Regular and Chaotic Oscillations of a Rotating Pendulum

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Intention

One reason of the great success of classical physics is the ability to predict the evolution of a system from which the dynamics (equation of motion) and the initial values are known. But this ability falls with chaotic systems. Because of the exponential increase of small errors in the initial conditions of a chaotic system every prediction of its behaviour becomes impossible in shortest time.

For a long time physicists thought that the chaotic behaviour of a system is due to its complexity. But recently, one found that very simple systems may become chaotic, too. As important as this realisation is the manner of the transition from order to chaos. This transition follows some general patterns: the system announces the breakdown of the deterministic behaviour. Of course, the knowledge of these patterns is of great practical interest.

The rotating pendulum presented here allows to study the transitions between regular and chaotic motions by means of computational simulations. Thereby, complete Feigenbaum scenarios and other transitions may be obtained. The numerical results are described in more detail in [1].

Description of the system

The system simulated by us is a linear pendulum whose bob is supported not by a string, which could become slack, but by a light rigid rod pivoted to a support rotating with an angular velocity $\omega$ at a distance $r$ around a vertical axis (you may imagine the gondola of a round-about.). Its plane of oscillation is spanned by the vertical axis and the “gallows”.

In the with rotating system of coordinates there are two forces acting on the gondola along its path: the tangent components of the gravitational and the centrifugal force. Therefore, the equation of motion reads

\[ l \ddot{\delta} = -g \sin \delta + \omega^2 (r + l \sin \delta) \cos \delta \]

\[ \ddot{\delta} = -\omega^2 \sin \delta + \omega^2 (\alpha + \sin \delta) \cos \delta \]

with $\omega^2 = \frac{g}{l}$, $\alpha = \frac{r}{l}$

By multiplying with $\delta$ and integrating once one gets the effective potential $U(\delta)$:

Fig. 1: Effective potential for four different values of $\Omega^2$.
The pictures of this paper are obtained with $\Omega^2 = 2.5$.  

Fig. 2: The rotating pendulum: coordinates and derivation of the equation of motion

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1 In: G. Marx (Ed.): Chaos in Education II. Vesprem (Hungary) 1987, pp. 312 - 317.
\[ \frac{1}{2} \frac{d}{dt} \delta^2 + \frac{d}{dt} \left[ -\frac{d}{dt} \cos \delta + \Omega^2 \frac{d}{dt} \right] \left( \alpha + \frac{1}{2} \sin \delta \right) \sin \delta = 0 \quad \text{or} \]
\[ U = \omega_s^2 \left( 1 - \cos \delta \right) + \Omega^2 \omega_s^2 \]
\[ \left( \alpha + \frac{1}{2} \sin \delta \right) \sin \delta \cdot \left( \Omega = \frac{\omega_s}{\omega} \right) \]

To be more realistic, we take into account the friction in the support. In order to get stationary oscillations we therefore have to introduce an additional driving force. The equation of motion thus reads:
\[ \ddot{\delta} = -\rho \dot{\delta} - \omega_s^2 \sin \delta + \Omega^2 \omega_s^2 (\alpha + \sin \delta) \cos \delta + f \sin \omega t \]

There are three interesting limiting cases which may be studied:
- \( f = 0 \): free oscillations of the pendulum,
- \( \Omega = 0 \): normal pendulum,
- \( \alpha = 0 \): rotating pendulum without gallows.

**Method of calculation**

For Integration the equation of motion is transformed into a system of three differential equations of first order:
\[
\begin{align*}
  x := \delta \\
y := \dot{\delta} \\
z := \omega t
\end{align*}
\]
\[
\begin{align*}
y &= \dot{x} = \dot{y} \\
z &= \omega t \\
y &= -\rho y - \omega_s^2 \sin x + \Omega^2 \omega_s^2 (\alpha + \sin x) \cos x + f \sin z
\end{align*}
\]

which is solved by the Runge-Kutta method of fourth order (see appendix). In most of the cases considered a step width of 50 steps/period turned out to be sufficient. The program is written in PASCAL for a ATARI 1040 personal computer. The pictures of this paper are obtained with the following parameters:
\[ \omega_s = 1s^{-1}, \, \omega = 1s^{-1}, \, \alpha = 0, \, \Omega^2 = 2.5, \, \rho = 0.85s^{-1}. \]

**The attractor concept**

For the description of dissipative systems the concept of attractors is fundamental. An attractor is a figure in phase space attracting the behaviour of a system with increasing time. Although defined in the abstract phase space it allows to describe and to develop chaotic behaviour by means of geometric methods.

For instance, without driving force the rotating pendulum will come to rest at the equilibrium angle irrespective of the initial conditions. In phase space the trajectory will contract to a point. That’s why this point is called a *point attractor* (fig. 3). A periodic driving force causes a steady oscillation of the pendulum; the trajectory in phase space will be a closed loop: now, the attractor is a limit cycle (fig. 4).

The pendulum possesses different coexistent limit cycles: starting with different initial conditions often leads to different final behaviour, sometimes (as in fig. 4) but not always according to the two “valleys” of the potential. The set of initial conditions evolving to a certain attractor is called the *catchment region* or the *basin of attraction* of that attractor. For instance, our "Julia" at the top of this paper shows the catchment region of a point attractor coexisting with a chaotic attractor (see below) obtained with a system very similar to that of this paper (see at the end of this paper).

In 1963 Lorenz 12) found an additional final behaviour: the *chaotic attractor* also called a *strange attractor*. Its strangeness is due to the global stability and the local instability of its structure: Because of the exponential increase of errors different initial conditions may cause a totally different behaviour on the microscopic level but the phase trajectories fill the same region in phase space.
Period doubling

With small values for the amplitude \( f \) of the driving force the pendulum reaches a simple limit cycle (fig. 5 a). However, increasing \( f \) above 0.7445 two alternating amplitudes of oscillation are observed in the \((\delta, t)\)-diagram (fig. 5 b). In phase space two driving cycles are needed for the trajectory to be

Fig. 5: Transition from order to chaos with increasing driving amplitude. a) \( f = 0.72 \), b) \( f = 0.752 \), c) \( f = 0.775 \), d) \( f = 0.782 \), e) \( f = 0.809 \). Each motion is demonstrated by the \((\delta, t)\)-diagram, the phase projection \((\delta, \dot{\delta})\) and the corresponding fourier spectrum.
closed. In the fourier spectrum an additional peak appears at one half of the driving frequency. Further increase of the driving amplitude causes further period doublings (fig. 5 c, d) until at about \( f = 0.78 \) the regular motion breaks down: the chaotic attractor occurs for the first time (fig. 5 e).

**Feigenbaum scenario**

To get an overview over all period doublings the displacement angles are stroboscopically registered synchronously with the driving force \((\sin z = 0)\) and plotted against the driving amplitude after the transients have settled to the final behaviour. The Feigenbaum diagram constructed in this way (fig. 6) is of great importance not one for the System presented here. As Feigenbaum [3] showed it is typical of the ordered way into chaos of a large class of systems.

The method of reducing the dimensionality of the phase space by stroboscopic registration of the system coordinates was introduced by Poincaré [4]. Therefore, it is called an Poincaré ” section of the phase space. Fig. 7 shows a Poincaré section of a chaotic attractor.

Until now, we restricted our interest to oscillations with small amplitudes around one of the equilibrium angles and oscillation minimas near the maximum of the potential at \( \delta = 0^\circ \). Therefore, the behaviour of the pendulum described so far represents only a small part of all its possibilities. Further increasing of the driving force causes the oscillation to “splash” over into the other valley of the potential at about \( f = 0.81 \) (fig. 8). In the following parameter interval \((0.8 < f < 1.7)\) an alternating sequence of chaotic and regular behaviour is obtained in which different kinds of transitions (besides period doubling e.g. intermittency [5]) may be observed.

**Experimental realisation**

The rotating pendulum allows to study transitions between order and chaos which are of importance for a large class of systems. However, It has the disadvantage to be difficult to realize experimentally. That’s why we recently investigated the behaviour of another kind of rotating pendulum in more detail [6]: a wheel rotating around its axis and bound elastically to an equilibrium position (“Pohlsches Rad”). By means of an additional mass eccentricly fixed to the wheel the restoring force becomes nonlinear and the potential takes on a form very similar to that shown in fig. 2. The results obtained by simulation of that system are very similar

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Fig. 6: Transition from order to chaos an Feigenbaum diagram: 100 values of the displacement angle \( \delta \) at the times defined by \( \sin z = 0 \) are plotted for 550 values of \( f \) after the decay of transients (after 50 driving periods). The framed picture shows a magnification of the \( f \)-Interval \([0.79400, 0.79461]\) (200 values of \( \delta \) after 200 driving periods).

Fig. 7: Poincaré section of a chaotic attractor obtained at \( f = 1.35 \)

Fig. 8: The behaviour of the pendulum in a wide parameter range. The region of fig. 7 in framed.
to those presented in this paper. Moreover, the numerical results can be verified experimentally for some important parameter intervals [7].

APPENDIX: Listing of the Integration Procedure (PASCAL) and program options

The procedure listed below demonstrates how to calculate the following state \((x, y, z, t)\) of the pendulum from a given one by integrating the equation of motion by means of the Runge-Kutta method of fourth order.

**The following variables must be declared globally:**
- \(f\),
- \(\rho\) \(\equiv\) \(\rho\),
- \(\omega_0\) \(\equiv\) \(\omega_0^2\),
- \(\omega\) \(\equiv\) \(\omega^2\),
- \(\omega_A\) \(\equiv\) \(\omega_A\),
- \(\alpha\) \(\equiv\) \(\alpha\), and
- \(\delta\) (stepwidth of integration \(=\) period/step where period \(=\) \(2\pi/\omega_A\) and step \(=\) number of integration steps per period of the driving force).
- \(\delta_h\) and \(\delta_s\) (the second and the sixth part of \(\delta\)).

**PROCEDURE RungeKutta** (VAR \(x, y, z, t:\) REAL);

**TYPE** chg = ARRAY [1..2] OF REAL;

VAR k1, k2, k3, k4 : chg;

**PROCEDURE** Change (x, y, z: REAL; VAR k: chg);

VAR Sinus: REAL;

BEGIN (* Change *)

\[
\begin{align*}
k[1] & : = -\omega_0^2 y + \omega_0 \omega \cos(x) + f \sin(z);  \\
\text{(* equation of motion *)}  \\
\end{align*}
\]

END; (* Change *)

BEGIN (* RungeKutta *)

\[
\begin{align*}
\text{Change}(x+k1[1]*\delta_h, y+k1[2]*\delta_h, z, k2);  \\
\text{Change}(x+k2[1]*\delta_h, y+k2[2]*\delta_h, z, k3);  \\
z: = z + \omega_A \delta_h;  \\
\text{Change}(x+k3[1]*\delta_h, y+k3[2]*\delta_h, z, k4);  \\
t: = t + \delta;  \\
x: = x + (k1[1] + k2[1] + k3[1] + k4[1]) \delta_s;  \\
\end{align*}
\]

END; (* RungeKutta *)

The term \(\sin(z)\) takes only \(2^\text{step+1}\) different values. The program may be accelerated by storing these values in an array.

Our program offers the following options:
- Plot of the effective potential for different values of \(\alpha\) and \(\Omega\),
- different presentations of the motion:
  - movielike plotting,
  - diagram plotting the displacement angle against time
  - attractor in the three-dimensional phase space \((\delta, \dot{\delta}, t)\),
  - two-dimensional phase projection \((\delta, \dot{\delta})\), and
  - the possibility to change the presentation during the calculation.
- Before performing the calculation the following parameters may be chosen:
  - amplitude of driving force \(f\),
  - length of the gallows \(\alpha\),
  - coefficient of friction \(\rho\),
  - Initial values of displacement \(\delta_0\), and velocity \(\dot{\delta}_0\),
  - stepwidth for integration,
  - number of periods presented at once on the screen and
  - number of table the calculated values are to be stored in.
- During the calculation it is possible to change
  - the presentation of the motion,
  - the dimensions of the diagram and
  - the parameters \(f, \alpha\) and \(\rho\).
- Little additional programs offer further possibilities:
  - the plot of Poincaré sections (see fig. 7),
  - later presentation of calculated tables,
  - fourier transformations of calculated tables (see fig. 5)
  - calculation of Feigenbaum diagrams plotting the displacement $\delta$ when the phase of the driving force vanishes against the amplitude of the driving force (see fig. 6, 8); during calculation all values of $\delta$ are stored,
  - later study of Feigenbaum diagrams with additional acoustic demonstration of the oscillations for arbitrary values of $f$.

Literature


