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The weak solution theory of the Porous Media Equation

BACHELOR'S THESIS

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1 Introduction

It is often said that mathematics is the language of the universe and nowhere this can be seen clearer than in the area of (partial) differential equations. As Steven Strogatz said in an article in the New York Times [Strogatz [2009]]:

> "In the 300 years since Newton, mankind has come to realize that the laws of physics are always expressed in the language of differential equations."

The heat equation (see below) and the wave equation are famous examples for this

$$u_t = \triangle u.$$

The heat equation hereby often finds application in the modelling of heat flow, but also can be used to describe the flow of matter like the diffusion of a substance. In this thesis, we will get to know another partial differential equation that has an origin in this field, namely the porous media equation (PME)

$$u_t = \triangle u^m$$

and its generalization

$$u_t = \triangle \Phi(u).$$

Originally, this equation has been derived to describe the flow of a gas through a porous medium like sand or soil. This is where it got its name from.

It is clear that this equation coincides with the heat equation when $m = 1$, where the theory is well known. In this thesis, we will therefore deal with the case that $m \neq 1$. Then, the equation is nonlinear and the classical theory for the existence and uniqueness of solutions can not be applied. Furthermore, we will see that solutions to this equation greatly differ from solutions to the heat equation.

The goal of this thesis is to develop an existence and uniqueness theory for solutions to the (generalized) porous media equation. To build a first intuition the second chapter will start with the classical physical derivation of the PME for the flow of a gas through porous media. We will then give a formal definition of the PME and the generalized PME.

Chapter 3 will then deal with a special solution to the PME called the Barenblatt or fundamental solution. As with the heat equation there exists a solution originating from a point source or dirac distribution. From this solution, the main differences to the heat equation can be explored.

Starting with Chapter 4, we will focus on the generalized PME and it will begin with a recap of the classical theory of quasilinear PDEs. We will see that under some assumptions the classical theory can be applied to the generalized PME as well, giving us classical solutions. Next, we will prove some estimates that hold for those classical solutions. Both will be very important in the then following chapter.

Chapter 5 is the main part and the highlight of this thesis. Here, we will prove the existence and uniqueness of a generalized form of solutions to the GPME called weak solutions. The background and definition will be specified there.

2 Derivation and physical application

As has been said in the introduction, the porous media equation has originally been derived to describe the flow of a gas through a porous medium like sand. This is also the approach we will take in this chapter and it will be guided by chapter III of Muskat [1937] (p. 121-136). We want to construct a PDE for the density ρ of the gas flowing through porous media. There are two physical equations we need.

The continuity equation: Notice that since we are dealing with gases, which are compressible (in contrast to e.g. water), the density of the liquid is not constant. In this way, it is possible for a fluid to flow from all directions into an infinitesimal volume element, necessarily changing the density of the fluid inside. The continuity equation now connects the excess flux of the fluid into an infinitesimal volume element with the change of the density of the fluid in this volume element [Muskat [1937], p. 121]:

$$\operatorname{div}(\rho v) = \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial y}(\rho v_y) = -\varepsilon \frac{\partial \rho}{\partial t}. \tag{2.1}$$

v is the velocity vector of the fluid and $v_{x/y/z}$ are the scalar velocities in the corresponding directions. ε is hereby a function describing the porosity of the medium. We will assume in the following that the medium is homogeneous such that $\varepsilon = \text{const}$.

This is a PDE of two unknown functions ρ and v. What we now need is an identity somehow relating those functions such that we get a PDE where ρ is the only unknown function.

Darcys Law: In 1856, the french engineer Darcy found an empirical law describing the flow of water through sand. It has since then been generalized for arbitrary fluids, media and three dimensions [Muskat [1937], p. 128] and takes on the form

$$v = -\frac{k}{\mu}\nabla p. \tag{2.2}$$

It states that the velocity v of a fluid in porous media is directly proportional to the pressure gradient ∇p. k is a scalar function describing the permeability of the medium. In our case it is once again constant thanks to the homogeneousity of the medium. μ is the viscosity constant of the fluid. We hereby have neglected any external force acting on the fluid. In particular notice that since we are dealing with gases, gravity can be ignored.

Inserting (2.2) into (2.1) we obtain

$$\operatorname{div}(\rho\frac{k}{\mu}\nabla p) = \varepsilon\frac{\partial\rho}{\partial t}. \tag{2.3}$$

At last we can express the pressure p through the density ρ, assuming that we are dealing with an ideal gas. Then the ideal gas equation holds [Muskat [1937], p. 124]

$$p = \frac{\rho RT}{M} \tag{2.4}$$

with R the universal gas constant, T the temperature and M the molar mass of the gas.

We now can consider two cases. The first one is an **isothermal flow** where $T = \text{const}$. In this case ρ is the only non constant property in (2.4). (2.3) then can be written as

$$\underbrace{\frac{kRT}{\varepsilon M\mu}}_{:=c}\operatorname{div}(\rho\nabla\rho) = \frac{c}{2}\operatorname{div}(\nabla\rho^2) = \frac{c}{2}\triangle\rho^2 = \frac{\partial\rho}{\partial t}. \tag{2.5}$$

The constant $\frac{c}{2}$ can be scaled out and we obtain the porous media equation with $m = 2$.

For the case of an **adiabatic flow** i.e., there is no heat exchange between the gas and the surrounding, we have the identity [Muskat [1937], p. 124]

$$p = \left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{2.6}$$

ρ_0 is the reference density and γ is the specific heat ratio. γ takes on real values greater than 1. For example for an ideal diatomic gas γ can be derived as $\gamma = 1.4$ [Spektrum Akademischer Verlag, Heidelberg [1998]]. Air consists mainly of Oxygen and Nitrogen, two diatomic gases, and can be regarded as a an ideal gas at standard conditions. Here, (2.3) is transformed to

$$\frac{k}{\varepsilon\rho_0^{\gamma}\mu}\operatorname{div}(\rho\nabla\rho^{\gamma}) = \underbrace{\frac{k\gamma}{\varepsilon\rho_0^{\gamma}\mu(\gamma+1)}}_{:=\tilde{c}}\operatorname{div}(\nabla\rho^{\gamma+1}) = \frac{\partial\rho}{\partial t} \tag{2.7}$$

The constant $\tilde{c}$ can once again be factored out and the PME with $m = 1 + \gamma$ is received.

This derivation only leads to cases where $m \geq 2$ but our theory can be safely done for $m > 0$.

Definition 2.1 (The Porous Media Equation)**.**
We call equation (2.8) the **porous media equation** (PME)

$$u_t = \triangle(u^m),\ m > 0. \tag{2.8}$$

Solutions to this equation are generally assumed to be positive. Considering also signed solutions, the PME equation is often changed to the **signed porous media equation**

$$u_t = \triangle(|u|^{m-1}u),\ m > 0. \tag{2.9}$$

This equation can be generalized by changing $|u|^{m-1}u$ to another continuous strictly increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(\pm\infty) = \pm\infty$. For simplicity we use the normalization $\Phi(0) = 0$, but a constant can always be added or subtracted. Another function f, also called forcing term, is added, which in physical means accounts for mass sources and sinks in the medium. We call this PDE the **generalized porous media equation (GPME)**

$$u_t = \triangle(\Phi(u)) + f. \tag{2.10}$$

In the derivation above we assumed the dimension of the space to be 3, but the definition also works for all dimensions $n \geq 1$.

Our theory of solutions will be based around the generalized porous media equation. It is clear that the PME is a special case of the GPME.

3 The Barenblatt solution

This chapter will deal with a special solution to the porous media equation, called the Barenblatt solution.

Definition 3.1. For an arbitrary constant $C > 0$, n the dimension of the space, we call

$$U(x,t) = t^{-\frac{n}{n(m-1)+2}} \left(C - \frac{m-1}{2nm(m-1)+4m} (|x| t^{-\frac{1}{n(m-1)+2}})^2 \right)_+^{\frac{1}{m-1}}. \tag{3.1}$$

the **Barenblatt solution** or **fundamental solution to the porous media equation** (2.8) [Vazquez [2006], p. 60].

It is constructed similarly to the fundamental solution to the heat equation (3.2) [Vazquez [2006], p. 59]:

$$V(x,t) = \left(\frac{1}{4t\pi}\right)^{n/2} \exp\left(\frac{-|x|^2}{4t}\right). \tag{3.2}$$

Both are radialsymmetric solutions with the dirac distribution as an initial condition. But as we will see both solutions are actually pretty different and we will discuss those differences in the second part of the chapter.

We will start with the derivation of the Barenblatt solution as in section 4.4.2 of Vazquez [2006] (p. 62-64).

3.1 Derivation of the Barenblatt solution

As for the heat equation we search for a selfsimilar solution, i.e. a solution of the form

$$U(x,t) = t^{-\alpha} f(\eta), \text{ with } \eta = xt^{-\beta} \tag{3.3}$$

with α and β being positive constants called similarity exponents and f a function called self-similar profile. Those have to be determined s.t. U is a solution to the PME. We want to apply the PME (2.8) ($U_t = \triangle(U^m)$) to (3.3). Calculating both sides independently, we obtain

$$\begin{aligned} U_t &= -\alpha t^{\alpha-1} f(xt^{-\beta}) + t^{-\alpha} \nabla f(xt^{-\beta}) \cdot (-\beta x t^{-\beta-1}) \\ &= -t^{-\alpha-1}(\alpha f(\eta) + \beta \eta \cdot \nabla f(\eta)) \end{aligned}$$

and

$$\begin{aligned}\triangle(U^m) &= t^{-\alpha m}\triangle(f^m(xt^{-\beta})) = t^{-\alpha m}\mathrm{div}[\nabla(f^m(xt^{-\beta}))]\\ &= t^{-\alpha m}t^{-\beta}\mathrm{div}[\nabla(f^m)(xt^{-\beta})] = t^{-\alpha m-2\beta}\triangle(f^m)(xt^{-\beta})\\ &= t^{-\alpha m-2\beta}\triangle(f^m)(\eta).\end{aligned}$$

Putting this into the partial differential equation (2.8) we recieve:

$$-t^{-\alpha-1}(\alpha f(\eta)+\beta\eta\cdot\nabla f(\eta)) = t^{-\alpha m-2\beta}\triangle(f^m)(\eta) \tag{3.4}$$

$$\Longleftrightarrow -t^{\alpha(m-1)+2\beta-1}(\alpha f(\eta)+\beta\eta\cdot\nabla f(\eta)) = \triangle(f^m)(\eta). \tag{3.5}$$

Since the right side only depends on x and t through η, this must hold for the left side as well. Therefore $t^{\alpha(m-1)+2\beta-1}$ has to be equal to one. We obtain a relation between α and β:

$$\alpha(m-1)+2\beta-1=0. \tag{3.6}$$

(3.5) then yields

$$\alpha f(\eta)+\beta\eta\cdot\nabla f(\eta)+\triangle(f^m)(\eta)=0. \tag{3.7}$$

Another relation between α and β can be obtained by using the physical law conservation of mass. Remember that a solution describes the density of a gas at a specific point. Integrating the density in space yields the mass s.t. in this context conservation of mass means $\int_{\mathbb{R}^n} U(x,t) = \text{const.}$ for every t. We recieve

$$\int_{\mathbb{R}^n} U(x,t) = t^{-\alpha}\int_{\mathbb{R}^n} f(xt^{-\beta})dx = t^{-\alpha}t^{\beta n}\int_{\mathbb{R}^n} f(\eta)d\eta = \text{const.}$$

where we have used the substitution $\eta = xt^{-\beta}$ in the second equality. Since the constant does not depend on t the left side does neither and $t^{-\alpha}t^{\beta n}$ has to be equal to 1. Therefore it must hold that

$$\alpha = n\beta. \tag{3.8}$$

(3.6) and (3.8) can now be solved such that the exponents finally can be written in terms of n and m:

$$\beta = \frac{1}{n(m-1)+2},\ \ \alpha = \frac{n}{n(m-1)+2}. \tag{3.9}$$

We now have determined α and β. We still need to specify the function f. Since we search for a radial symmetric solution let $f(\eta)$ be of the form $f(\eta) = f(r) = f(|\eta|)$.

(3.7) then can be written as

$$\alpha f(r) + \beta r f'(r) + \triangle(f^m)(r) = n\beta f + \beta r f' + \frac{1}{r^{n-1}}(r^{n-1}(f^m)')' = 0 \tag{3.10}$$

since the laplacian in spherical coordinates simplifies for a radial symmetric function ϕ to [Schweizer [2013], p. 169]

$$\triangle\phi = \frac{1}{r^{n-1}}(r^{n-1}\phi')'.$$

We now want to solve the ordinary differential equation (3.10) to recieve the function f. Notice that (3.10) is equivalent to

$$(r^{n-1}(f^m)' + \beta r^n f)' = 0$$

and can be simply integrated to

$$r^{n-1}(f^m)' + \beta r^n f = C.$$

For the constant, we choose $C = 0$ s.t. $f \to 0$ when $r \to \infty$. This ODE can easily be solved. Since

$$\begin{aligned} & r^{n-1}(f^m)' + \beta r^n f = 0 \\ \iff \quad & (f^m)' + \beta r f = 0 \\ \iff \quad & m f^{m-1} f' + \beta r f = 0 \\ \iff \quad & \left(\frac{m}{m-1} f^{m-1}\right)' = -\beta r, \end{aligned}$$

integrating leads to

$$\frac{m}{m-1} f^{m-1} = -\frac{\beta}{2} r^2 + C$$

and

$$f^{m-1} = -\frac{\beta(m-1)}{2m} r^2 + \tilde{C} \tag{3.11}$$

with $\tilde{C}$ a new constant obtained by integration. To solve for f we have to ensure that the right side of (3.11) is positive. We will do this by only taking the positive part. Finally

$$f = \left(\tilde{C} - \frac{\beta(m-1)}{2m} r^2\right)_+^{\frac{1}{m-1}} \tag{3.12}$$

and considering the expression for α and β (3.9) and inserting everything into (3.3) leads to our final term for the solution.

With the construction of U (since we only consider the positive part in (3.11)), we have that U is a solution to the PME (2.8) whenever $U > 0$. But we do not necessarily get an everywhere differentiable function (see Figure (3.3) s.t. the Barenblatt solution can not be a solution to the PME in the classical sense.

In the next section, we will see that this function despite not being smooth still describes a flow of a gas as we would imagine s.t. we would still like to interpret the Barenblatt solution as a solution to the PME. This motivates the definition of a weak solution, a new concept of a solution to the (G)PME that does not need the classic regularity, i.e. be two times differentiable. In Chapter 5, we will see that for $t > 0$ U actually is a weak solution to the PME.

3.2 Properties of the Barenblatt solution

In this section, we will state a few interesting properties of the Barenblatt solution [Vazquez [2006] p. 59-61]. This will be done intuitively based on graphs of the fundamental solution. We will restrict ourselves to $n = 1$ and $C = 1$.

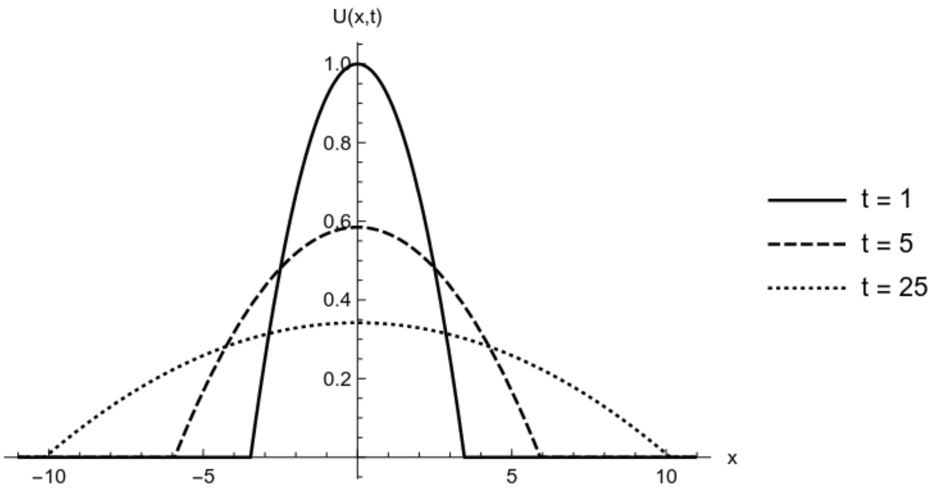

Figure 3.1: The Barenblatt solution U (3.1) with $C, n = 1$ and $m = 2$ drawn for times $t = 1, 5, 25$.

Figure 3.1 shows the Barenblatt solution for $m = 2$ drawn at different times. It can be seen that with increasing time the mass gets much more distributed as the graph becomes wider. One can also spot that when $t \to 0$ the solution approaches infinity at $x = 0$ and 0 everywhere else and we can guess that the solution actually approaches a dirac $\delta-$distribution. This can also be proven. This is a behavior we recognize from the fundamental solution to the heat equation and matches our intuition of a flow of a gas.

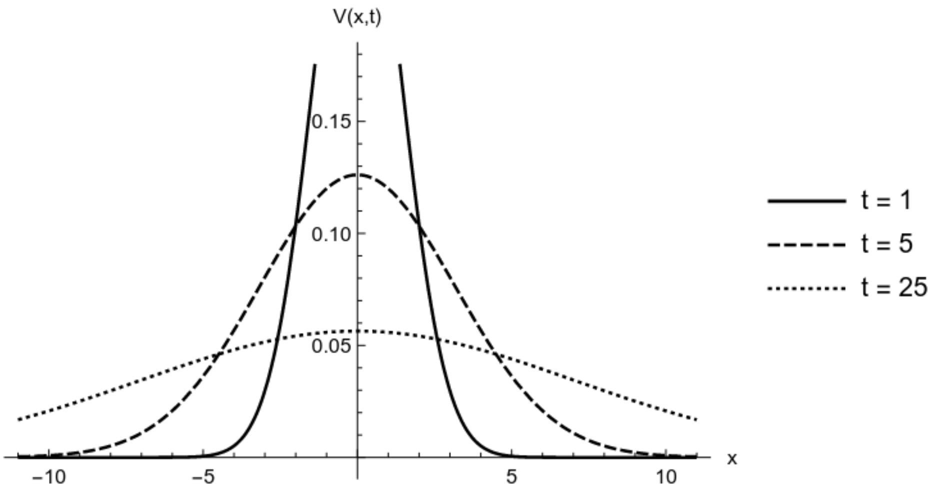

Figure 3.2: The fundamental solution to the heat equaiton V (3.2) drawn for times $t = 1, 5, 25$.

Different to the heat equation, we see that the Barenblatt solution has compact support for every t. This implies that we do not have an infinite propagation speed, which is a well known and important feature of the heat equation. This is one of the most peculiar and important differences of the PME to the heat equation. Imagining a flow of a gas through a porous media, a finite propagation speed seems much more appropriate.

One can also see that the fundamental solution to the heat equation is smooth, while the Barenblatt solution is not differentiable.

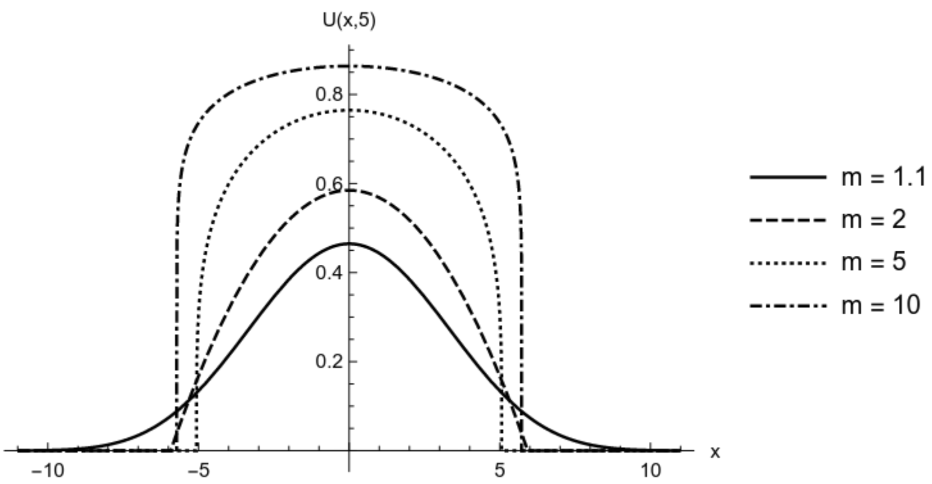

Figure 3.3: The Barenblatt solution U (3.1) with $C, n = 1$ and drawn at time $t = 5$ for $m = 1.1, 2, 5, 10$.

Another interesting property is that for $m \to 1$ the Barenblatt solution approaches the fundamental solution to the heat equation. This can be seen in Figure 3.3.

4 Basic results for quasilinear PDEs

In this chapter, we will first state a basic existence result for parabolic quasilinear PDEs in divergence form. This will then be used to construct - under some assumptions - solutions to the GPME that will help us build the theory to the (generalized) porous media equation in the next chapter. Furthermore, some estimates to the solutions will be proven, which will also be needed in the existence proof.

4.1 Quasilinear parabolic PDEs

For the announced existence result we first need to know what a parabolic quasilinear PDE in divergence form even is and what problem we want to solve. We will look at PDEs that can be written in the following way:

$$\begin{aligned}\mathcal{P}u :&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) + f(x,t,u,\nabla u) - u_t \\ &= \operatorname{div}(a(x,t,u,\nabla u)) + f(x,t,u,\nabla u) - u_t \\ &= 0. \end{aligned} \tag{4.1}$$

We will say that the quasilinear operator $\mathcal{P}$ or the PDE is in divergence form and $a(x,t,u,p)$ and $f(x,t,u,p)$ are called structural functions. Almost all quasilinear PDEs can be written in this form and some interesting results can be proven for this type of PDE. We want to find solutions to this equation that also fulfill the so called Dirichlet problem, a boundary and initial value problem. Let us first establish some notation.

Remark. We want to solve PDEs on a subset of n-dimensional space $\Omega \subset \mathbb{R}^n$ and for times $t < T$. The cylindrical subset of spacetime will be called $Q_T = \Omega \times (0,T)$ and Q if $T = \infty$. In this thesis Ω will always be a bounded and open subset of $\mathbb{R}^n$. The boundary of Ω will be denoted with $\Gamma = \partial\Omega$ and the lateral boundary with $\Sigma_T = \Gamma \times (0,T)$ or Σ if $T = \infty$. We will assume that Γ can be locally described as the graph of a Lipschitz function.

Definition 4.1 (The Dirichlet problem)**.**
Let $\mathcal{P}$ be a quasilinear operator in divergence form. For the **Dirichlet problem** we search for a function u such that (4.1) holds and further

$$u(x,0) = u_0(x) \text{ in } \Omega. \tag{4.2}$$

We call the function u_0 the initial data. Since $\Omega \subsetneq \mathbb{R}^n$, we further need data on the lateral boundary:

$$u(x,t) = g(x,t) \text{ in } \Sigma_T. \tag{4.3}$$

In **homogeneous Dirichlet problems**, the lateral boundary data has to be equal to 0, i.e. $g(x,t) \equiv 0$. Those two conditions can be combined in the following way:

$$u(x,t) = \rho(x,t) \text{ on } \Omega \times \{0\} \cup \Sigma_T, \tag{4.4}$$

where ρ coincides with u_0 and g on the boundary. We will call ρ the initial and boundary condition, IBC for short.

We will focus on quasilinear PDEs of parabolic type which have to satisfy the following uniform parabolicity condition.

Definition 4.2 (Quasilinear parabolic PDEs)**.**
PDEs of the form (4.1) which moreover satisfy the **uniform parabolicity condition**, i.e. constants $0 < c_1 < c_2 < \infty$ exist s.t. $\forall \xi \in \mathbb{R}^n, (x,t) \in \bar{Q}_T$, arbitrary $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$ inequality (4.5) holds:

$$c_1|\xi|^2 \leq \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial a_i}{\partial p_j}(x,t,u,p)\xi_i\xi_j \leq c_2|\xi|^2 \tag{4.5}$$

are called **quasilinear parabolic PDEs.**

With a few more assumptions on a, f and the boundary and initial conditions the Dirichlet problem to those PDEs can be uniquely solved with classical quasilinear theory.

We will now state the classical result of the quasilinear theory that we will be using. This is Thm. 6.2 of Ladyženskaja et al. [1968], where some generalities have been simplified with stricter assumptions.

Theorem 4.3. *Consider a Dirichlet problem as in Definition 4.1. Suppose that the following conditions hold:*

a) For $(x,t) \in \bar{Q}_T$, arbitrary $u \in \mathbb{R}$ it holds that

$$\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial a_i}{\partial p_j}(x,t,u,0)\xi_i\xi_j \geq 0 \textit{ for arbitrary } \xi \in \mathbb{R}^n \textit{ and} \tag{4.6}$$

$$A(x,t,u,0)u \geq -b_1u^2 - b_2 \tag{4.7}$$

for constants $b_i \geq 0$ and $A = -f - \sum_{i=1}^{n} \frac{\partial a_i}{\partial u} p_i - \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}$.

b) For $(x,t) \in \bar{Q}_T$ and arbitrary u and p the functions $a_i(x,t,u,p)$ and $f(x,t,u,p)$ are continuous and the $a_i(x,t,u,p)$ are differentiable with respect to x, u and p. Furthermore, the uniform parabolicity condition (4.5) holds and moreover

$$\sum_{i=1}^{n}\left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|\right)(1+|p|) + \sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial x_j}\right| + |f| \leq \mu(1+|p|)^2 \tag{4.8}$$

for a constant $\mu > 0$.

c) For $(x,t) \in \bar{Q}_T$ and arbitrary u and p the functions $a_i, f, \frac{\partial a_i}{\partial p_j}, \frac{\partial a_i}{\partial u}$ and $\frac{\partial a_i}{\partial x_i}$ are Hölder continuous functions in x, t, u and p with Hölder constants $\beta, \beta, \beta/2$ and β respectively.

d) For the IBC on the boundary it holds that $\rho|_{\Sigma_T} \in C^{2,1}(\Sigma_T)$, i.e. two times continuously differentiable in x and once in t. Furthermore $\sup_{x\in\Omega}|\rho_x(x,0)| < \infty$ and in general the IBC has to be Hölder continuous in x and t with Hölder constants γ, $\frac{\gamma}{2}$ respectively: $\rho \in C^{\gamma,\gamma/2}(\bar{Q}_T)$.

e) The boundary Γ can be locally described as a C^2-function: $\Gamma \in C^2$.

Then there exists a solution u to the Dirichlet problem (Def. 4.1) with $u \in C^{\alpha,\alpha/2}(\bar{Q}_T)$. The space derivatives ∇u are bounded in $\bar{Q}_T$ and the time derivative and the second space derivatives are in $C^{\beta,\beta/2}(Q_T)$. The solution is unique if $f(x,t,u,p)$ is locally Lipschitz continuous in u and p.

Proof. See [Ladyženskaja et al. [1968], p. 457-459]. □

Notice that, with the Hölder continuity of the derivatives, they can be continuously extended to $\bar{Q}_T$ s.t. we can assume that $u \in C^{2,1}(\bar{Q}_T)$.

Another interesting result for parabolic equations is the comparison principle, which will also be of great use.

Theorem 4.4 (Comparison principle)**.**
Let $\mathcal{P}$ be a quasilinear operator of the form (4.1) and parabolic, i.e. (4.5) holds. Let furthermore a and f be continuously differentiable with respect to u and p. If for $C^{2,1}(\bar{Q}_T)$-functions u and v holds that $\mathcal{P}u \geq \mathcal{P}v$ and $u \leq v$ on $\Omega \times \{0\}$ and Σ_T then $u \leq v$ in Q_T.

Proof. See Thm. 9.7 of Liebermann [2005] (p.222). □

We will often use it in the following form, where we can compare two solutions to a Dirichlet problem based on their initial and boundary data.

Remark. Let u, $v \in C^{2,1}(\bar{Q}_T)$ be two solutions to a Dirichlet problem with $\mathcal{P}_1 u = 0$ and $\mathcal{P}_2 v = 0$, the $\mathcal{P}_{1/2}$ as in the theorem with $\mathcal{P}_1 = a + f_1$, $\mathcal{P}_1 = a + f_2$. If for the data holds $u_0 \leq v_0$ and $g_1 \leq g_2$ and $f_1 \leq f_2$, then $u \leq v$.

Proof. With Thm. 4.4 it suffices to show that $\mathcal{P}_1 u \geq \mathcal{P}_1 v$:

$$\mathcal{P}_1 v = (a + f_1)(v) = (a + f_2 - f_2 + f_1)(v) = \underbrace{\mathcal{P}_2 v}_{=0} + \underbrace{(f_1 - f_2)}_{\leq 0}(v) \leq 0 = \mathcal{P}_1 u.$$

□

4.2 Classical solutions to the GPME

The (generalized) PME is also a quasilinear PDE and in the case Φ is differentiable, we can write it in the form of (4.1):

$$u_t = \operatorname{div}(\Phi'(u)\nabla u) + f(x,t)$$

s.t. $a(x,t,u,\nabla u) = \Phi'(u)\nabla u$, where Φ' denotes the derivative of Φ. We have that $\Phi' \geq 0$ since Φ is strictly increasing. Analogous to Def. 4.1, we can define the Dirichlet Problem for the GPME. The goal of this thesis will be to solve the homogeneous problem.

> *Remark* (The hom. Dirichlet problem for the GPME).
> For the GPME Def. 4.1 simplifies to
>
> $$u_t = \triangle(\Phi(u)) + f \text{ or } u_t = \operatorname{div}(\Phi'(u)\nabla u) + f(x,t) \text{ for differentiable } \Phi, \tag{4.9}$$
> $$u(x,0) = u_0(x) \text{ in } \Omega, \tag{4.10}$$
> $$u(x,t) = 0 \text{ on } \Sigma_T. \tag{4.11}$$

In the case of the PME we get:

$$u_t = \operatorname{div}(mu^{m-1}\nabla u), \tag{4.12}$$

but it can not be a uniform parabolic equation because the uniform parabolic condition (4.5) does not hold:

$$\begin{aligned} c_1|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j}(x,t,u,p)\xi_i\xi_j = \sum_{i,j=1}^{n} \frac{\partial}{\partial p_j}(\Phi'(u)p_i)\xi_i\xi_j = \sum_{j=1}^{n} \Phi'(u)\xi_j\xi_j \\ = \Phi'(u)|\xi|^2 = mu^{m-1}|\xi|^2 \end{aligned} \tag{4.13}$$

is only true if $c_1 = 0$ ((4.13) is equal to zero if $u = 0$) which is forbidden for uniform parabolicity. We call this broadening of the concept of parabolicity **degenerate parabolicity**. For degenerate parabolic problems, the above mentioned existence and uniqueness theorem does not work and this is the reason we have to build a new existence and uniqueness theory around the GPME.

Notice that the reason for the degeneracy is that the derivative of Φ can be equal to zero. If on the other hand $\Phi' \geq c > 0$, this is not the case and in (4.13) we have a lower bound that is greater than 0 and the equation is therefore non-degenerate.

The in (4.5) demanded upper bound is also not clear, since mu^{m-1} can get arbitrarily large. But other than the degeneracy this is not a huge problem as we will see.

We will now show that under some assumptions, one of them the positivity of Φ' s.t. the equation is non-degenerate, the GPME has a unique classical solution with the classical quasilinear theory. In the next chapter, we will then prove the existence of (weak) solutions to the (degenerate) GPME with general Φ by approximating them with classical solutions to non-degenerate problems of the following form.

Theorem 4.5 (Classical solutions to the generalized PME)**.**
Let $\Gamma \in C^2$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be as in Definition 2.1 with an additional smoothness assumption. Let furthermore $\Phi'(u) \geq c > 0$ for $u \in \mathbb{R}$, $f : \bar{Q}_T \to \mathbb{R}$ be a smooth and bounded function and the initial data $u_0 : \Omega \to \mathbb{R}$ a smooth function with compact support, i.e. $u_0 \in C_c^\infty(\Omega)$.

Then there exists a unique classical solution u to the problem (4.9)-(4.11) with the regularity of Thm. 4.3 and

$$-M_1 - N_1 t \leq u_n(x,t) \leq M_2 + N_2 t \text{ in } \bar{Q}_T \tag{4.14}$$

with $M_1 = \sup_{(x,t)\in\bar{Q}_T}(-u_0)$, $M_2 = \sup_{(x,t)\in\bar{Q}_T}(u_n)$, $N_1 = \sup_{(x,t)\in\bar{Q}_T}(-f)$ and $N_2 = \sup_{(x,t)\in\bar{Q}_T}(f)$.

Proof. With smooth Φ the GPME can be written in divergence form

$$u_t = \operatorname{div}(\Phi'(u)\nabla u) + f(x,t).$$

In the notation of (4.1), we therefore have $a(x,t,u,p) = \Phi'(u)p$. We now have to prove that the problem satisfies all the assumptions of Thm. 4.3.

One problem we immediately notice is that Φ' must not be bounded s.t. the uniform parabolicity condition must not hold. To fix this, we define a new function $\tilde{\Phi}'(u)$ with $\tilde{\Phi}'(u) = \Phi'(u)$ on the interval $[-M_1 - N_1 T, M_2 + N_2 T]$ with the constants as in the theorem and extend it smoothly to $\pm\infty$ s.t. $\tilde{\Phi}'(u)$ and all derivatives are bounded.

We first will show that this new hom. Dirichlet problem where Φ' has been replaced by $\tilde{\Phi}'$ (u_0 and f stay the same) satisfies Thm. 4.3 and then prove that the solution u only takes values in $[-M_1 - N_1T, M_2 + N_2T]$ where $\tilde{\Phi}'(u) = \Phi'(u)$ s.t. it also solves the problem with Φ' instead of $\tilde{\Phi}'$.

a): (4.6) is clearly true since $\Phi' \geq c > 0$ (as in (4.13)). (4.7) holds as well since

$$A(x,t,u,0)u = \left(-f - \sum_{i=1}^{n} \frac{\partial a_i}{\partial u} \cdot 0 - \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}\right) u = -f(x,t)u \geq -\sup(f)(u^2+1).$$

b) f is continuous with our assumptions and $a = \tilde{\Phi}'(u)p$ is smooth and therefore continuous and differentiable with respect to x, u and p. As in (4.13) the uniform parabolicity condition in our case is the following:

$$c_1|\xi|^2 \leq \tilde{\Phi}'(u)|\xi|^2 \leq c_2|\xi^2|$$

for arbitrary $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Since $0 < c \leq \Phi'$ and $\tilde{\Phi}'$ is bounded, we can find constants $c_1, c_2 > 0$ s.t. the condition holds. Further holds (4.8):

$$\begin{aligned}
&\sum_{i=1}^{n}\left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|\right)(1+|p|) + \sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial x_j}\right| + |f| \\
&= \sum_{i=1}^{n}\left(|\tilde{\Phi}'(u)p_i| + |\tilde{\Phi}''(u)p_i|\right)(1+|p|) + 0 + |f| \\
&= \underbrace{\left(|\tilde{\Phi}'(u)| + |\tilde{\Phi}''(u)|\right)\|p\|_1(1+|p|)}_{:=A} + |f|.
\end{aligned}$$

It suffices to show that $A \leq \mu(1+|p|)^2$ since f is bounded and therefore

$$A + |f| \leq \mu(1+|p|)^2 + \sup f \leq (\mu + \sup f)(1+|p|)^2.$$

Then

$$\begin{aligned}
& & \left(|\tilde{\Phi}'(u)| + |\tilde{\Phi}''(u)|\right)\|p\|_1(1+|p|) \leq \mu(1+|p|)^2 \\
&\Longleftrightarrow & \left(|\tilde{\Phi}'(u)| + |\tilde{\Phi}''(u)|\right)\|p\|_1 \leq \mu(1+|p|) \\
&\Longleftarrow & (|\tilde{\Phi}'(u)| + |\tilde{\Phi}''(u)|)(1+\|p\|_1) \leq \mu(1+|p|)
\end{aligned}$$

and since in $\mathbb{R}^n$ all norms are equivalent, we only need to show that $(|\tilde{\Phi}'(u)| + |\tilde{\Phi}''(u)|)$ is bounded. This is the case for $|\tilde{\Phi}'(u)|$ with our choice of $\tilde{\Phi}'$ and we can also choose $\tilde{\Phi}'$ in a way that $|\tilde{\Phi}''(u)|$ is also bounded. This shows (4.8).

c): We have to show the Hölder continuity in x, t, u and p for arbitrary u and p and $(x,t) \in \bar{Q}_T$ of the following functions:

$$a_i = \tilde{\Phi}'(u)p_i, \qquad \frac{\partial a_i}{\partial p_j} = \begin{cases} \tilde{\Phi}'(u), & i = j \\ 0, & \text{else} \end{cases},$$

$$\frac{\partial a_i}{\partial u} = \tilde{\Phi}''(u)p_i, \qquad \frac{\partial a_i}{\partial x_i} = 0,$$

and f.

This is clear for the first four functions since $\Phi'(u)$ and $\Phi''(u)$ are smooth and therefore Lipschitz continuous on compact sets. Furthermore, $\tilde{\Phi}'(u)$ is equal to $\Phi'(u)$ on a compact set and outside we can extend it in a way that all derivatives are bounded. f is Hölder continuous with our assumptions.

d): The IBC ρ can be constructed from u_0 for example with $\rho(x,t) = u_0(x)$. Then $\rho|_{\Sigma_T} = 0$ s.t. we deal with a hom. Dirichlet Problem. Furthermore, since $u_0 \in C_c^\infty(\Omega)$, $\rho|_{\Sigma_T}$ obviously is in $C^{2,1}(\Sigma_T)$. As the derivative of u_0 is bounded (the derivative is also in $C_c^\infty(\Omega)$), we have that $\sup_{x\in\Omega} |\rho_x(x,0)| < \infty$. ρ is also Hölder continuous since u_0 is in $C_c^\infty(\Omega)$ and therefore in particular Lipschitz.

Finally, for **e)** we notice that $\Gamma \in C^2$ with our assumption in the theorem and we conclude that the problem has a solution u. Moreover, since f is smooth and therefore Lipschitz on compact sets the solution is unique.

We still have to show that the solution u only takes on values in $[-M_1 - N_1 T, M_2 + N_2 T]$ such that we also get a solution to the original problem.

Consider the corresponding Dirichlet problem to the PDE $\bar{u}_t = \operatorname{div}(\tilde{\Phi}'(\bar{u})\nabla\bar{u}) + \bar{f}$ with initial data $\bar{u}_0 \equiv M_2 = \sup u_0$, $\bar{f} \equiv N_2 = \sup f$ and boundary data $\bar{g} = M_2 + tN_2$. Then $\bar{u} = M_2 + tN_2$ is the unique solution to the problem. With the comparison principle (Theorem 4.4) follows that $u(x,t) \leq \bar{u}(x,t) = M_2 + tN_2$ for all $(x,t) \in Q_T$. Similarly, with $M_1 = \sup(-u_0)$ and $N_1 = \sup(-f)$ we obtain $u(x,t) \geq -M_1 - tN_1$. This shows that u only takes values in the set where $\Phi' = \tilde{\Phi}'$ such that it is a solution to the original problem and the theorem is proven. With this last step we also receive identity (4.14). □

4.3 Estimates for classical solutions

As already mentioned, the existence theorem in the next chapter will be based on approximating the degenerate problem with non-degenerate problems in the form of Theorems 4.3 and 4.5. Transitioning to the limit will then be based on estimates of the approximate solutions.

4.3.1 The energy estimate

To pass to the limit we will need to estimate the derivatives of a solution in some way. This first estimate is for a function of the space derivatives and we will see that the appropriate function to estimate will be $\Phi(u)$. The energy estimate ensures that for classical solutions $\|\nabla\Phi(u)\|^2_{L^2(Q_T)}$ can be controlled by the data.

Theorem 4.6 (The energy estimate)**.** *(as in 3.2.4 of Vazquez [2006], p.38, 39)*
For a classical solution u to the homogeneous Dirichlet problem as in Thm. 4.5 the following identity holds:

$$\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \int_\Omega \Psi(u(x,T))dx = \int_\Omega \Psi(u_0(x))dx + \iint\limits_{Q_T} f\Phi(u)dxdt, \tag{4.15}$$

where

$$\Psi(s) = \int_0^s \Phi(\sigma)d\sigma \tag{4.16}$$

is the primitive of Φ. Furthermore, if $f \neq 0$ holds the inequality

$$\frac{1}{2}\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \int_\Omega \Psi(u(x,T))dx \leq \int_\Omega \Psi(u_0(x))dx + C\iint\limits_{Q_T} f^2 dxdt, \tag{4.17}$$

where C is a constant only depending on Ω.

Notice that with the assumption of Thm. 4.5 and the proven regularity ($u \in C^{2,1}(\bar{Q}_T)$), all of those integrals are finite. Ψ is a non-negative function.

Proof. Because u is a classical solution, equation (4.9) holds and we can multiply it by $\Phi(u)$ and integrate in Q_t:

$$\begin{aligned}
&\iint\limits_{Q_T} u_t\Phi(u)dxdt = \iint\limits_{Q_T} \triangle(\Phi(u))\ \Phi(u)dxdt + \iint\limits_{Q_T} f\Phi(u)dxdt \\
\iff &\iint\limits_{Q_T} \frac{\partial}{\partial t}\Psi(u)dxdt = \int_0^T\int_\Gamma \underbrace{\Phi(u)}_{\Phi(u)=0 \text{ on } \Gamma}\ \nabla(\Phi(u))\cdot\nu dAdt - \iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt \\
&\qquad + \iint\limits_{Q_T} f\Phi(u)dxdt,
\end{aligned}$$

where we have used partial integration in x and ν denotes the outer unit normal vector on Γ. Integrating in t on the left side yields

$$\int_\Omega \Psi(u(x,T)) - \Psi(u(x,0))dx = -\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \iint\limits_{Q_T} f\Phi(u)dxdt,$$

from what follows the first identity through rearranging.

To acquire the inequality we first use Hölder's inequality on the last therm to get

$$\iint\limits_{Q_T} f\Phi(u)dxdt \leq \left(\iint\limits_{Q_T} f^2 dxdt\right)^{\frac{1}{2}} \left(\iint\limits_{Q_T} \Phi(u)^2 dxdt\right)^{\frac{1}{2}} \leq c\iint\limits_{Q_T} f^2 dxdt + \frac{1}{4c}\iint\limits_{Q_T} \Phi(u)^2 dxdt,$$

where we have used Young's inequality in the last step with arbitrary $c > 0$.

This last term can be bounded with the Poincaré inequality (Thm. 4.7 of Alt [2012]) since ∇u is bounded and $u = 0$ on Σ_T:

$$\frac{1}{4c}\iint\limits_{Q_T} \Phi(u)^2 dxdt \leq \frac{C}{4c}\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt.$$

Putting it all together, we get

$$\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \int_\Omega \Psi(u(x,T))dx \leq \int_\Omega \Psi(u_0(x))dx + c\iint\limits_{Q_T} f^2 dxdt + \frac{1}{2}\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt$$

with a constant c that only depends on Ω. Once again the claim follows through rearranging.

□

4.3.2 An estimate for the derivative in time

The next estimate will at first look a bit random, but for the existence proof we will also need an estimate of the time derivative of a solution or of a function of it and this turns out to be the appropriate way. This will become clearer in the next chapter.

Theorem 4.7 (The time derivative estimate). *(as in 3.2.5 of Vazquez [2006], p. 40, 41) For a classical solution u as in Thm. 4.5 the following identity holds*

$$\iint\limits_{Q_T} \zeta\Phi'(u)(u)_t^2 dxdt = \iint\limits_{Q_T} \frac{\zeta_t}{2}|\nabla(\Phi(u))|^2 - (f\zeta)_t\Phi(u)\ dxdt, \tag{4.18}$$

with $\zeta : [0,T] \to \mathbb{R}$ being a smooth cutoff function with $\zeta(0) = \zeta(T) = 0$.

Proof. Since u is a classical solution equation (4.9) holds for all $(x,t) \in Q_T$ and we can multiply it by $w_t = (\Phi(u))_t$ obtaining $w_t u_t = w_t \triangle w + w_t f$.

Integrating in Ω gives us

$$\begin{aligned}\int_\Omega w_t u_t dx &= \int_\Omega w_t \triangle w dx + \int_\Omega f w_t dx \\ &= \int_\Gamma \underbrace{w_t}_{=0 \text{ on } \Gamma} \nabla w \cdot \nu dA - \int_\Omega \underbrace{\nabla w \cdot \nabla w_t}_{=\frac{1}{2}\frac{\partial}{\partial t}|\nabla w|^2} dx + \int_\Omega f w_t dx\end{aligned}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla w|^2 dx + \int_\Omega f w_t dx,$$

where we have used partial integration in the second step.

Now we multiply the equation with a smooth cutoff function $\zeta : [0,T] \to \mathbb{R}$ with $\zeta(0) = \zeta(T) = 0$ and integrate in time. We use the cutoff function so that we lose the boundary terms of the partial integration in t:

$$\begin{aligned}
\iint\limits_{Q_T} \zeta w_t u_t dx &= -\int_0^T \frac{\zeta}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla w|^2 dx dt + \iint\limits_{Q_T} \zeta f w_t dx \\
&= -\left[\frac{\zeta(t)}{2}\int_\Omega |\nabla w(t)|^2 dx\right]_0^T + \iint\limits_{Q_T} \frac{\zeta_t}{2}|\nabla w|^2 dx dt \\
&\quad + \left[\int_\Omega \zeta(t) f(t) w(t) dx\right]_0^T - \iint\limits_{Q_T} (\zeta f)_t w dx dt \\
&= \iint\limits_{Q_T} \frac{\zeta_t}{2}|\nabla w|^2 dx dt - \iint\limits_{Q_T} (\zeta f)_t w dx dt,
\end{aligned}$$

what gives us the sought-after identity. □

4.3.3 The L^1-contraction principle

Lastly, we have the L^1-contraction principle, an interesting result on its own. It is for example possible to prove the stability of the problem with this result, but we will use it in the existence proof.

Theorem 4.8 (L^1-contraction principle)**.** *(Prop. 3.5 of Vazquez [2006])*
Let u and $\hat{u}$ be two solutions to the hom. Dirichlet problem for the GPME of the form of Thm. 4.5 with the same Φ. On the other hand, the initial data u_0, $\hat{u}_0$ and forcing terms f and $\hat{f}$ must not be equal. Then for every $0 \le \tau < t \le T$ holds

$$\int_\Omega (u(x,t) - \hat{u}(x,t))_+ dx \le \int_\Omega (u(x,\tau) - \hat{u}(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f - \hat{f})_+ dx ds \tag{4.19}$$

and

$$\|u(t) - \hat{u}(t)\|_{L^1(\Omega)} \le \|u(\tau) - \hat{u}(\tau)\|_{L^1(\Omega)} + \int_\tau^t \|f(s) - \hat{f}(s)\|_{L^1(\Omega)} ds. \tag{4.20}$$

Proof. Let $p \in C^1(\mathbb{R})$ be as such that $0 \le p \le 1$, $p(s) = 0$ for $s \le 0$ and $p'(s) > 0$ for $s > 0$. Let furthermore $w = \Phi(u) - \Phi(\hat{u})$ which vanishes on Σ_T since there $u = \hat{u}$.

Since u and $\hat{u}$ are solutions to (4.9), we can subtract those equations and get

$$u_t - \hat{u}_t = \triangle(\Phi(u)) - \triangle(\Phi(\hat{u})) + f - \hat{f} = \triangle w + f - \hat{f}.$$

We now can multiply with $p(w)$ and integrate in Ω:

$$\int_\Omega (u-\hat{u})_t p(w)dx = \int_\Omega \triangle w p(w)dx + \int_\Omega (f-\hat{f})p(w)dx.$$

With integration by parts we have for the right side

$$\begin{aligned}\int_\Omega \triangle w p(w)dx + \int_\Omega (f-\hat{f})p(w)dx &= \int_\Gamma p(w)\nabla w\cdot \nu dA \\ &- \int_\Omega \nabla w\cdot\nabla w p'(w)dx + \int_\Omega (f-\hat{f})p(w)dx \\ &= -\int_\Omega |\nabla w|^2 p'(w)dx + \int_\Omega (f-\hat{f})p(w)dx\end{aligned}$$

as $p(w)=0$ on Γ. Since $p'\geq 0$, the first term is smaller or equal to zero. We further notice that $(f-\hat{f})p(w)\leq (f-\hat{f})_+$ s.t

$$\int_\Omega (u-\hat{u})_t p(w)dx \leq \int_\Omega (f-\hat{f})_+ dx.$$

Now consider a monotone sequence of functions $(p_n)_{n\in\mathbb{N}}$ of the same form of p converging to the ${\text{sign}_0}^+$ function

$${\text{sign}_0}^+(s) = \begin{cases} 0, & s\leq 0 \\ 1, & s>0 \end{cases}.$$

Since

$$\frac{\partial}{\partial t}(u-\hat{u})_+ = {\text{sign}_0}^+(u-\hat{u})\frac{\partial}{\partial t}(u-\hat{u})$$

and also

$${\text{sign}_0}^+(u-\hat{u}) = {\text{sign}_0}^+(\Phi(u)-\Phi(\hat{u}))$$

as Φ is strictly monotone, it follows with monotone convergence

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega (u-\hat{u})_t p_n(w)dx &= \int_\Omega (u-\hat{u})_t {\text{sign}_0}^+(\Phi(u)-\Phi(\hat{u}))dx \\ &= \int_\Omega (u-\hat{u})_t {\text{sign}_0}^+(u-\hat{u})dx \\ &= \int_\Omega \frac{\partial}{\partial t}(u-\hat{u})_+ dx \\ &= \frac{d}{dt}\int_\Omega (u-\hat{u})_+ dx \leq \int_\Omega (f-\hat{f})_+ dx.\end{aligned}$$

If we integrate the last inequality in t from τ to t, we get with the fundamental theorem of calculus

$$\int_\Omega (u(x,t)-\hat{u}(x,t))_+ dx \le \int_\Omega (u(x,\tau)-\hat{u}(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f-\hat{f})_+ dxds$$

which is the first part of the statement.

Switching the roles of u and $\hat{u}$ leads to

$$\int_\Omega (\hat{u}(x,t)-u(x,t))_+ dx \le \int_\Omega (\hat{u}(x,\tau)-u(x,\tau))_+ dx + \int_\tau^t \int_\Omega (\hat{f}-f)_+ dxds.$$

If we add both inequalities, we get

$$\begin{aligned} &\int_\Omega (u(x,t)-\hat{u}(x,t))_+ + (\hat{u}(x,t)-u(x,t))_+ dx \\ \le &\int_\Omega (u(x,\tau)-\hat{u}(x,\tau))_+ + (\hat{u}(x,\tau)-u(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f-\hat{f})_+ + (\hat{f}-f)_+ dxds \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_\Omega |u(x,t)-\hat{u}(x,t)| dx \le \int_\Omega |u(x,\tau)-\hat{u}(x,\tau)| dx + \int_\tau^t \int_\Omega |f-\hat{f}| dxds \\ \Longleftrightarrow\ &\|u(t)-\hat{u}(t)\|_{L^1(\Omega)} \le \|u(\tau)-\hat{u}(\tau)\|_{L^1(\Omega)} + \int_\tau^t \|f-\hat{f}\|_{L^1(\Omega)} ds. \end{aligned}$$

This is the second part of the statement. □

Remark. Notice that with $\hat{u}=0$ we obtain

$$\int_\Omega (u(x,t))_+ dx \le \int_\Omega (u(x,\tau))_+ dx + \int_\tau^t \int_\Omega (f(x,s))_+ dxds \tag{4.21}$$

and

$$\|u(t)\|_{L^1(\Omega)} \le \|u(\tau)\|_{L^1(\Omega)} + \int_\tau^t \|f(x,s)\|_{L^1(\Omega)} ds. \tag{4.22}$$

5 Existence and uniqueness of weak solutions to the homogeneous Dirichlet problem

In this chapter, we will first get to know the concept of weak solutions to the GPME and then prove the uniqueness and existence in a subclass called weak energy solutions, i.e. solutions that satisfy an energy estimate as in chapter 4.

5.1 Weak solutions

In chapter 2, we discussed the Barenblatt solution to the PME and recognized that it can not be a classical solution to the PME because it is not differentiable. The goal of weak solutions is to expand our concept of solutions to the (G)PME in a way such that functions that do not have the necessary regularity can also be viewed as solutions. This is especially relevant in the physical context: The PME models the flow of a gas and one can imagine that the density distribution of a gas in fact must not be smooth (e.g Figure 3.1).

A common procedure to construct a concept of weak solutions is to start with the classic PDE, multiply it with a test function and integrate by parts to pull the derivatives from the solution to the test function. Therefore, consider a classical solution u to the homogeneous Dirichlet Problem (4.9)-(4.11) for the GPME on $Q_T = \Omega \times (0, T)$. Then

$$u_t = \triangle(\Phi(u)) + f.$$

We can multiply this equation with a test function $\eta \in C^1(\bar{Q}_T)$ which vanishes on Σ_T and for $t = T$. We see why this choice makes sense if we integrate:

$$\begin{aligned} &\iint\limits_{Q_T} u_t \eta dxdt = \iint\limits_{Q_T} \triangle(\Phi(u))\eta dxdt + \iint\limits_{Q_T} f\eta dxdt \\ \Leftrightarrow &\int_\Omega [u\eta]_0^T \, dx - \iint\limits_{Q_T} u\eta_t dtdx = \int_0^T \int_\Gamma \eta \nabla(\Phi(u)) \cdot \nu dAdt - \iint\limits_{Q_T} \nabla(\Phi(u)) \cdot \nabla\eta dxdt + \iint\limits_{Q_T} f\eta dxdt. \end{aligned}$$

Here, we used partial integration in t for $u_t\eta$ and partial integration in x for $\triangle\Phi(u)\eta$. Because of the definition of η ($\eta = 0$ on Σ_T and for $t = T$), we finally get

$$\iint\limits_{Q_T} \nabla\Phi(u) \cdot \nabla\eta - u\eta_t dxdt = \int_\Omega u(x, 0)\eta(x, 0)dx + \iint\limits_{Q_T} f\eta dxdt.$$

Notice that for this equation we don't need the second derivative of u in x and the derivative of u in t. If we use this equation instead of (4.9) to define a solution, we therefore get a bigger class of solutions, which we call weak solutions. Moreover, the initial condition is build in if we change $u(x,0)$ to $u_0(x)$. We only have to make some assumptions on u, u_0 and f such that the equation makes sense, i.e. the integrals are finite. Furthermore, we need to ensure the homogeneousity, what will be done by demanding that u has to be equal to 0 on the boundary.

Definition 5.1 (Weak solutions to the homogeneous Dirichlet problem of the GPME)**.** (Def. 5.4 of Vazquez [2006])
A function u is called a weak solution to the homogeneous Dirichlet problem of the GPME (4.9)-(4.11) with $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$ if

1. $u \in L^1(Q_T)$,
2. $\Phi(u) \in L^1((0,T) : W_0^{1,1}(\Omega))$ and
3. u satisfies the following identity

$$\iint\limits_{Q_T} \nabla(\Phi(u)) \cdot \nabla\eta - u\eta_t dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint\limits_{Q_T} f\eta dxdt \tag{5.1}$$

for all test functions $\eta \in C^1(\bar{Q}_T)$ which vanish on Σ_T and for $t = T$.

$W_0^{1,1}$ denotes hereby the Sobolev space (Def. 1.27 of Alt [2012]). Notice that the homogeneousity condition $u|_{\Sigma_T} = 0$ is part of the function class $\Phi(u(t)) \in W_0^{1,1}$: $\Phi(u)|_{\Sigma_T} = 0$ iff $u|_{\Sigma_T} = 0$ as Φ is strictly increasing and $\Phi(0) = 0$ (s. Def. (2.1)).

Obviously, every classical solution of the form of Thm. 4.5 is also a weak solution, but there also exist weak solutions that are not classical:

Remark (A weak solution that is not a classical solutions).
In chapter 2 we got to know the Barenblatt solution to the PME ($u_t = \triangle(u^m)$). This is actually not a weak solution as of Def. 5.1 since the initial data is a point source and as such not in $L^1(\Omega)$. Furthermore, it spreads to the whole space s.t. the boundary data can not be 0 for all times.

It is however possible to construct a weak solution from the Barenblatt solution if we simply start at $t_0 > 0$. For simplicity we assume that $0 \in \Omega$. Then

$$\hat{U}(x,t) = U(x, t + t_0) \tag{5.2}$$

is a weak solution to the PME for all intervals $(0,T)$ where the support of $\hat{U}$ still lies inside Ω and the boundary data is as such equal to zero.

Proof. We write (5.2) in the form

$$\hat{U}(x,t) = (t_0+t)^{-\alpha}\left(C - \kappa(|x|(t_0+t)^{-\beta})^2\right)_+^{\frac{1}{m-1}}$$

with $\kappa = \frac{\beta(m-1)}{2m}$.

The radius r of the support of $\hat{U}$ can be written as a function of t:

$$\begin{aligned} & (t_0+t)^{-\alpha}\left(C - \kappa(|x|(t_0+t)^{-\beta})^2\right)_+^{\frac{1}{m-1}} > 0 \\ \Longleftrightarrow\quad & C > \kappa(|x|(t_0+t)^{-\beta})^2 \\ \Longleftrightarrow\quad & \frac{C(t_0+t)^{2\beta}}{\kappa} > |x|^2. \end{aligned}$$

And because of this: $r(t) := \sqrt{\frac{C(t_0+t)^{2\beta}}{\kappa}}$. Now we can define $Q_1 := \{(x,t) \in Q_T|\ |x| < r(t)\}$ as the subset of Q_T where $\hat{U} > 0$ and $Q_2 := Q_T \setminus Q_1$ where $\hat{U} = 0$.

To prove 1. of Def. 5.1 notice that $\hat{U}$ is bounded by $(t_0+t)^{-\alpha}C^{\frac{1}{m-1}}$ for all $t > 0$ and therefore uniformly bounded in Q_T and since Q_T is bounded, $\hat{U}$ is in $L^1(Q_T)$.

For 2. notice that $\Phi(\hat{U}) = \hat{U}^m$ is also bounded and in $L^1(Q_T)$. For the weak gradient of $\hat{U}^m$ we observe that $\hat{U}^m(t) \in C^1(\Omega)$ for every t because $\hat{U}^m$ is C^1 in Q_1,

$$\nabla(\hat{U}^m)|_{Q_1} = \left(\frac{(t_0+t)^{-\alpha m}m}{m-1}\left(C - \kappa(|x|(t_0+t)^{-\beta})^2\right)^{\frac{1}{m-1}}\left(\kappa 2(t_0+t)^{-2\beta}x_i\right)\right)_{i=1,\dots,n}.$$

and $\nabla(\hat{U}^m)|_{Q_1} \to 0 = \nabla\hat{U}|_{Q_2}$ for $|x| \to r(t)$ s.t. $\nabla(\hat{U}^m)(t)$ is continuous. Since $\nabla(\hat{U}^m)$ is also uniformly bounded, we have $\hat{U}^m \in L^1((0,T) : W_0^{1,1}(\Omega))$.

Finally, we have to prove 3. Starting with the left side of (5.1), we can restrict the domain to Q_1 because everywhere else $\nabla(\hat{U})^m$ and $\hat{U}$ are equal to 0. We recall that

$$\hat{U} > 0 \Leftrightarrow |x| < \underbrace{\sqrt{\frac{C(t_0+t)^{2\beta}}{\kappa}}}_{:=B_x(t)}$$

and that for $\hat{U} > 0$ it is necessary that

$$t \geq \underbrace{\max\left\{\left(\frac{|x|^2\kappa}{C}\right)^{\frac{1}{2\beta}} - t_0, 0\right\}}_{:=B_t(x)}.$$

Then

$$\begin{aligned}\iint\limits_{Q_T} \nabla(\hat{U}^m)\cdot\nabla\eta - \hat{U}\eta_t dxdt &= \iint\limits_{Q_1} \nabla(\hat{U}^m)\cdot\nabla\eta - \hat{U}\eta_t dxdt \\ &= \int_0^T \int_{\{|x|<B_x(t)\}} \nabla(\hat{U}^m)\cdot\nabla\eta dxdt - \int_\Omega \int_{B_t(x)}^T \hat{U}\eta_t dtdx.\end{aligned}$$

We then can use partial integration in x on the first term and in t on the second (since we look at Q_1, where $\hat{U}$ is a classical solution):

$$\begin{aligned}&= \underbrace{\int_0^T \int_{\{|x|=B_x(t)\}} \eta\nabla(\hat{U}^m)\cdot\nu dA\, dt}_{\nabla(\hat{U}^m)=0 \text{ when } |x|=B_x(t)} - \iint\limits_{Q_1} \triangle(\hat{U}^m)\eta dxdt - \int_\Omega \underbrace{\left[\hat{U}\eta\right]_{B_t(x)}^T}_{\eta(T)=0} dx + \iint\limits_{Q_1} \hat{U}_t\eta dxdt \\ &= \iint\limits_{Q_1} \hat{U}_t\eta - \triangle(\hat{U}^m)\eta dxdt + \int_\Omega \underbrace{(\hat{U}\eta)(B_t(x))}_{\hat{U}(B_t(x))=0 \text{ for } B_t(x)\neq 0} dx \\ &= \iint\limits_{Q_1} (\hat{U}_t - \triangle(\hat{U}^m))\eta dxdt + \int_\Omega \hat{U}(0)\eta(0)dx.\end{aligned}$$

Since $\hat{U}$ is a classical solution on Q_1, the first term is equal to zero and we finally get identity (5.1) for the Barenblatt solution, recall that $f = 0$:

$$\iint\limits_{Q_T} \nabla(\hat{U}^m)\cdot\nabla\eta - \hat{U}\eta_t dxdt = \int_\Omega \hat{U}(0)\eta(0)dx.$$

□

5.2 Existence and uniqueness of weak solutions

Now that we have defined weak solutions, the thing left to do is proving the existence and uniqueness of those weak solutions to the homogeneous Dirichlet problem.

5.2.1 Energy solutions

To proof the existence of solutions by approximating the equation with non-degenerate equations, we need a way to pass to the limit. In order to do that an energy identity as in (4.17) will be very helpful. We will see that for weak solutions this will actually be an inequality:

$$\iint\limits_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \int_\Omega \Psi(u(x,T))dx \leq \int_\Omega \Psi(u_0(x))dx + \iint\limits_{Q_T} f\Phi(u)dxdt.$$

This can only hold if $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ with $H_0^1 = W_0^{1,2}$ and furthermore lets us assume that $\int_\Omega \Psi(u_0(x))dx < \infty$, i.e. $\Psi(u_0) \in L^1(\Omega)$ with Ψ the primitive of Φ. We will say that $u_0 \in L_\Psi(\Omega)$.

Moreover, we assume that $f \in L^2(Q_T)$ such that $\iint_{Q_T} f\Phi(u)dxdt$ exists. Under the conditions of Def. 5.1 we will call weak solutions that also fit those additional assumptions **weak energy solutions**.

5.2.2 Existence of weak solutions

In addition to the standard assumptions on Φ (Φ is continuous, strictly increasing and $\Phi(\pm\infty) = \pm\infty$), we will assume for simplicity that Φ is in $W_{loc}^{1,1}(\mathbb{R})$. The following proof is a much more detailed version of the proof of Thm. 5.7 of Vazquez [2006], where possible uncertainties have been cleared up.

Theorem 5.2 (Existence of weak energy solutions)**.**
Let $u_0 \in L^1(\Omega)$ and $u_0 \in L_\Psi(\Omega)$ and $f \in L^2(Q)$. Then there exists a weak solution u to the hom. Dirichlet problem of the GPME (Def. 5.1) in Q_T for arbitrary $T > 0$. Furthermore holds that $u \in L^\infty((0,T) : L_\Psi(\Omega))$ and $u \in L^\infty((0,T) : L^1(\Omega))$ and $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$. The energy inequality

$$\iint_{Q_T} |\nabla(\Phi(u))|^2 dxdt + \int_\Omega \Psi(u(x,T))dx \leq \int_\Omega \Psi(u_0(x))dx + \iint_{Q_T} f\Phi(u)dxdt \tag{5.3}$$

and the comparison principle holds: Consider two solutions u_1, u_2 with initial data u_{i_0} and forcing terms f_i, $i = 1, 2$. If $u_{1_0} \leq u_{2_0}$ and $f_1 \leq f_2$, then $u_1 \leq u_2$.

Proof. The idea of this proof consists of changing the non-linearity Φ to functions Φ_n with positive derivatives $\Phi_n' > 0$ that converge to Φ s.t. the problem will be non-degenerate. Let therefore $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of functions such that

i) $\Phi_n \in C^\infty(\mathbb{R})$ with $\Phi_n'(u) \geq c_n > 0$ for all $n \in \mathbb{N}$, $u \in \mathbb{R}$ and constants c_n,

ii) $\Phi_n \to \Phi$ uniformly on compact sets and in $W^{1,1}(\mathbb{R})$ on compact sets and

iii) $\Phi_n(0) = 0$ for all $n \in \mathbb{N}$.

The proof will be done in several lemmata with decreasing assumptions on u_0, f, Γ and Φ.

Let $T > 0$ be arbitrary.

Lemma 1: *The theorem holds under the additional assumptions that u_0, f and f_t are bounded, $\Gamma \in C^2$ and Φ is locally Lipschitz continuous.*

Proof. Let $f_n \to f$ be a uniformly bounded sequence of smooth functions converging to f in $L^p(Q_T)$ for all $p < \infty$. We can find such a sequence since the set of smooth functions is dense in L^p and f is bounded. Let furthermore $(u_{n_0})_{n\in\mathbb{N}} \subset C_c^\infty(\bar{\Omega})$ be a uniformly bounded sequence of functions converging to u_0 in $L_\Psi(\Omega)$ and $L^1(\Omega)$.

We get the following homogeneous Dirichlet problem

$$\begin{aligned} &u_t = \triangle(\Phi_n(u_n)) + f_n \text{ in } Q_T, \\ &u_n(x,0) = u_{n_0}(x) \text{ in } \bar{\Omega}, \\ &u_n(x,t) = 0 \text{ on } \Sigma_T. \end{aligned}$$

This problem can be solved with Thm. 4.5 and has a unique solution u_n with the regularity of Thm. 4.3. Since the comparison principle holds for those solutions, we can derive as in (4.5) that u_n is uniformly bounded in Q_T:

$$-M_1 - N_1 t \leq u_n(x,t) \leq M_2 + N_2 t \text{ in } \bar{Q}_T \ \forall n \in \mathbb{N}$$

with

$$M_1 = \sup_{(x,t)\in\bar{Q}_T, n\in\mathbb{N}} (-u_{n_0})$$

and $M_2 = \sup(u_{n_0})$, $N_1 = \sup(-f_n)$, $N_2 = \sup(f_n)$ accordingly, which are smaller than infinity with the choice of the sequences.

We get a uniformly bounded sequence u_n of solutions and we want to show that this sequence converges to a weak solution u to the GPME.

For u to be a weak energy solution the following must hold

i) $u \in L^1(Q_T)$

ii) $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ and

iii) u satisfies the following identity

$$\iint_{Q_T} \nabla(\Phi(u)) \cdot \nabla\eta - u\eta_t dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint_{Q_T} f\eta dxdt$$

for all functions $\eta \in C^1(\bar{Q}_T)$ which vanish on Σ_T and for $t = 0$.

We will start with ii). Let $w_n = \Phi_n(u_n)$ and we will show that w_n is bounded in H^1 giving us an in L^2 convergent subsequence. From that we will be able to conclude that a subsequence of u_n and w_n converges in L^2 and a.e.

We first estimate ∇w_n with an energy estimate as in (4.17). Since u_n is a classical solution of the form of Thm. 4.5 we can apply Thm. 4.6 and obtain

$$\iint\limits_{Q_T} |\nabla w_n|^2 dxdt + \int_\Omega \Psi_n(u_n(x,T))dx = \int_\Omega \Psi_n(u_{n0}(x))dx + \iint\limits_{Q_T} f_n w_n dxdt$$

with Ψ_n being the primitive of Φ_n, $\Psi_n(s) = \int_0^s \Phi_n(\sigma)d\sigma$.

Thm. 4.6 moreover gives the inequality

$$\frac{1}{2}\iint\limits_{Q_T} |\nabla w_n|^2 dxdt + \int_\Omega \Psi_n(u_n(x,T))dx \le \int_\Omega \Psi_n(u_{n0}(x))dx + C\iint\limits_{Q_T} f_n^2 dxdt$$

where C is a constant only depending on Ω.

Notice that $u_{n0} \to u_0 \in L_\Psi(\Omega)$ and $f_n \to f \in L^2(Q_T)$ and u_{n0} and f_n are therefore uniformly bounded in $L_\Psi(\Omega)$ respectively in $L^2(Q_T)$. Thus, the right hand side is bounded independently of n and it follows that ∇w_n is uniformly bounded in $L^2(Q_T)$ for arbitrary T.

Next, we will estimate the time derivative $w_{n_t} = \Phi_n'(u_n)(u_n)_t$. For that we first use Thm. 4.7 since we are dealing with classical solutions and get

$$\iint\limits_{Q_T} \zeta\Phi_n'(u_n)(u_{n_t})^2 dxdt = \iint\limits_{Q_T} \frac{\zeta_t}{2}|\nabla(\Phi_n(u_n))|^2 - (f_n\zeta)_t\Phi_n(u_n)dxdt,$$

with $\zeta \in C^\infty([0,T])$ being a cut off function and $\zeta(0) = \zeta(T) = 0$. Once again the right hand side is bounded independently of n: The first term since $\nabla(\Phi_n(u_n)) = \nabla w_n$ is bounded in $L^2(Q_T)$ with the energy estimate above and the second term since f and f_t are bounded s.t. f_n, f_{n_t} are uniformly bounded; $\Phi_n(u_n)$ is uniformly bounded as well since u_n is, Φ_n is continuous and $\Phi_n \to \Phi$ uniformly on compacts.

Since ζ was an arbitrary cutoff function and T was arbitrary, too, we have that for every $T < \infty$ and $\tau > 0$ the integral $\int_\tau^T \int_\Omega \Phi_n'(u_n)(u_{n_t})^2 dxdt$ is uniformly bounded.

To get from this estimate to the derivative $w_{n_t} = \Phi_n'(u_n)(u_n)_t$, we remember that the u_n are uniformly bounded by a constant C in Q_T and Φ is Lipschitz continuous on compact sets such that the derivatives of the approximations $\Phi_n'(s)$ are uniformly bounded for $|s| < C$.

In this way, we can multiply the integral $\int_\tau^T \int_\Omega \Phi_n'(u_n)(u_{n_t})^2 dxdt$ with the uniformly bounded $\Phi_n'(u(x,t))$ to infer that

$$\int_\tau^T \int_\Omega (\Phi_n'(u_n)(u_{n_t}))^2 dxdt = \int_\tau^T \int_\Omega (w_{n_t})^2 dxdt$$

is uniformly bounded and as such w_{n_t} in $L^2(Q_T^\tau)$ with $Q_T^\tau = \Omega \times (\tau, T)$.

We now can conclude since ∇w_n is uniformly bounded in $L^2(Q_T)$ and $(w_n)_t$ in $L^2(Q_T^\tau)$ that $(w_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H^1(Q_T^\tau)$ and we get a subsequence converging to a weak limit $w_{n_k} \rightharpoonup w$ in $H^1(Q_T^\tau)$ with the theorem of Banach and Alaoglu (Thm. III.3.7 of Werner [2018]). The Sobolev embedding theorem (Thm. 8.9 of Alt [2012]) now ensures that $H^1(Q_T^\tau)$ is compactly embedded in $L^2(Q_T^\tau)$, i.e. every bounded sequence in $H^1(Q_T^\tau)$ has a convergent subsequence in $L^2(Q_T^\tau)$ since L^2 is also complete.

We therefore get (after passing to another subsequence) that $w_{n_k} \to w$ also strongly in $L^2(Q_T^\tau)$. After passing to another subsequence, we can assume that w_{n_k} is also pointwise convergent a.e. since every in L^2 convergent sequence has an a.e. convergent subsequence. Furthermore we also get that $w \in L^2(Q_T)$ since w_n is also uniformly bounded in $L^2(Q_T)$.

Since Φ is continuous and bijective and Φ_n uniformly convergent, we also have the a.e. convergence of u_{n_k} to a function u as follows:

$$|u_{n_k}(x,t) - \bar{u}| \to 0 \iff |\Phi(u_{n_k})(x,t) - \bar{w}| \to 0, \text{ for some } \bar{u} \text{ and } \bar{w}$$

and

$$\begin{aligned} |\Phi(u_{n_k})(x,t) - \bar{w}| &= |\Phi(u_{n_k})(x,t) - \Phi_{n_k}(u_{n_k})(x,t) + \Phi_{n_k}(u_{n_k})(x,t) - \bar{w}| \\ &\leq \underbrace{|\Phi(u_{n_k})(x,t) - \Phi_{n_k}(u_{n_k})(x,t)|}_{\to 0 \text{ as } \Phi_{n_k} \text{ loc. unif. conv. and } u_{n_k} \text{ bounded}} + \underbrace{|\Phi_{n_k}(u_{n_k})(x,t) - \bar{w}|}_{\text{a.e. convergent}} \overset{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

with $\hat{w} = w(x,t)$. We define $u(x,t) = \lim_{k\to\infty} u_{n_k}(x,t)$.

Furthermore $w = \Phi(u)$ a.e. For that we write for arbitrary $(x,t) \in Q_T$:

$$\begin{aligned} |w_{n_k}(x,t) - \Phi(u)(x,t)| &= |\Phi_{n_k}(u_{n_k})(x,t) - \Phi(u)(x,t)| \\ &\leq \underbrace{|\Phi_{n_k}(u_{n_k})(x,t) - \Phi(u_{n_k})(x,t)|}_{\to 0 \text{ as above}} + |\Phi(u_{n_k})(x,t) - \Phi(u)(x,t)| \\ &\overset{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

since $\Phi_n \to \Phi$ uniformly on compact sets, Φ is continuous and $u_{n_k} \to u$ a.e. With the weak convergence of $w_{n_k} \rightharpoonup w$ in $H^1(Q_T^\tau)$ follows that $\nabla w = \nabla\Phi(u)$, the weak derivative of $\Phi(u)$.

We finally note that $w = 0$ on Σ_T s.t. $w = \Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ because $w_{n_k} = \Phi_{n_k}(u_{n_k})$ $= 0$ on Σ_T for all k. This shows ii).

To get i) we observe that $u_{n_k} \to u$ in $L^p(Q_T)$ for every $p \geq 1$ since the u_n are uniformly bounded s.t. the convergence follows with the a.e. convergence and Lebesgue's dominated convergence theorem.

We still have to show iii), i.e. that u is a weak solution to the problem and the identity (5.1) holds. Since u_{n_k} are classical solutions the identity holds for them and we have

$$\iint\limits_{Q_T} \nabla(\Phi_{n_k}(u_{n_k})) \cdot \nabla\eta - u_{n_k}\eta_t dxdt = \int_\Omega u_{n_{k0}}(x)\eta(x,0)dx + \iint\limits_{Q_T} f_{n_k}\eta dxdt$$

for every function $\eta \in C^1(\bar{Q}_T)$ which vanishes on Σ_T and for $t = 0$.

Since we have shown that $\nabla(\Phi_n(u_n))$ converges weakly in $L^2(Q_T)$, u_{n_k} does so in $L^2(Q_T)$ and η and $\nabla\eta$ are bounded and thus in L^2 and we have picked the sequences u_{n_0} and f_n to converge to their limits in L^1 respectively L^2, we can pass to the limit in the identity above to receive (5.1). u consequently is a weak solution to the problem.

For the energy inequality, notice that we can not directly go to the limit in the first term of

$$\iint\limits_{Q_T} |\nabla w_{n_k}|^2 dxdt + \int_\Omega \Psi_{n_k}(u_{n_k}(x,T))dx = \int_\Omega \Psi_{n_k}(u_{n_{k0}}(x))dx + \iint\limits_{Q_T} f_{n_k} w_{n_k}(x)dxdt$$

since we only have weak convergence of $\nabla\Phi_n(u_n)$. But thanks to the weak semi-continuity of $\|\cdot\|_{L^2}$ (Prop. 4.6 of Schweizer [2013]) we have that

$$\|\nabla(\Phi(u))\|^2_{L^2(Q_T)} \leq \liminf_{k\to\infty} \|\nabla(\Phi_{n_k}(u_{n_k}))\|^2_{L^2(Q_T)}.$$

The $\liminf_{k\to\infty}$ is controlled by the right hand side, where the limit exists, s.t. the equality becomes an inequality in the limit and we obtain (5.3) in this way.

Finally, for the comparison principle consider two sets of data $u_0 \leq \hat{u}_0$ and $f \leq \hat{f}$. Our approximation process then yields $u_{n_0} \leq \hat{u}_{n_0}$ and $f_n \leq \hat{f}_n$ s.t. the classical comparison principle holds and $u_n \leq \hat{u}_n$ for every $n \in \mathbb{N}$. Passing to the limit we get $u \leq \hat{u}$. ☐

Lemma 2: *The theorem holds under the additional assumptions that u_0, f and f_t are bounded, $\Gamma \in C^2$. Φ does not need to be locally Lipschitz continuous.*

Proof. As before we approximate the initial problem with Φ_n, u_{0_n} and f_n and obtain non-degenerate quasilinear problems that can be solved with Thm. 4.5 and solutions u_n. Those solutions are uniformly bounded as in Lemma 1.

When the Lipschitz condition on Φ is eliminated, we find that Φ' must not be bounded on compact sets and we can not conclude as in the Lemma above that $(w_n)_t$ is uniformly bounded in $H^1(Q_T^\tau)$ and that therefore exists an a.e. to a function w convergent subsequence.

What we will do in this step instead, is composite u and w with continuous strictly increasing functions in a way that we can conclude with Lemma 1 that those compositions converge a.e.

We then will be able to obtain the desired convergence of the solutions with the inverse of those functions.

For that consider the function Z defined through (here we need that $\Phi \in W^{1,1}_{loc}(\mathbb{R})$)

$$Z(s) = \int_0^s \min\{1, \Phi'(\sigma)\} d\sigma$$

and the approximations

$$Z_n(s) = \int_0^s \min\{1, \Phi_n'(\sigma)\} d\sigma.$$

It holds that $Z_n \to Z$ locally uniformly since $\Phi_n' \to \Phi'$ in $L^1_{loc}(\mathbb{R})$ and therefore

$$\min\{1, \Phi_n'\} \to \min\{1, \Phi'\} \text{ in } L^1_{loc}(\mathbb{R})$$

s.t. for an arbitrary compact interval $[a, b]$

$$\begin{aligned}
\sup_{s\in[a,b]} |Z_n(s) - Z(s))| &= \sup_{s\in[a,b]} \left| \int_0^s \min\{1, \Phi_n'(\sigma)\} - \min\{1, \Phi'(\sigma)\} d\sigma \right| \\
&\leq \sup_{s\in[a,b]} \int_0^s |\min\{1, \Phi_n'(\sigma)\} - \min\{1, \Phi'(\sigma)\}| d\sigma \\
&= \int_0^b |\min\{1, \Phi_n'(\sigma)\} - \min\{1, \Phi'(\sigma)\}| d\sigma \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

The Z_n and Z are obviously strictly increasing since Φ and Φ_n are strictly increasing and also bijective since Φ and Φ_n are as well. Since the Z_n and Z are also continuous, they have continuous inverse functions we will call $\Lambda_n = Z_n^{-1}$ which also converge locally uniformly to a continuous and strictly increasing function Λ. This can be seen as follows: For a compact interval $[Z(a), Z(b)]$ (every interval can be written in this form because Z is bijective and continuous) exists with the local uniform convergence of Z_n and monotonicity of Z an N s.t. $[Z(a), Z(b)] \subset [Z_n(a - 1), Z_n(b + 1)]$ for all $n \geq N$.

Then

$$\begin{aligned}
&\sup_{s\in[Z(a),Z(b)]} |\Lambda_n(s) - \Lambda(s)| \leq \sup_{s\in[Z_n(a-1),Z_n(b+1)]} |\Lambda_n(s) - \Lambda(s)| \\
&= \sup_{s\in[a-1,b+1]} |\Lambda_n(Z_n(s)) - \Lambda(Z_n(s))| = \sup_{s\in[a-1,b+1]} |s - \Lambda(Z_n(s))| \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

since a uniform convergent function composited with a continuous function is uniform convergent.

Considering that the derivatives Z' and Z_n' are also bounded by 0 and 1, Z and the Z_n are uniformly Lipschitz continuous and we have $|Z_n(s)| \leq |s|$ and $|Z_n(s)| \leq |\Phi_n(s)|$.

Now we consider the function $z_n(x,t) = Z_n(u_n(x,t)) = \int_0^{u_n(x,t)} \min\{1, \Phi_n'(s)\}ds$. The z_n are uniformly bounded in Q_T as

$$|z_n(x,t)| = |Z_n(u_n(x,t))| \leq |u_n(x,t)|$$

and u_n is uniformly bounded.

Seeing that

$$\begin{aligned} |\nabla z_n| &= \left|\nabla \int_0^{u_n(x,t)} \min\{1, \Phi_n'(s)\}ds\right| = |\nabla u_n(x,t) \; \min\{1, \Phi_n'(u_n(x,t))\}| \\ &\leq |\nabla u_n(x,t)| \end{aligned}$$

what is uniformly bounded in $L^2(Q_T)$ with the energy estimate as in Lemma 1, we also have the uniform boundedness of ∇z_n in $L^2(Q_T)$.

Recognizing also that $\min\{a,b\}^2 < ab$ with $a, b > 0$ it follows

$$\begin{aligned} |(z_n)_t|^2 &= \left|\frac{\partial}{\partial t} \int_0^{u_n(x,t)} \min\{1, \Phi_n'(s)\}ds\right|^2 = ((u_n)_t(x,t))^2 \min\{1, \Phi_n'(u_n(x,t))\}^2 \\ &\leq ((u_n)_t(x,t))^2 \Phi_n'(u_n(x,t)). \end{aligned}$$

The estimate is also uniformly bounded, this time in $L^2(Q_T^\tau)$, with the estimate of the time derivative as in Lemma 1.

With the last three estimates, we conclude the uniform boundedness of z_n in $H^1(Q_T^\tau)$ such that we once more have an in $L^2(Q_T^\tau)$ and a.e. convergent subsequence $z_{n_k} \to z$ which we will call z_n for simplicity in the following.

To receive the a.e convergence of u_n and w_n from this result, we note that $u_n = \Lambda_n(z_n)$ and $w_n = \Phi_n(\Lambda_n(z_n))$. Since Φ_n and Λ_n are locally uniformly convergent and z_n a.e. convergent and bounded, we get as in Lemma 1 the a.e. convergence of $u_n \to u = \Lambda(z)$ and $w_n \to w = \Phi(\Lambda(z)) = \Phi(u)$. Furthermore, u_n and w_n are uniformly bounded in Q_T s.t. we get the convergence in $L^p(Q_T)$ for all $p < \infty$ with the dominated convergence theorem. i) is therefore proven.

For ii) we still need to show that the weak gradient $\nabla(\Phi(u))$ exists and is in $L^2(Q_T)$. Since ∇w_n is uniformly bounded in $L^2(Q_T)$, it exists a weakly convergent subsequence $\nabla w_n \rightharpoonup \xi$ in $L^2(Q_T)$.

We now want to show that ξ is the weak derivative, i.e. $\iint_{Q_T} \Phi(u)\text{div}(\eta)dxdt = -\iint_{Q_T} \xi\cdot\eta dxdt$ for every test function $\eta : Q_T \to \mathbb{R}^n \in C_0^\infty$. For w_n there exists a strong derivative ∇w_n s.t. this obviously holds for w_n for every $n \in \mathbb{N}$ and if the limit exists

$$\lim_{n\to\infty} \iint\limits_{Q_T} w_n \text{div}(\eta)dxdt = \lim_{n\to\infty} -\iint\limits_{Q_T} \nabla w_n \cdot \eta dxdt.$$

Since $\nabla w_n \to \xi$ weakly in $L^2(Q_T)$, we have just shown that $w_n \to w = \Phi(u)$ in $L^2(Q_T)$ and as such also weakly in $L^2(Q_T)$ and $\eta,\ \text{div}(\eta) \in L^2(Q_T)$ we can conclude that

$$\iint\limits_{Q_T} \Phi(u)\text{div}(\eta)dxdt = -\iint\limits_{Q_T} \xi \cdot \eta dxdt$$

and ξ is the weak gradient of $\Phi(u)$, $\nabla\Phi(u) = \xi$.

To show that u satisfies (5.1) and therefore iii), we can proceed exactly as in Lemma 1 as we have just proven the necessary convergences. We have once more constructed a weak solution to the problem. The energy inequality and the comparison principle follow analogous as in Lemma 1. $\square$

Lemma 3: *The theorem holds still when the boundedness assumption on u_0, f and f_t is removed. We still assume that $\Gamma \in C^2$.*

Proof. We approximate the potentially unbounded functions u_0 and f by functions that meet the assumptions of Lemma 2, i.e. u_{n_0}, f_n and $(f_n)_t$ are bounded. Because $u_0 \in L_\Psi(\Omega)$ and $u_0 \in L^1(\Omega)$ it is possible to find a uniformly in $L_\Psi(\Omega)$ bounded sequence of functions u_{n_0} with $u_{n_0} \to u_0$ in $L^1(\Omega)$.

For f_n we choose a sequence that is uniformly bounded in $L^2(Q_T)$ and $f_n \to f$ in $L^2(Q_T)$. Once again remember the assumption $f \in L^2(Q)$. Notice that Φ does not get approximated and does not change with n.

The resulting Dirichlet problem can be solved with Lemma 2 and solutions u_n.

Notice that the L^1-contraction principle (Thm. 4.8) also holds for weak solutions of the form of Lemma 2 since it holds for the approximation in Lemma 2 and we can pass to the limit because we have the necessary convergences of u_{n_0}, f_n and u_n respectively $\hat{u}_{n_0}$, $\hat{f}_n$ and $\hat{u}_n$.

We then can infer for $u_n(t)$ and $u_m(t)$ for arbitrary $t \in (0,T)$ that

$$\|u_m(t) - u_n(t)\|_{L^1(\Omega)} \leq \|u_{m_0} - u_{n_0}\|_{L^1(\Omega)} + \int_0^t \|f_m(s) - f_n(s)\|_{L^1(\Omega)} ds.$$

Since $u_{n_0} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^2(Q_T)$, we can pass to the limit in n and m. It follows that $u_n(t)$ is a Cauchy Sequence for all $t \in (0,T)$ and therefore converges to a function $u(t)$ in $L^1(\Omega)$ as $L^1(\Omega)$ is complete.

Furthermore, it follows that

$$\begin{aligned}\operatorname*{ess\ sup}_{t\in(0,T)} \|u_n(t) - u(t)\|_{L^1(\Omega)} &\leq \operatorname*{ess\ sup}_{t\in(0,T)} \left(\|u_{n_0} - u_0\|_{L^1(\Omega)} + \int_0^t \|f_n(s) - f(s)\|_{L^1(\Omega)} ds \right) \\ &= \|u_{n_0} - u_0\|_{L^1(\Omega)} + \int_0^T \|f_n(s) - f(s)\|_{L^1(\Omega)} ds \\ &\overset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

with the convergence of u_{n_0} and f_n s.t. $u_n \to u$ also in $L^\infty((0,T) : L^1(\Omega))$ and, after passing to a subsequence, almost everywhere.

The energy inequality holds for all $n \in \mathbb{N}$ and we get as before with the Poincaré inequality ($f_n \in L^2(Q_T)$ and $\Phi(u_n) \in L^2((0,T) : H^1(Q_T))$):

$$\frac{1}{2} \iint\limits_{Q_T} |\nabla(\Phi(u_n))|^2 dxdt + \int_\Omega \Psi(u_n(x,T))dx \leq \int_\Omega \Psi(u_{n_0}(x))dx + C \iint\limits_{Q_T} f_n^2 dxdt.$$

$\int_\Omega \Psi(u_n(x,T))dx$ is therefore uniformly bounded since u_{n_0} and f_n are uniformly bounded in $L_\Psi(\Omega)$ respectively in $L^2(Q_T)$. Moreover it follows that u_n is uniformly bounded in $L^\infty((0,T) : L_\Psi(\Omega))$ s.t. $u \in L^\infty((0,T) : L_\Psi(\Omega))$ since T was arbitrary and therefore

$$\int_\Omega \Psi(u_n(x,t))dx \leq \int_\Omega \Psi(u_{n_0}(x))dx + C \iint\limits_{Q_t} f_n^2 dxdt \leq \int_\Omega \Psi(u_{n_0}(x))dx + C \iint\limits_{Q_T} f_n^2 dxdt$$

for all $t \in (0,T)$ and the right side does not depend on n or t. This shows i).

We can also conclude as before that $\nabla\Phi(u_n)$ is uniformly bounded in $L^2(Q_T)$ and since

$$\iint\limits_{Q_T} |\nabla(\Phi(u_n))|^2 dxdt \geq C \iint\limits_{Q_T} \Phi(u_n)^2 dxdt$$

with the Poincaré inequality and a constant only depending on Ω, $\Phi(u_n)$ is also uniformly bounded in $L^2(Q_T)$. This implies that $\nabla(\Phi(u_n))$ and $\Phi(u_n)$ have weakly in $L^2(Q_T)$ convergent subsequences we will also call $\Phi(u_n) \rightharpoonup \zeta$ and $\nabla(\Phi(u_n)) \rightharpoonup \xi$ for simplicity. Since Φ is continuous, we have $\Phi(u_n) \to \Phi(u)$ a.e. and also $\zeta = \Phi(u)$ a.e.:

For this, we remember that with Mazur's Lemma (Korollar III.3.9 of Werner [2018]) since $\Phi(u_n) \rightharpoonup \zeta$ in $L^2(Q_T)$, there exist convex combinations of $\Phi(u_n)$, i.e for every $n \in \mathbb{N}$ exist $(\lambda_i^{(n)})_{n \leq i \leq N(n)}$ with $\sum_{i=n}^{N(n)} \lambda_i^{(n)} = 1$ such that

$$\sum_{i=n}^{N(n)} \lambda_i^{(n)} \Phi(u_i) \overset{n\to\infty}{\longrightarrow} \zeta \text{ strongly in } L^2(Q_T)$$

and therefore a.e. (on a subsequence). Since $\Phi(u_n) \to \Phi(u)$ a.e the convex combinations also converge a.e. to $\Phi(u)$ s.t. $\zeta = \Phi(u)$.

To prove that $\xi = \nabla(\Phi(u))$, we can proceed exactly as in Lemma 2 and we finally receive $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ and ii) is proven.

As before identity (5.1) holds for the approximate solutions and we can pass to the limit with the proven convergences s.t. u is a weak solution to the GPME with the regularity as in the theorem. The energy inequality and the comparison principle still hold as before. We can furthermore pass to the limit in the L^1-contraction principle so that it also holds in the limit. $\square$

End of the proof: Γ is not in C^2 but Lipschitz.

For that, we approximate Ω with an increasing sequence of open sets $\Omega_n \subset\subset \Omega_{n+1}$ with boundary $\Gamma_n \in C^2$. Let $\zeta_n(x)$ be cutoff functions s.t. $\zeta_n = 1$ on Ω_{n-1} and $\operatorname{supp}(\zeta_n) \subset \Omega_n$.

We define $u_{n_0} = u_0 \zeta_n$ and $f_n = f\zeta_n$ and obtain the following Dirichlet problem

$$\begin{aligned} (u_n)_t &= \triangle(\Phi(u_n)) + f_n \text{ in } Q_T^n = \Omega_n \times (0,T) \\ u_n(x,0) &= u_{n_0}(x) \text{ in } \bar{\Omega}_n \\ u_n(x,t) &= 0 \text{ on } \Sigma_n = \Gamma_n \times (0,T). \end{aligned}$$

This can be solved with Lemma 3 as $\Gamma_n \in C^2$ what yields weak solutions u_n which we extend by 0 on $\Omega \setminus \Omega_n$. It holds that u_{n_0} is uniformly bounded in $L_\Psi(\Omega)$ and $u_{n_0} \to u_0$ in $L^1(\Omega)$, $f_n \to f$ in $L^2(Q_T)$ with dominated convergence. We now can conclude analogous to Lemma 3 with the L^1-contraction principle which also holds in this case and the energy inequality that $u_n \to u$ in $L^\infty((0,T) : L^1(\Omega))$ and $u \in L^\infty((0,T) : L_\Psi(\Omega))$ and . For a subsequence then follows $u_n \to u$ a.e. and we get i).

To show ii), i.e. $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ one can proceed exactly as in Lemma 3.

To see that u finally is a weak solution we can pass to the limit in the identity (5.1) for u_n s.t. it also holds for u as before because we have just shown the necessary convergences. This concludes the proof and we have found a solution u with the regularity stated in the theorem. The energy inequality and the comparison principle still apply as before. □

We have therefore succeeded in our goal of showing the existence of solutions to the homogeneous Dirichlet problem with sufficient generality. Similarly, one can do this for the inhomogeneous problem, but this won't be done in this thesis. We refer the reader to [5.7 of Vazquez [2006], p. 103-106].

5.2.3 Uniqueness of weak solutions

For simplicity to show the uniqueness of a weak energy solution u to the GPME, we will assume that $u \in L^2(Q_T)$. Notice that in Lemma 3 and the end of the proof of Thm. 5.2 we have only proven that the solutions are in $L^\infty((0,T) : L_\Psi(\Omega))$ and do not need to be L^2-functions.

Theorem 5.3 (Uniqueness of weak energy solutions). *(Thm. 5.3 of Vazquez [2006])*
If for a weak solution u to the hom. Dirichlet problem of the GPME (Def. 5.1) holds that $u \in L^2(Q_T)$ and $\Phi(u) \in L^2((0,T) : H_0^1(\Omega))$ it is the unique solution to the problem.

Proof. Let u_1 and u_2 be two weak solutions with $w_i := \Phi(u_i)$. Both solutions satisfy condition (5.1) such that by subtracting both equations we get for all test functions $\eta \in C^1(\bar{Q}_T)$ with $\eta|_{\Sigma_T} = 0$ and $\eta(T) = 0$ that

$$\iint\limits_{Q_T} \nabla(w_1 - w_2) \cdot \nabla\eta - (u_1 - u_2)\eta_t dxdt = 0. \tag{$*$}$$

The idea is to use a suitable test function s.t. $(*)$ can only hold if $u_1 = u_2$. For that consider the function

$$\eta(x,t) = \begin{cases} \int_t^T w_1(x,s) - w_2(x,s)ds & \text{, if } 0 < t < T \\ 0 & \text{, if } t \geq T \end{cases}.$$

Notice that for $t < T$

$$\eta_t = \frac{\partial}{\partial t}\int_t^T w_1(x,s) - w_2(x,s)ds = -(w_1(x,t) - w_2(x,t)).$$

η_t is therefore a function in $L^2(Q_T)$ since we assumed that $w_i = \Phi(u_i) \in L^2((0,T) : H_0^1(\Omega))$. Furthermore

$$\nabla\eta = \int_t^T \nabla(w_1(x,s) - w_2(x,s))ds$$

what is also in $L^2(Q_T)$ because

$$
\begin{aligned}
&\iint_{Q_T} \left| \int_t^T \nabla(w_1(x,s) - w_2(x,s))ds \right|^2 dxdt \\
&\leq \iint_{Q_T} (T-t) \int_t^T |\nabla(w_1(x,s) - w_2(x,s))|^2 \, dsdxdt \\
&\leq T \int_0^T \underbrace{\int_\Omega \int_0^T |\nabla(w_1(x,s) - w_2(x,s))|^2 \, dsdx}_{=C<\infty,\text{ since } \nabla(\Phi(u_{1/2}))\in L^2(Q_T)} dt = T^2 C < \infty,
\end{aligned}
$$

where we have used Jensen's inequality [Thm. VI.1.3 of Elstrodt [2018]] in the second step. η is therefore in $H^1(Q_T)$ and obviously $\eta = 0$ on Σ_T and for $t = T$. η may not be in $C^1(\bar{Q}_T)$ but we can approximate η with smooth functions η_n converging to η in $H^1(Q_T)$ with $\eta_n = 0$ on Σ_T and for $t = 0$ (since the smooth functions are dense in H^1). For those functions $(*)$ holds and we can pass to the limit with the convergence in $H^1(Q_T)$ (since $u_i \in L^2(Q_T)$) and we get $(*)$ for our defined η.

Now

$$
\begin{aligned}
0 &= \iint_{Q_T} \nabla(w_1 - w_2) \cdot \nabla\eta - (u_1 - u_2)\eta_t dxdt \\
&= \iint_{Q_t} \underbrace{\nabla(w_1 - w_2) \cdot \int_t^T \nabla(w_1(x,s) - w_2(x,s))ds}_{= \, -v_t \cdot v \, = \, -\left(\frac{1}{2}|v|^2\right)_t, \; v = \int_t^T \nabla(w_1(x,s) - w_2(x,s))ds} \, dxdt + \iint_{Q_T} (u_1 - u_2)(w_1 - w_2)dxdt \\
&= \int_\Omega \int_0^T -\frac{1}{2}(|v|^2)_t dtdx + \iint_{Q_T} (u_1 - u_2)(w_1 - w_2)dxdt \\
&= \frac{1}{2} \int_\Omega -|v(T)|^2 + |v(0)|^2 dx + \iint_{Q_T} (u_1 - u_2)(w_1 - w_2)dxdt \\
&= \frac{1}{2} \int_\Omega \left| \int_0^T \nabla(w_1(x,s) - w_2(x,s))ds \right|^2 dt + \iint_{Q_T} (u_1 - u_2)(w_1 - w_2)dxdt.
\end{aligned}
$$

The first term is obviously non-negative and the second as well because $w_1 - w_2 > 0$ iif $u_1 - u_2 > 0$ as Φ is strictly increasing. Because the term can only be equal to zero if both parts are zero, we can conclude that $u_1 = u_2$ a.e. $\square$

6 Conclusion and outlook

In this thesis, we have become acquainted with a new type of partial differential equation, the (generalized) porous media equation and its weak solution theory. For this, we have derived the PME in its original physical way. To motivate the weak solution theory, we have discussed the Barenblatt solution, which is a classical solution to the PME whenever it is positive, but we have also seen that different to the fundamental solution to the heat equation this solution originating from a point source isn't differentiable on the whole space, implying that it can not be a classical solution everywhere.

In the next chapter, we then got to know the main difference of the GPME to the well studied quasilinear parabolic equations, namely the possible degeneracy of the equation. Treating this difficulty to prove the existence of weak solutions was done by approximating the degenerate problem by non degenerate problems of which we could conclude the existence of solutions from the classical theory we stated. For these classical solutions we have proven some important estimates we used extensively in the existence proof.

In this thesis, we have restricted ourselves to the homogeneous Dirichlet problem but the same can be done to the inhomogeneous problem or even the Neumann problem. The weak solutions defined in this thesis can be even further generalized to very weak solutions where $\Phi(u)$ is only in L^2 s.t. we don't need $\nabla\Phi(u)$ for the definition of the solution (6.2 of Vazquez [2006], p. 130,131).

The PME finds application in various fields of science. The classical application of the flow of a gas through a porous medium we have seen in this thesis. Further applications we find for example in the modelling of the diffusion of a biological population (s. Gurtin and MacCamy [1977]). Here the degeneracy of the PME is one argument for the accuracy of the model because it ensures a finite propagation speed and it is even possible for the populated area to stay bounded for all time. Another fun example of this is the paper by Newman and Sagan [1981], where the spread of a galactic civilization in space is modeled by the porous media equation.

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Declaration of Academic Integrity

I hereby confirm that this thesis on "The weak solution theory of the porous media equation" is solely my own work and that I have used no sources or aids other than the ones stated. All passages in my thesis for which other sources, including electronic media, have been used, be it direct quotes or content references, have been acknowledged as such and the sources cited.

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