Visualization of quantum entanglement

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Abstract

A simple visualization technique for quantum entanglement is presented. The method is based on a generalization of Feynmans approach for the visualization of wave functions and is appropriate for undergraduate and even high school students.

I. INTRODUCTION

Quantum entanglement is a topic of increasing importance but difficult to understand. While the practical realization of Feynman's vision of a quantum computer[1] appears to be only a matter of time, respective teaching concepts appropriate for high school and undergraduate students are still in need.

Though, in view of today's crowded science curricula, the question whether quantum entanglement is among the topics students have to learn nowadays is legitimate, the variety of technological applications in the field of quantum communication that are within reach of commercial application, alone, would justify the necessity to teach quantum entanglement. The fact that the numerous philosophical questions which arise in the context of quantum entanglement invite students to re-conceptualize their common understanding of the world they live in might yet be of even greater pedagogic importance, for even fundamental concepts such as that of "geometric distances" turn out to be inappropriate for the description of entangled photons.

In view of the complexity of the phenomenon, we developed a visual an thus very concrete approach to the topic, which is based on a generalization of visualizations proposed by Feynman[2].

Consider a simple experiment with a beam splitter as shown in Fig. 1. After passing the beamsplitter, the wave function of a single photon with energy $E = \hbar\omega$ may be described by the superposition

$$|\Psi_{\text{Position}}\rangle = \frac{1}{\sqrt{2}} e^{i\omega t} |\text{trans.}\rangle + \frac{1}{\sqrt{2}} e^{i\omega t} |\text{ref.}\rangle \tag{1}$$

The amplitudes for transmission and reflection are equal, given by $\langle\Psi|\text{trans.}\rangle = \langle\Psi|\text{ref.}\rangle = \frac{1}{\sqrt{2}} e^{i\omega t}$. Before the measurement, the amplitudes for both possibilities are in *superposition*. With a probability of $p_{\text{trans.}} = |\langle\Psi|\text{trans.}\rangle|^2 = 1/2$, the photon is transmitted. With a probability of $p_{\text{ref.}} = |\langle\Psi|\text{ref.}\rangle|^2 = 1/2$ it is reflected.

The amplitude of the photon field is visualized as a "rotating clock" $\sqrt{p}\ e^{i\phi(t)}$, with $p = |\psi|^2$ being the detection probability and $\phi(t)$ the "rotating phase". The *result* of a measurement, however, cannot be described by the amplitude. Either the photon is detected - or not. The measured result must therefore be described by the binary information "0" (not detected) or "1" (detected) rather than by the means of a wave function. We have

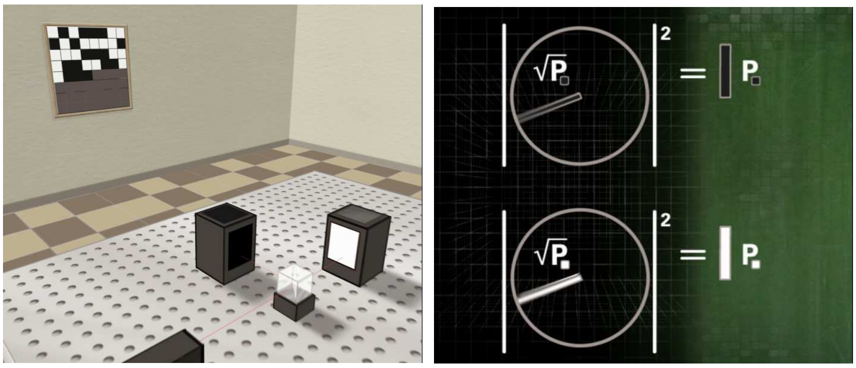

FIG. 1: *Left: Single photons propagate towards a beam splitter. Each photon is detected either as transmitted ("white box") or reflected ("black box"). After many measurements, a random black-and-white pattern emerges. Right: Visualization of the amplitude* $\frac{1}{\sqrt{2}}e^{i\omega t}$ *and the corresponding probability* $p = \frac{1}{2}$ *for transmission and reflection, respectively.*

to distinguish between the mathematical description of the photon before the measurement and the result of a measurement.

The measurement result *transmission* is visualized as a *white* box (□), the result *reflection* as a *black* box (■). After many measurements, a random black-and-white pattern is obtained.

By using variations of these visualization techniques, we will show that quantum entanglement can be illustrated both in theory and experiment. Note that these visualizations are not just simplified illustrations. Each picture can be translated back to the corresponding equation and can be considered as an alternative "language" which can be used to teach the topic. Advantages and limitations of the graphical approach will be discussed in the conclusions.

Experimental data and an interactive version of all the relevant quantum optics experiments discussed in the present paper have been realized in the group of Prof. Jan-Peter Meyn at the university of Erlangen and are accessible at www.quantumlab.de[4].

This paper is structured as follows: In section II, we discuss the visualization of linear polarized light. In section III, we proceed to the polarization of entangled photons. Visualizations both of theory and experiment are presented. In the conclusion (section IV), we address advantages and limitations of our visual approach for physics education.

FIG. 2: *Left: At the polarizing beam splitter, the polarization state of the photon* $|\Omega_{\text{Polarisation}}\rangle = e^{i\omega t}\cos\theta|\hat{V}\rangle - e^{i\omega t}\sin\theta|\hat{H}\rangle$ *is split into the transmitted component* $\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle = -e^{i\omega t}\sin\theta$ *and the reflected component* $\langle\hat{V}|\Omega_{\text{Polarisation}}\rangle = e^{i\omega t}\cos\theta$*. Here,* $|\hat{H}\rangle$ *and* $|\hat{V}\rangle$ *are the horizontal and vertical axis at the polarizing beam splitter. Middle: Measured transmission and reflection results for single photons, visualized as random black-and-white patterns. At the rotatable half wave plate the angle* θ *is increased in intervals of* $\theta = \pi/12$ *relative to the horizontal axis* $|\hat{H}\rangle$*. Right: Resulting transmission and reflection probabilities* $p_{\hat{H}}(\theta) = \cos^2\theta$ *and* $p_{\hat{V}}(\theta) = \sin^2\theta$*.*

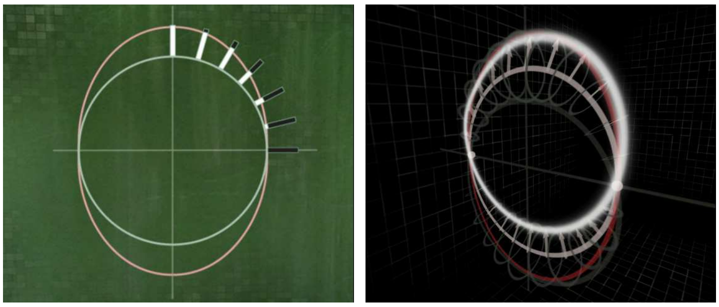

FIG. 3: *Left: Visualization of the angular dependency of the transmission probability* $p_{\hat{H}}(\theta) = |\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle|^2$ *of vertical polarized light. After many measurements, Malus law is recovered, indicated as red line. Right: Visualization of the transmission amplitude* $\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle = -\sin(\theta)e^{i\omega t} = \sqrt{p}e^{i\omega t}$.

II. POLARIZATION OF LIGHT

The photon wave function can be described as product of the position and polarization state: $|\Psi_{\text{Position}}\rangle \times |\Omega_{\text{Polarisation}}\rangle$. Different types of beam splitters act either on the position state $|\Psi_{\text{Position}}\rangle$ or the polarization state $|\Omega_{\text{Polarisation}}\rangle$. In the latter case, the beam splitter is called polarizing beam splitter (Fig. 2). At the polarizing beam splitter, the polarization wave function $|\Omega_{\text{Polarisation}}\rangle = \alpha|\hat{H}\rangle + \beta|\hat{V}\rangle$ is split into the transmitted horizontal amplitude $\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle = \alpha$ and the reflected vertical amplitude $\langle\hat{V}|\Omega_{\text{Polarisation}}\rangle = \beta$. The transmission probability is given by $p_{\square} = |\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle|^2 = |\alpha|^2$, the reflection probability by $p_{\blacksquare} = |\langle\hat{V}|\Omega_{\text{Polarisation}}\rangle|^2 = |\beta|^2$.

We will now discuss the visualization of vertically polarized light, described by the wave function

$$|\Omega_{\text{Polarisation}}\rangle = e^{i\omega t}|V\rangle \tag{2}$$

The vertical axis of the incoming photon is arbitrary and differs from the basis $|\hat{H}\rangle, |\hat{V}\rangle$ given by the polarizing beam splitter. The polarization state is determined in the basis $|\hat{H}\rangle, |\hat{V}\rangle$, which is rotated by an angle θ relative to the basis $|H\rangle, |V\rangle$

$$\begin{aligned} |H\rangle &= \cos\theta|\hat{H}\rangle + \sin\theta|\hat{V}\rangle \\ |V\rangle &= \cos\theta|\hat{V}\rangle - \sin\theta|\hat{H}\rangle \end{aligned} \tag{3}$$

In the basis of the polarizing beam splitter, the wave function is given by

$$|\Omega_{\text{Polarisation}}\rangle = \cos\theta e^{i\omega t}|\hat{V}\rangle - \sin\theta e^{i\omega t}|\hat{H}\rangle \tag{4}$$

The amplitudes are $\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle = -\sin\theta e^{i\omega t}$ for transmission (□) and $\langle\hat{V}|\Omega_{\text{Polarisation}}\rangle = \cos\theta e^{i\omega t}$ for reflection (■). The probability that the detector "white" detects the photon is $p_{\square} = \sin^2\theta$. The probability that the detector "black" detects the photon is $p_{\blacksquare}(\theta) = \cos^2\theta$.

In the experiment, the polarization $|\Omega_{\text{Polarisation}}\rangle$ of the incident photon is rotated at the half wave plate in steps of $\theta = \pi/12$. The experimental results for single photons are shown in Fig. 2. In Fig. 3, we show transmission probabilities $p_{\square} = |\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle|^2$ and transmission amplitudes $\langle\hat{H}|\Omega_{\text{Polarisation}}\rangle$. Since the photon has angular momentum one in units of $\hbar$, the wave function has one nodal line. For vertically polarized light, the nodal line in the transmission amplitude is horizontal.

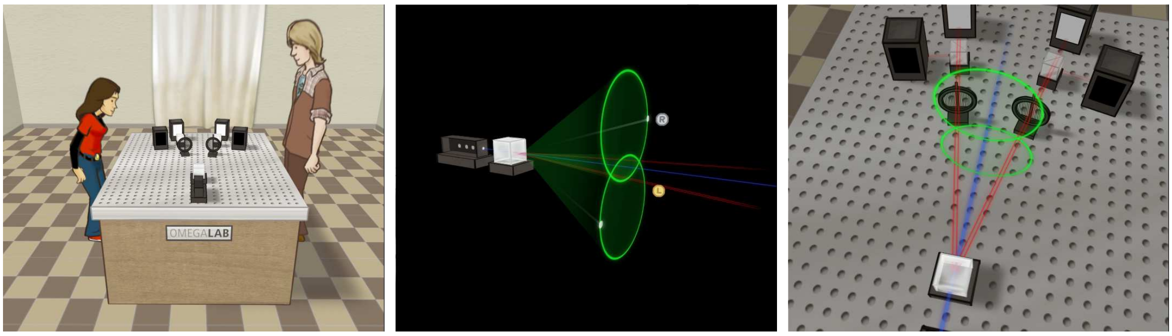

FIG. 4: *Left: "Alice and Bob" and their quantum optics experiment with pairs of photons. Middle: The photon pairs exit the birefringent BBO-crystal on opposite sides of two cones. Entanglement can occur on the two intersection lines of the cones. Right: The polarization state of each photon of the pair is detected on Alice and Bobs side, respectively. Both can choose their measurement axis* (α, β) *independently. For each combination of angles* (α, β) *of the polarizers, four combinations of measurement results can occur.*

III. POLARIZATION ENTANGLEMENT

A. The experimental setup

The experimental realization of polarization entanglement is a delicate thing. The most difficult part is the superposition of a pair of photons such that photon 1 cannot be distinguished from photon 2. In 1995, a new high-intensity source of polarization-entangled photon pairs has been realized[3]. Here, we only mention the fundamental necessities for the experimental realization and observation of polarization entangled photon pairs. The detailed description of the experiment can be found elsewhere[4]. The main idea is the following: In the birefringent BBO-crystal, a short ultraviolet laser pulse is converted by type-II parametric down-conversion to a pair of photons with orthogonal polarization. Let these polarization directions be $|H\rangle, |V\rangle$. In Fig. 4, this is illustrated by the means of two cones: The horizontally polarized photon is emitted somewhere on the upper cone, the photon in vertical polarization is emitted somewhere on the lower cone. Due to conservation of momentum , both photons lie on opposite sides of the pump beam. The intersection of the cones forms two lines.

Since *two* photons are observed, we need two detector pairs for transmission and reflection, called A (Alice) and B (Bob), each observing the polarization state of one photon. When the detectors are placed in the direction of the two *intersection* lines of the two cones, two possibilities emerge: Either detector A observes the photon with polarization $|H\rangle$ and detector B the photon with polarization $|V\rangle$, or vice versa. When both possibilities are indistinguishable, their superposition is the Bell state

$$|\psi^+\rangle := \frac{1}{\sqrt{2}}|H_1, V_2\rangle + \frac{1}{\sqrt{2}}|V_2, H_1\rangle \tag{5}$$

In the birefringent BBO-crystal, the propagation velocities of the horizontal and vertical polarized waves are different. Only when this time delay is compensated, an entangled state is created. Otherwise, both possibilities become distinguishable and the superposition breaks down.[3]

The polarization of the photons is *detected* using a rotatable half wave plate and a polarizing beam splitter (Fig. 2) both for detector A (Alice) and detector B (Bob) (see Fig. 4). Both Alice and Bob can choose a basis for the measurement. Let $|H_A\rangle, |V_A\rangle$ be the basis chosen by Alice's half wave plate, and $|H_B\rangle, |V_B\rangle$ the basis chosen by Bob's half wave plate. The measurement results in a given basis can be visualized by a combination of two random black-white patterns: One for the results found by Alice in her basis ($|H_A\rangle$, transmission $\equiv \square$, and $|V_A\rangle$, reflection $\equiv \blacksquare$), one for the results found by Bob in his basis ($|H_B\rangle \equiv \square$, $|V_B\rangle \equiv \blacksquare$). In any basis, one of the four possible results $\{(\blacksquare, \blacksquare), (\blacksquare, \square), (\square, \blacksquare), (\square, \square)\}$ is obtained in each experiment. The probabilities for each of these four combinations can be calculated from the wave function $|\psi^+\rangle$.

B. A classical analogy

For illustration, we propose a classical analogy. We replace the detectors A and B by two persons, called Alice and Bob. Both observe *one* coin, but from different perspectives (Fig. 5): Alice observes the coin from above the table, Bob from the beneath the table. The two sides of the coin are called $\mathcal{P}_1$ and $\mathcal{P}_2$. If Alice observes $\mathcal{P}_1$, Bob will get $\mathcal{P}_2$, and vice versa. Before the measurement, both possibilities have probability 1/2:

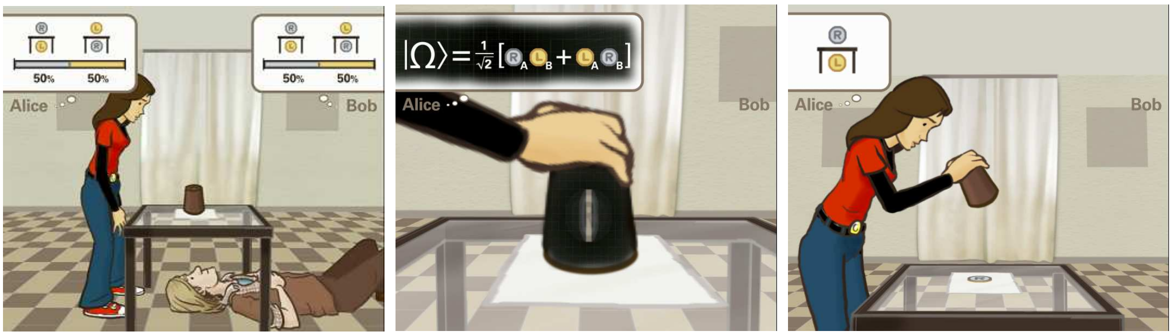

FIG. 5: *Left: Before the measurement, the polarization state is not yet determined. The two possibilities* $\{\mathcal{P}_1^{Alice}, \mathcal{P}_2^{Bob}\}$ *and* $\{\mathcal{P}_2^{Alice}, \mathcal{P}_1^{Bob}\}$*, visualized as two sides of a coin, have equal probabilities. Here, we choose the singlet Bell state* $|\phi^+\rangle := \frac{1}{\sqrt{2}}|R_1, L_2\rangle + \frac{1}{\sqrt{2}}|L_1, R_2\rangle$ *given as superposition of circular polarized light with* $\mathcal{P}_1 = R_1$ *for the "first" photon, and* $\mathcal{P}_2 = L_2$ *for the "second" photon. Different Bell states can be described, depending on the choice of* $\{\mathcal{P}_1, \mathcal{P}_2\}$*. Middle: In quantum mechanics, both possibilities are in superposition - visualized as a spinning coin. Right: Only after detection, the superposition breaks down. If Alice measures in the circular basis right-circularly polarized light, then Bob measures in the same basis left-circularly polarized light.*

$$\begin{aligned} 50\% &: \{\mathcal{P}_1^{Alice}, \mathcal{P}_2^{Bob}\} \\ 50\% &: \{\mathcal{P}_2^{Alice}, \mathcal{P}_1^{Bob}\} \end{aligned} \tag{6}$$

A measurement of the state of the coin does not change its state. Therefore, no interference is possible and the description of the two states of the coin in terms of probabilities is sufficient. In this respect, the coin is different from a quantum state.

In quantum mechanics, the state of the "coin" before the measurement is undetermined. If both possibilities are indistinguishable, a superposition state is created. Only after the measurement, one realization is observed. In our analogy this may be represented by a spinning coin, whose rotation is stopped "instantaneously" by a swift move of the palm of Alice's or Bob's hand. Since Alice's measurement changes the state of the coin and the possible results for Bob, Alice's measurement influences the possible outcome of Bob's measurement.

From the classical/quantum analogy "coin $\rightarrow$ spinning coin", we proceed to the entangled state by replacing probabilities by quantum amplitudes, $p \rightarrow \sqrt{p}e^{i\phi}$. We obtain the entangled

Bell-state

$$|\Omega_{\text{Polarisation}}(1,2)\rangle \propto \frac{1}{\sqrt{2}}|\mathcal{P}_1,\mathcal{P}_2\rangle + \frac{1}{\sqrt{2}}|\mathcal{P}_2,\mathcal{P}_1\rangle \tag{7}$$

Depending on the polarization states $\mathcal{P}_1, \mathcal{P}_2$ we choose for the "two sides of the coin", different types of entanglement can be created.

C. Visualization of polarization entanglement

With a slightly modified experimental setup, not only the entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}|H_1,V_2\rangle + \frac{1}{\sqrt{2}}|V_2,H_1\rangle$, but also any other Bell state can be created. We choose the singlet state with zero angular momentum as an example:

$$|\phi^+\rangle = \frac{e^{i\omega t}}{\sqrt{2}}|R_1\rangle|L_2\rangle + \frac{e^{i\omega t}}{\sqrt{2}}|L_1\rangle|R_2\rangle \tag{8}$$

This Bell states may be interpreted as the "two sides of the same coin" in the basis $|R\rangle, |L\rangle$. As a straightforward generalization of the visualization of the polarization of a single photon, we can visualize entangled states. Since the total angular momentum is zero, the state has zero nodal lines. Indeed, as shown in Fig. 6, $|\phi^+\rangle$ is rotation invariant.

The state $|\phi^+\rangle$ can be expressed in any basis, not only $|R\rangle, |L\rangle$. Since Alice and Bob measure linear polarization, we express $|\phi^+\rangle$ as

$$\begin{aligned} |\phi^+\rangle &\propto \frac{1}{\sqrt{2}}|R_1\rangle|L_2\rangle + \frac{1}{\sqrt{2}}|L_1\rangle|R_2\rangle \\ &= \frac{1}{2\sqrt{2}}(|H_1\rangle + i|V_1\rangle)(|H_2\rangle - i|V_2\rangle) + \frac{1}{2\sqrt{2}}(|H_1\rangle - i|V_1\rangle)(|H_2\rangle + i|V_2\rangle) \\ &= \frac{1}{\sqrt{2}}|H_1\rangle|H_2\rangle + \frac{1}{\sqrt{2}}|V_1\rangle|V_2\rangle \end{aligned} \tag{9}$$

The visualization of this equation is shown in Fig. 6. The state $|\phi^+\rangle$ is rotation invariant. Therefore, any rotation of the coordinate system $|H\rangle, |V\rangle$ leaves $|\phi^+\rangle$ invariant:

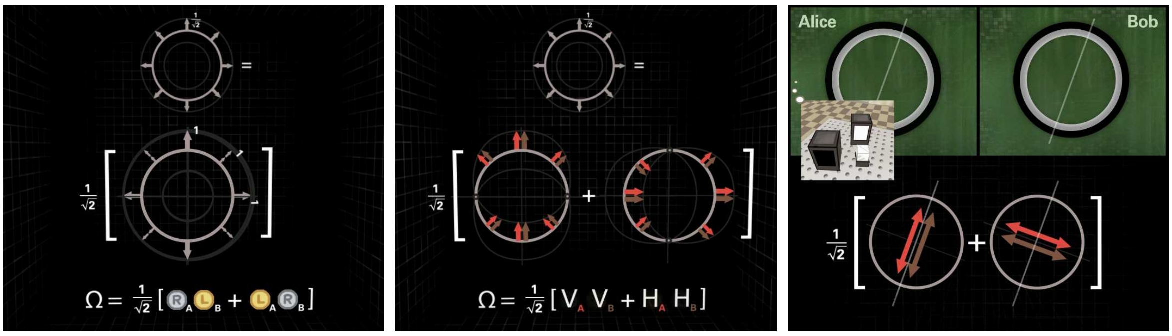

FIG. 6: *Left: The superposition* $|\phi^+\rangle = \frac{1}{\sqrt{2}}|R_1, L_2\rangle + \frac{1}{\sqrt{2}}|L_1, R_2\rangle$ *of left- and right circularly polarized light is the rotation invariant singlet state with zero angular momentum and no nodal lines. Middle: The same state can be described in the basis of linear polarized light as* $\frac{1}{\sqrt{2}}|V_1, V_2\rangle + \frac{1}{\sqrt{2}}|H_1, H_2\rangle$*. Right: Alice chooses her basis* $(\hat{H}, \hat{V})$*. Due to rotation invariance, the wave function is given by* $|\phi^+\rangle = \frac{1}{\sqrt{2}}|\hat{V}_1, \hat{V}_2\rangle + \frac{1}{\sqrt{2}}|\hat{H}_1, \hat{H}_2\rangle$ *in any basis. The corresponding transmission and reflection probabilities, which are* 50% *in any basis at Alice and Bobs side, are shown in the upper half of the figure.*

$$
\begin{aligned}
|\phi^+\rangle &\propto \frac{1}{\sqrt{2}}|H_1\rangle|H_2\rangle + \frac{1}{\sqrt{2}}|V_1\rangle|V_2\rangle \\
&= \frac{1}{\sqrt{2}}(\cos(\theta)|H_1\rangle - \sin(\theta)|V_1\rangle)(\cos(\theta)|H_2\rangle - \sin(\theta)|V_2\rangle)) \\
&+ \frac{1}{\sqrt{2}}(\cos(\theta)|V_1\rangle + \sin(\theta)|H_1\rangle)(\cos(\theta)|V_2\rangle + \sin(\theta)|H_2\rangle)) \\
&= \frac{1}{\sqrt{2}}|\hat{H}_1\rangle|\hat{H}_2\rangle + \frac{1}{\sqrt{2}}|\hat{V}_1\rangle|\hat{V}_2\rangle.
\end{aligned} \tag{10}
$$

The meaning of this equation is obvious: Indeed, the state $|\phi^+\rangle$ is rotation invariant.

D. Visualization of the measurement process

Let Alice choose a basis $|H_A\rangle, |V_A\rangle$, and Bob a basis $|H_B\rangle, |V_B\rangle$. In Alice (respectively, Bobs) basis, the rotation invariant Bell state is given by

$$
|\phi^+\rangle = \frac{1}{\sqrt{2}}|H_A\rangle|H_B\rangle + \frac{1}{\sqrt{2}}|V_A\rangle|V_B\rangle \tag{11}
$$

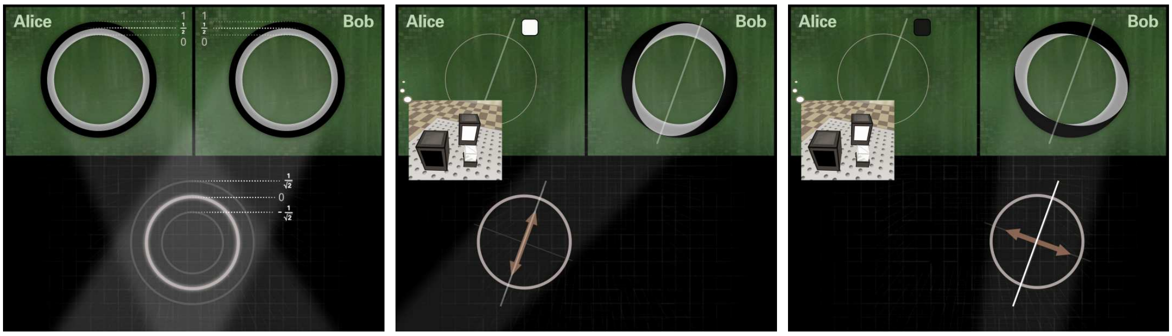

FIG. 7: *Left: Before the interaction with the detectors, Alice and Bob share the same wave function. Middle: If Alice photon is transmitted, the polarization of Bob's photon is horizontal relative to Alice's axis. Note that Alice's choice of basis has immediate influence on Bob's photon. Right: If Alice's photon is reflected, then the polarization of Bob's photon is orthogonal relative to Alice axis.*

While, we will argue from Alice's point of view in the following, the roles of Alice and Bob are interchangeable. When a *linear* polarization state is measured by detector Alice, either the result H_A (□) or V_A (■) is detected. The two-photon wave function $|\phi^+\rangle$ is reduced to the single-photon wave function $|V_B\rangle$ or $|H_B\rangle$ as shown in Fig 7. The superposition of two photons with angular momentum zero (zero nodal lines) is destroyed, and a single photon with angular momentum one remains (one nodal line). Note that at Bob's detector, the position of the nodal line of *his* polarization state depends on the basis *Alice* chooses for her measurement. Therefore, Alice's choice of basis has immediate consequences for Bob, although Bob might not be able to notice that in the first place. If at Alice side, the detector "white" □ received the photon, Bob's wave function in his respective basis is

$$|\psi_B\rangle = e^{i\omega t}|H_A\rangle = e^{i\omega t}\cos(\beta-\alpha)|H_B\rangle + e^{i\omega t}\sin(\beta-\alpha)|V_B\rangle \tag{12}$$

The nodal line of Bob's photon is parallel to Alice axis. If at Alice side, the detector "black" (■) received the photon, Bob's wave function in his respective basis is

$$|\psi_B\rangle = e^{i\omega t}|V_A\rangle = e^{i\omega t}\cos(\beta-\alpha)|V_B\rangle - e^{i\omega t}\sin(\beta-\alpha)|H_B\rangle \tag{13}$$

The nodal line of Bob's photon is orthogonal to Alice axis. Note that $Alice'$ choice of axis determines the position of the nodal line of the photon at $Bob's$ side. The conditional probability $p(H_B|H_A)$ that Bobs detects the result transmission (□) given that Alice got

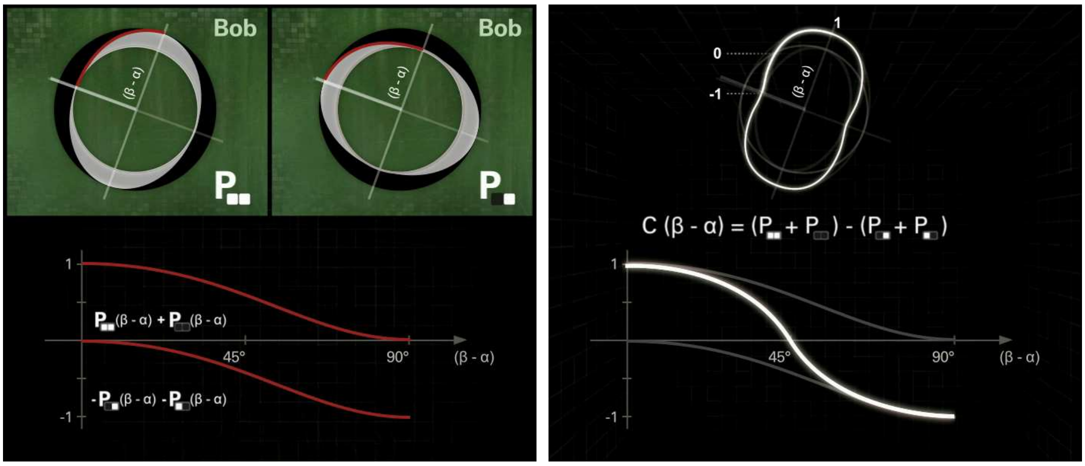

FIG. 8: *Left: Visualization of the conditional probabilities* $p(H_B|H_A) = \cos^2(\beta-\alpha)$ *and* $p(H_B|V_A) = \sin^2(\beta-\alpha)$. *Right: Visualization of the quantum mechanical prediction of the correlation function* $C_{\text{theory}}(\alpha,\beta) = \cos^2(\beta-\alpha) - \sin^2(\beta-\alpha) = \cos[2(\beta-\alpha)]$.

transmission (□) in her basis is given by

$$p(H_B|H_A) = \cos^2(\beta-\alpha) \tag{14}$$

The conditional probability $p(V_B|H_A)$ that Bobs detects the result reflection (■) is given by

$$p(V_B|H_A) = \sin^2(\beta-\alpha) \tag{15}$$

The same argument holds if Alice and Bob change their roles: then, $p(H_A|H_B) = \cos^2(\beta-\alpha)$ and $p(V_A|H_B) = \sin^2(\beta-\alpha)$. These equations are visualized in Fig. 8.

As long as Bob doesn't know Alice's results, he cannot distinguish between the results (H_A, V_A) on Alice's side. Therefore, he cannot measure the conditional probability, but only the sum

$$p(H_B) = p(H_B|H_A)p(H_A) + p(H_B|V_A)p(V_A) = p_{\square\square} + p_{\square\blacksquare} = \cos^2(\beta-\alpha)\frac{1}{2} + \sin^2(\beta-\alpha)\frac{1}{2} = \frac{1}{2} \tag{16}$$

Indeed, the probability for Bob to measure the result transmission (□) is 50% in any basis.

From the wave function $|\phi^+\rangle$, we can deduce the correlation function

$$\begin{aligned} C_{\text{theory}}(\beta-\alpha) &= [p(V_B|V_A)p(V_A)+p(H_B|H_A)p(H_A)] - [p(H_B|V_A)p(V_A)+p(V_B|H_A)p(H_A)] \\ &= p_{\square\square}+p_{\blacksquare\blacksquare}-p_{\square\blacksquare}-p_{\blacksquare\square} = \cos^2(\beta-\alpha)-\sin^2(\beta-\alpha) \\ &= \cos[2(\beta-\alpha)] \end{aligned} \tag{17}$$

Correlation is immediately shared between Alice and Bob via the "quantum channel" - *information* is not: Even if the wave function at Bob's detector is influenced immediately by Alice's measurement, Bob can recognize this only if he knows Alice result. This information is passed from Alice to Bob's side as digital information ("classical channel"). This cannot be done immediately, but is limited to the speed of light.

E. Comparison of theory and experiment

Next, we discuss the experimental results. A very convenient visualization of the *experimental* correlation function $C_{\text{exp.}}$ can be obtained using the random black-and-white patterns obtained from the detectors A (Alice) and B (Bob) as shown in Fig. 9. Two cases may be distinguished: Either Alice and Bob obtain the same result ($\{\square\square\}, \{\blacksquare\blacksquare\}$), or different results ($\{\square\blacksquare\}, \{\blacksquare\square\}$). The normalized difference between coinciding and non-coinciding results defines the experimental correlation function C

$$C_{\text{exp.}} = \frac{N_{\square\square}+N_{\blacksquare\blacksquare}-N_{\square\blacksquare}-N_{\blacksquare\square}}{N_{\square\square}+N_{\blacksquare\blacksquare}+N_{\square\blacksquare}+N_{\blacksquare\square}} \tag{18}$$

Depending on the basis chosen by Alice and Bob, the measured correlation $C_{\text{exp.}}(\beta-\alpha)$ changes dramatically. The measurement results for the state $|\phi^+\rangle$ are shown in Fig 9. For each combination of angles, we selected 64 measurements. The visualization of the correlation function for different combinations of angles (α, β) is shown in Fig 10.

The deviations of experimental and theoretical values are caused by fluctuations due to the small number of measurements which have been taken into account ($N = 64$). In the real experimental situation more than $N \simeq 100.000$ photon pairs per second are detected. In this case, the agreement between theory and experiment is overwhelming.

The astonishing point in quantum theory is the fact that *one* wave function simultaneously interacts with *two* detectors at arbitrary distances, as long as the superposition holds.

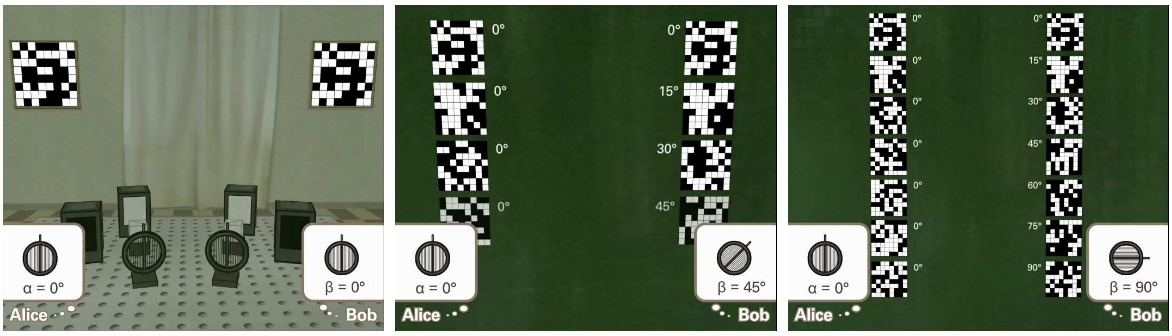

FIG. 9: *Left: If the linear polarization basis chosen by Alice and Bob coincides, their random black-and-white patterns will coincide. Both photons are completely correlated. Middle: Bob rotates his linear polarization basis in steps of* $\pi/12$. *For each combination* (α, β), *Alice and Bob observe different random black-and-white patterns. Right: If Bob's basis is rotated* $\pi/2$ *relative to Alice's, their results are completely anticorrelated.*

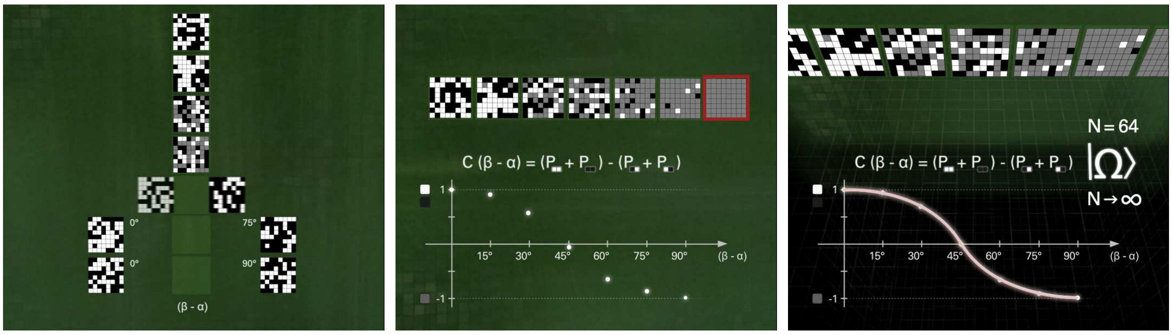

FIG. 10: *Left: Visualization of the experimental correlation function: Alice's and Bob's random black-and-white patterns are compared. Non-coinciding results are shown in grey. The respective correlation function is* $C_{\text{exp.}} = 1 - 2\frac{N_{\text{grey}}}{N}$ *with* $N = 64$. *Right: For* $N \to \infty$ *experimental and theoretical correlation function are in excellent agreement. Quantum mechanics correctly reproduces the experimental results using the assumption of non-local correlations.*

Correlation is shared immediately, even if the information about the nature of this correlation is revealed only after comparing the digital data exchanged between the detectors Alice and Bob.

IV. CONCLUSION

The visual approach to quantum entanglement proposed in this paper gives access to the basic ideas, even if the mathematical formalism behind the visualizations is not explained in detail, as it would probably be the case at high school and undergraduate levels of physics education. Yet, if we discuss the many advantages of visualization in the physics classroom, we must also be aware of its limitations: The antisymmetric Bell state $|\phi^-\rangle = \frac{1}{\sqrt{2}}|R_1, L_2\rangle - \frac{1}{\sqrt{2}}|L_1, R_2\rangle$, for example, has no appropriate classical analogy like a spinning coin.

In spite of that, we are convinced that visualizations like those presented in this paper will expand the topical range of today's physics courses, both at the high school, as well as at the undergraduate level of physics education. All figures shown in this papers are stills from animations of an interactive DVD-ROM which has been created for educational purposes in cooperation with a team of professional designers lead by M. Tewiele. In fact, the methods presented here can be generalized to all Bell states, to GHZ-states, a visualization of Bell-type inequalities and to computations of a quantum computer[5]. Even if simple classical analogies are not accessible in all cases, visualizations based on the generalization of Feynmans approach as proposed in Fig. 1 can be realized. We believe the use of advanced computer graphics to be a key for future developments in science education.

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