tric charges seen as existing at a point, i.e., as particles having no size. No change is required for particles of finite size having a distribution of electric charge since all that matters is the quantity of charge crossing a small area per second. The argument would also apply to a continuous flow of electricity, assuming such to be physically possible, merely by regarding it as the limiting case when the elementary particles become contiguous. In this case, the loop size can be an infinitesimal and so the field quantities need no longer be seen as macroscopic quantities.

VI. CONCLUSION

It is argued that the key to Maxwell's equations was provided by Faraday. His opposition to the current beliefs about action at a distance focused attention on the state of the space between source, whether charge or current, and the remote body acted upon. Maxwell's success in expressing in mathematical form the interaction of the various electromagnetic quantities is all the more surprising because he had little option other than to rely on some of the mathematical paraphernalia which Faraday had rejected. Fortunately, his intuition was on a par with Faraday's.

The basic laws of classical electromagnetism uncovered by the combined efforts of Faraday and Maxwell stem from recognition of electric and magnetic intensities as vector fields. Since a vector field is completely determined by its divergence and curl, the specification of these quantities becomes of paramount importance. That the divergence of magnetic intensity is zero and the divergence of electric intensity is proportional to electric charge density are the first two axioms.

The Faraday law that the curl of the electric intensity is proportional to the time rate of change of the magnetic intensity and the dual to that law, which has been called here the Maxwell law, complete the definitions and are to be regarded as the final two axioms of electromagnetism. They have the supreme importance of determining the fundamental inter-relationship of the two fields showing their essential wave nature.

The argument put forward here is that no other axioms are required. The magnetic effect of a current, so long regarded as one of the axioms of electromagnetism, is shown to be a consequence of the Faraday view of electric flux and the Maxwell law. This not only offers an explanation of the Ampere law $\nabla \times H = J$, but shows why that law does not depend upon the velocity of the charges constituting the current. Moreover, it clears up uncertainties about how to interpret the famous Maxwell equation involving the confusing concept of "displacement current."

To set the seal on Faraday's expectations it turns out that Maxwell's equations are integrable, permitting a true account to be given of action at a distance. It is to the Special Theory of Relativity that one must look to obtain a physical understanding of the consequences of such integrations, and at the same time to be provided with surprisingly simple techniques for solving such difficult integration problems as finding the distant fields of a moving charge.

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[3] R. S. Elliott, *Electromagnetics* (McGraw-Hill, New York, 1966), pp. 264–272.

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A catastrophic toy

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We discuss a simple mechanical toy and show that it exhibits a catastrophe which is quite similar to a first-order phase transition.

I. INTRODUCTION

Recently, Prigo[1] described the so-called Christmas tree toy. His main intention was to explain an unusual wobbling of the tree sections superposing their rotation. In contrast to Prigo we could not observe this oscillatory behavior.[2] Instead, we found it more surprising that the opening and closing of the four shell sections do not arise continuously with increasing rotation rate, but arise rather abruptly.

Weak pushes of the thumb plunger producing only small rotation rates leave the shell closed, whereas strong pushes give rise to a sudden opening. Conversely, if the thumb plunger is released the rotation rate slows down and the shell closes as abruptly as it was opened. Closer observation reveals that the critical rotation rate at which the toy opens is higher than the rate at which it closes. This behavior

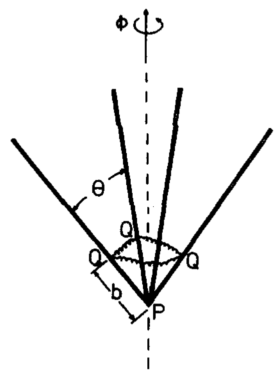

Fig. 1. The model system involves four narrow rods hinged at a common point P, held together by four springs attached at points Q, at a distance b from point P. The generalized coordinates θ and ϕ describe the polar rod angle and the rotational position, respectively (Fig. from Ref. 1).

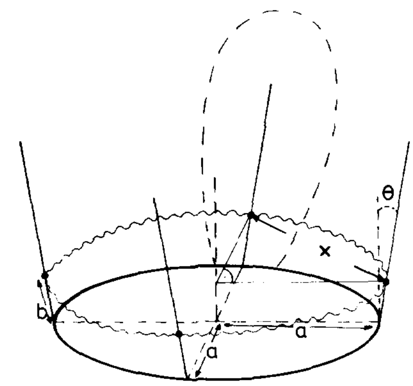

Fig. 2. The circular base of radius a to which the four shell sections are hinged. They are held together by (rubber) springs of length x.

seems to be similar to a catastrophe in the sense of Thom[3] (cusp-type catastrophe) or to a Landau discontinuous phase transition.

II. A SIMPLE MODEL

The calculation of Prigo could not reveal this phenomenon since he assumed conservation of angular momentum. This is less realistic because, by pushing the thumb plunger, angular momentum is added, and is withdrawn when the rotation rate slows down by friction.

Therefore, assuming that not the angular momentum, but only the rotation rate of the shell can be controlled, within the simple model of Prigo the Lagrangian-effective potential $U(\theta)$ as function of the angle θ between the shell sections and the vertical symmetry axis takes the form[4]:

$$U(\theta) = 4kb^2 \sin^2\theta - 2I\dot{\phi}^2 \cdot \sin^2\theta, \quad (1)$$

where b is the distance from the point P (at which the sections are hinged) to the point Q (to which the springs are attached; see Fig. 1). k denotes the stiffness of the springs and I the moment of inertia of each shell section relative to the point P. For simplicity, we ignored the gravitational potential energy, which is negligible compared to the elastic energy of the springs.

The effective potential [Eq. (1)] shows immediately the existence of stable equilibrium positions: It has two minima, one at the polar angle

$$\theta_0 = 0^\circ, \quad \text{if } \dot{\phi}^2 < 4kb^2/2I, \quad (2a)$$

and another at

$$\theta_0 = 90^\circ, \quad \text{if } \dot{\phi}^2 > 4kb^2/2I. \quad (2b)$$

Thus starting with low rotation rates the shell sections are fixed at the stable equilibrium position $\theta_0 = 0^\circ$ until the critical angular velocity $\dot{\phi}_c = (4kb^2/2I)^{1/2}$ is reached. At this moment the shell sections are driven into the other equilibrium position $\theta_0 = 90^\circ$, i.e., the shell opens. Conversely, when the angular velocity is slowing down due to friction the shell stays open until at $\dot{\phi}_c$ it closes again, jumping back into the state with $\theta_0 = 0^\circ$.

This model reflects the observed behavior in that it predicts two equilibrium positions of $U(\theta)$ for the closed and the opened shell. But, this is not exactly what one observes: Actually the closing and opening take place at different angular velocities $\dot{\phi}_c$ and $\dot{\phi}_{c'}$, with $\dot{\phi}_{c'}$, thus exhibiting a hysteresis behavior.

III. AN IMPROVED MODEL

To get the hysteresis and thereby a closer similarity to a discontinuous phase transition we have to take into account that the rubber springs connecting the four shell sections are already stretched when the polar angle θ is zero. Assuming that the unstretched springs have a vanishing length, a corresponding modification may be done if one considers that the sections are not fastened to a common point but to the circumference of the circular base (see Fig. 2).

Thus denoting the radius of the base with a, the springs are stretched by an amount of $\sqrt{2}a$ at the position $\theta = 0^\circ$ (closed shell sections), whereas at $\theta > 0^\circ$ they have an elongation $x = \sqrt{2}(a + b\sin\theta)$.

Inserting this expression into the term $4 \cdot \frac{1}{2}kx^2$ of the spring potential energy Eq. (1) takes the form

$$U(\theta) = U(0) + 8kab \cdot \sin\theta + 4kb^2 \sin^2\theta - 2I\dot{\phi}^2 \sin^2\theta, \quad (3)$$

with $U(0) = 4ka^2$ and a moment of inertia I, which now refers to the improved model.

In Fig. 3 we depict some plots of $U(\theta) - U(0)$ versus the angle θ for increasing values of $\dot{\phi}$ [Fig. 3(a)–(e)], showing the main features of the phase transitionlike behavior of the toy: For small $\dot{\phi}$ the effective potential U increases monotonically with the angle θ, showing an absolute minimum at $\theta = 0^\circ$ [Fig. 3(a)]. This minimum represents the equilibrium position of the shell sections for small rotation rates.

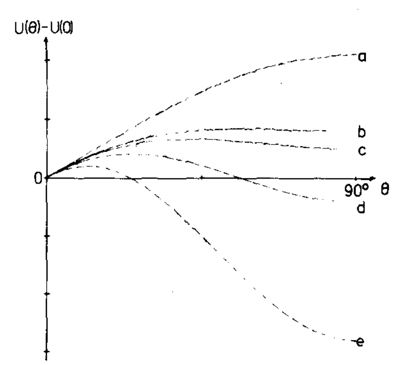

Fig. 3. The effective potential $U(\theta) - U(0)$ versus θ for increasing values of the rotation rate $\dot{\phi}$ from (a)–(e).

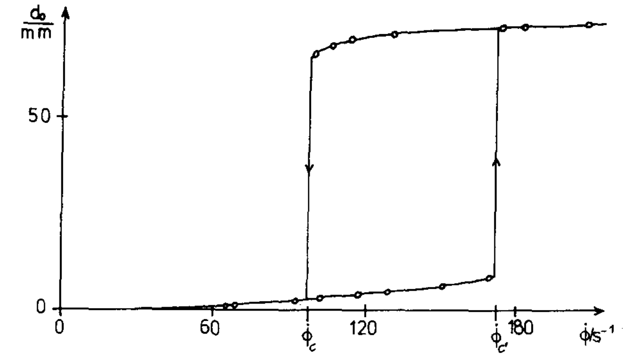

Fig. 4. The measured diameter d_0 of the opening circle of the shell sections for different rotation rates $\dot\phi$. The two critical points $\dot\phi_c$ and $\dot\phi_{c'}$ give rise to a hysteresis.

For greater $\dot\phi$ a local minimum appears at $\theta = 90°$ [Fig. 3(b)]. With further increasing $\dot\phi$ this minimum becomes deeper and deeper [Fig. 3(c)–(e)]. At the same time the potential barrier separating the two minima decreases until it becomes so low that small perturbations may drive the shell sections from the $\theta = 0°$ state to the $\theta = 90°$ state. Conversely, when $\dot\phi$ is decreased by friction, $U(\theta) - U(0)$ will pass through the sequence e $\rightarrow$ a (in Fig. 3).

Now, the system is fixed to the $\theta = 90°$ state until, again, fluctuations suffice to overcome the potential barrier. But this inverse transition will happen at a lower value of $\dot\phi$ than in the opposite direction, as is apparent from Fig. 3.

Thus our discussion of the opening and closing of the shell sections reveals a hysteresislike behavior and by this the same phenomena as the discontinuous phase transition of a thermodynamic system.

A minor weakness of our model may be seen in the fact that the potential barrier will disappear exactly only for $\dot\phi \rightarrow \infty$, so that the $\theta = 0°$ state is still metastable in the situation of Fig. 3(e). But, due to the permanent presence of perturbations, there will be no practical consequence of this feature inherent in our model. On the other hand, further improvements of our model (e.g., taking into account the nonlinearity of the rubber springs) should be possible.

IV. EXPERIMENTAL INVESTIGATIONS

There are no difficulties in investigating the phase transitionlike behavior of the toy experimentally.[5] Here, we shall show only two important results:

(a) The direction of the shell sections can easily be measured by the diameter d_0 of their opening circle as it is visible from above. This was done for different angular velocities $\dot\phi$ adjusted by a variable-speed motor. Figure 4 shows our results and verifies the theoretically predicted hysteresis.

(b) The stability of the equilibrium position of the shell sections has been investigated: Actually, we measured the deviation δ of the diameter d from the equilibrium value d_0 due to a definite force F applied to the shell sections for different angular velocities $\dot\phi$ (for experimental details see Ref. 5). In Fig. 5 our measurements are represented by the quantity δ/F for different values of $\dot\phi$ in the vicinity of the critical point $\dot\phi_c$. δ/F may be interpreted as a response function and shows a divergent behavior which is well known from phase transitions.

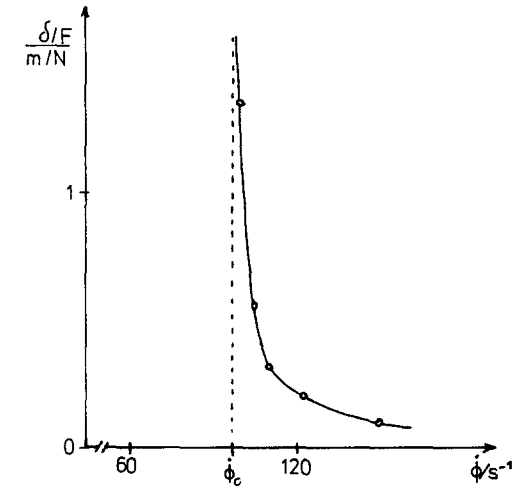

Fig. 5. The response function δ/F for different values of $\dot\phi > \dot\phi_c$.

V. CONCLUSIONS

The bird-in-shell toy may be considered as a simple, cheap, and fascinating device to represent the main mechanical phenomena which are analogous to phase transitions.[6] The possibility of a direct experience of what may be called the sensory aspect of the phase transitionlike behavior, together with the simplicity of the mathematical modeling, may give a rather clear intuition of phenomena which very often are only accessible by sophisticated reasoning.

[1]R. B. Prigo, Am. J. Phys. **52**, 335 (1984).

[2]Instead of the Christmas tree investigated by Prigo we discuss the so-called "bird in shell," which, according to Prigo, should exhibit the same behavior.

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[4]Thus we adopt a different physical situation which is reflected by the different sign in Eq. (1) as compared to the corresponding expression in Ref. 1.

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