < \epsilon$

Value Iteration for $t = 1, 2$ of the Box World Example

- With increasing iterations, the sequence of actions considered gets longer

$vCase^1(s) =$

$\exists b.BoxIn(b, paris, s)$	19.0
$\neg\text{“} \wedge [\exists c.BoxOn(b, t, s) \wedge TruckIn(t^*, paris, s)]$	8.1
$\neg\text{“}$	0.0

$vCase^2(s) =$

$\exists b.BoxIn(b, paris, s)$	26.1
$\neg\text{“} \wedge [\exists b, t.BoxOn(b, t, s) \wedge TruckIn(t, paris, s)]$	15.4
$\neg\text{“} \wedge [\exists b, c, t.BoxOn(b, t, s) \wedge TruckIn(t, c, s)]$	7.3
$\neg\text{“}$	0.0

if $(\exists b.BoxIn(b, paris))$ **then**
  do *noop*
else if $(\exists b^*, t^*.TruckIn(t^*, paris) \wedge BoxOn(b^*, t^*))$
  do $unload(b^*, t^*)$
else if $(\exists b, c, t^*.BoxOn(b, t^*) \wedge TruckIn(t, c))$ **then**
  do $drive(t^*, paris)$
else if $(\exists b^*, c, t^*.BoxIn(b^*, c) \wedge TruckIn(t^*, c))$ the
  do $load(b^*, t^*)$
else if $(\exists b, c_1^*, t^*, c_2.BoxIn(b, c_1^*) \wedge TruckIn(t^*, c_2$
  do $drive(t^*, c_1^*)$
else do *noop*

First-order Algebraic Decision Diagrams (FOADDs)

- We want to compactly represent arbitrary case statements
  - E.g.,

$$case(s) = \begin{array}{|c|c|} \hline \exists x.\,[A(x) \vee \forall y.\,A(x) \wedge B(x) \wedge \neg A(y)] & 1 \\ \hline \neg(\exists x.\,[A(x) \vee \forall y.\,A(x) \wedge B(x) \wedge \neg A(y)]) & 0 \\ \hline \end{array}$$

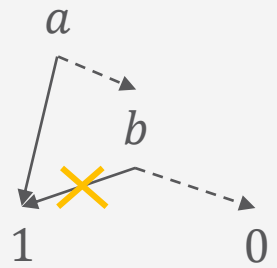
- Push down quantifiers, expose propositional structure → convert into FOADD

$$[\exists x.\,A(x)] \vee ([\exists x.\,A(x) \wedge B(x)] \wedge [\forall y.\,\neg A(y)])$$

Variable	Variable ⇔ FOL KB
a	$\equiv [\exists x.\,A(x)]$
b	$\equiv [\exists x.\,A(x) \wedge B(x)]$

$$case(s) = \begin{array}{|c|c|} \hline a \vee (b \wedge \neg a) & 1 \\ \hline \neg\big(a \vee (b \wedge \neg a)\big) & 0 \\ \hline \end{array}$$

$case(s) =$



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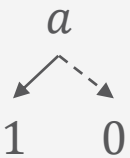
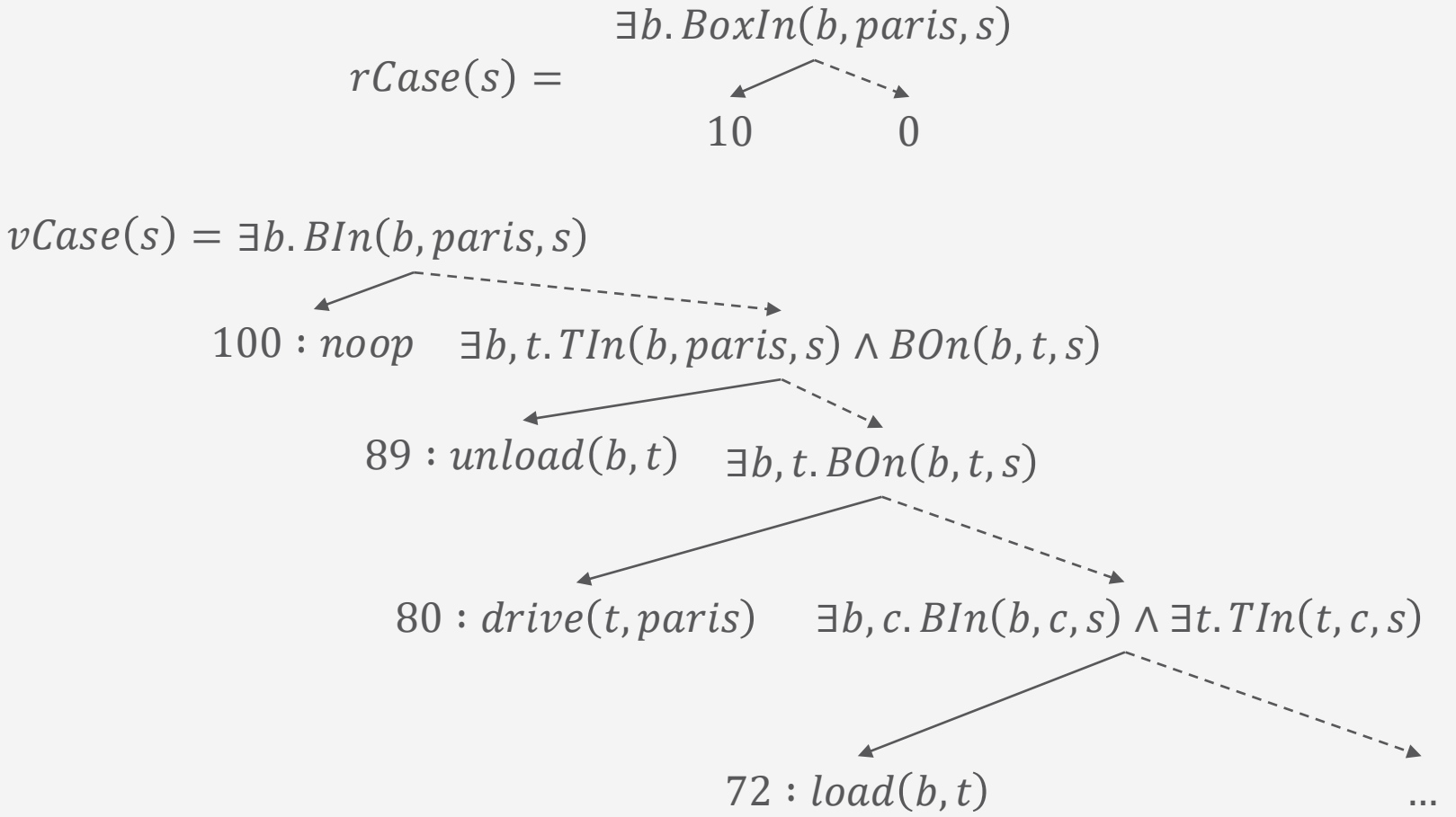


First-order context-specific independence

Results for SDP with FOADDs

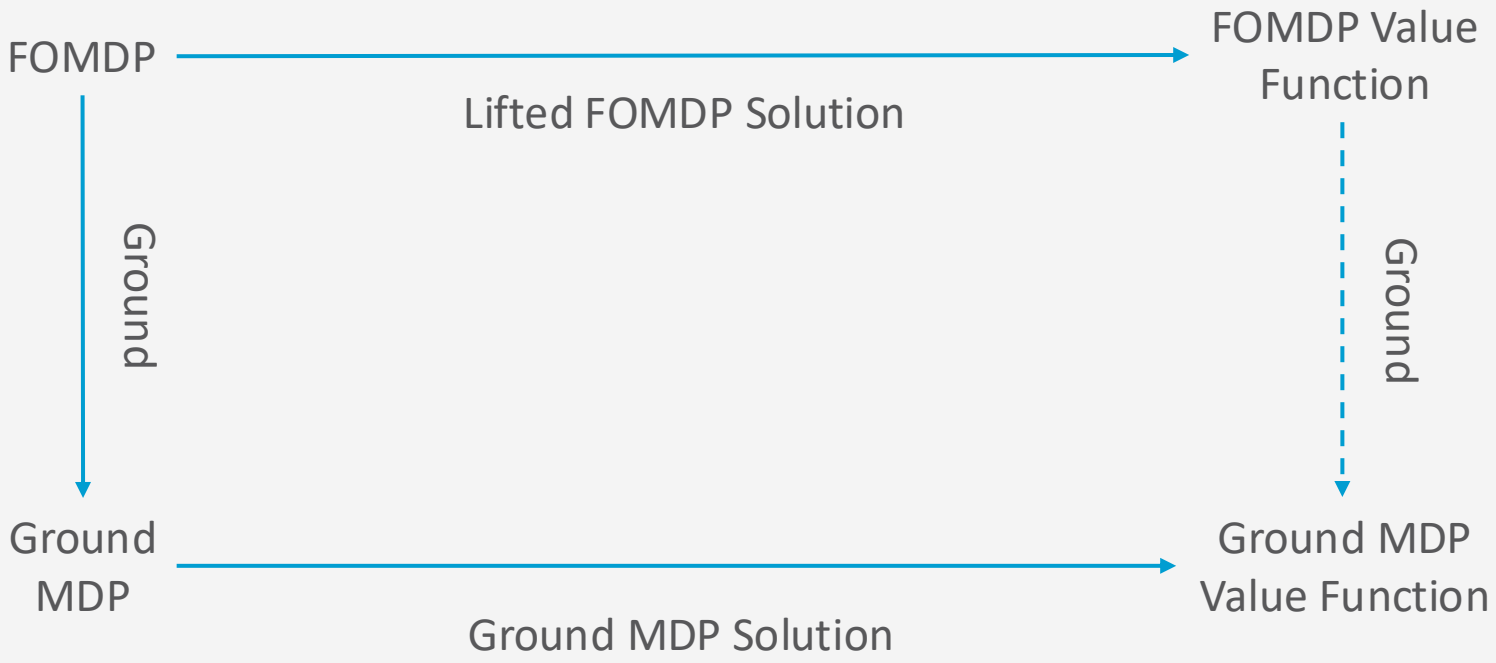
- Encode case statements with FOADDs
  - Solid line: true case
  - Dotted line: false case
- Use FOADD operations for structured SDP
  - E.g., Box World
    - Using $\gamma = 0.9$



Factored SDP for factored FOMDPs [Sanner and Boutilier, 2007]

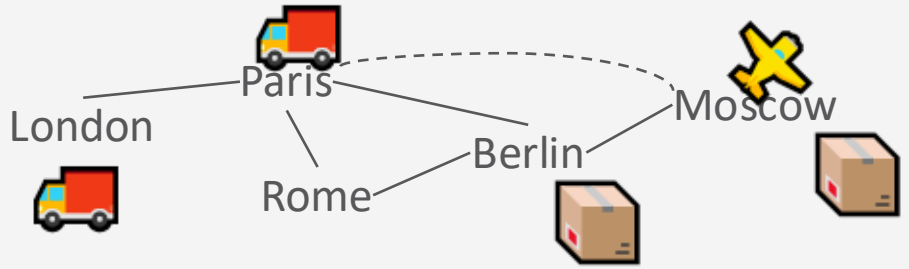
Correctness of SDP

- Show SDP for FOMDPs is correct w.r.t. ground MDP



Caveats of First-order Planning

- Many problems have topologies
  - E.g., reachability constraints in logistics



- If topology not fixed a priori
  - First-order solution must consider ∞ topologies
  - In general case, leads to ∞ values / policies
    - Universal rewards
    - Value function must distinguish ∞ cases
    - Policy will also likely be ∞

$$rCase(s) = \begin{cases} \forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s) & 1 \\ \neg\big(\forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s)\big) & 0 \end{cases}$$

$$V^t(s) = \begin{cases} \forall b, c.\, Dest(b,c) \Rightarrow BoxIn(b,c,s) & 1 \\ \text{One box not at destination} & \gamma \\ \text{Two boxes not at destination} & \gamma^2 \\ \vdots & \vdots \\ t-1 \text{ boxes not at destination} & \gamma^{t-1} \end{cases}$$

Caveats of First-order Planning

- Unreachable states
  - PDDL domains often under-constrained
    - E.g., logistics: one box cannot be in two cities at once
  - Constraints implicitly obeyed in initial state
    - Action effects cannot violate constraints
      - Reachable legal states are small subset of all states
    - But (P)PDDL does not constrain legal states

Suggests need for hybrid first-order / search-based approaches

- If no background theory to restrict legal states
  - First-order planning must solve for all states
    - When initial state unknown
  - Where majority of states are actually illegal
- First-order planning w/o initial state solves more difficult problem than search-based solutions
  - Initial state contains implicit constraint information
  - Reachable state space is small subset of all states

A Note on First-order Modelling in Reinforcement Learning

- Novel propositional situations worth exploring may be instances of a well-known context in the relational setting → *exploitation* promising
  - E.g., household robot learning water-taps
    - Having opened one or two water-taps in a kitchen, one can expect other water-taps in kitchens to work similarly

    ⇒ Priority for exploring water-taps in kitchens in general reduced

    ⇒ Information gathered likely to carry over to water-taps in other places

    ❖ Hard to model in propositional setting: each water-tap is novel

Interim Summary

- FOMDPs are one model for lifted decision-theoretic planning
  - Exploit state and action abstraction for MDPs
- Use situation calculus specified action theory
- Use case statements to represent reward, probabilities
- Symbolic dynamic programming = lifted DP
  - Use FOADDs to compactly represent case statements
  - First-order context-specific independence to compactify FOADDs

Outline: Decision Making – Structure

Structure by Groups in the Agent Set

- Agent types
- Partitioned decPOMDPs

Structure by Relations in the State Space

- Situation calculus
- First-order MDPs

Structure by Features in the State Space

- Dynamic Bayesian networks
- Factored MDPs

State Space

- Often has structure
  - Not just a black box with n states, leading to a probability distribution $T(s', s, a) = P(s'|s, a)$
  - Can be structured by relations (FO MDPs), regularities (robot navigation; harbour management)
- Generally, n different features that describe a state space
  - Encode in n individual random variables S_i with respective domains $\mathrm{dom}(S_i) = \{v_1, \ldots, v_{d_i}\}$
    - State space size then describable as $|S| = \prod_i d_i \leq d^n, d = \max_i d_i$
      - I.e., exponential in the number of random variables
- Given (conditional) independences between different S_i, factorisation of probability distributions in model possible
  - Applicable to MDPs, POMDPs, DecPOMDPs, partitioned DecPOMDPs
  - Most work exists for factored MDPs (also the simplest case to consider)

Factorisation in General

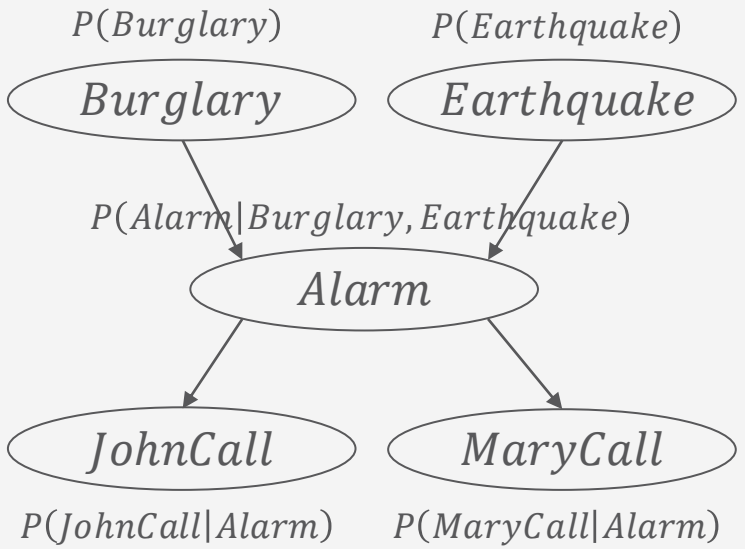
- (Conditional) independences:
  - $A \perp B$ (A, B independent) $\Leftrightarrow P(A, B) = P(A) \cdot P(B)$
  - $A \perp B \mid C$ (A, B conditionally independent given C) $\Leftrightarrow P(A, B|C) = P(A|C) \cdot P(B|C)$
    - Alternate version: $A \perp B \mid C \Leftrightarrow P(A|B, C) = P(A|C)$
- (Conditional) independences allow for factorising a distribution into smaller factors
  - In general: Factorisation of a full joint probability distribution $P(S_1, \dots, S_n)$ into m factors over subsets $\boldsymbol{C}$ of random variables that form $P(S_1, \dots, S_n)$ after multiplication (and normalisation):

$$P(S_1, \dots, S_n) = \frac{1}{Z} \prod_{j=1}^{m} \phi(\boldsymbol{C}_j)$$

    - Where $\boldsymbol{C}_j$ refers to sets of random variables that are mutually dependent on each other
    - Memory complexity: $\mathcal{O}(d^n)$ vs. $\mathcal{O}\left(m \cdot d^{|\boldsymbol{C}_{max}|}\right)$

Probabilistic Graphical Models (PGMs)

- PGMs use a graph structure to represent dependences
  - Common formalism: Bayesian network (BN) B
    - Directed acyclic graph
      - Nodes: random variables S_i
      - Edges: if S_i depends on S_j, edge $S_j \longrightarrow S_i$
    - Factors: conditional probability distributions (CPDs) $\forall i\ P(S_i|\mathrm{pa}(S_i))$
      - Roots: $\mathrm{pa}(S_i) = \emptyset \rightarrow$ Prior distributions $P(S_i)$
      - Usually not depicted in graph; have to be denoted somewhere
      - Semantics: $P(S_1, \dots, S_n) = \prod_{i=1}^{n} P(S_i|\mathrm{pa}(S_i))$
  - Not further considered here:
    Undirected version with potential functions ϕ as factors:
    - Factor graphs, Markov networks
    - Same semantics, different graphical representation



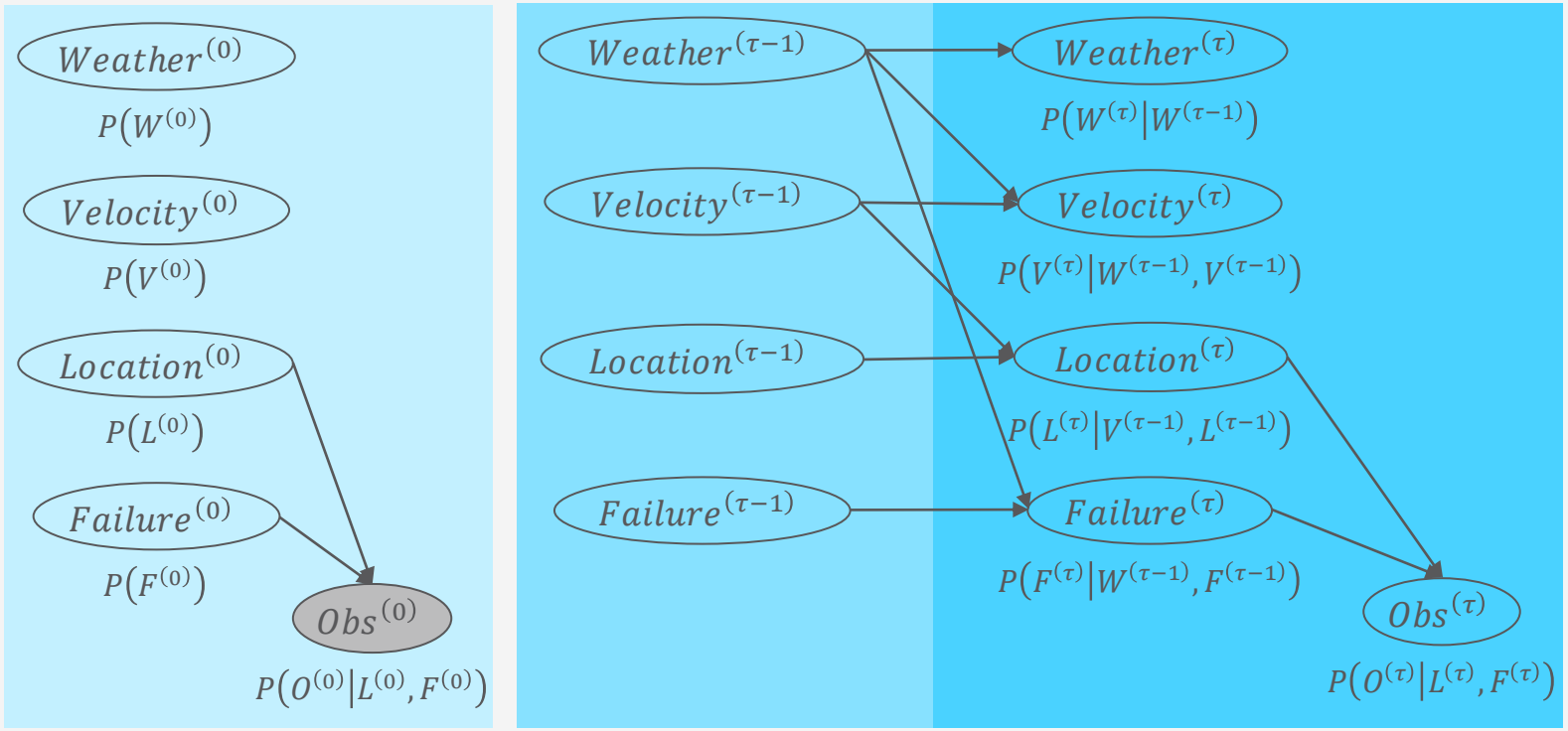
Full joint probability distribution size: d^5
Sizes of CPDs: $d + d + d^3 + d^2 + d^2$
Given $d = 2$: $2^5 = 32$ vs. 20
(As probabilities add to 1:
size -1 for each probability distribution in each CPD,
i.e., $1 + 1 + 4 + 2 + 2 = 10$)

Dynamic Bayesian Networks

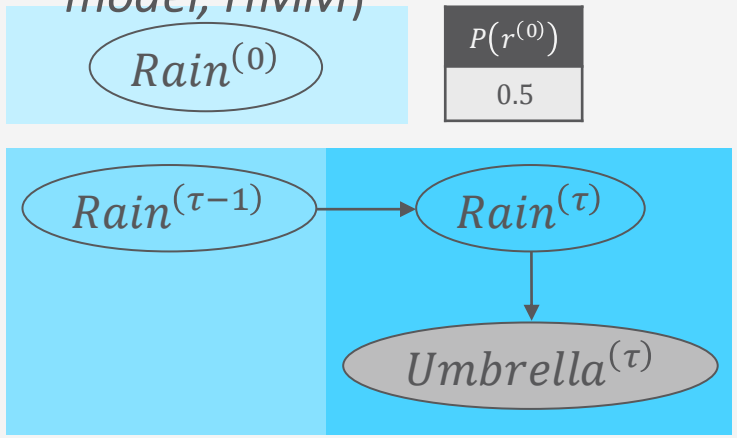
- MDP models a sequential, i.e., temporal, stationary, Markovian probabilistic setting
  - Factorisation also needs to encode a sequential, stationary, Markovian probabilistic setting
- Popular modeling formalism used:
  Dynamic BN (DBN) is a two-tuple $\left(B^{(0)}, B^{(\rightarrow)}\right)$
  - Template variables S_i indexed by time step τ in BNs
    $\rightarrow$ Can be instantiated for particular time steps t
  - BN $B^{(0)}$ for time step 0 to encode
    - If set to uniform distributions or using DBN for fix point calculations, can be safely ignored
  - BN $B^{(\rightarrow)}$ for time step τ with connections from time step $\tau - 1$ (copy pattern)
    - Markov-1 $\rightarrow$ Only connections from $\tau - 1$ to τ
    - Stationary $\rightarrow$ $B^{(\rightarrow)}$ identical for all $t \in \{1, \ldots\}$
  - Semantics: unroll for T time steps and multiply

Dynamic Bayesian Networks: Example

- Left: vehicle localization task, where a moving car tries to track its current location using the data obtained from a, possibly faulty, sensor



- Right: Toy example of a special case of a DBN with one latent and one observable variable (*hidden Markov model, HMM*)



$P(r^{(0)})$
0.5

$R^{(t-1)}$	$P(r^{(t)}\|R^{(t-1)})$
true	0.7
false	0.3

$R^{(t)}$	$P(u^{(t)}\|R^{(t)})$
true	0.9
false	0.2

Factored MDPs

- MDP with its state space S structured according to $S_1, \dots, S_n$, which in general means that
  - Transition probability distribution $T(S', S, A) = P(S'|S, A)$ is given by $T(S'_1, \dots, S'_n, S_1, \dots, S_n, A) = P(S'_1, \dots, S'_n|S_1, \dots, S_n, A)$
    - Or using the template notation: $T\left(S^{(\tau)}, S^{(\tau-1)}, A^{(\tau-1)}\right) = P\left(S^{(\tau)}\middle|S^{(\tau-1)}, A^{(\tau-1)}\right)$ is given by $T\left(S_1^{(\tau)}, \dots, S_n^{(\tau)}, S_1^{(\tau-1)}, \dots, S_n^{(\tau-1)}, A^{(\tau-1)}\right) = P\left(S_1^{(\tau)}, \dots, S_n^{(\tau)}\middle|S_1^{(\tau-1)}, \dots, S_n^{(\tau-1)}, A^{(\tau-1)}\right)$
    - Note that the overall size of T does not increase as the state space size is identical
  - Given that $S_1, \dots, S_n$ represent features of (hopefully weakly) connected parts of a system, T can be factored according to (conditional) independences → often represented using a DBN
    - Factorisation of T:

$$T(S', S, A) = P(S'_1, \dots, S'_n|S_1, \dots, S_n, A) = \prod_{i=1}^{n} P(S'_i|\mathrm{pa}(S'_i)) =: T_B$$

Factored MDPs: Actions and Rewards

- To be correct, the DBN just described is a standard DBN extended with random variable nodes for actions, whose assignment we want to determine, and reward nodes to denote that a reward function outputs a reward depending on the state (and action)
  - BN extended with so-called decision and utility nodes called influence or decision diagram

Side note: Since the state in MDPs is fully observable, every random variable in a DBN is observable, which is not the general case for DBNs, where usually there is a set of latent variables, which are never observed and as such often queried, and a set of evidence variables, which are usually observed (save for sensor failures).

Factored MDPs: Actions and Rewards

- What about rewards?
  If the reward remains a function over the complete state space without any factorisation, we have not gained much
- But remember: Multi-attribute utility theory
  - Reward function with preference independence between subsets of random variables
    → additive reward function
    - Factorisation of R:

$$R(S) = R(S_1, \dots, S_n) = \sum_{j=1}^{m} R_j(\boldsymbol{C}_j)$$

      - Best case $R(S_1, \dots, S_n) = \sum_{i=1}^{n} R_i(S_i)$
    - Compare factorisation of T: $T(S', S, A) = P(S'_1, \dots, S'_n | S_1, \dots, S_n, A) = \prod_{i=1}^{n} P(S'_i | \mathrm{pa}(S'_i))$

Factored MDPs: Space Complexity

- With a structured state space, representation size down
  - Given
    - State space with n features and a maximum domain size of d
    - DBN over n features and a maximum domain size of d, with $c = \max_{i \in \{1,\dots,n\}} |\mathrm{pa}(S_i)| + 1$
    - Given action space of size a
  - Space complexity
    - Transition function $T(S', S, A)$: $\mathcal{O}(d^n \cdot a)$ vs. $\mathcal{O}(n \cdot d^c \cdot a)$
    - Reward function $R(S)$: $\mathcal{O}(d^n)$ vs. $\mathcal{O}(n \cdot d^c)$

Solving Factored MDPs

- Bellman equation:

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s' \in \mathrm{dom}(S)} P(s'|a,s)U(s')$$

  - Becomes

$$U(S_1, \dots, S_n) = \sum_{j=1}^{m} R_j(\boldsymbol{C}_j) + \gamma \max_{a \in A(S_1,\dots,S_n)} \sum_{s_1' \in \mathrm{dom}(S_1)} \dots \sum_{s_n' \in \mathrm{dom}(S_n)} \prod_{i=1}^{N} P\left(s_i^{(\tau)} \middle| \mathrm{pa}\left(s_i^{(\tau)}\right)\right) U(s_1', \dots, s_n')$$

- Unfortunately, a factored MDP does not induce a factored value function U
  - One way to go: concentrate on value functions that have a factored representation
    - Approximate the unfactored value function with a factored one

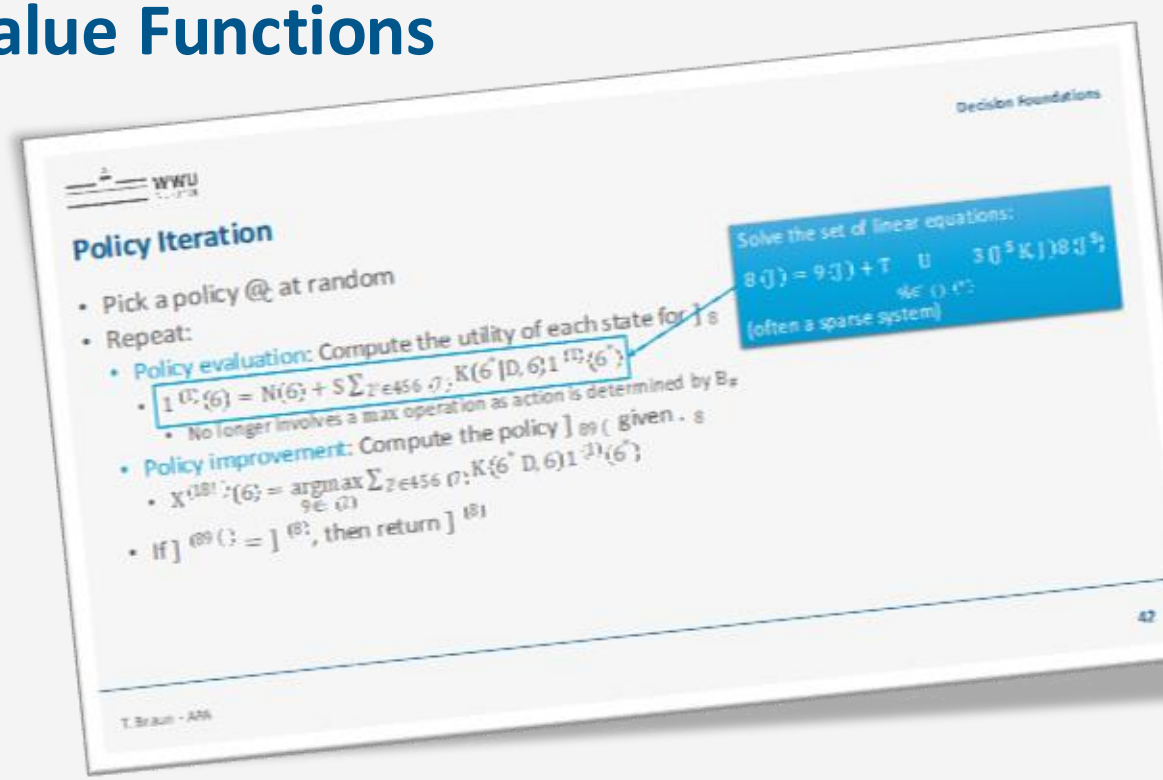
Linear Value Functions

- Linear value function $\mathcal{V}$ over a set of basis functions $H = \{h_1, \dots, h_k\}$
  - Function $\mathcal{V}$ that can be written as $\mathcal{V}(s_1, \dots, s_n) = \sum_{j=1}^{k} w_j \cdot h_j(s_1, \dots, s_n)$ for some coefficients $\mathrm{w} = (w_1, \dots, w_k)'$
    - Let $\mathcal{H}$ be the linear subspace of $\mathbb{R}^n$ spanned by H
    - Let H be an $n \times k$ matrix whose columns are the k basis functions viewed as vectors
    - Then, $\mathcal{V}$ can be written as Hw
  - Equivalent expressive power to, e.g., single layer neural network
    - Features corresponding to the basis functions
    - Optimise the coefficients w to obtain a good approximation for true value function
  - Separates the problem of defining a reasonable space of features and the induced space $\mathcal{H}$, from the problem of searching within the space
    - Former problem is typically purview of domain experts, latter is focus of analysis + algorithmic design

Carlos Guestrin, Daphne Koller, Ronald Parr, and Shobha Venkataraman: Efficient Solution Algorithms for Factored MDPs. In *Journal of Artificial Intelligence Research*, 2003.

Approximate Policy Iteration with Linear Value Functions

- Restrict policy iteration algorithm to only use value functions $\mathcal{V}$ within the provided $\mathcal{H}$
  - Policy improvement as before
  - Policy evaluation changes
    - Whenever policy iteration takes a step that results in a $\mathcal{V}$ outside of $\mathcal{H}$, project result back into $\mathcal{H}$ by finding a value function within $\mathcal{H}$ closest to $\mathcal{V}$
- Projection operator Π
  - Mapping $\Pi : \mathbb{R}^n \rightarrow \mathcal{H}$
  - Π is said to be a projection w.r.t. a norm $\|\cdot\|$ if $\Pi\mathcal{V} = \mathrm{H}\mathrm{w}^*$ such that $\mathrm{w}^* \in \arg\min_{\mathrm{w}} \|\mathrm{H}\mathrm{w} - \mathcal{V}\|$
    - Π is the linear combination of the basis functions that is closest to $\mathcal{V}$ w.r.t. chosen norm



Carlos Guestrin, Daphne Koller, Ronald Parr, and Shobha Venkataraman: Efficient Solution Algorithms for Factored MDPs. In *Journal of Artificial Intelligence Research*, 2003.

Approximate Policy Iteration with Linear Value Functions

- Policy evaluation for a policy $\pi^{(t)}$
  - Value function — the value of acting according to the current policy $\pi^{(t)}$ — is approximated through a linear combination of basis functions
- Given $\pi^{(t)}$, i.e., actions are fixed,
  - $T(S', S, A) = T\left(S', S, \pi^{(t)}\right) = T(S', S)$
- Policy evaluation can be written in terms of matrices and vectors
  - $\mathcal{V}$ and R as n-dimensional vectors and T as an $n \times n$-dimensional matrix, denoted $\mathrm{V}, \mathrm{R}, \mathrm{T}$
  - Then, $\mathcal{V} = \mathrm{R} + \gamma \mathrm{T}\mathcal{V}$
    - System of linear equations with one equation for each state → approximate solution within $\mathcal{H}$:

$$\mathrm{w}^{(t)} = \arg\min_{\mathrm{w}} \|\mathrm{Hw} - (\mathrm{R} + \gamma \mathrm{THw})\| = \arg\min_{\mathrm{w}} \left\|(\mathrm{H} - \gamma \mathrm{TH})\mathrm{w}^{(t)} - \mathrm{R}\right\|$$

      - Problem: How to choose $\|\cdot\|$ wisely, i.e., providing error bounds?

Approximate Policy Iteration with Linear Value Functions

- Convergence and error analysis for MDPs use max-norm ($\mathcal{L}_\infty$)
  → Tie projection operator to $\mathcal{L}_\infty$ norm
- Minimising the $\mathcal{L}_\infty$ norm studied in optimisation literature as the problem of finding the Chebyshev solution to an overdetermined linear system of equations
  - I.e., finding w^* such that $\mathrm{w}^* \in \arg\min_{\mathrm{w}} \|C\mathrm{w} - b\|_\infty$
    - $C = (\mathrm{H} - \gamma \mathrm{TH})$, $b = R$
  - Algorithm due to Stiefel (1960) solves problem by linear programming:
    - Variables: $w_1, \ldots, w_k, \phi;$
    - Minimise: $\phi;$
    - Subject to: $\phi \geq \sum_{j=1}^{k} c_{ij} \cdot w_j - b_i$ and
      $\phi \geq b_i - \sum_{j=1}^{k} c_{ij} \cdot w_j, \quad i = 1, \ldots, n.$
      - At solution (w^*, ϕ^*), w^* is the Chebyshev solution and ϕ^* is the $\mathcal{L}_\infty$ projection error

Only $k + 1$ variables but $2n$ constraints: Impractical in general but in factored MDPs with linear value functions, constraints can be represented efficiently → tractable

Factored Value Functions

- Efficient computation of value function using $h_1, \dots, h_k$ ($\boldsymbol{s} = s_1, \dots, s_n$) using Q value function

$$Q(\boldsymbol{s}, a) = R(\boldsymbol{s}, a) + \gamma \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a)\mathcal{V}(\boldsymbol{s}) = R(\boldsymbol{s}, a) + \gamma \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a) \sum_i w_i h_i(\boldsymbol{s}')$$

  - Define $G(\boldsymbol{s}, a)$ with $g_i(\boldsymbol{s}, a) \coloneqq \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a) h_i(\boldsymbol{s}')$

$$G(\boldsymbol{s}, a) := \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a) \sum_i w_i h_i(\boldsymbol{s}') = \sum_i w_i \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a) h_i(\boldsymbol{s}') = \sum_i w_i g_i(\boldsymbol{s}, a)$$

    - Can compute each basis function separately
- Factored (linear) value function
  - Linear function over the basis set $h_1, \dots, h_k$ where scope of each basis function h_i restricted to some subset of variables $\boldsymbol{C}_i \subset S$
  - Goal: scopes of $h_1, \dots, h_k$ correspond to cliques in graph of DBN representing transition model T

Factored Value Functions: Use in Q Value Function

- Consider $g(\boldsymbol{s}, a) \coloneqq \sum_{\boldsymbol{s}' \in \boldsymbol{S}} P(\boldsymbol{s}'|\boldsymbol{s}, a) h(\boldsymbol{s}') = T_B h$
  - $P(\boldsymbol{s}'|\boldsymbol{s}, a)$ factored as a DBN T_B
  - h has restricted scope over $\boldsymbol{C}$
- Sum over $\boldsymbol{C}'$ conditioned on ancestors $\boldsymbol{R} = \text{anc}(\boldsymbol{C}')$ of $\boldsymbol{C}'$ in T_B

$$g_i(\boldsymbol{s}, a) = \sum_{\boldsymbol{s}' \in \boldsymbol{S}'} P(\boldsymbol{s}'|\boldsymbol{s}, a) h_i(\boldsymbol{s}') = \sum_{\boldsymbol{s}' \in \boldsymbol{S}'} P(\boldsymbol{s}'|\boldsymbol{s}, a) h_i(\boldsymbol{c}')$$

$$= \sum_{\boldsymbol{c}' \in \boldsymbol{C}'} P(\boldsymbol{c}'|\boldsymbol{s}, a) h_i(\boldsymbol{c}') \underbrace{\sum_{\boldsymbol{r}' \in \boldsymbol{S}' \setminus \boldsymbol{C}'} P(\boldsymbol{r}'|\boldsymbol{s}, a)}_{= 1} = \sum_{\boldsymbol{c}' \in \boldsymbol{C}'} P(\boldsymbol{c}'|\boldsymbol{r}, a) h_i(\boldsymbol{c}')$$

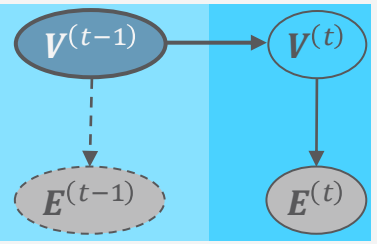
  - Depends on the number of values $\boldsymbol{R}$ can take, which depends on $\boldsymbol{C}'$ and complexity of dynamics represented in T_B, i.e., connectivity of graph B

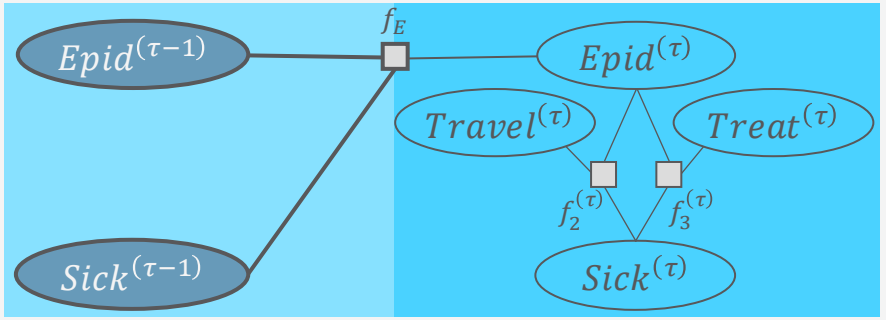
Factored Value Functions: Use in LP with Exponentially Many Constraints

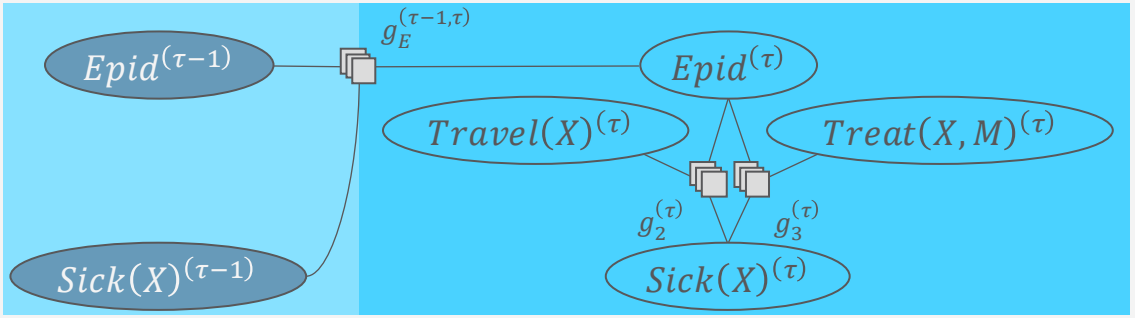
- Constraints of form $\phi \geq \sum_i w_i c_i(\boldsymbol{s}) - b(\boldsymbol{s}), \forall \boldsymbol{s} \in \boldsymbol{S}$
  - $\phi, w_1, \ldots, w_k$ free variables
  - $\boldsymbol{s}$ ranges over all states
- Can be replaced by one equivalent non-linear constraint $\phi \geq \max_{\boldsymbol{s}} \sum_i w_i c_i(\boldsymbol{s}) - b(\boldsymbol{s})$
  - Tackle problem of representing non-linear constraint by
    - Computing maximum assignment for a fixed set of weights
      - Simpler problem: Given fixed weights w_i, compute $\phi^* = \max_{\boldsymbol{s}} \sum_i w_i c_i(\boldsymbol{s}) - b(\boldsymbol{s})$
    - Representing non-linear constraint by small set of linear constraints using a construction called factored LP

Relations and Symmetries in a Factored State Space

- Can add additional assumptions about relations and symmetries in the DBN
  - DBN may describe objects and relations among them
  - Objects may be considered indistinguishable
- Possible to group (and then count) objects in the transition and reward functions
  - Basis function for each group







Interim Summary: *Structure by Features in the State Space*

- State space characterised by set of attributes
  - (Conditional) independences allow for factorisation of functions in MDP
  - Probabilistic graphical models represent such factorisations
- Factored MDP: MDP with a DBN as a representation of the transition model
  - Reduction in space complexity
  - Factored transition function does not lead to factored value function
- Factored (linear) value functions over a set of basis functions
  - Enable computing policy evaluation efficiently
- Approximate policy iteration
  - Project results outside of subspace spanned by basis functions back into subspace

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- Partitioned decPOMDPs

Structure by Relations in the State Space

- Situation calculus
- First-order MDPs

Structure by Features in the State Space

- Dynamic Bayesian networks
- Factored MDPs

$\Longrightarrow$ Next: Human-awareness