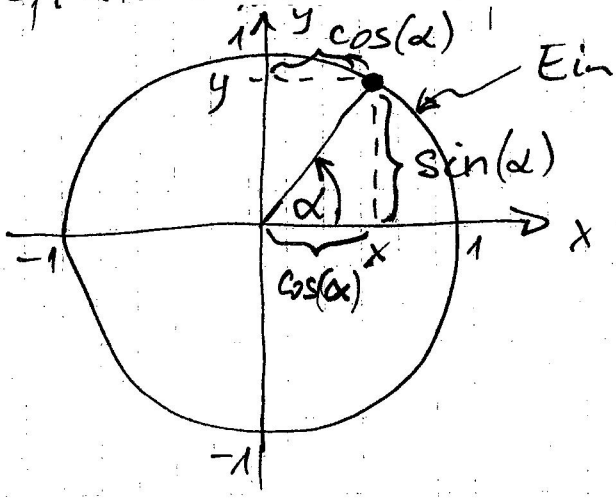
2. Trigonometrie / Geometrie

- „Definition" von sin / cos / tan:

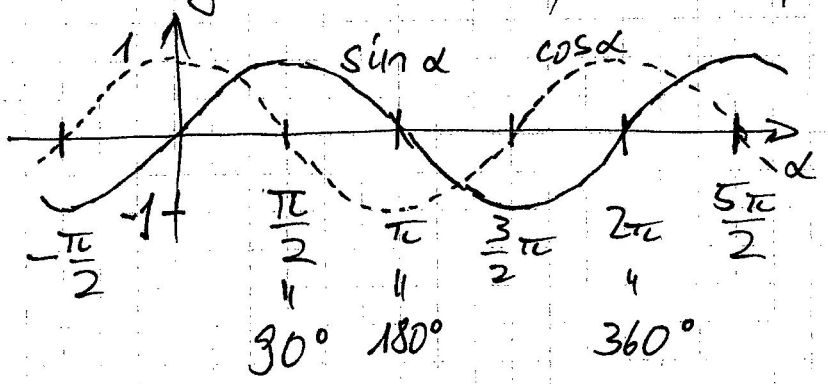


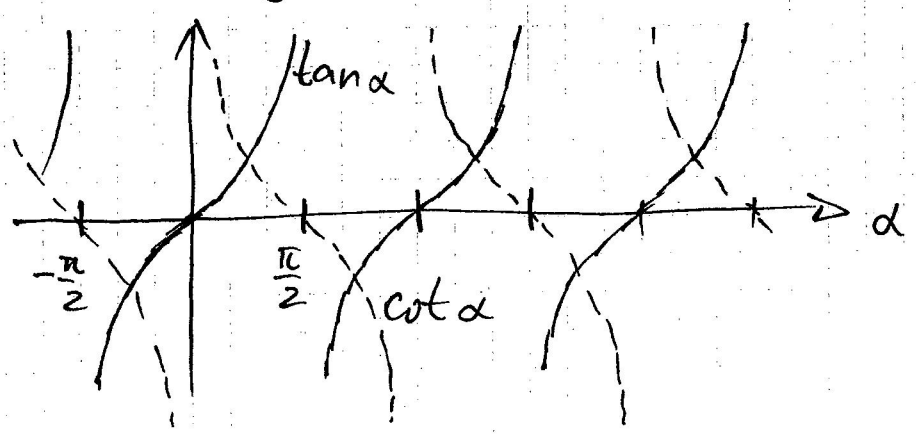
- α = Winkel zur x-Achse
  - oft in Bogenmaß (≙ 360° ≙ 2π)
  - gegen den Uhrzeigersinn ≡ „math. positiv"
  - nicht auf 0…360° / 0…2π beschränkt

$$\left.\begin{aligned} x &= \cos\alpha \\ y &= \sin\alpha \end{aligned}\right\} \Rightarrow \frac{y}{x} = \frac{\sin\alpha}{\cos\alpha} =: \tan\alpha,$$

$$x/y = \cos\alpha/\sin\alpha = \frac{1}{\tan\alpha} =: \cot\alpha$$

- „Wiederholung" nach 360° / 2π ⇒ periodische Funktionen



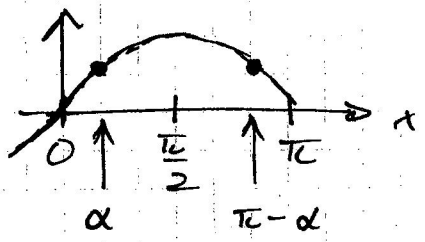


$$\left.\begin{aligned}\sin(\alpha+2\pi) &= \sin(\alpha)\\ \cos(\alpha+2\pi) &= \cos(\alpha)\end{aligned}\right\}\ \text{Periodizität}$$

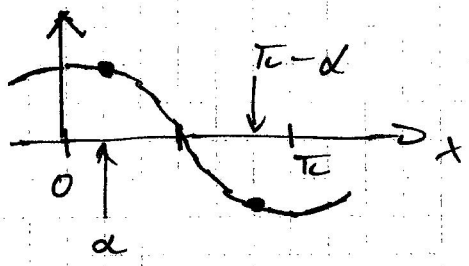
$$\left.\begin{aligned}\cos(\alpha) &= \sin\left(\alpha+\tfrac{\pi}{2}\right),\\ \sin(\alpha) &= \cos\left(\alpha-\tfrac{\pi}{2}\right)\end{aligned}\right\}\ \text{Verschiebung gegeneinander}$$

$$\left.\begin{aligned}\sin(-\alpha) &= -\sin(\alpha)\\ \cos(-\alpha) &= \cos(\alpha)\end{aligned}\right\}\ \begin{matrix}\text{ungerade}\\ \text{gerade}\end{matrix}\ \text{Funktion}$$

$$\sin(\alpha) = \sin(\pi-\alpha)$$



$$\cos(\alpha) = -\cos(\pi-\alpha)$$

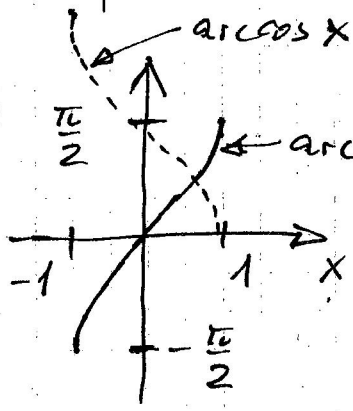


wichtige spezielle Werte:

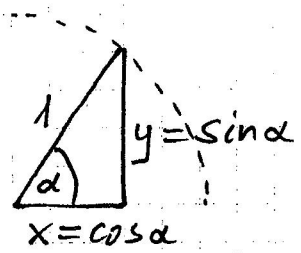
Winkel α	$\sin(\alpha)$	$\cos(\alpha)$
$0° \,/\, 0$	0	1
$30° \,/\, \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
$45° \,/\, \pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60° \,/\, \pi/3$	$\frac{1}{2}\sqrt{3}$	$1/2$
$90° \,/\, \pi/2$	1	0

- Umkehrfunktionen: $x = \cos\alpha \Rightarrow \alpha = \arccos x$

$y = \sin\alpha \Rightarrow \alpha = \arcsin y$



- Einschränkung des Arguments auf $[-1, 1]$



Rechtwinkliges Dreieck

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow \boxed{\cos^2\alpha + \sin^2\alpha = 1}$

$\Rightarrow \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

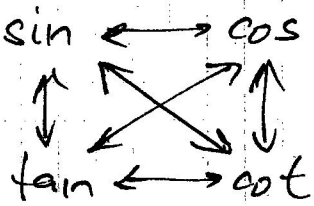
↑ Vorzeichen: je nach Wert von α

$\Rightarrow \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \pm\dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$)

ferner: $\tan^2 = \dfrac{\sin^2}{1-\sin^2} \Rightarrow \sin^2 = \dfrac{\tan^2}{1+\tan^2}$

$\Rightarrow \sin\alpha = \pm\dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$

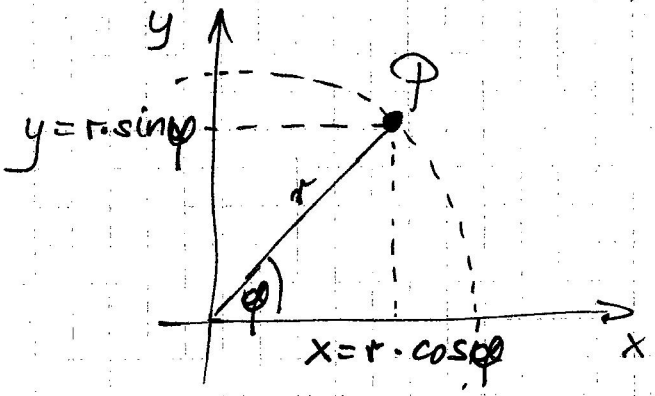
und weitere Umrechnungsvorschriften



- Additionstheoreme (hier ohne Beweis → komplexe Zahlen)

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

- Polarkoordinaten: falls der betrachtete Kreis nicht den Radius 1, sondern r hat:



→ Punkt P kann nicht nur durch die kartesischen Koordinaten (x, y) angegeben werden, sondern auch durch die (ebenen) Polarkoordinaten (r, φ):

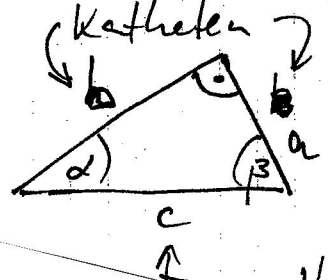
r = Abstand zum Koordinatenursprung

φ = Winkel zur x-Achse

Zusammenhang: $(r,\varphi) \to (x,y)$: $x = r\cdot\cos\varphi$, $y = r\cdot\sin\varphi$

bzw. $(x,y) \to (r,\varphi)$: $r = \sqrt{x^2+y^2}$, $\varphi = \arctan\frac{y}{x}$

- Rechtwinkliges Dreieck: $a^2 + b^2 = c^2$, $\alpha+\beta = 90° = \frac{\pi}{2}$

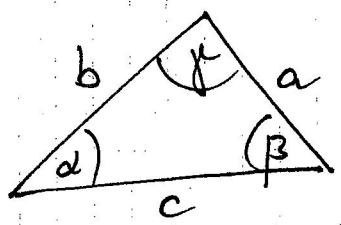


$b = c\cdot\cos\alpha = c\cdot\sin\beta$

$a = c\cdot\sin\alpha = c\cdot\cos\beta$

$\frac{a}{b} = \frac{c\cdot\sin\alpha}{c\cdot\cos\alpha} = \tan\alpha = \cot\beta$

• Allgemeines Dreieck: $\alpha+\beta+\gamma = 180° = \pi$

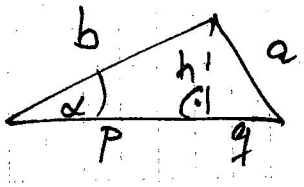


Kosinus-Satz, z.B. bzgl. α: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$(= b^2 + c^2$, falls $\alpha = 90°)$

(und analog für die anderen Seiten)

Bew. zum Kosinus-Satz:



$h^2 + p^2 = b^2$

$h^2 + q^2 = a^2$

$\Rightarrow a^2 = b^2 - p^2 + q^2$

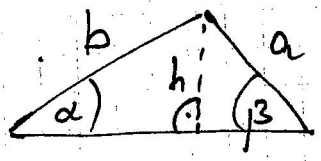
ferner $c = p + q$,

d.h. $q^2 = (c-p)^2 = c^2 + p^2 - 2cp \Rightarrow a^2 = b^2 - p^2 + c^2 + p^2 - 2cp$

$= b^2 + c^2 - 2cp$

und: $p = b\cdot\cos\alpha \Rightarrow \boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha}$

Sinus-Satz:

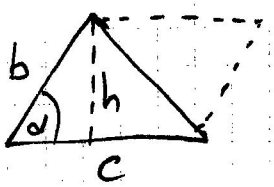


$h = b\cdot\sin\alpha = a\cdot\sin\beta$

$\Rightarrow \boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$

dann auch

• Und noch ein paar Flächen, Volumina etc.:

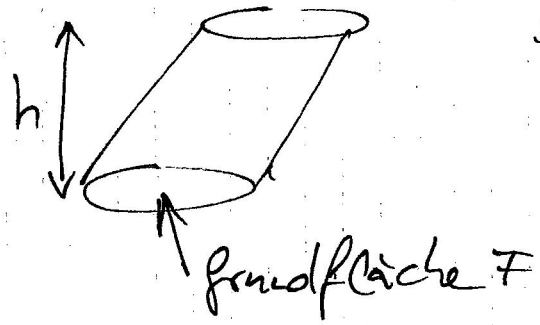


• Fläche des Parallelogramms:

$F_P = c\cdot h = c\cdot b\cdot\sin(\alpha)$

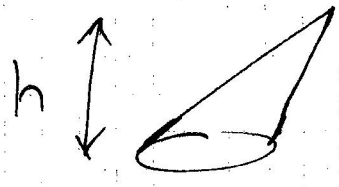
• Fläche des Dreiecks:

$F_D = \frac{1}{2}F_P = \frac{1}{2}cb\sin(\alpha)$

- 

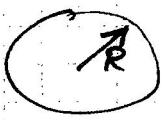
Zylinder (gerade oder schief, Grundfläche beliebig):

$$V = F \cdot h$$

- 

Kegel / Pyramide (gerade oder schief, Grundfläche beliebig):

$$V = \frac{1}{3} F \cdot h$$

- Kreis: 

Durchmesser $D = 2R$

Umfang: $2\pi R = \pi \cdot D$

Fläche: πR^2

- Kugel:

Oberfläche: $4\pi R^2$

Volumen: $\frac{4}{3}\pi R^3$