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Statistical Relational AI

Exploiting Symmetries

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Agenda

1. Introduction [Tanya]
2. **Exploiting Symmetries in Probabilistic Graphical Models [Marcel]**
    - Identifying Most Likely Sources of Events
    - Introducing Time
    - Approximating Symmetries over Time to Keep Reasoning Polynomial
3. Exploiting Symmetries in Conditional Knowledge Bases [Marco]
4. Summary [Tanya]

Who did it? – Identifying the Most Likely Sources of Events

Using Symmetries in Inference for a new Query Type

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Who did it? – Identifying the Most Likely Sources of Events

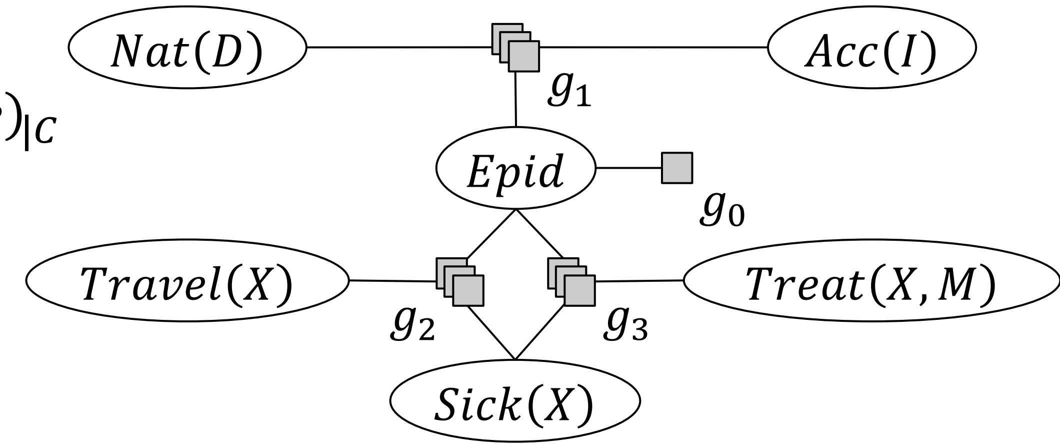
- Under-specified evidence, i.e., set of observations, without a known source: $R(X) = r$
  - $Sick(X') = true, |dom(X')| = 1000$
  - NB: Not possible in propositional setting as an observation can only belong to a specific random variable without additional information about relations or types
- Optimisation problem for a single logical variable in the evidence:
  Given evidence $\boldsymbol{e}$ with known source, find a domain C for X such that the probability of the evidence without a source is maximal under the domain, written as

$$\arg\max_{C} P(R(X) = r|\boldsymbol{e})_{|C}$$

  - Use C as source for evidence
  - Example from above: If no further evidence and only one group of indistinguishable instances represented by X in model, then any 1000 instances represented by X will do

Marcel Gehrke, Ralf Möller, and Tanya Braun. Who did it? Identifying the Most Likely Origins of Events. In *PGM-22 Proceedings of the 11th* International Conference on Probabilistic Graphical Models, 2022.

Who did it? – Combinatorial Problem

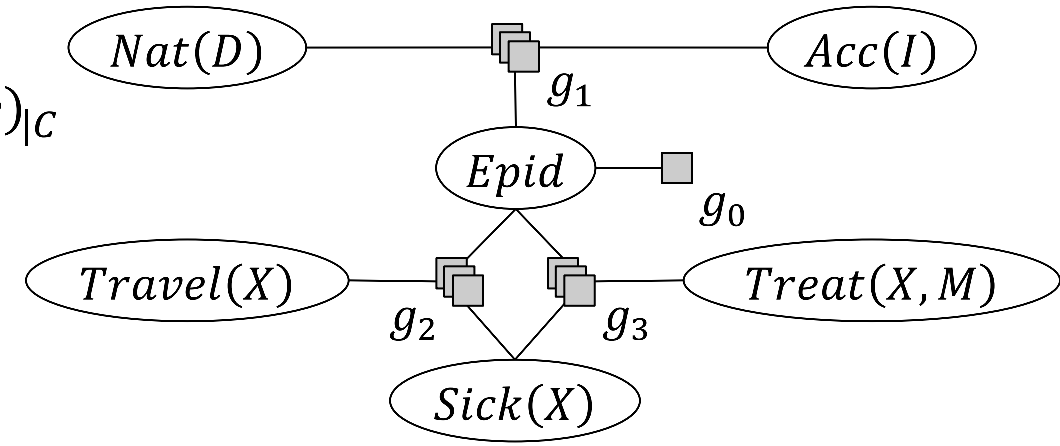


- Optimisation problem:

$$\arg\max_{C} P(R(X) = r|\boldsymbol{e})_{|C}$$

- What if we have under-specified evidence for two different PRVs
  - $Travel(X) = true, |dom(X)| = 10$
  - $Sick(Y) = true, |dom(Y)| = 10$
    - The same 10 constants exhibit both observations at the same time (overlapping) and
    - that 10 constants exhibit $DoR(X) = true$ and 10 other constants exhibit $Rep(X) = true$ (disjoint)
  - Assuming we could identify the corresponding propositional random variables and $|\mathcal{D}(X)| = 100$ in our model
    - $\binom{100}{10} \approx 3 \cdot 10^{26}$ possibilities

Who did it? – Adding Constraint Sets

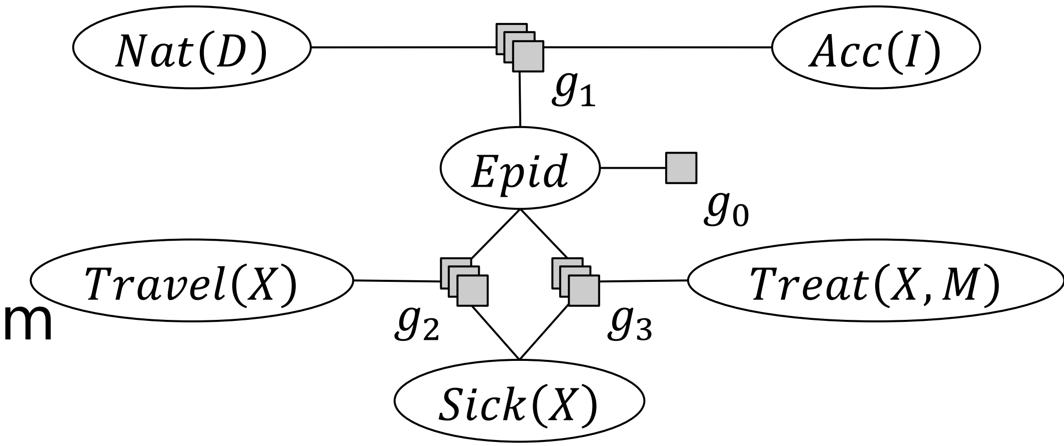


- Optimisation problem:

$$\arg\max_{C} P(R(X) = r \mid \boldsymbol{e})_{|C}$$

- However, it gets complicated once you have more sets of unknown sources or sets of known sources to consider as well
  - $Travel(X) = true, |dom(X)| = 1000$
  - $Treat(Y, m1) = true, |dom(Y)| = 500$
  - $Sick(Z) = true, dom(Z) = \{x_1, \dots, x_{100}\}$
  - Various domain assignments possible from full overlap to complete disjoint sets
    - $dom(Z) = dom(Y') = dom(X') = \{x_1, \dots, x_{100}\}$, $dom(Y'') = dom(X'') = \{x_{101}, \dots, x_{500}\}$, $dom(X''') = \{x_{501}, \dots, x_{1000}\}$
    - …
    - $dom(Z) = \{x_1, \dots, x_{100}\}$ $dom(Y) = \{x_{101}, \dots, x_{600}\}$ $dom(X) =$ $\{x_{601}, \dots, x_{1600}\}$

Who did it? – Adding even more Constraint Sets



- Number of (distinguishable) events (from known and unknown sources) influence the number possibilities
  - # of balls and urns of combinatorial problem depend on # of distinguishable events
  - Remains a combinatorial problem even with indistinguishability
- In practice number of different events limited
  - Groups of indistinguishable individuals assumed to behave approximately the same
  - No contradicting events for one constant
  - Computable in many practical cases

Interim Summary

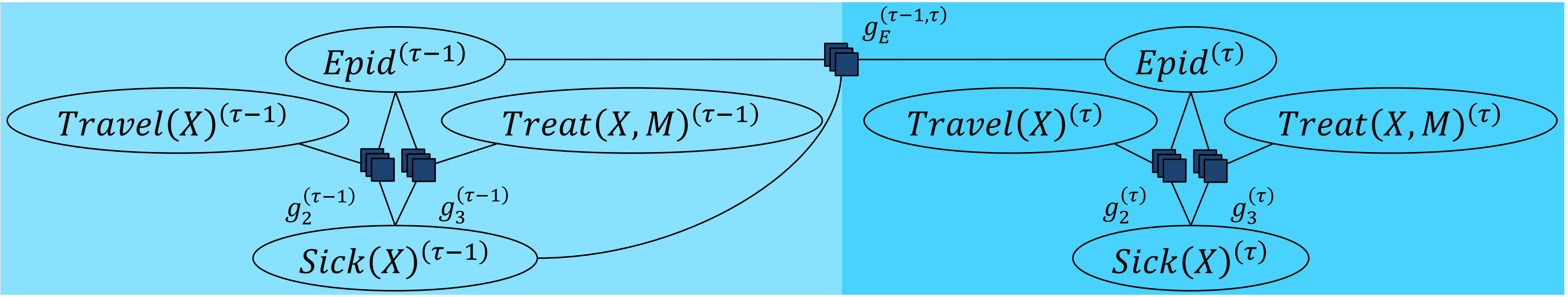
- New type of query based on logical variables
- Identifying the most likely sources of events is a combinatorial problem
  - Having indistinguishability allows us to tackle the combinatorial problem (fewer urns and balls to consider)
  - With a propositional representation not applicable
  - Without indistinguishability already for small domain sizes too many possibilities

Temporal Lifted Inference

Using Symmetries in Temporal Inference

Dynamic Parfactor-based Model

- Marginal distribution query: $P(A_\pi^i \mid E_{0:t})$ w.r.t. the model:
  - Hindsight: $\pi < t$ (Was there an epidemic $t - \pi$ days ago?)
  - Filtering: $\pi = t$ (Is there currently an epidemic?)
  - Prediction: $\pi > t$ (Is there an epidemic in $\pi - t$ days?)
- MPE, MAP on temporal sequence



2-slice Model & 1.5-slice Model

Partial specification of a sequential model (without $P(\boldsymbol{R}^{(0)})$)

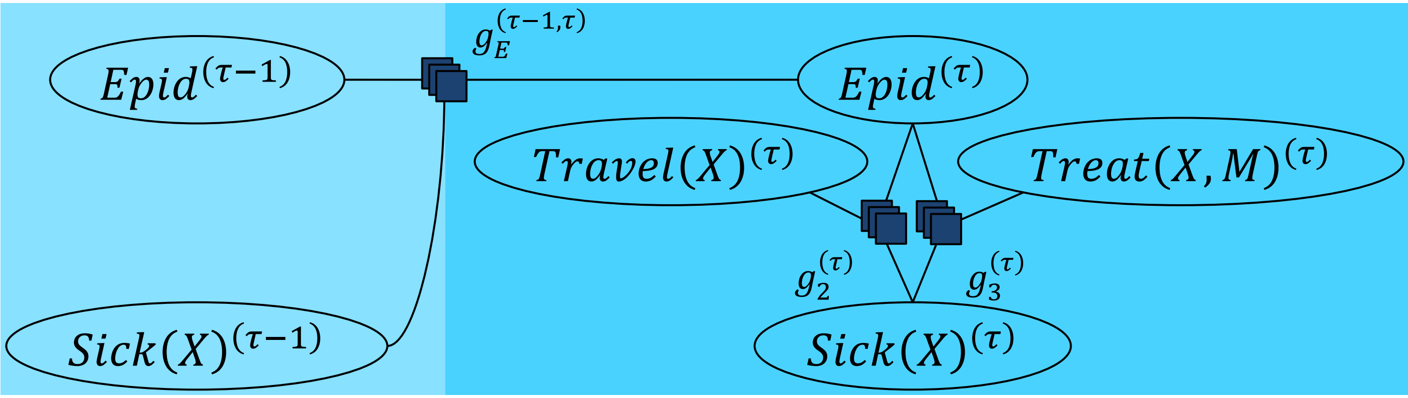
- 2-slice model (in effect only a 1.5-slice model)
  - Specification of a transition model $P(\boldsymbol{R}^{(\tau)} | \boldsymbol{R}^{(\tau-1)})$ based on a factorisation
    - Template model that gets instantiated by replacing τ with a (time) step t
    - Sufficient: Intra-slice description for τ, Inter-slice description from $\tau - 1$ to τ
  - For parfactor-based models: $G^{\rightarrow} = \left\{g_i^{(\tau)}\right\}_{i=1}^{n} \cup \left\{g_j^{(\tau-1,\tau)}\right\}_{j=1}^{m}$ with
    - Intra-slice parfactors

$$g_i^{(\tau)} = \phi_i^{(\tau)}\left(A_1^{(\tau)}, \dots, A_{k_i}^{(\tau)}\right)_{|C_i^{(\tau)}}$$

    - Inter-slice parfactors

$$g_j^{(\tau-1,\tau)} = \phi_j^{(\tau-1,\tau)}\left(A_1^{(\nu)}, \dots, A_{k_j}^{(\nu)}\right)_{|C_j^{(\tau-1,\tau)}}$$

      - $\nu \in \{\tau, \tau - 1\}$

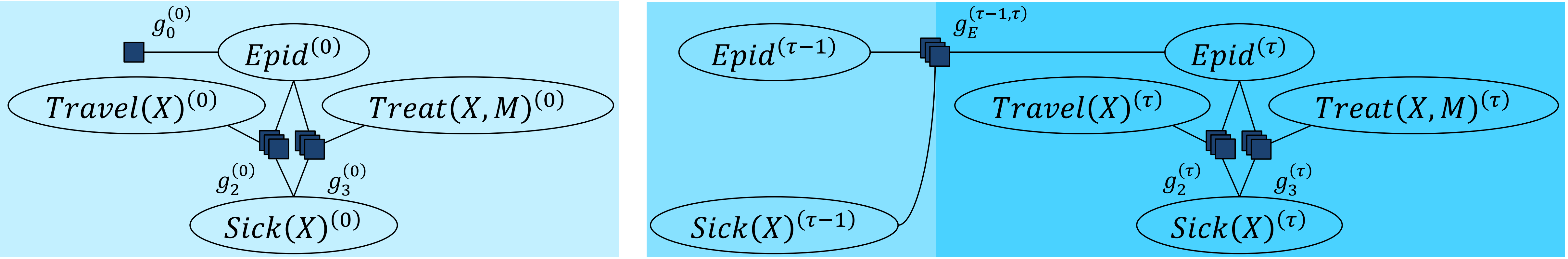


Dynamic Parfactor-based Model

- Dynamic parfactor-based model $(G^0, G^{\rightarrow})$ with
  - $G^0 = \left\{g_i^{(0)}\right\}_{i=1}^{n_0}, g_i^{(0)} = \phi_i^{(0)}\left(\mathcal{A}_1^{(0)}, \dots, \mathcal{A}_{k_i}^{(0)}\right)$
  - $G^{\rightarrow}$ a 2-slice model, i.e., $G^{\rightarrow} = \left\{g_i^{(\tau)}\right\}_{i=1}^{n} \cup \left\{g_j^{(\tau-1,\tau)}\right\}_{j=1}^{m}$
  - Example: $G^0 = \left\{g_2^{(0)}, g_3^{(0)}\right\} \cup \left\{g_0^{(0)}\right\}$, $G^{\rightarrow} = \left\{g_2^{(\tau)}, g_3^{(\tau)}\right\} \cup \left\{g_E^{(\tau-1,\tau)}\right\}$
    - No assumptions about observability

G^0 often consists of intra-slice parfactors of $G^{\rightarrow}$ with $\tau = 0$, possibly extended by pseudo prior distributions, i.e.,

$$\left\{g_i^{(0)}\right\}_{i=1}^{n_0} = \left\{g_i^{(\tau)|\tau=0}\right\}_{i=1}^{n} \cup \left\{g_i^{(0)}\right\}_{i=1}^{n'}$$



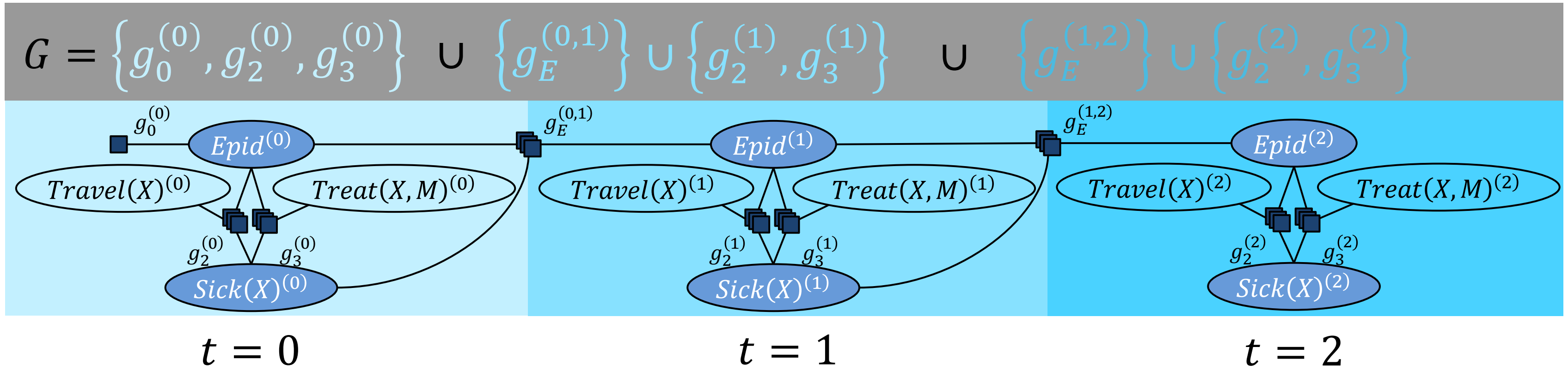
Sequential Models

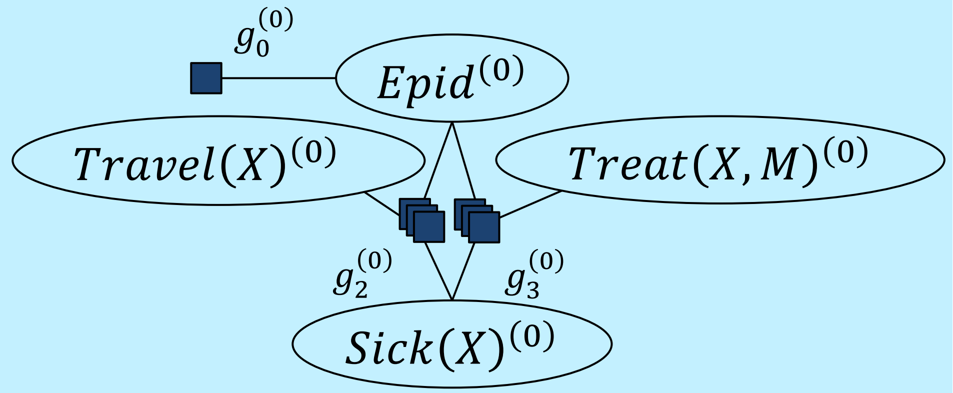
- Assumptions: Markov-1, stationary process

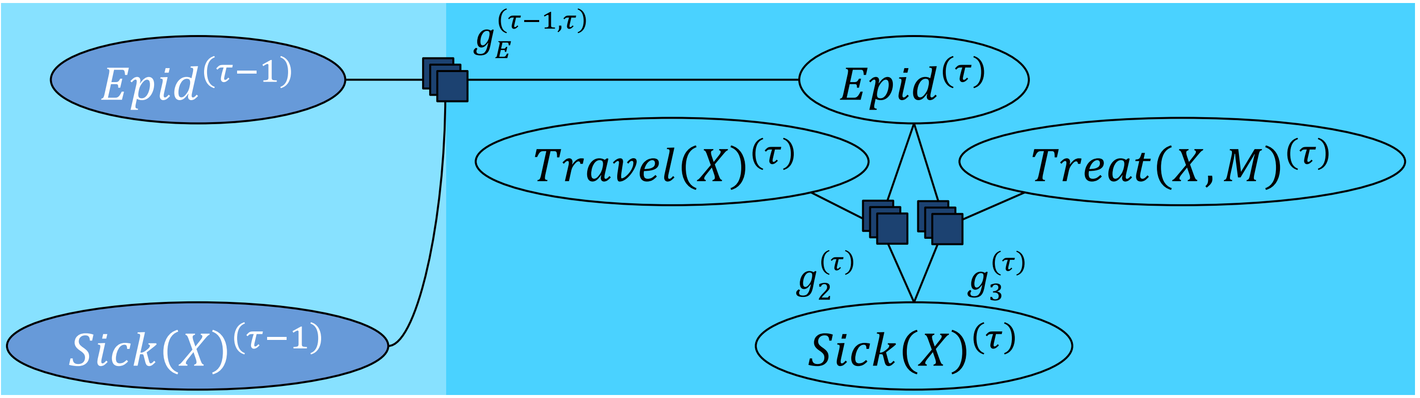
- Typical definition: Sequential model is a tuple $(M^0, M^\rightarrow)$, where
  - M^0 describes the behaviour at step 0 (only intra-slice behaviour as no previous step)
  - $M^\rightarrow$ is a 2-slice model describing intra- and inter-slice behaviour
  - Existing model types: dynamic BNs, HMMs, dynamic parfactor models, dynamic MLNs, …
- Frequent connection between $M^0, M^\rightarrow$
  - In factor-based models: Intra-slice behaviour in $M^\rightarrow$ for step τ equal to behaviour in M^0, i.e., intra-slice parfactors in $M^\rightarrow$ with $\tau = 0$
  - M^0 may also contain prior information encoded in distributions
    - In BN-based models: prior distributions in M^0

Independences in Dynamic Models

- *Observation*:
  All paths from one step to the next go through the PRVs $A^{(\tau-1)}$ of the inter-slice parfactors $g^{(\tau-1,\tau)}$ in $M^{\rightarrow}$

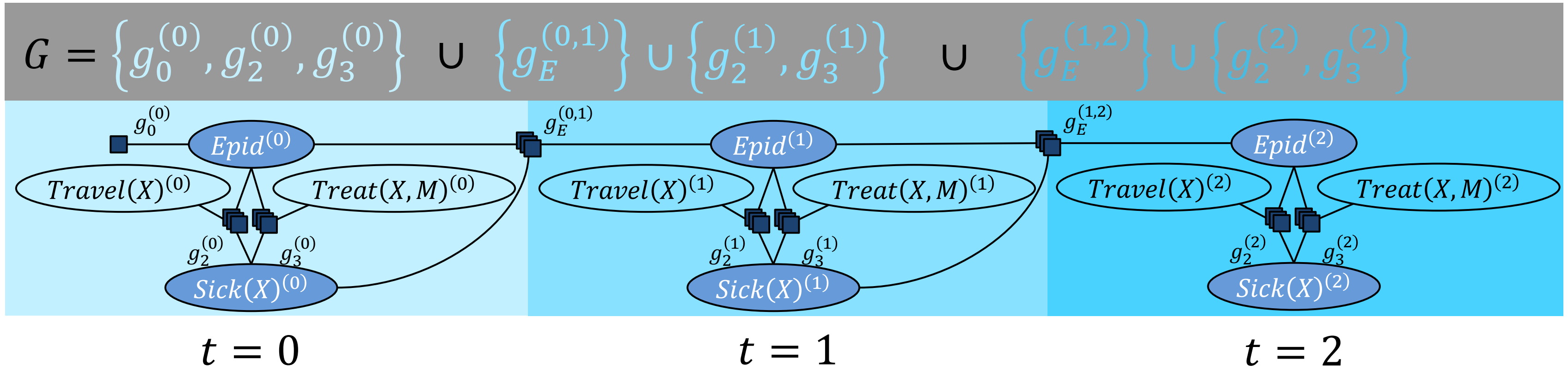


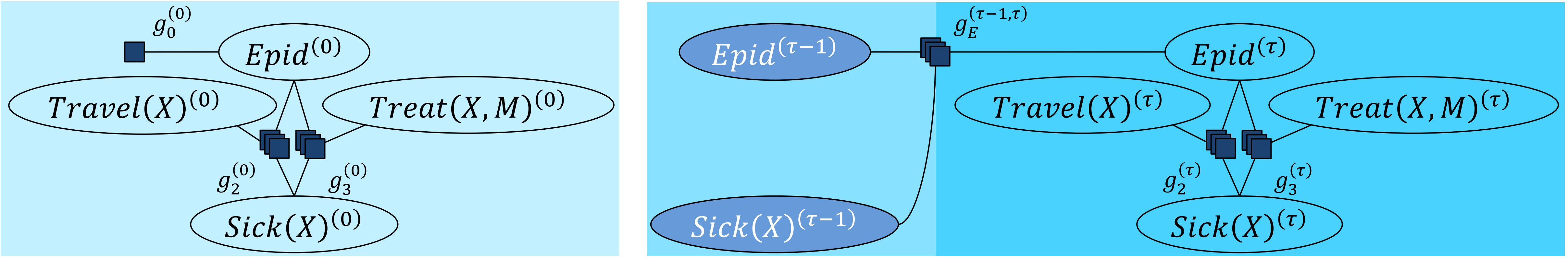




Independences in Dynamic Models

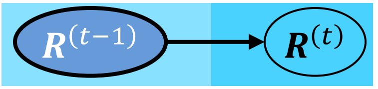
- Fact: Conditioning on these PRVs makes a current step independent of the previous steps
  - Collect information on these PRVs as a message before progressing in time

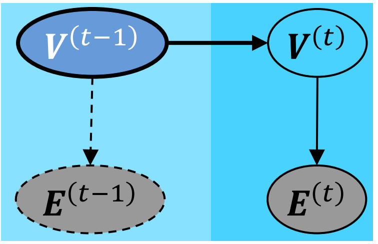


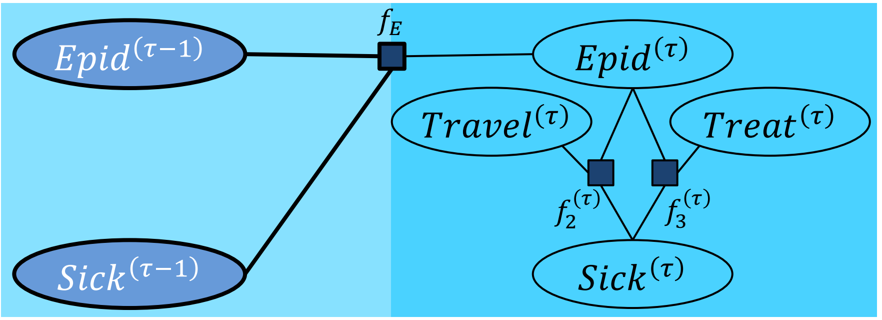


Interfaces

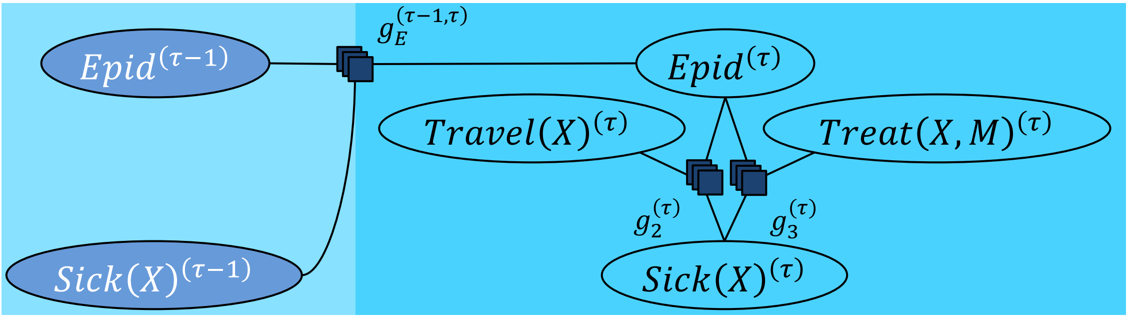
- Forward interface $\boldsymbol{I}^{(\tau-1)}$:
  Separating subset between $M^{(\tau-1)}$ and $M^{(\tau)}$
  - Trivial separators, usually not minimal
    - Maximal interface: $\boldsymbol{R}$
    - If $\boldsymbol{R} = \boldsymbol{V} \cup \boldsymbol{E}$: maximal interface $\boldsymbol{V}$
  - Set of PRVs with index $\tau - 1$ in $M^{\rightarrow}$
    - $\boldsymbol{I}^{(\tau-1)} = \{A^{(\tau-1)} | A^{(\tau-1)} \in \mathrm{rv}(M^{\rightarrow})\}$
      - Can only appear in inter-slice parfactors or MLN formulas







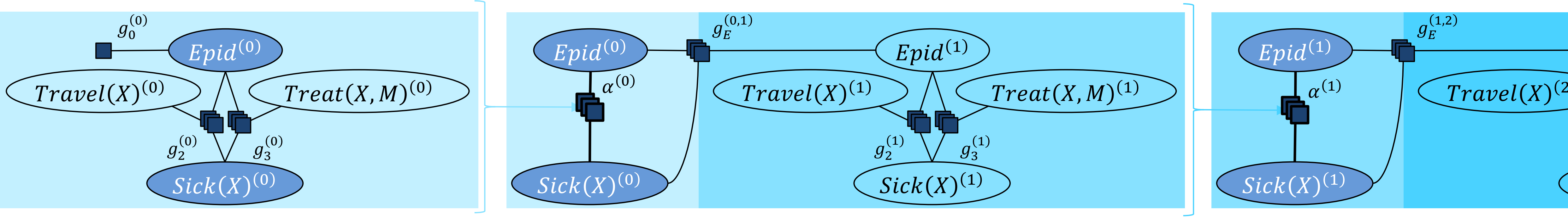
$\boldsymbol{I}$ also separates backwards from $M^{(\tau)}$ to $M^{(\tau-1)}$;
there is also an explicit *backward interface*,
which is determined in direction of τ to $\tau - 1$
[More details in Kevin Murphy's dissertation, 2002]



Interfaces To Move Through Time

- When inference for step t done
  - Query for $\boldsymbol{I}^{(t)}$ in current model $M^{(t)}$
    - Comparable to messages in FO jtrees: Calculate message $\alpha^{(t)}$ over separator $\boldsymbol{I}^{(t)}$
  - ...model $M^{(t+1)}$
    - ...$^{1} \cup \alpha^{(t)}$
    - ...connection to $M^{(0:t)}$, as all information of $M^{(0:t)}$ encoded in $\alpha^{(t)}$
- Use FO jtrees and LJT for moving through time

Helper structure and lifted inference algorithm to answer multiple queries

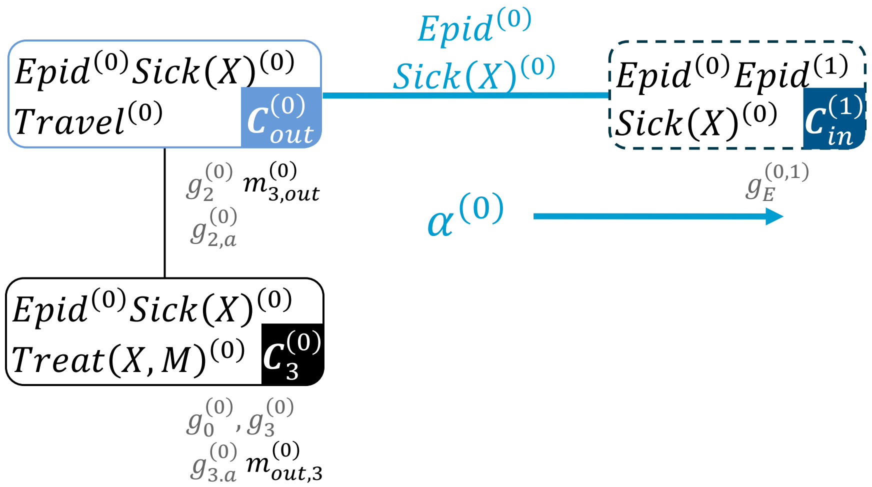


Progressing in Time: Example

- $t = 0$
  - Current FO jtree
  - Enter evidence $\boldsymbol{e}^{(0)} = \{sick(alice)^{(0)}\}$
  - Send intra-slice messages: $m^{(0)}_{3,out}$, $m^{(0)}_{out,3}$
  - Answer queries $P\left(R^{(0)}_i \middle| \boldsymbol{e}^{(0)}\right)$
- Inter-slice message: $\alpha^{(0)}$
  - Eliminate all non-separator PRVs, here $Travel(X)^{(0)}$, from the local model $G^{(0)}_{out} = \{g^{(0)}_2, g^{(0)}_{2,a}\}$ and $m^{(0)}_{3,out}$
  - Send result as message $\alpha^{(0)}$ to $\boldsymbol{C}^{(1)}_{in}$

$\alpha^{(0)}$ contains all information from $G^{(0)}$ including $\boldsymbol{e}^{(0)}$, which makes slice 0 and 1 independent.

Next: Instantiate an FO Jtree for $\tau = 1$ and add $\alpha^{(0)}$ to the local model of $\boldsymbol{C}^{(1)}_{in}$.

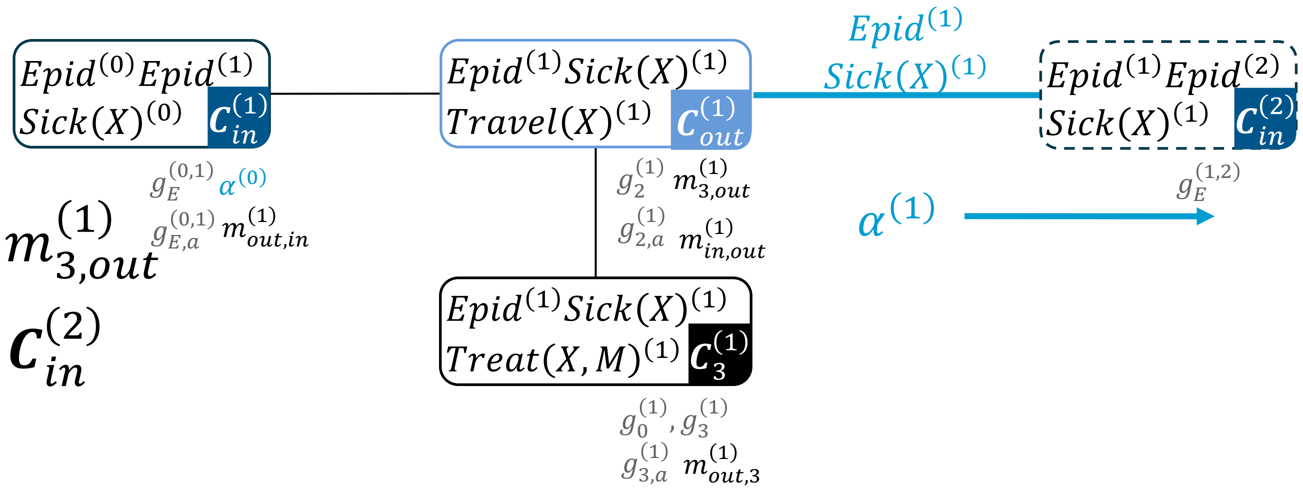


Progressing in Time: Example

- $t = 1$
  - Current FO jtree with $\alpha^{(0)}$
  - Enter evidence $\boldsymbol{e}^{(1)} = \{sick(alice)^{(1)}\}$
  - Send intra-slice messages: $m_{3,out}^{(1)}, m_{out,3}^{(1)}, m_{in,out}^{(1)}, m_{out,in}^{(1)}$
  - Answer queries $P\left(R_i^{(1)} \middle| \boldsymbol{e}^{(0:1)}\right)$
- Inter-slice message: $\alpha^{(1)}$
  - Eliminate $Travel(X)^{(1)}$ from $G_{out}^{(1)} = \{g_2^{(1)}, g_{2,a}^{(1)}\}$, $m_{in,out}^{(1)}$, and $m_{3,out}^{(1)}$
  - Send result as message $\alpha^{(1)}$ to $\boldsymbol{C}_{in}^{(2)}$

The information in $\alpha^{(0)}$ is distributed during the intra-slice message passing and therefore also included in $\alpha^{(1)}$. $\alpha^{(1)}$ now contains all information from $G^{(0:1)}$ including $\boldsymbol{e}^{(0:1)}$, which makes slice 1 and 2 independent.

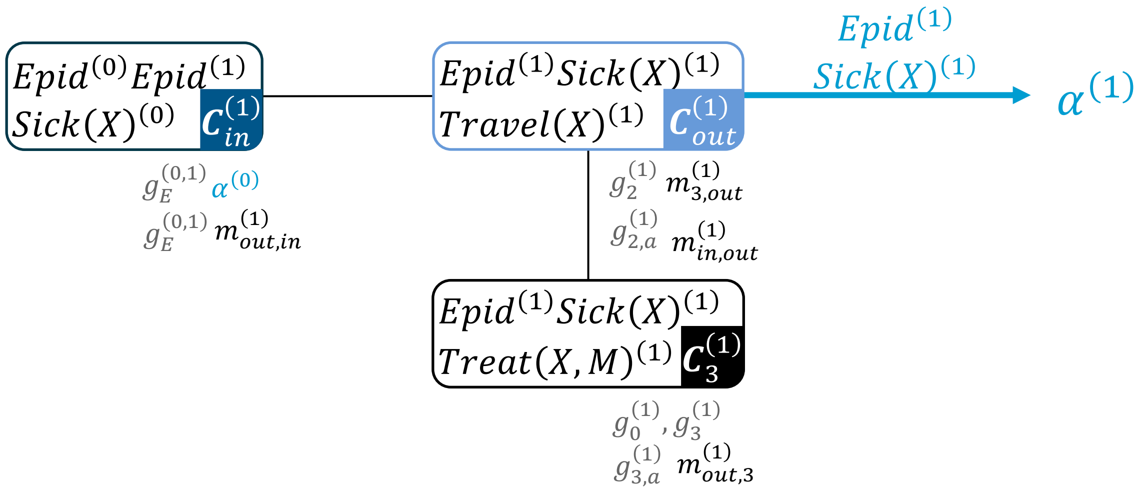
Next: Instantiate an FO Jtree for $\tau = 2$ and add $\alpha^{(1)}$ to the local model of $\boldsymbol{C}_{in}^{(2)}$.



Filtering Queries

- Queries one can answer in this way: *Filtering* queries
  - $P\left(R^{(t)} \middle| \boldsymbol{e}^{(0:t)}\right)$
  - Queries for random variables / grounded instances of PRVs / parameterised queries of the current slice t given all evidence $\boldsymbol{e}^{(0:t)}$ up until t
  - Advantages
    - Only a current FO jtree necessary
    - One additional message ($\alpha^{(t)}$) to progress in time

- What about *prediction* and *hindsight* queries?
  - $P\left(R^{(\pi)} \middle| \boldsymbol{e}^{(0:t)}\right)$, $\pi \neq t$



Complexity

- LJT complexity for message passing and query answering

$$O_{MP} = O\left(n_J \cdot \log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$$

$$O_{QA} = O\left(\log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$$

Lifted calculations Largest parfactor size

  - n_T, n_J number of nodes in FO dtree/jtree
  - n largest domain size
  - r largest range size, $r_\#$ largest range size of a PRV in a CRV
  - w_g largest ground width
  - $w_\#$ largest counting width
- LDJT: Moving forward in time
  - Uses message passing within each time step → O_{MP}
  - Uses LJT query answering for inter-slice message → O_{QA}

Complexity

- Given a maximum number T occurring over all M queries ($M = \sum_{t=0}^{T} m_t$, $m = \frac{1}{T}\sum_{t=0}^{T} m_t$)
  - Moving forward: $T \cdot \left(O_{MP} + O_{QA}\right)$
  - *Best* worst case at a time step $\tau = \tau' \in \{0, \dots, T\}$
    - Filtering query for τ', each with O_{QA}
  - Overall,

$$T \cdot \left(O_{MP} + O_{QA}\right) + M \cdot O_{QA} = \left(T \cdot n_J + T + M\right) \cdot O\left(\log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$$

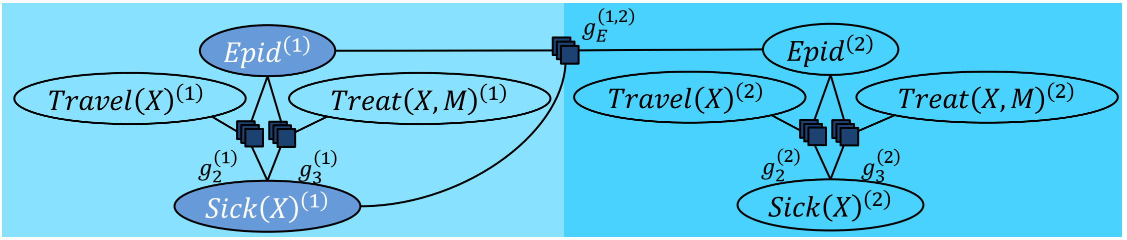
$$= O\left(\left(T \cdot n_J + M\right) \cdot \log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$$

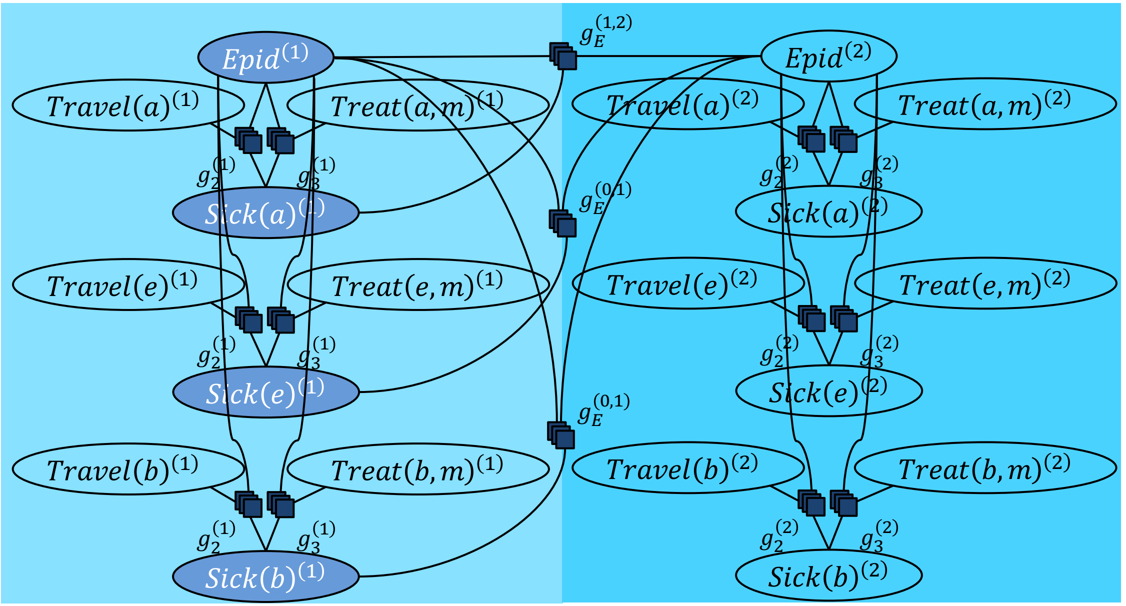
$$= O\left(\left(T \cdot n_J + T \cdot m\right) \cdot \log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$$

Compare LJT complexity
$O\left(\left(n_J + m\right) \cdot \log_2(n) \cdot r^{w_g} \cdot n^{r_\# w_\#}\right)$

Comparison to Ground Inference

- Earning of LVE vs. VE
  - $n_{gr(T)} \gg n_T$ (with large domains)
  - $w \gg (w_g + w_\#)$ (with count conversions)
    - Without count conversions, $w = w_g$, $w_\# = 0$
- In sequential case,
  - Earnings because of lifting the interface
    - Even without count conversions, $w \gg w_g$
  - Example: Grounding with
    - $\mathrm{dom}(X) = \{a, e, b\}$
    - $\mathrm{dom}(M) = \{m\}$





Interim Summary

- Interfaces to separate past from present and present from future
  - Inter-slice messages
    - Forward message to propagate all information up to current slice to next slice
    - Backward message to propagate all information down to current slice to previous slice
- LDJT algorithm
  - LJT algorithm for intra-slice inference
  - Inter-slice messages to move in time
  - Reduced complexity in terms of lifted interface

Temporal Lifted Inference

Keeping Symmetries in Temporal Inference

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The Problem with Evidence

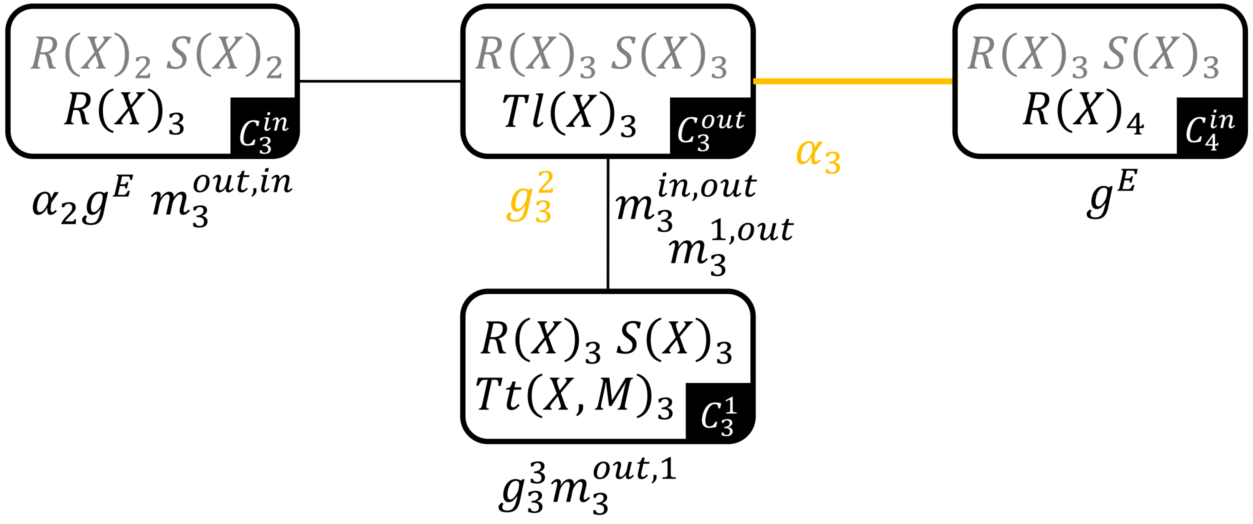
- Evidence can ground a model over time
- Non-symmetric evidence
  - Observe evidence for some instances in one time step
  - Observe evidence for a subset of these instances in another time step
  - Splits a logical variable slowly over time
- Vanilla junction trees for each time step
  - Without any splits
- Forward message carries over splits, leading to slowly grounding a model over time

Marcel Gehrke, Ralf Möller, and Tanya Braun. Lifted Taming Reasoning in Temporal Probabilistic Relational Models. In *ECAI-20 Proceedings of the 24th European Conference on Artificial Intelligence*, 2020.

Evidence over Time

- Slight variation of example
  - Replace $Epid_t$ with $R(X)_t$
- Evidence: $Tl_3(x_1) = true$
  - Split g_3^2 into
    - $g_3^{2=1}$ for x_1 and
    - $g_3^{2\neq 1}$ for $X \neq x_1$

- α_3 consists of
  - $m_3^{in,out}$
  - $m_3^{1,out}$
  - $g_3^{2=1}$ and $g_3^{2\neq 1}$ with $Tl_3(X)$ eliminated

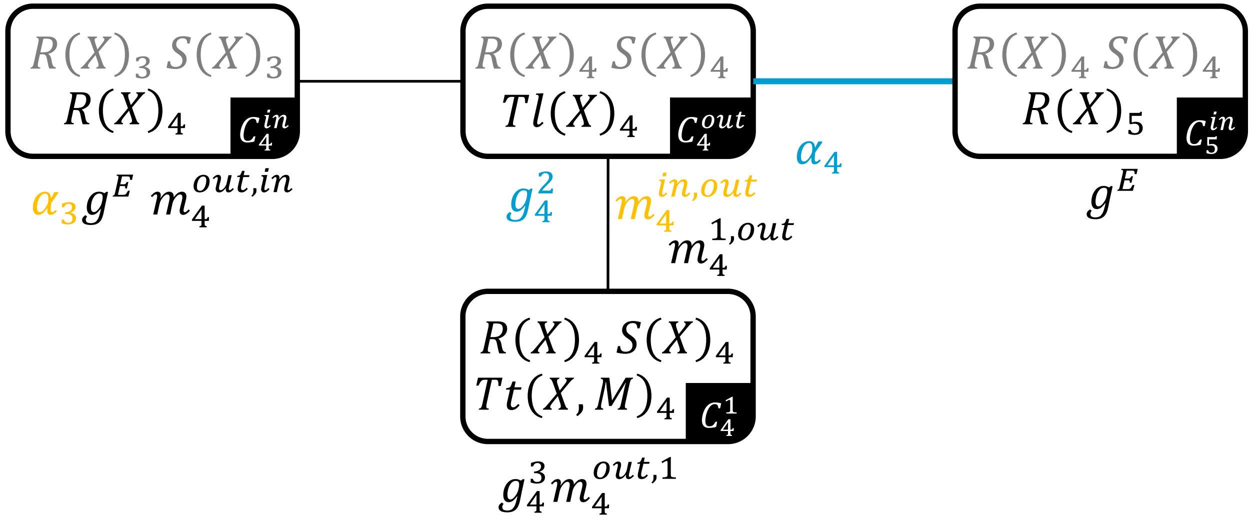


Evidence over Time

- Next step $\tau = 4$
- Evidence: $Tl_4(x_2) = true$
  - Split g_4^2 into
    - $g_4^{2=2}$ for x_2 and
    - $g_4^{2\neq 2}$ for $X \neq x_2$

X is slowly grounded

- α_3 contains
  - $g_3^{2=1}$ and $g_3^{2\neq 1}$ with $Tl_3(X)$ eliminated
  - For $m_4^{in,out}$, X is split w.r.t. x_1
- In α_4, X is split w.r.t. x_1 and x_2



Undoing Splits

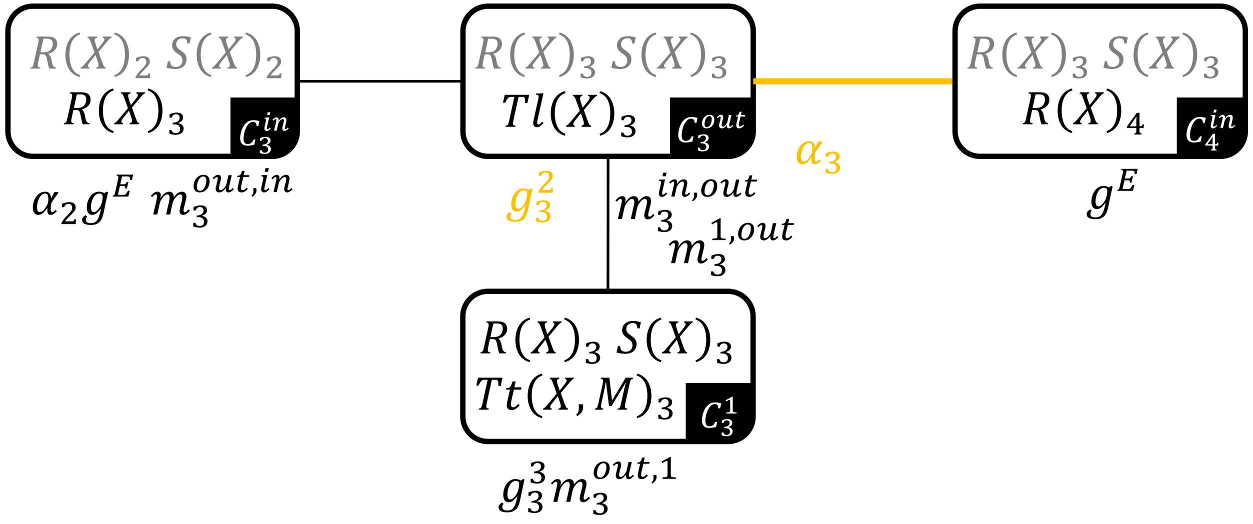
- Need to undo splits to

  keep reasoning polynomial w.r.t. domain sizes

1. Where can splits be undone efficiently?
2. How to undo splits?
3. Is it reasonable to undo splits?
   - Effect of slight differences in evidence?
   - Impact of evidence vs. temporal behaviour of model?

1. Where Can Splits Be Undone Efficiently?

- Evidence causes splits in a logical variable in the same way in all factors in a model
- LDJT always instantiates a vanilla junction tree
- Forward message carries over splits



2. How to Undo Splits?

- The colour passing algorithm can efficiently identify exact symmetries
- But
  - Evidence causes differences in distributions

- Need to *find* approximate symmetries to undo splits caused by evidence
- Need a way to *merge* factors

Parag Singla and Pedro Domingos: Lifted First-order Belief Propagation. In *AAAI-08 Proceedings of the 23rd AAAI Conference on Artificial Intelligence*, 2008.
Kristian Kersting, Babak Ahmadi, and Sriraam Natarajan. Counting Belief Propagation. In *UAI-09 Proceedings of the 25th Conference on Uncertainty in Artificial Intelligence*, 2009.
Babak Ahmadi, Kristian Kersting, Martin Mladenov, and Sriraam Natarajan. Exploiting Symmetries for Scaling Loopy Belief Propagation and Relational Training. In *Machine Learning*, 92(1):91-132, 2013.

Comparing Parfactors: Approaches

- Comparing all marginals is expensive
- Comparing the joint distribution over the complete interface is expensive
- Comparing marginals of a *subset* of PRVs can determine non-similar factors similar
  - E.g.,
    - $\mathrm{dom}(X) = \{x_1\}$

$R(X)$	$S(X)$	g
$false$	$false$	0
$false$	$true$	7
$true$	$false$	4
$true$	$true$	1

$R(X)$	$S(X)$	g
$false$	$false$	2
$false$	$true$	4
$true$	$false$	2
$true$	$true$	4

    - $P(S(x_1) = true)$: $\frac{2}{3}$ $\frac{2}{3}$
    - $P(R(x_1) = true)$: $\frac{5}{12}$ $\frac{1}{2}$

Comparing Parfactors

- Potentials determine distributions
- Similar ratios in potentials lead to similar marginals and similar factors
  - E.g.,
    - $\mathrm{dom}(X) = \{x_1\}$

$R(X)$	$S(X)$	g
$false$	$false$	4
$false$	$true$	3
$true$	$false$	2
$true$	$true$	1

$R(X)$	$S(X)$	g
$false$	$false$	3.9
$false$	$true$	3.1
$true$	$false$	2.1
$true$	$true$	0.9

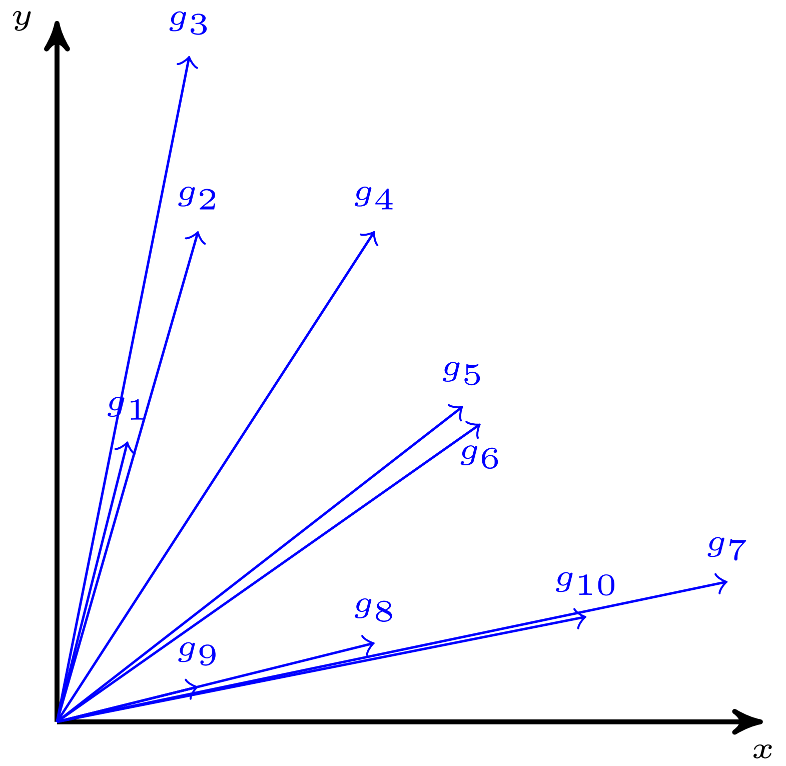
    - $P(S(x_1) = true)$: $\frac{4}{10}$ $\frac{4}{10}$
    - $P(R(x_1) = true)$: $\frac{3}{10}$ $\frac{3}{10}$
    - $P(S(x_1) = true, R(x_1) = true)$: $\frac{1}{10}$ $\frac{0.9}{10}$

Identifying Similar Groups

- Groups are equal if they have the same full joint distribution
- Full joint distribution computationally hard to get

→ Use parfactors as vector

→ If vectors of two groups point in same direction, they have a similar full joint distribution



Find Approximate Symmetries

- Cosine similarity for similarity of vectors

$$\cos(\theta) = \frac{\sum_{i=1}^{n} A_i \cdot B_i}{\sqrt{\sum_{i=1}^{n} A_i^2} \cdot \sqrt{\sum_{i=1}^{n} B_i^2}}$$

- E.g.,

$Tl(X)$	$S(X)$	g
$false$	$false$	0
$false$	$true$	7
$true$	$false$	4
$true$	$true$	1

$Tl(X)$	$S(X)$	g
$false$	$false$	2
$false$	$true$	4
$true$	$false$	2
$true$	$true$	4

  – $\cos(\theta) = \frac{0 \cdot 2 + 7 \cdot 4 + 4 \cdot 2 + 1 \cdot 4}{\sqrt{0+49+16+1} \cdot \sqrt{4+16+4+16}} \sim 0.7785$

Find Approximate Symmetries

- Cosine similarity for similarity of vectors

$$\cos(\theta) = \frac{\sum_{i=1}^{n} A_i \cdot B_i}{\sqrt{\sum_{i=1}^{n} A_i^2} \cdot \sqrt{\sum_{i=1}^{n} B_i^2}}$$

- E.g.,

$Tl(X)$	$S(X)$	g
$false$	$false$	4
$false$	$true$	3
$true$	$false$	2
$true$	$true$	1

$Tl(X)$	$S(X)$	g
$false$	$false$	3.9
$false$	$true$	3.1
$true$	$false$	2.1
$true$	$true$	0.9

  – $\cos(\theta) = \frac{4 \cdot 3.9 + 3 \cdot 3.1 + 2 \cdot 2.1 + 1 \cdot 0.9}{\sqrt{16+9+4+1} \cdot \sqrt{15.21+9.61+4.41+0.81}} \sim 0.9993$

Find Approximate Symmetries

- Cosine similarity for similarity of vectors

$$\cos(\theta) = \frac{\sum_{i=1}^{n} A_i \cdot B_i}{\sqrt{\sum_{i=1}^{n} A_i^2} \cdot \sqrt{\sum_{i=1}^{n} B_i^2}}$$

- E.g.,

$Tl(X)$	$S(X)$	g
$false$	$false$	4
$false$	$true$	3
$true$	$false$	2
$true$	$true$	1

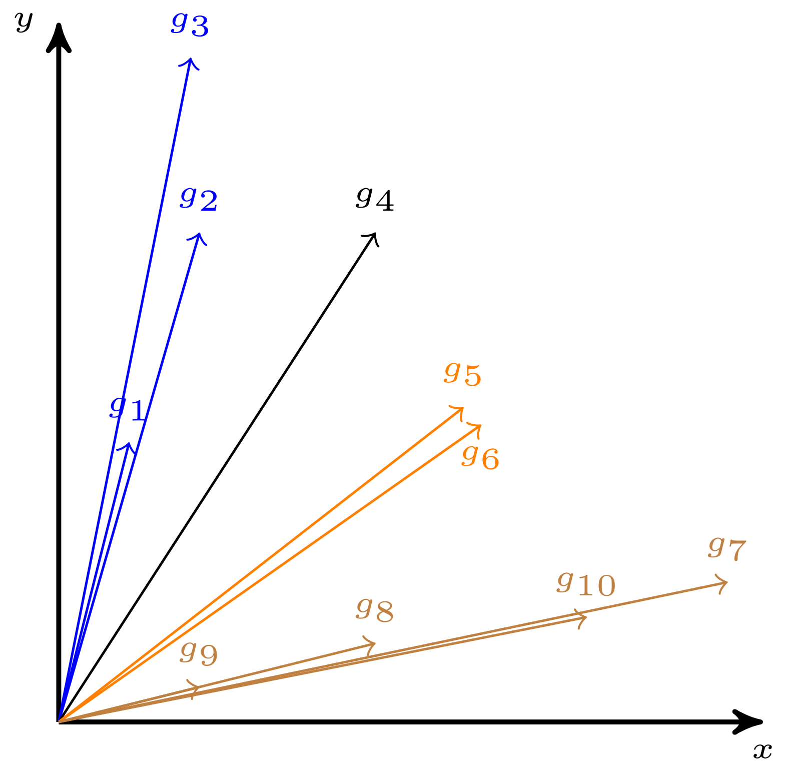
$Tl(X)$	$S(X)$	g
$false$	$false$	8
$false$	$true$	6
$true$	$false$	4
$true$	$true$	2

  - $\cos(\theta) = \frac{4 \cdot 8 + 3 \cdot 6 + 2 \cdot 4 + 1 \cdot 3}{\sqrt{16+9+4+1} \cdot \sqrt{64+36+16+4}} = 1$

Use $1 - \cos(\theta)$ as distance function in clustering algorithm

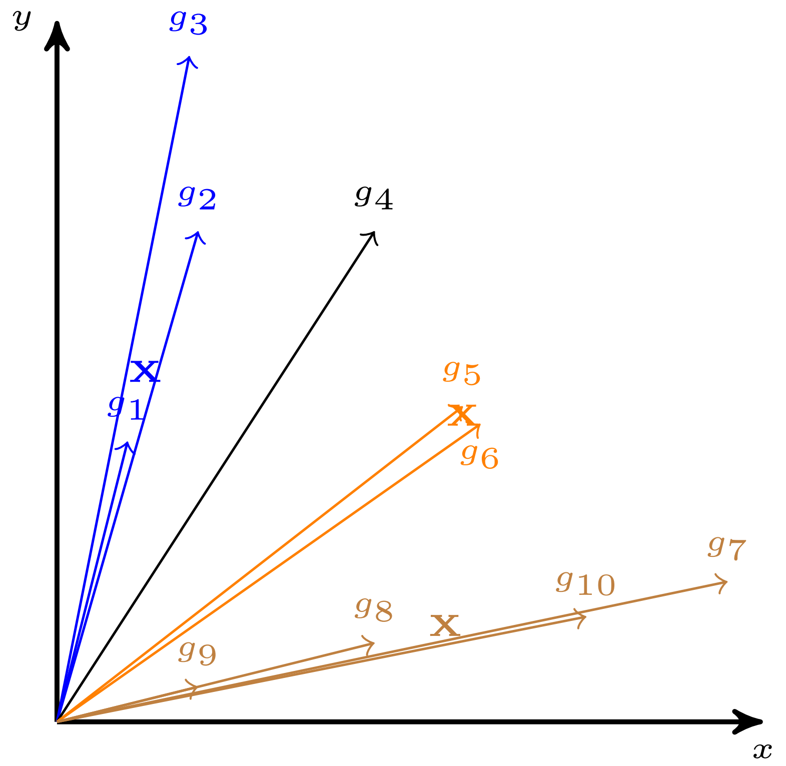
Cluster Groups

- Density-based clustering as unknown number of clusters
  - E.g., DBSCAN
- Cosine similarity as distance function
- Use ANOVA for testing fitness of clustering
  - Hypothesis testing
  - Do the cluster means sufficiently distinguish the clusters?



Merge Groups

- Merge groups of cluster by calculating mean of cluster while accounting for groundings
- Replace old groups with merged group in forward message



Merging Parfactors

- Merge similar parfactors, *accounting for groundings*

$|\text{dom}(X)| = 4$

$Tl(X)$	$S(X)$	g
$false$	$false$	4
$false$	$true$	3
$true$	$false$	2
$true$	$true$	1

$|\text{dom}(X')| = 4$

$Tl(X')$	$S(X')$	g
$false$	$false$	7.9
$false$	$true$	6
$true$	$false$	3.9
$true$	$true$	2.1

$|\text{dom}(X'')| = 2$

$Tl(X'')$	$S(X'')$	g
$false$	$false$	15.7
$false$	$true$	12.2
$true$	$false$	8.1
$true$	$true$	3.8

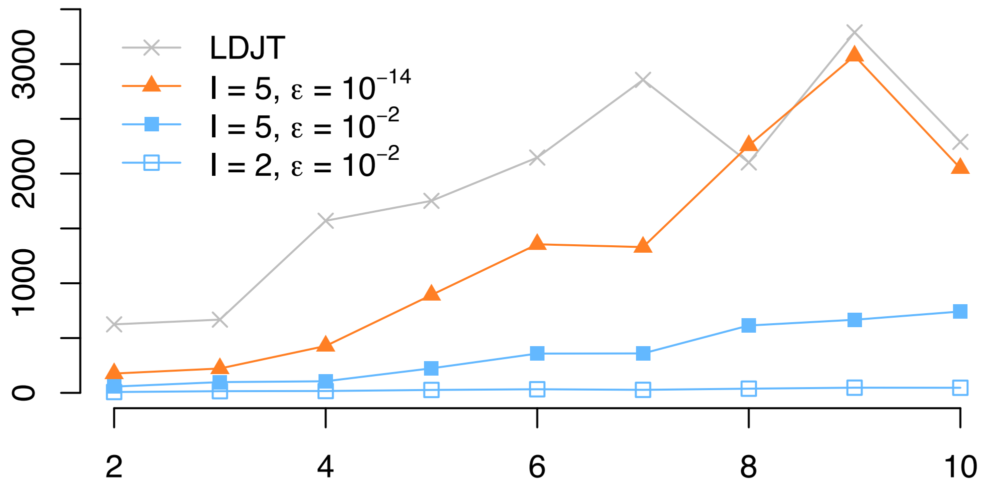
$\rightarrow$

$|\text{dom}(X)| = 10$

$Tl(X)$	$S(X)$	g
$false$	$false$	$\frac{(4 \cdot 4 + 7.9 \cdot 4 + 15.7 \cdot 2)}{10} = 7.9$
$false$	$true$	$\frac{(3 \cdot 4 + 6 \cdot 4 + 12.2 \cdot 2)}{10} = 6.04$
$true$	$false$	$\frac{(2 \cdot 4 + 3.9 \cdot 4 + 8.1 \cdot 2)}{10} = 3.98$
$true$	$true$	$\frac{(1 \cdot 4 + 2.1 \cdot 4 + 3.8 \cdot 2)}{10} = 2$

Runtimes and Error

- DBSCAN for Clustering (neighbourhood: $\varepsilon, n = 2$)
- ANOVA for checking fitness of clusters ($\alpha = 0.005$)
- Right: Runtimes in seconds
  - I denotes how often TAMe is run (every I'th run)
  - ε parameter for DBSCAN
- Below: prediction error of LDJT with TAMe compared to LDJT without TAMe

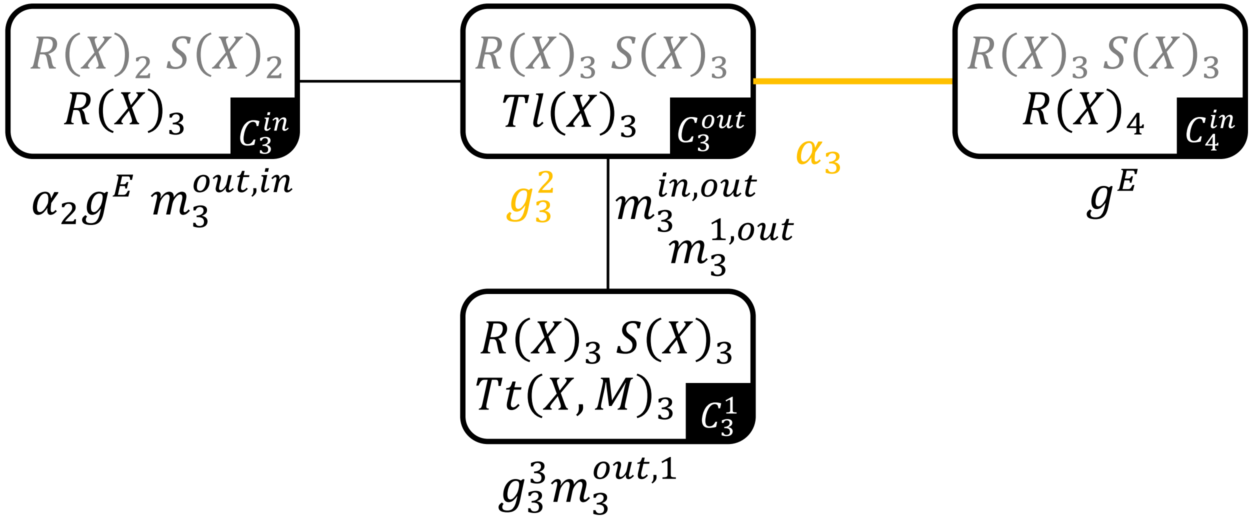


π	**Max**	**Min**	**Average**
0	0.0001537746121	0.0000000001720	0.0000191206488
2	0.0000000851654	0.0000000000001	0.0000000111949
4	0.0000000000478	0	0.0000000000068

Is It Reasonable to Undo Splits?

- *Approximate* forward message
- For each time step, sequential behaviour is multiplied onto the forward message
- Indefinitely bounded error due to sequential behaviour

Without any new evidence, the model would become fully lifted again!



Interim Summary

- Need to undo splits to
  keep reasoning polynomial w.r.t. domain sizes

1. Where can splits be undone efficiently?
   - Undo splits in a forward message
2. How to undo splits?
   - Find approximate symmetries
   - Merge based on groundings
3. Is it reasonable to undo splits?
   - Yes, due to the temporal model behaviour (indefinitely bounded error)
     - Achieve fully lifted model again without new evidence