Exact recovery in TV regularization

Bachelor's Thesis

by

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1 Introduction

In our technologically advanced age, imaging has become indispensable in many areas. It is needed for health diagnostics through CT and MRI, for monitoring environmental changes through satellite images, for examining the smallest structures through electron microscopy in nanotechnology and for monitoring through thermal imaging in defence and security. The range of imaging applications is large and extensive, providing data for discoveries, analyses and decisions.

However, all of these areas face a common challenge: images are altered during capture, processing and storage, which can have potentially damaging effects. In healthcare, such alterations can obscure important information and lead to misdiagnosis. In environmental science, they can obscure crucial environmental changes. In nanotechnology, they prevent accurate observation of tiny structures. As we rely more and more on these technologies, correcting these image changes is critical to maintaining the accuracy and reliability of the data.

To better understand the timing and circumstances of such image distortions, we consider below a simplified representation of a photosensor image processing workflow diagram (Figure 1 below).

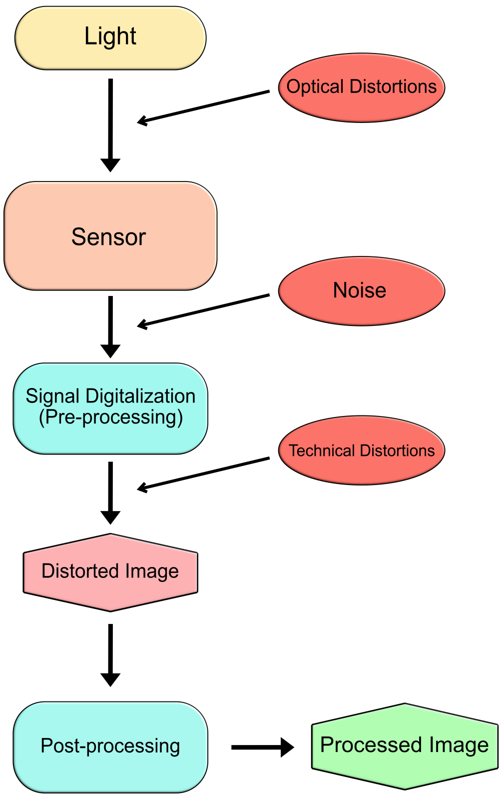

Figure 1: Simplified representation of the signal processing in a photo sensor.

1.1 Types of Image Distortions: Illustrative Examples

It becomes evident that the distortions we have mentioned can be categorized into three distinct groups. To illustrate this point, we will explore concrete examples from each of these categories. The examples shown in this section can be reproduced if necessary using the provided MATLAB Livescript (`ExactRecovery.mlx`).

Optical distortions

These image alterations take place before the actual capture process, that is, before the light reaches the sensor. Some common examples include:

- Motion blur: Occurs when the object being sensed moves significantly during the capture process.
- Optical aberrations: Distortions induced by certain types of lenses that affect the image quality.
- Defocus blur: Arises from incorrect focusing, resulting in a loss of image sharpness.

An example of defocus blur is shown in Figure 2 below. In this case, a Gaussian filter was used to create the blur effect, and more details on this technique will be provided later in this work.

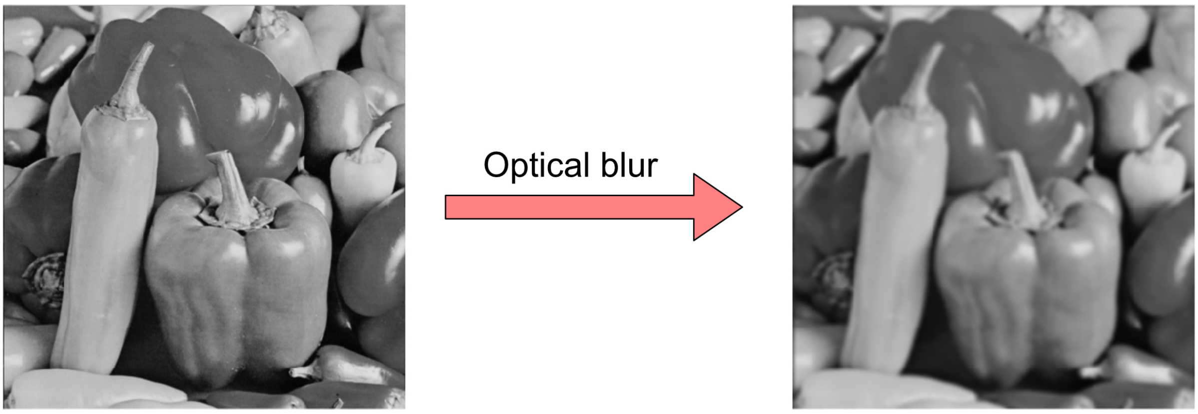

Figure 2: On the left side, the original image from [21], on the right side an blurred version.

Noise

Image noise is the random variation of brightness or colour information in images. It appears as grainy or speckled patterns that can degrade the quality of the captured image. Noise is primarily caused by the natural randomness of light and defects in the camera's sensor. In low light, the light particles hitting the sensor can vary greatly, resulting in more noticeable noise. In addition, when the sensor converts these particles of light into electrical signals, it can introduce additional irregularities, that can make the noise in the final image even more noticeable. An example of such noise can be seen in figure 3 below.

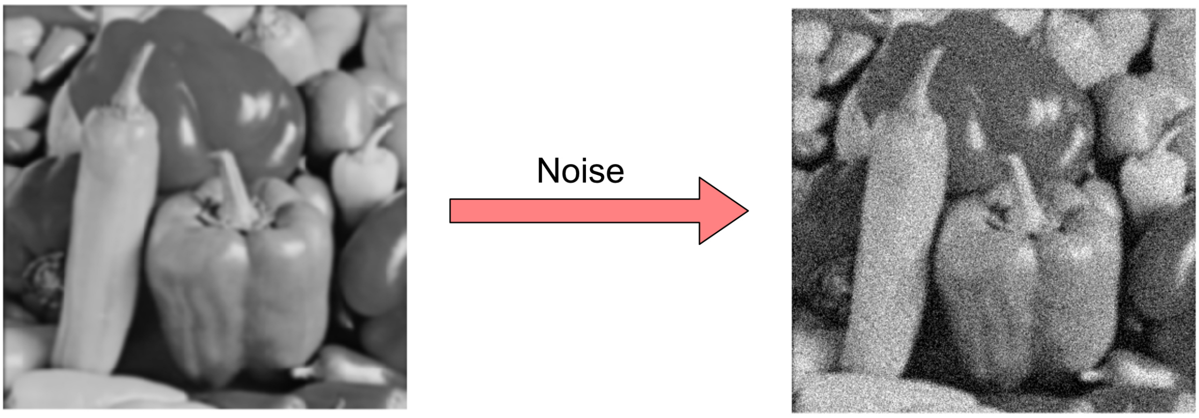

Figure 3: On the left side, the blurred image resulting from figure 2, on the rigth side, the same image with added noise can be seen.

Technical distortions

To create a digital representation of an image, the light intensity and colour of the measured image are evaluated within a fixed grid. to reduce the final size of the image, the original image is now evaluated on a coarser grid. This can result in loss of detail in the image. In addition, overlapping of individual frequencies in the image may occur. This is called aliasing and can result in what are known as moiré patterns. These are interference created when high frequency details in the image interact with the pixel grid, creating a wavy or rippled effect. An example of sampling on a coarser grid, called subsampling, is shown in Figure 4.

Therefore, when comparing our original image in figure 2 with the result in figure 4, we

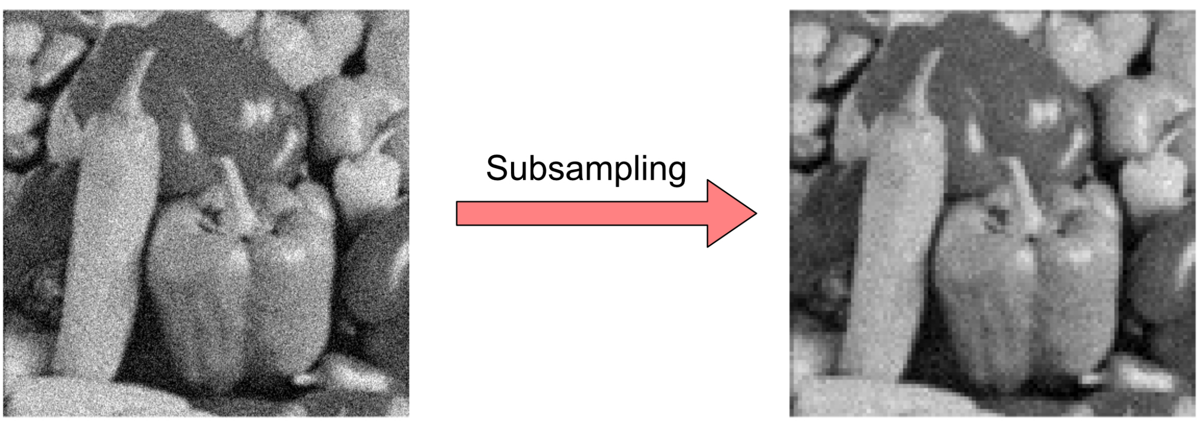

Figure 4: On the left side, the result from Figure 3 is shown, while on the right, the application of subsampling from 400 x 400 pixels to 100 x 100 is displayed.

can observe considerable distortion due to these three mentioned influencing factors.

1.2 Overview of Image Processing Algorithms

Recent advances in image post-processing have helped to overcome problems caused by distortions in imaging technology. This section provides a brief overview of the various methods that can be used to address these problems.

- **Linear filtering:** Basic smoothing, such as the Gaussian filter shown in Figure 2, can reduce noise but can also blur the image. Details on how to use this filter are covered later.
- **Non-linear filtering:** Techniques such as median filtering use statistical measures (such as the median) from around a pixel. These filters preserve edge sharpness better than linear filters.
- **Adaptive filtering:** An improvement on the previous filters, adaptive filters change their settings based on the characteristics of the image. In contrast to standard filters, which treat the entire image uniformly, adaptive filters adapt their process to specific areas of the image.
- **Wavelet-based methods:** These methods decompose an image into wavelet coefficients, each of which represents different frequency aspects at different scales. Unlike the Fourier transform, which focuses only on frequency, wavelets capture both frequency and location details. By reducing the wavelet coefficients, we can retain important elements and remove minor components such as noise when reconstructing the image.
- **Regularization methods:** In techniques such as Tikhonov or Total Variation regularization, a so-called regularization term is added to the problem formulation for image reconstruction. The nature of the added term and its weighting can influence the properties of the reconstructed images. We will examine this in more detail in section 2.
- **Machine learning/Deep learning:** The latest approaches use patterns learned from large data sets to identify and correct distortions. These methods use the power of data-driven learning to effectively deal with different types of image distortion. However, a drawback is the high computational effort required to train such models.

1.3 Objectives and structure of the thesis

Given this information, we are now in a position to specify the objectives of this thesis. Our main focus will be on Total Variation (TV-) regularization. More specifically, a significant part of this work will be devoted to the investigation of a theoretical aspect intrinsic to TV regularization (for a more detailed explanation, see 2.3). In order to do so, we will elaborate and detail the results presented in [9]. In this pursuit, we aim to refine

and provide a more in-depth analysis of the proofs of lemmas, propositions and theorems from [9], incorporating supplementary material such as definitions from secondary literature and illustrative examples. Furthermore, in order to enhance the understanding of the results, we will address the characteristics and efficacy of TV regularization, providing relevant examples.
The structure of this thesis is as follows:

- In Section 2, we will gain an overview of TV regularization and derive the problem statement from [9]. This chapter is particularly recommended for those who are new to TV regularization, as it aims to provide an intuitive understanding of the algorithm. However, those already experienced in this field may choose to proceed directly to Section 2.3.

- Section 3 provides the theoretical foundations for further argumentation. In each subsection, we offer additional literature for those who wish to delve deeper into the specific topics discussed.

- In Section 4, we will derive the so-called Fenchel dual problem using our example from Section 2.1 and discuss properties of its solutions. Here too, we refer to further literature for in-depth study.

- Section 5 narrows down the solution set of the problem presented in 2.1 and examines certain exposed faces (a mathematical concept, see 5.1 for its definition) of this set. We will draw theoretical parallels to k-sparsity.

- Following this, in Section 6, we further transform the dual representation of our problem into the so-called prescribed curvature problem (minimizing a 'shape functional'), which we then analyze for convergence and stability.

- Finally, in Section 7, we present the main results from [9], based on the assumption of the so-called non-degenerate source condition (see 7.4). We will then demonstrate through a simple example the conditions under which this property is satisfied.

Unless explicitly stated otherwise, we will refer to [9] as our source in the following sections. If a reference to [9] is given, it indicates that no other source could be found for the particular statement.

2 TV-regularization: an introduction

TV regularization was first introduced for image denoising and reconstruction in a seminal paper by Rudin, Osher, and Fatemi in 1992 [20], and it is known for its ability to effectively reduce noise while preserving the existing edges in an image. In this section, we aim to provide an insight into the reconstruction of distorted, noisy images using this method. To do so, we will first define the theoretical problem associated with TV regularization, followed by an exploration of the characteristics and limitations of the algorithm. We will then address the primary question of our theoretical investigation in this thesis.

2.1 Theoretical problem statement

To arrive at a general problem formulation, we consider the unknown function u_0 in the space $L^2(\mathbb{R}^2)$, representing the image we aim to reconstruct. The problem at hand is that, as previously described, we only have access to a distorted version of u_0.

Noiseless case

For the noiseless case we only have access to images y_0, where $y_0 = \Phi u_0$ and Φ is a linear operator mapping from $L^2(\mathbb{R}^2)$ into a separable Hilbert space $\mathcal{H}$. As shown in figure 2, Φ can be, for example, a Gaussian filter (blur), which is the convolution with a Gaussian kernel. The minimization problem to reconstruct u_0 from y_0 using TV regularization is then formulated as follows

$$\inf_{u \in L^2(\mathbb{R}^2)} \mathrm{TV}(u) \quad \text{subject to} \quad \Phi u = y_0, \qquad (\mathcal{P}_0(y_0))$$

where $\mathrm{TV}(u)$ represents the total (gradient) variation of u . Informally, $\mathrm{TV}(u)$ measures the total extent of fluctuations or "oscillations" in u, disregarding the direction of these changes (see Section 3.4 for a precise definition). Consequently, in this reconstruction approach, the objective is to ascertain the 'smoothest' function u that also preserves sharp edges and conforms to the observed data.

Noisy case

To account for the effect of noise in the observations, the recovery of u_0 is adapted from y_0 to $y_0 + w$, where $w \in \mathcal{H}$ represents additive noise. (For an example, see the end result of figure 4). For the noisy case where $y = y_0 + w$ and $\lambda > 0$, the problem is formulated as follows:

$$\inf_{u \in L^2(\mathbb{R}^2)} \left(\frac{1}{2} \|\Phi u - y\|_{\mathcal{H}}^2 + \lambda \mathrm{TV}(u) \right). \qquad (\mathcal{P}_\lambda(y))$$

Here, an image $u_{\lambda,w}$ is reconstructed from a (linearly) distorted and noisy y by searching for an u that is as close as possible to y (minimizing the first term), while ensuring that the total variation of u remains small (minimizing the second term). The influence of the latter part is determined by the regularization parameter λ.

2.2 Examples

In this section, we demonstrate the characteristics of TV regularization through a series of simulations. We aim to highlight the strengths and weaknesses of this algorithm in comparison to other noise reduction methods. Following this, we will provide the motivation for the theoretical investigation undertaken in this work. The illustrations for this section can be reproduced with the MATLAB Livescript (`ExactRecovery.mlx`) mentioned earlier.

2.2.1 Comparison of TV regularization with other algorithms

In this section, we compare TV regularization with a simple Gaussian filter and a wavelet-based denoising algorithm. We focus on the denoising case ($\Phi = id$) without subsequent subsampling. For the TV regularization, we utilize an algorithm from [7, p. 1269, p. 1273] with regularization parameter $\lambda = 0.1$. The Gaussian filter is implemented as a convolution with a Gaussian kernel, standard deviation 2. The wavelet-based algorithm stems from a method included in a MATLAB toolbox (see [17]).

Comparison: Image of bell peppers

First, let us consider the familiar image from Section 1, with additive noise w such that $||w|| = 30$. In Figure 5, we observe that all three algorithms effectively remove the noise. However, the wavelet-based algorithm performs the best, eliminating noise while maintaining image sharpness and detail. The Gaussian filter, as expected, reduces noise at the expense of overall sharpness. The TV regularization yields a completely different outcome, where many details of the original image are lost. As previously described, this algorithm reduces the total variation in the image, leading to the creation of flat zones. We will delve deeper into this characteristic in the following example.

Figure 5: Original image from [21] with noise (Picture 1), processed using wavelet algorithm (Picture 2), Gaussian filter (Picture 3), and TV regularization (Picture 4).

Comparison: Fine structures

In this example, we once again compare the previously mentioned algorithms, this time using an image of a wooden fence from [21], with the same level of noise intensity as before. Observing Figure 6, the clear weakness of TV regularization becomes apparent. Fine details cannot be preserved by this algorithm (refer to the left side of the depicted fence in Figure 6). Therefore, for noise reduction in such images, it is advisable to prefer an alternative method. Even the Gaussian filter, with its significantly lower computational complexity compared to TV regularization, offers a much better alternative in this scenario.

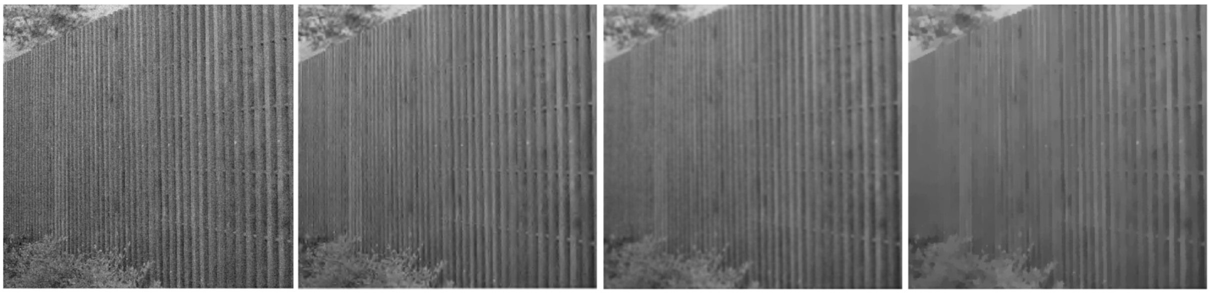

Figure 6: Original image from [21] with noise (Picture 1), processed using wavelet algorithm (Picture 2), Gaussian filter (Picture 3), and TV regularization (Picture 4).

Comparison: Piecewise constant image

For our final comparison of the three algorithms, we examine the reconstruction of a piecewise constant image with an increased noise intensity of $||w|| = 80$ and a modified regularization parameter in the TV algorithm of $\lambda = 0.3$. As observed in Figure 7, in such images, the previously mentioned weakness of TV regularization becomes its strength. The algorithm's propensity to create piecewise constant areas aids in effectively separating the image from noise while preserving sharp edges. Consequently, we will henceforth focus on using the TV regularization algorithm exclusively for such images and will later explain how these images can be formally characterized.

Figure 7: Original image from with noise (Picture 1), processed using wavelet algorithm (Picture 2), Gaussian filter (Picture 3), and TV regularization (Picture 4).

2.2.2 Results with variation of the regularization parameter

In order to better understand the impact of the regularization parameter λ in the reconstruction process using TV regularization, we will examine solutions of $\mathcal{P}_\lambda(y)$ with different values of λ in the following analysis. In this case, we consider the problem $\mathcal{P}_\lambda(y)$, where Φ represents a convolution with a Gaussian kernel, followed by subsampling on a 100 x 100 grid. For this new problem, we now use a different TV-regularization algorithm, based on [7, p. 1269, p. 1275].

$\lambda \to 0$:

In this limit, we have:

$$\inf_{u\in L^2(\mathbb{R}^2)} \left(\frac{1}{2}\|\Phi u - y\|_{\mathcal{H}}^2 + \underbrace{\lambda \mathrm{TV}(u)}_{\to 0} \right).$$

Here, the problem approaches a least squares problem where the goal is to find a u that minimizes $\|\Phi u - y\|_{\mathcal{H}}^2$. As observed from Figure 8, this leads to overfitting of Φu to the noise w. Additionally given that Φ is a convolution with a Gaussian kernel, $\Phi(u)$ is continuous, resulting in overfitting-induced oscillations across the entire reconstructed image.

$\lambda \to \infty$:

In this case, we derive:

$$\begin{aligned} &\inf_{u\in L^2(\mathbb{R}^2)} \left(\frac{1}{2}\|\Phi u - y\|_{\mathcal{H}}^2 + \lambda \mathrm{TV}(u) \right) \\ \Longleftrightarrow &\inf_{u\in L^2(\mathbb{R}^2)} \left(\underbrace{\frac{1}{2\lambda}\|\Phi u - y\|_{\mathcal{H}}^2}_{\to 0} + \mathrm{TV}(u) \right). \end{aligned}$$

It is evident that the total variation term of u becomes dominant in the minimization. This results, as seen in Figure 8, in excessive smoothing of the original image and loss of important structures. Overall, it becomes clear that the choice of regularization parameter is not trivial and must be carefully selected according to the application area.

Figure 8: Blurred image with noise, subsampled on a 100 x 100 grid (Picture 1), TV-reconstructed with $\lambda = 0.01$ (Picture 2), TV-reconstructed with $\lambda = 0.2$ (Picture 3), and TV-reconstructed with $\lambda = 1.5$ (Picture 4).

2.3 Motivation: Behaviour for decreasing noise

In this section, we delve into the motivation for the theoretical investigation presented in this work. Let us revisit the piecewise constant image from the previous section (see figure 9). This image can be interpreted as a top view of the graph of the function $u_0 = \sum_{i=1}^4 a_i \chi_{E_i}$, where the E_i represent the shapes visible in $\mathbb{R}^2$.

Examining the solutions $u_{\lambda,w}$ of Problem $\mathcal{P}_\lambda(y)$ as $w \to 0$ with a suitably chosen λ for the unknown image u_0 (see figure 10), we observe that all reconstructions $u_{\lambda,w}$ exhibit a similar structure to u_0. Specifically, $u_{\lambda,w}$ displays the same number of shapes as u_0, and these shapes appear to converge towards those in u_0.

We aim to address the question of whether the indications suggested by these simulations can be theoretically substantiated. More precisely, whether for such a u_0, under certain conditions, a solution $u_{\lambda,w}$ of $\mathcal{P}_\lambda(y)$ actually satisfies: $u_{\lambda,w} = \sum_{i=1}^4 \tilde{a}_i \chi_{\tilde{E}_i}$, with both $\tilde{a}_i \to a_i$ and $\tilde{E}_i \to E_i$ as $w \to 0$.

Parallels to sparse representation

The advantage of such properties for solutions is evident. Instead of storing each pixel of the reconstructed image in a fixed pixel grid (a matrix of fixed size), the aforementioned representation of $u_{\lambda,w}$ allows us to store the data of the reconstructed image much more efficiently. Furthermore, it enables evaluation on arbitrarily large pixel grids without loss of image quality. In Section 5, we will see that our problem naturally leads to a sparse representation of the solutions, where sparsity in mathematics deals with the problem of describing a given dataset (in our case, the original image u_0) using a suitable basis with as few representatives as possible.

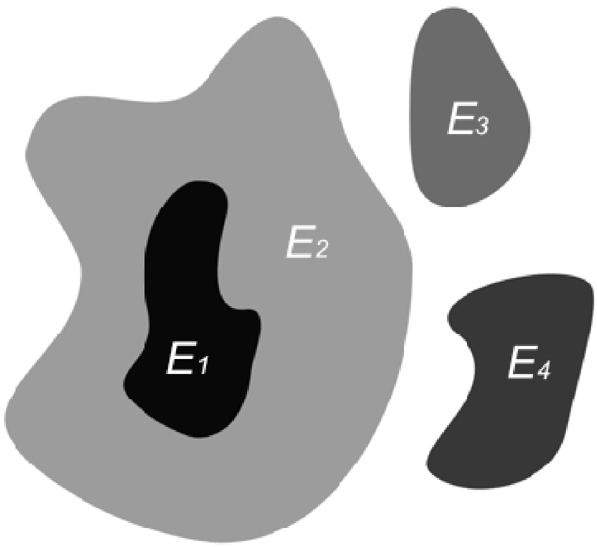

Figure 9: Simple shapes interpreted as sets in $\mathbb{R}^2$.

Figure 10: Original image (Picture 1), processed using TV regularization with $||w|| = 1$ (Picture 2), with $||w|| = 10$ (Picture 3), with $||w|| = 20$ (Picture 4).

3 Preliminaries

3.1 Sets locally described by C^k functions

As indicated in section 2, our analysis focuses primarily on examining the level sets of solutions to $\mathcal{P}_0(y_0)$ and $\mathcal{P}_\lambda(y)$. Given the strong dependence of our arguments on the boundary regularity of these sets, we will introduce a definition of a 'smooth' set, together with a concept of convergence applicable to such sets. We present here a slightly modified version of a smooth set from [10, p. 78ff.], which is restricted to the use of open squares.

Definition 3.1 (Smooth set): [9, p. 5f.] Let E be a subset of $\mathbb{R}^2$ such that $E \neq \emptyset$. Define a square in $\mathbb{R}^2$ with center $x \in \mathbb{R}^2$, sidelength $2r > 0$, and axis oriented along a unit vector $v \in S^1$ by:

$$C(x,r,v) := x + R_v(C(0,r)), \tag{1}$$

where $R_v \in SO(2)$ rotates the point $(0,1)$ to v, and $C(0,r) := (-r,r)^2$.

We say that E is of class C^k if for every $x \in \partial E$ there exists $r_x > 0$, $v_x \in S^1$, and a function $u_x \in C^k([-r_x, r_x])$ such that:

$$\begin{cases} \operatorname{int} E \cap C(x,r_x,v_x) = x + R_{v_x}(\text{hypograph}(u_x) \ \cap \ C(0,r_x)), \\ \quad \partial E \cap C(x,r_x,v_x) = x + R_{v_x}(\text{graph}(u_x) \ \cap \ C(0,r_x)) \end{cases}$$

where:

$$\text{hypograph}(u_x) := \{(z,t) \in C(0,r_x) \mid t < u_x(z)\},$$
$$\text{graph}(u_x) := \{(z,t) \in C(0,r_x) \mid t = u_x(z)\}.$$

and $x + A$ represents the translation of A by x. Furthermore, it holds that there exist r_x and v_x, such that $u_x(0) = 0$ and $u_x'(0) = 0$. Given the previously established properties of the set E and its boundary ∂E, we can now define the outward unit normal at a point $(z, u_x(z))$ on ∂E, which is denoted by:

$$\nu_E(z,u_x(z)) := \frac{1}{\sqrt{1 + u_x'(z)^2}} \begin{pmatrix} -u_x'(z) \\ 1 \end{pmatrix} \tag{2}$$

then the signed curvature at this point is given by:

$$H_E(z,u_x(z)) := \left(\frac{-u_x'}{\sqrt{1+u_x'^2}}\right)'(z) = \frac{-u_x''(z)}{(1+u_x'(z)^2)^{3/2}}, \tag{3}$$

which is intrinsic to the geometry of E and invariant under changes in r, x, or u_x.
In less formal terms, a smooth set is defined as a set whose boundary exhibits a certain regularity. This means that for every segment of the boundary, there exists a correspondingly regular function such that, barring rotation and translation, the graph of this function accurately aligns with that particular boundary segment (as illustrated below in Figure 11).

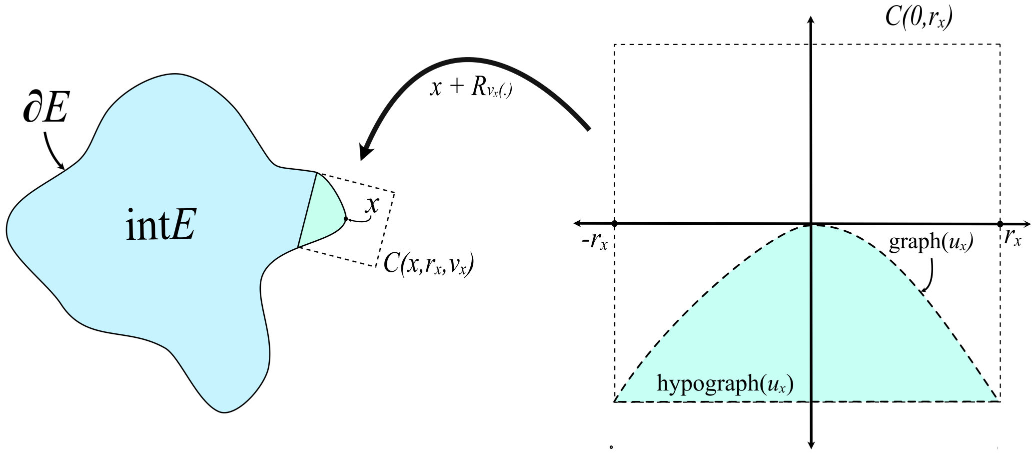

Figure 11: Illustration of the local representation for a smooth set E

Proposition 3.2: [10, Thm. 5.2, Def. 5.1]

Let ∂E be a compact boundary. Then, the selection of r_x (as in definition 3.1) can be made independent of x. Under these conditions, the set $\{u_x\}_{x\in\partial E}$ is uniformly equicontinuous, which means:

$$\forall\epsilon>0,\ \exists\delta>0,\ \forall u_x\in\{u_x\}_{x\in\partial E},\ \forall x,y\in[-r,r]: |x-y|<\delta \implies |u(x)-u(y)|<\epsilon.$$

Given this insight, we can now define a concept of convergence.

Definition 3.3 (Convergence of sets of class C^k): [9, p. 6f.] Suppose E is a C^k class set with a compact boundary ∂E. A sequence $(E_n)_{n\in\mathbb{N}}$ is said to converge to E in C^k if there exists constants $r>0$ and $n_0\in\mathbb{N}$ such that:

(i) For all $n\geq n_0$, $\partial E_n\subseteq\bigcup_{x\in\partial E}C(x,r,\nu_E(x))$.

(ii) For each $n\geq n_0$ and every $x\in\partial E$, there are functions $u_{n,x}\in C^k([-r,r])$ for which:

$$\begin{cases} \partial E_n\cap C(x,r,\nu_E(x)) = x+R_{\nu_E(x)}(\mathrm{graph}(u_{n,x})\ \cap\ C(0,r)), \\ \mathrm{int}\,E_n\cap C(x,r,\nu_E(x)) = x+R_{\nu_E(x)}(\mathrm{hypograph}(u_{n,x})\ \cap\ C(0,r). \end{cases}$$

(iii) Letting $(u_x)_{x\in\partial E}$ be a set of functions that adhere to:

$$\begin{cases} \partial E \cap C(x,r,\nu_E(x)) = x + R_{\nu_E(x)}(\text{graph}(u_x) \ \cap \ C(0,r)), \\ \text{int}\, E \cap C(x,r,\nu_E(x)) = x + R_{\nu_E(x)}(\text{hypograph}(u_x) \ \cap \ C(0,r)) \end{cases}$$

and the following convergence is guaranteed:

$$\sup_{x\in\partial E} \|u_{n,x} - u_x\|_{C^k([-r,r])} \xrightarrow[n\to\infty]{} 0.$$

Remark 3.4 (Parameterization of the boundary of a smooth set): Since we will often work with functions defined on the boundary of a set E of class C^k, we provide an insight into how one can move from previously described local representations through C^k functions to a global representation of ∂E by a function $\gamma : [0,L] \to \partial E$, known as a parameterization of ∂E.
Consider $(C(x,r_x,v_x))_{x\in\partial E}$ as in Definition 3.1, this collection obviously forms an open covering of ∂E. Assuming ∂E is compact, in this case, there exists a finite sub-cover $(C(x_i,r_{x_i},v_{x_i}))_{i\in I}$ of ∂E. For demonstration purposes, assume $I = \{1,2,3\}$. Hence, we can construct compact sets $D_i \subseteq \mathbb{R}$ such that we obtain functions $u_i \in C^k(D_i,\mathbb{R}^2)$ $(i \in \{1,2,3\})$ with:

$$u_1(D_1) \cup u_2(D_2) \cup u_3(D_3) = \partial E$$

and an $L > 0$ exists such that:

$$D_1 \cup D_2 \cup D_3 = [0,L].$$

The individual D_i have the following form:

$$\begin{aligned} D_1 &= [0,b_1] \cup [a_1,L] \\ D_2 &= [a_2,b_2] \\ D_3 &= [a_3,b_3]. \end{aligned}$$

Given $a_2 < b_1, a_3 < b_2, a_1 < b_3$ and $u_1(0) = u_1(L)$, the definition of γ for $x \in [0,L]$ is:

$$\gamma(x) = \begin{cases} u_1(x), & \text{for } x \in [0,a_2] \\ \phi_1(x)u_1(x) + (1-\phi_1(x))u_2(x), & \text{for } x \in (a_2,b_1) =: A_1 \\ u_2(x), & \text{for } x \in [b_1,a_3] \\ \phi_2(x)u_2(x) + (1-\phi_2(x))u_3(x), & \text{for } x \in (a_3,b_2) =: A_2 \\ u_3(x), & \text{for } x \in [b_2,a_1] \\ \phi_3(x)u_3(x) + (1-\phi_3(x))u_1(x), & \text{for } x \in (a_1,b_3) =: A_3 \\ u_1(x), & \text{for } x \in [b_3,L] \end{cases}$$

where $\phi_i \in C^k(A_i)$ with $0 \leq \phi_i(x) \leq 1$, and ϕ_i tending to 1 at the start and to 0 at the end of each interval A_i. We observe that γ' is tangential to ∂E. We will see later how this property will be helpful for functions defined on ∂E. For a more detailed derivation in the general setting (not limited to $\mathbb{R}^2$), see [13, p. 212ff.].

3.2 Deformation in the normal direction

Upon defining a formal approach for defining smooth sets (shapes within the unknown image u_0), our next objective is to delineate a method for distorting these sets (shapes in the approximation solution $u_{\lambda,w}$ of $\mathcal{P}_\lambda(y)$). This method must preserve the boundary regularity of the original set. To realize this aim, we utilize a method termed normal deformation. Here, the deformation of the set occurs along the normals to its boundary, ensuring that the altered boundary of the set remains smooth. This approach is exemplified in Figure 12. The following lemma and proposition provide an overview of how this can be formally presented.

Lemma 3.5: [9, p. 7] Given a bounded set E of class C^k (where $k \geq 2$), there is a constant $C > 0$ ensuring that for each function φ in $C^{k-1}(\partial E)$, the mapping $\varphi\nu_E$ can be expanded to $\xi_\varphi \in C^{k-1}(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\|\xi_\varphi\|_{C^{k-1}(\mathbb{R}^2,\mathbb{R}^2)} \leq C\|\varphi\|_{C^{k-1}(\partial E)}.$$

Proposition 3.6: [9, p. 7] Assuming E is a bounded open set of class C^k (with $k \geq 2$), and for every function φ in $C^{k-1}(\partial E)$ that satisfies $\|\varphi\|_{C^{k-1}(\partial E)} \leq c$ (for some $c > 0$), a unique bounded open set of class C^{k-1}, denoted as E_φ, exists. This set complies with:

$$\partial E_\varphi = (Id + \varphi\nu_E)(\partial E). \tag{4}$$

Furthermore, an extended function ξ_φ of $\varphi\nu_E$ exists, ensuring that

$$E_\varphi = (Id + \xi_\varphi)(E)$$

with

$$\|\xi_\varphi\|_{C^{k-1}(\mathbb{R}^2,\mathbb{R}^2)} < 1.$$

As a special mention, E_φ is diffeomorphic to E.

As we conclude this chapter, we reconnect to the previously mentioned smooth sets. The next proposition allows us to establish a direct link between convergence in the C^k sense as in Definition 3.3 and decreasing deformation φ.

Proposition 3.7: [9, p. 7] Let $(E_n)_{n>0}$ be a sequence that converges to a bounded set E in C^k with $k > 2$. If n is sufficiently large, then there exists $\varphi_n \in C^{k-1}(\partial E)$ such that $E_n = E_{\varphi_n}$, and $\|\varphi_n\|_{C^{k-1}(\partial E)} \xrightarrow[n\to\infty]{} 0$.

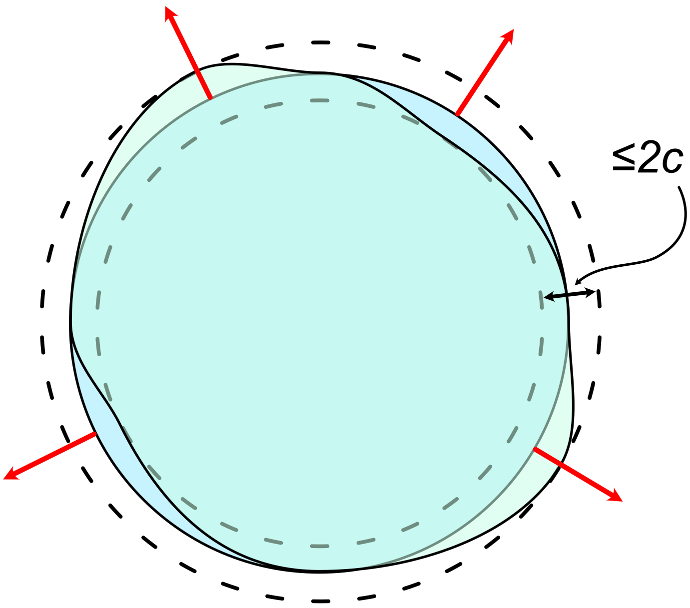

Figure 12: Illustration of a ball B in $\mathbb{R}^2$ (shown in light blue) deformed along its outward normal (indicated in red) with the resulting set B_φ depicted in light green.

3.3 Measures in Euclidean spaces

This chapter is included for the sake of completeness, and aims to remind us of some basic concepts of measure theory that are necessary for further argumentation. As we will see later, the total variation (TV) of a function is essentially a measure on $\mathbb{R}^n$.

Definition 3.8 (Borel $\sigma-$Algebra): [16, p. 14] The Borel σ-Algebra on $\mathbb{R}^n$, denoted $\mathcal{B}(\mathbb{R}^n)$, is defined as:

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}),$$

where $\mathcal{O}$ is the collection of all open subsets of $\mathbb{R}^n$, $\sigma(\cdot)$ represents the generation of the smallest σ-algebra containing its argument, in this case, $\mathcal{O}$. Such a σ-algebra includes the entire set, is closed under countable unions and intersections, and retains the complements of its members.

Definition 3.9 (Borel and Radon measure): [1, Def. 1.40] Let X be a locally compact separable metric space with its Borel σ-algebra, $\mathcal{B}(X)$.

(a) A measure on X is called a *Borel measure*. It is termed a *positive Radon measure* if it is positive and finite on the compact subsets of X.

(b) A set function on X that is a measure on every relatively compact Borel subset is called a *Radon measure*. If $\mu : \mathcal{B}(X) \to \mathbb{R}^m$ is such a measure, we say it is a *finite Radon measure*.

Definition 3.10 (Total variation of a measure): [1, Def. 1.4b]
Let $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ be a signed measure on $\mathbb{R}^n$ (This means μ can take both positive and negative values).

The *total variation* $|\mu|$ of μ is itself a measure, defined by:

$$|\mu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a disjoint sequence of sets in } \mathcal{B}(\mathbb{R}^n) \text{ with } E = \bigcup_{i=1}^{\infty} E_i\right\}$$

for every set E in $\mathcal{B}(\mathbb{R}^n)$.
If additionally μ is a finite Radon measure, then according to [1, Prop. 1.47], for every open set A it holds that

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{m} \int_{\mathbb{R}^n} u_i \, d\mu_i \,\middle|\, u \in [C_c(A)]^m,\ \|u\|_\infty \leq 1\right\}.$$

Definition 3.11 (s-Dimensional Hausdorff Measure): [16, 1, p. 5, Def. 1.65] Let E be a subset of $\mathbb{R}^n$ and s be a real number in the interval $[0,\infty]$. The s-dimensional Hausdorff measure of E, denoted by $\mathcal{H}^s(E)$, is defined as

$$\mathcal{H}^s(E) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E),$$

where $\mathcal{H}^s_\delta(E)$ for a given $\delta \in [0,\infty]$ is given by

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_i (\operatorname{diam} U_i)^s : \{U_i\} \text{ is a countable covering of } E \text{ such that } \operatorname{diam} U_i < \delta\right\}.$$

Furthermore, if F is a subset of $\mathbb{R}^n$, the restriction of $\mathcal{H}^s(E)$ to F is denoted as

$$\mathcal{H}^s(E)\llcorner_F = \mathcal{H}^s(E \cap F).$$

In the following, we will primarily focus on the 1-dimensional Hausdorff measure, denoted by $\mathcal{H}^1(\partial E)$.

Definition 3.12 (Hausdorff-metric): [1, p. 320] Let $A, B \subseteq \mathbb{R}^n$ be non-empty and compact. The *Hausdorff metric* $\mathcal{H}(A,B)$ between the sets A and B is defined as:

$$\mathcal{H}(A,B) = \max\left\{\sup_{a\in A}\Big(\inf_{b\in B}\|a-b\|\Big),\ \sup_{b\in B}\Big(\inf_{a\in A}\|a-b\|\Big)\right\}.$$

Definition 3.13 (Lebesgue measure): [1, Def. 1.52] The Lebesgue measure, m, on $\mathbb{R}^N$ assigns to each Lebesgue measurable set E a non-negative number representing its "volume". It's defined by:

$$m(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{vol}(B_i) : E \subset \bigcup_{i=1}^{\infty} B_i\right\}$$

where

$$B_i = [a_{i,1}, b_{i,1}] \times [a_{i,2}, b_{i,2}] \times \cdots \times [a_{i,n}, b_{i,n}] = \{x \in \mathbb{R}^n \mid a_{i,j} \leq x_j \leq b_{i,j}\ ,\ j \in \{1, \ldots, N\}\}$$

and

$$\mathrm{vol}(B_i) = \prod_{j=1}^{n} (b_{i,j} - a_{i,j}).$$

In the following we will restrict ourselves to the case $\mathbb{R}^2$. For brevity, we will denote the area of a set E (i.e. $m(E)$) by $|E|$.

Definition 3.14 (Lebesgue density of smooth sets): [9, p. 6] Let E be an open subset of $\mathbb{R}^2$ of class C^1 and let x belong to $\mathbb{R}^2$. The *Lebesgue density* of E at x is defined as

$$\theta_E(x) := \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} = \begin{cases} 1 & \text{if } x \in E, \\ 1/2 & \text{if } x \in \partial E, \\ 0 & \text{if } x \in \mathbb{R}^2 \setminus \bar{E}. \end{cases} \tag{5}$$

A measurable set $\tilde{E}$ in $\mathbb{R}^2$ is considered equivalent to a C^1 open set E when E is uniquely identified by its Lebesgue points, represented as $\{x \in \mathbb{R}^2 \mid \theta_{\tilde{E}}(x) = 1\}$. In other words, $\tilde{E}$ and E differ only on a set of measure zero, and thus, they belong to the same Lebesgue equivalence class.

In the discussions that follow, when E has a C^k class representative, we will denote the boundary of E as the topological boundary of this representative, using the notation ∂E.

3.4 Functions with bounded variation

In this chapter, our goal is to study the theory of functions with bounded variation. We will show that for such functions, their total variation in the setting of section 2 describes a finite measure on $\mathbb{R}^2$. We will then introduce some terms related to this measure to avoid any misunderstandings in later arguments. If further information is desired, [1, p. 116ff.] provides a comprehensive list of properties of such functions.

Definition 3.15 (The space BV): [1, Def. 3.1] Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in L^1(\Omega)$. We say that u is a *function of bounded variation* in Ω if the distributional derivative of u is representable by a finite Radon measure in Ω. For each test function $\phi \in C_c^\infty(\Omega)$ and each $i = 1, \ldots, N$, we have:

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi d D_i u$$

where $Du = (D_1 u, \ldots, D_N u)$ is an $\mathbb{R}^N$-valued measure in Ω. The vector space of all functions of bounded variation in Ω is denoted by $\mathrm{BV}(\Omega)$.

Definition 3.16 (The total variation): In addressing our problem, we focus on functions denoted by u belonging to the space $\mathrm{BV}(\mathbb{R}^2)$. We can then define the total variation of u as follows:

$$TV(u) := |Du|(\mathbb{R}^2) = \sup_{\substack{z \in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2) \\ \|z\|_\infty \leq 1}} \left(-\int_{\mathbb{R}^2} u\, div\, z\right). \tag{6}$$

Remark 3.17: Consider $z \in C_c^\infty(\mathbb{R}^2)$ satisfy $\|z\|_\infty \leqslant 1$. Let us examine the expression

$$\begin{aligned}
-\int_{\mathbb{R}^2} u \operatorname{div} z\, dx &= -\sum_{i=1}^{2} \int_{\mathbb{R}^2} u \frac{\partial z}{\partial x_i}\, dx \\
&\overset{*}{=} \sum_{i=1}^{2} \int_{\mathbb{R}^2} \frac{\partial u}{\partial x_i} z\, dx \\
&= \sum_{i=1}^{2} \int_{\mathbb{R}^2} D_i u\, z\, dx \\
&= \sum_{i=1}^{2} \langle D_i u, z \rangle_{L^2(\mathbb{R}^2)},
\end{aligned}$$

($*$ results from integration by parts, and the boundary term vanishes due to the compact support of z.)

If the given expression is bounded, the analysis of the properties of $\langle D_i u, \cdot \rangle_{L^2(\mathbb{R}^2)}$ suggests that the prerequisites for the Riesz representation theorem (see [1, Thm. 1.54]) are met. Consequently, this infers the existence of a finite Radon measure $Du = (D_1 u, D_2 u)$ on $\mathbb{R}^2$. Adhering to definition 3.10, we deduce:

$$|Du|(\mathbb{R}^2) = \sup_{\substack{z \in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2) \\ \|z\|_\infty \leq 1}} \left(\sum_{i=1}^{2} \langle D_i u, z \rangle_{L^2(\mathbb{R}^2)} \right) = \sup_{\substack{z \in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2) \\ \|z\|_\infty \leq 1}} \left(-\int_{\mathbb{R}^2} u \operatorname{div} z \right).$$

Thus, the distributional derivative of u is representable by a finite Radon measure, as characterized in Definition 3.15.

Definition 3.18 (Weak* convergence): [1, Def. 3.11] Consider a sequence u_h and a function u, both belonging to $BV(\Omega)$. The sequence u_h is said to converge weakly* to u within $BV(\Omega)$ provided that:

- u_h converges to u in $L^1(\Omega)$, thus $\|u_h - u\|_{L^1(\Omega)} \longrightarrow 0$,
- Du_h converges weakly* to Du in Ω. This can be written as

  $$\lim_{h\to\infty} \int_\Omega \phi\, dDu_h = \int_\Omega \phi\, dDu$$

  for every test function $\phi \in C_0(\Omega)$.

Definition 3.19 (Strict convergence): [1, Def. 3.14] Let $u, u_h \in BV(\Omega)$. We define the sequence (u_h) to *strictly converge* in $BV(\Omega)^m$ to u if and only if the following two conditions are satisfied:

1. The sequence (u_h) converges to u in $L^1(\Omega)$,
2. $|Du_h|(\Omega)$ converges weakly* to $|Du|(\Omega)$ as $h \to \infty$.

3.5 Convex sets and functions

As we will see in Chapter 5, a significant portion of this work is based on the analysis of the convex set $\{TV \leq 1\} = \{u \mid TV(u) \leq 1\}$. In this context, we will elucidate some fundamentals of convex analysis and subsequently apply these principles specifically to $\mathrm{TV}(u)$. In the following, let X be a vector space.

Definition 3.20 (Convex Set): [19, p. 10]
A set $C \subseteq X$ is called convex if, for any two points $A, B \in C$ and any λ with $0 \leq \lambda \leq 1$, the point $\lambda A + (1-\lambda)B$ also lies in C. In other words, every line segment connecting two points in C is entirely contained within C.

Definition 3.21 (Extreme points of a convex set): [19, p. 162] An extreme point of a convex set $C \subseteq X$ is a point $x \in C$ such that there are no two distinct points $y, z \in C$, $y \neq z$, and no λ with $0 < \lambda < 1$ such that $x = \lambda y + (1-\lambda)z$. Essentially, an extreme point of C cannot be expressed as a convex combination of any other two points in C. For a more intuitive understanding, we illustrate this with an arbitrary convex set on $\mathbb{R}^2$ in figure 13 below.

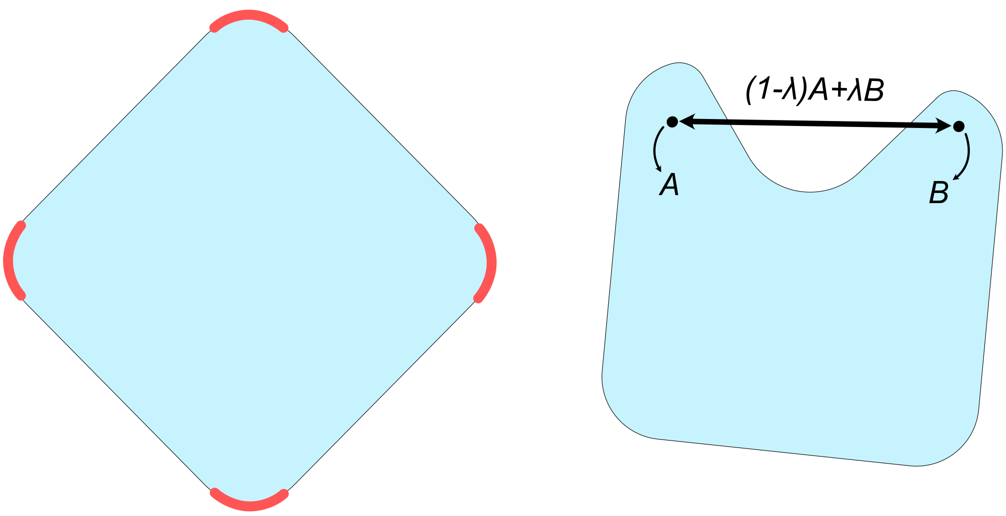

Figure 13: Figure depicts a convex set on the left, with its extreme points highlighted in red. Conversely, the right side presents an illustration of a non-convex set.

Having clarified convexity in relation to sets, we can now extend the concept to functions. For this purpose, let X be a Banach space in the following discussion.

Definition 3.22 (Convex / Support Function): [5, p. 111] A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is termed *convex* if X is convex and for any $x, y \in X$ and λ in $[0,1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Let the support function on the dual space X^* for a set C be defined as follows:

$$\sigma_C(x^*) := \sup\{x^*(x) \mid x \in C\}.$$

Definition 3.23 (Convex subdifferential): [5, p. 117] Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The convex subdifferential of f at $x \in X$, denoted by $\partial f(x)$, is

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq x^*(y-x), \forall y \in X\},$$

Elements of $\partial f(x)$ are referred to as subgradients.

The subdifferential represents a generalization of the gradient for non-differentiable convex functions. We will illustrate this with the following example.

Example: The absolute value function

To comprehend how the subdifferential functions for non-differentiable scenarios, consider the absolute value function $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) := |x|$. In this case, the dual elements we are looking for in the subdifferential can be identified with elements from $\mathbb{R}$ (see Riesz representation theorem [18, Thm. 7.16]). We observe:

- **At differentiable points:** For $x \neq 0$, the function $f(x) = |x|$ is differentiable. It can be easily shown that:
  - $\partial f(x) = \{1\}$ for $x > 0$,
  - $\partial f(x) = \{-1\}$ for $x < 0$.

  Thus, at differentiable points, the subdifferential simply corresponds to the conventional derivative of the function.

- **At non-differentiable points:** The function is non-differentiable at $x = 0$. Consider the subdifferntial in this point:

  $$\partial f(0) = \{g \in \mathbb{R} \mid |x| \geq g \cdot x\}.$$

  It is straight forward to show that $\partial f(0) = [-1, 1]$.

Therefore, the subdifferential at the point 0 in this case can be visualized as encompassing all slopes between those of the tangents approaching the graph of f from the left and right at the point 0.

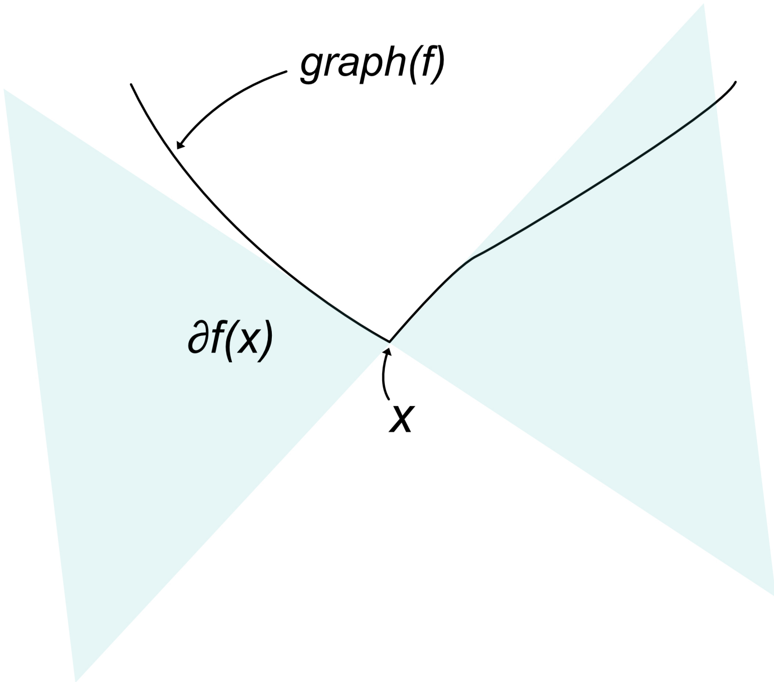

Figure 14: Illustration of the subdifferential (in light blue) of a function f (whose graph is depicted in black) at a non-differentiable point x.

Subdifferential of the total variation

We proceed to determine $\partial \mathrm{TV}(u)$ for a given $u \in L^2(\mathbb{R}^2)$. First, however, we must demonstrate that TV is convex. It is shown as follows ($x, y \in L^2(\mathbb{R}^2),\ \lambda \in [0,1]$):

$$\begin{aligned}
\mathrm{TV}(\lambda x + (1-\lambda)y) &= \sup\left\{ -\int_{\mathbb{R}^2} (\lambda x + (1-\lambda)y)\operatorname{div} z \mid z \in C_c^\infty(\mathbb{R}^2), \|z\|_\infty \leq 1 \right\} \\
&= \sup\left\{ -\lambda \int_{\mathbb{R}^2} x \operatorname{div} z - (1-\lambda)\int_{\mathbb{R}^2} y \operatorname{div} z \mid z \in C_c^\infty(\mathbb{R}^2), \|z\|_\infty \leq 1 \right\} \\
&\leq \lambda \sup\left\{ -\int_{\mathbb{R}^2} x \operatorname{div} z \mid z \in C_c^\infty(\mathbb{R}^2), \|z\|_\infty \leq 1 \right\} \\
&\quad + (1-\lambda) \sup\left\{ -\int_{\mathbb{R}^2} y \operatorname{div} z \mid z \in C_c^\infty(\mathbb{R}^2), \|z\|_\infty \leq 1 \right\} \\
&= \lambda \mathrm{TV}(x) + (1-\lambda)\mathrm{TV}(y).
\end{aligned}$$

Furthermore, it serves as the support function for the convex set C, defined as

$$C := \left\{ \operatorname{div} z \,|\, z \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2), \|z\|_\infty \leq 1 \right\}.$$

We can now observe that $\partial \mathrm{TV}(0)$ is the closure of C in $L^2(\mathbb{R}^2)$. This closure is denoted by

$$\begin{aligned}\overline{C} &= \Big\{\mathrm{div} z \,|\, \mathrm{div} z \in L^2(\mathbb{R}^2),\, \|z\|_\infty \le 1\Big\} \\ &= \Big\{x \in L^2(\mathbb{R}^2) \,|\, \forall u \in L^2(\mathbb{R}^2),\, \sigma_C(u) \ge \langle x, u\rangle_{L^2(\mathbb{R}^2)}\Big\} \\ &= \Big\{x^* \in (L^2(\mathbb{R}^2))^* \,|\, \forall u \in L^2(\mathbb{R}^2),\, TV(u) \ge x^*(u)\Big\} = \partial \mathrm{TV}(0).\end{aligned} \tag{7}$$

Since $L^2(\mathbb{R}^2)$ is a Hilbert space, by the Riesz representation theorem [18, Thm. 7.16], there exists an $x \in L^2(\mathbb{R}^2)$ such that $x^*(u) = \langle x, u\rangle_{L^2(\mathbb{R}^2)}$ for all $u \in L^2(\mathbb{R}^2)$.

Thus, we arrive at the identity:

$$\partial \mathrm{TV}(0) = \Big\{\eta \in L^2(\mathbb{R}^2) \,|\, \forall u \in L^2(\mathbb{R}^2),\, \Big|\int_{\mathbb{R}^2} \eta u\Big| \le \mathrm{TV}(u)\Big\}$$

and for any $u \in L^2(\mathbb{R}^2)$, the subdifferential of TV at u is given by

$$\partial \mathrm{TV}(u) = \Big\{\eta \in \partial \mathrm{TV}(0) \,|\, \int_{\mathbb{R}^2} \eta u = \mathrm{TV}(u)\Big\} \tag{8}$$

(see [14, Lem. 2]).

3.6 Sets of finite perimeter

In this section we establish a connection between the functions of bounded variation and sets with finite perimeter. We will show that the total variation of such functions can be described by their level sets, which we will define later. First, let us define what is known as a finite-perimeter set:

Definition 3.24 (Finite perimeter set): [16, p. 122] Consider a Lebesgue measurable set E within $\mathbb{R}^2$. We term E as having locally finite perimeter if, for any compact subset K of $\mathbb{R}^2$, the condition

$$\sup\Big\{\int_E \mathrm{div}\ z\, dx \mid z \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2),\ \mathrm{supp}(z) \subset K,\ \|z\|_\infty \le 1\Big\} < \infty$$

is satisfied. When the above expression remains bounded regardless of the choice of K, we identify E as a set with finite perimeter in $\mathbb{R}^2$. In such cases, we express the perimeter of E as

$$\begin{aligned}P(E) &:= \sup\Big\{\int_E \mathrm{div} z\, dx \mid z \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2),\ \|z\|_\infty \le 1\Big\} \\ &= \sup\Big\{\int_{\mathbb{R}^2} \chi_E\ \mathrm{div} z\, dx \mid z \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2),\ \|z\|_\infty \le 1\Big\} \\ &= TV(\chi_E) = |D\chi_E|(\mathbb{R}^2).\end{aligned}$$

Furthermore, we specify the relative perimeter of E within a set $F \subseteq \mathbb{R}^2$ as follows:

$$P(E;F) := |D\chi_E|(F).$$

Given E as an open set of class C^k ($k \geq 1$), the perimeter $P(E)$ can be straightforwardly identified as the length of its boundary, represented as $P(E) = H^1(\partial E)$ (see [16, Thm. 3.8]).

We observe that the total variation of a characteristic function over B from $\mathbb{R}^2$ can be represented by the perimeter of B. Through the subsequent theorem, we extend this representation to arbitrary functions in $L^1_{\text{loc}}(\mathbb{R}^2)$.

Theorem 3.25 (Coarea formula): [1, Thm. 3.40] Let u be a function in $L^1_{\text{loc}}(\mathbb{R}^2)$ and t be a real number. The *level sets* of u are defined as:

$$U^{(t)} := \begin{cases} \{x \in \mathbb{R}^2 \mid u(x) \geq t\}, & \text{if } t \geq 0, \\ \{x \in \mathbb{R}^2 \mid u(x) \leq t\}, & \text{otherwise.} \end{cases} \tag{9}$$

The *Coarea Formula* relates functions of bounded variation with sets of finite perimeter and is given by:

$$\forall u \in L^2(\mathbb{R}^2), \quad TV(u) = |Du|(\mathbb{R}^2) = \int_{-\infty}^{\infty} |D\chi_{U^{(t)}}|(\mathbb{R}^2)\, dt = \int_{-\infty}^{\infty} P(U(t))\, dt. \tag{10}$$

Finite perimeter sets possess the interesting property that they satisfy the following inequality.

Theorem 3.26 (Isoperimetric inequality): [16, Prop. 12.37] Given a set $E \subseteq \mathbb{R}^2$ with finite perimeter, the *isoperimetric inequality* asserts that:

$$\sqrt{\min\{|E|, |\mathbb{R}^2 \setminus E|\}} \leq c_2 P(E).$$

We can infer that for a set E with finite perimeter, it must hold that either E or its complement $\mathbb{R}^2 \setminus E$ has a finite measure. This aspect will be beneficial in subsequent discussions. Moreover, in the two-dimensional context, $c_2 := \frac{1}{\sqrt{4\pi}}$ is identified as the isoperimetric constant. The condition of equality is met exactly when E has the form of a circle in $\mathbb{R}^2$. Thus, in this case, the circle is the unique minimizer of the perimeter.

For our continued reasoning, we will introduce a somewhat weaker variant of the perimeter minimizer, a so called quasi-minimizer.

Definition 3.27 ((Λ, r_0)-perimeter minimizer): [16, p. 278] Let A be an open set and $E \subset \mathbb{R}^n$ ($n \geq 2$) a set with locally finite perimeter. We define E to be a (Λ, r_0)-perimeter minimizer in A if the support of the measure $|D\chi_E|$ coincides with ∂E, and there exist constants $\Lambda \geq 0$ and $r_0 > 0$ satisfying

$$P(E; B(x,r)) \leq P(F; B(x,r)) + \Lambda |E \Delta F|,$$

for all sets F where $E \Delta F \subset\subset B(x,r) \cap A$ and for all $r < r_0$.

At the end of this section, we give a definition of a restriction that can be applied to sets with finite perimeter. This will be used in chapter 5 in particular.

Definition 3.28 (Simple Set): [2, p. 52, Def. 3] Let E be a set of finite perimeter in $\mathbb{R}^2$. We define the following concepts:

- **Decomposable set:** A set E is termed decomposable if it can be partitioned into two subsets A and B, each with finite Lebesgue measures, in such a way that $P(E) = P(A) + P(B)$.
- **Indecomposable set:** Conversely, a set E is considered indecomposable if it cannot be partitioned into such subsets.
- **Simple set:** A set E is classified as simple if it is either identical to $\mathbb{R}^2$ or has a finite measure, and both the set E and its complement in $\mathbb{R}^2$ are indecomposable.

4 Fenchel dual representations

The aim of this section is to transform the problems $\mathcal{P}_0(y_0)$ and $\mathcal{P}_\lambda(y)$ from section 2 into their dual forms, known as the Fenchel representation. For the argumentation in this section, it is assumed that Φ is a linear continuous operator. We will see in the course of this thesis what exactly this transformation of the problem statement contributes to the analysis. But first, let us give an overview of the theory behind a dual representation.

4.1 Theoretical background

To proceed, let X and Y be Banach spaces. To formulate the dual problem, we first need the Fenchel conjugate of the functions involved in the problem, which is given by:

Definition 4.1 (Fenchel Conjugate): [5, p. 134] Consider a function $f : X \to [-\infty, +\infty]$. The conjugate of f, denoted as $f^* : X^* \to [-\infty, +\infty]$, is given by the following relationship:

$$f^*(x^*) = \sup_{x \in X} (x^*(x) - f(x)).$$

A notable characteristic of this function is that it remains convex without imposing any requirements on f. This is evident because for any $\lambda \in [0,1]$ and $x^*, y^* \in X^*$, the following holds:

$$\begin{aligned}
f^*(\lambda x^* + (1-\lambda)y^*) &= \sup_{x \in X} \Big((\lambda x^* + (1-\lambda)y^*)(x) - f(x)\Big) \\
&= \sup_{x \in X} \Big((\lambda x^* + (1-\lambda)y^*)(x) - (\lambda + 1 - \lambda)f(x)\Big) \\
&\leq \lambda \sup_{x \in X} \Big((x^*(x) - f(x)\Big) + (1-\lambda) \sup_{x \in X} \Big(y^*(x) - f(x)\Big) \\
&= \lambda f^*(x^*) + (1-\lambda)f^*(y^*).
\end{aligned}$$

Consequently, the weak formulation of Fenchel duality is established through the following theorem:

Theorem 4.2 (Fenchel weak duality): [5, Thm. 4.4.2] Consider two convex functions, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$, and a bounded linear transformation $A : X \to Y$. We define the primal objective p and the dual objective d as follows:

$$p = \inf_{x \in X} \left(f(x) + g(Ax) \right),$$

$$d = \sup_{x^* \in Y^*} \left(-f^*(A^* x^*) - g^*(-x^*) \right).$$

In this context, it always holds that $p \geq d$.

For the equality of both problems, that is, to have

$$\inf_{x \in X} \left(f(x) + g(Ax) \right) = \sup_{x^* \in Y^*} \left(-f^*(A^* x^*) - g^*(-x^*) \right),$$

we additionally require a $x \in X$ such that $f(x) < \infty$, $g(Ax) < \infty$ and that g is continuous at Ax (see [5, Thm. 4.4.3]). If this property is satisfied, it leads to what is known as *strong duality.*

In this context, we refer to p as the primal and d as the dual problem formulation. The advantages of this dual representation of the original problem are clear. For instance, if we minimize over an infinite-dimensional space in the original problem formulation, it can be very difficult or even impossible to a find a solution. However, by considering the dual representation, which possibly takes place over a finite-dimensional space, we can solve the problem much more easily.

Let us now transform the problem statements from 2.1 into their dual forms.

4.2 Problem $\mathcal{P}_0(y_0) \leftrightarrow \mathcal{D}_0(y_0)$:

As discussed in 2.1, we derive our primal problem for the noiseless case, $\mathcal{P}_0(y_0)$, through the following minimization:

$$\inf_{u \in L^2(\mathbb{R}^2)} TV(u) \quad \text{subject to} \quad \Phi u = y_0.$$

then its dual representation, as in theorem 4.2, is formulated as:

$$\sup_{p \in \mathcal{H}} \langle p, y_0 \rangle_{\mathcal{H}} \quad \text{subject to} \quad \Phi^* p \in \partial \mathrm{TV}(0). \tag{$\mathcal{D}_0(y_0)$}$$

Proof. It is evident that $\mathcal{P}_0(y_0)$ may also be expressed in the following manner:

$$\inf_{u \in L^2(\mathbb{R}^2)} \left(TV(u) + \iota_p(\Phi(u)) \right),$$

where ι_p is defined as:

$$\iota_p(z) := \begin{cases} 0, \text{ for } z = y_0 \\ \infty, \text{ otherwise.} \end{cases}$$

Since we already know from section 3.5 that TV is convex, we only need to show that this is also true for ι_p. To do this, consider $a, b \in \mathcal{H}$ with $a = b = y_0$. Then the following holds:

$$\begin{aligned} \iota_p(\lambda a + (1-\lambda)b) &= \iota_p(\lambda y_0 + (1-\lambda)y_0) \\ &= \iota_p(y_0) = 0 + 0 \\ &= \lambda \iota_p(a) + (1-\lambda)\iota_p(b). \end{aligned}$$

Additionally, when $a = y_0$ and $b \neq y_0$ (or vice versa), we observe:

$$\begin{aligned} \iota_p(\lambda a + (1-\lambda)b) &= \infty = 0 + \infty \\ &= \lambda \iota_p(a) + (1-\lambda)\iota_p(b). \end{aligned}$$

Thus in total, we obtain:

$$\iota_p(\lambda a + (1-\lambda)b) \leq \lambda \iota_p(a) + (1-\lambda)\iota_p(b).$$

Following theorem 4.2, the corresponding dual problem can then be expressed as

$$\begin{aligned} &\sup_{p^* \in \mathcal{H}^*} (-\mathrm{TV}^*(\Phi^* p^*) - \iota_p^*(-p^*)) \\ = &\sup_{p^* \in \mathcal{H}^*} (-\sup\{\Phi^* p^*(u) - \mathrm{TV}(u)) \mid u \in L^2(\mathbb{R}^2)\} - \sup\{-p^*(z) - \iota_p(z) \mid z \in \mathcal{H}\}). \end{aligned}$$

As before, using the Riesz representation theorem [18, Thm. 7.16], we can simplify the expression, leading to

$$= \sup_{p \in \mathcal{H}} (- \underbrace{\sup\{\langle \Phi^* p, u\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u)) \mid u \in L^2(\mathbb{R}^2)\}}_{a} - \underbrace{\sup\{\langle -p, z\rangle_{\mathcal{H}} - \iota_p(z) \mid z \in \mathcal{H}\}}_{b}).$$

Considering the term b and noting that $\iota_p(z) = \infty$ for $z \neq y_0$, we obtain

$$\begin{aligned} &\sup\{\langle -p, z\rangle_{\mathcal{H}} - \iota_p(z) \mid z \in \mathcal{H}\} = \langle -p, y_0\rangle_{\mathcal{H}} - \iota_p(y_0) = \langle -p, y_0\rangle_{\mathcal{H}} \\ &= -\langle p, y_0\rangle_{\mathcal{H}}. \end{aligned}$$

For the term a, the analysis yields

$$\underbrace{\sup\{\langle \Phi^* p, u\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u) \mid u \in L^2(\mathbb{R}^2)\}}_{=:\iota_D(\Phi^* p)} = \begin{cases} 0, & \text{for } \Phi^* p \in \partial \mathrm{TV}(0) \\ \infty, & \text{otherwise} \end{cases}.$$

This holds true because, for $\Phi^* p \in \partial \mathrm{TV}(0)$, it follows that for all $u \in L^2(\mathbb{R}^2)$:

$$\mathrm{TV}(u) \geq \langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} \Leftrightarrow \ 0 \geq \langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u).$$

Thus, it follows that:

$$\sup_{u \in L^2(\mathbb{R}^2)} (\langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u)) = 0.$$

Conversely, for $\Phi^* p \notin \partial \mathrm{TV}(0)$, we have:

$$\mathrm{TV}(u) < \langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} \Leftrightarrow \ 0 < \langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u).$$

Hence, there exists a sequence $(u_n)_{n \in \mathbb{N}} \subseteq L^2(\mathbb{R}^2)$ with $u_n := nu$ such that:

$$\begin{aligned} &\langle \Phi^* p, u_n \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u_n) \Leftrightarrow \ \langle \Phi^* p, nu \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(nu) \\ &\Leftrightarrow \ n(\langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u)) \xrightarrow[n \to \infty]{} \infty. \end{aligned}$$

Therefore:

$$\sup_{u \in L^2(\mathbb{R}^2)} (\langle \Phi^* p, u \rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u)) = \infty.$$

The dual problem $\mathcal{D}_0(y_0)$, corresponding to $\mathcal{P}_0(y_0)$, is thus given by:

$$\begin{aligned} &\sup_{p \in \mathcal{H}} (-\iota_D(\Phi^* p) - \langle p, y_0 \rangle_{\mathcal{H}}) \\ = &\sup_{p \in \mathcal{H}} (\langle p, y_0 \rangle_{\mathcal{H}} - \iota_D(\Phi^* p)) \\ = &\sup_{p \in \mathcal{H}} \langle p, y_0 \rangle_{\mathcal{H}} \quad \text{subject to} \quad \Phi^* p \in \partial \mathrm{TV}(0). \end{aligned}$$

Unfortunately, we cannot demonstrate the properties of strong duality from theorem 4.2 for this problem, but a proof for this problem is provided in [14, Thm.1]. Therefore, we can also assume strong duality for both problem formulations here.

□

Given the equivalence of both problem formulations, we can deduce that, if a solution p exists for $\mathcal{D}_0(y_0)$, then for any solution u of $\mathcal{P}_0(y_0)$, the following relationship is established:

$$\begin{aligned} &\langle p, y \rangle_{\mathcal{H}} = \mathrm{TV}(u) \\ \Longleftrightarrow \ &\langle p, \Phi u \rangle_{\mathcal{H}} = \mathrm{TV}(u) \\ \Longleftrightarrow \ &\langle \Phi^* p, u \rangle_{\mathcal{H}} = \mathrm{TV}(u). \end{aligned}$$

Thus, we obtain with (8)

$$\Phi^* p \in \partial \mathrm{TV}(u). \tag{11}$$

In the reverse direction, it holds that if $(u,p) \in L^2(\mathbb{R}^2) \times \mathcal{H}$ with $\Phi u = y_0$ and additionally satisfying (11), then it can be inferred that u, p also solve problems $\mathcal{P}_0(y_0)$ and $\mathcal{D}_0(y_0)$.
Considering our problem formulation from section 2.1, given an unknown image $u_0 \in L^2(\mathbb{R}^2)$ and linear measurements $y_0 = \Phi u_0$, it suffices to assume the existence of $p \in \mathcal{H}$ for which $\Phi^* p \in \partial \mathrm{TV}(u_0)$ to ensure that u_0 is a solution of $\mathcal{P}_0(y_0)$. This property is termed as the *source condition*.
Furthermore, we may deduce that if Φ is injective on the set $\{u \in L^2(\mathbb{R}^2) \mid \Phi^* p \in \partial \mathrm{TV}(u)\}$, then u_0 is the unique solution of problem $\mathcal{P}_0(y_0)$. This holds because, for u_1, u_2 that satisfy the source condition, the given Φ ensures that the equality $\Phi u_1 = y_0 = \Phi u_2$ implies $u_1 = u_2$.

4.3 Problem $\mathcal{P}_\lambda(y) \leftrightarrow \mathcal{D}_\lambda(y)$:

In the case involving noise, our primal problem $\mathcal{P}_\lambda(y)$ is stated as:

$$\inf_{u \in L^2(\mathbb{R}^2)} \left(\mathrm{TV}(u) + \frac{1}{2\lambda} \|\Phi u - y\|_{\mathcal{H}}^2 \right)$$

and its dual representation is obtained as follows:

$$\sup_{p \in \mathcal{H}} \left(\langle p, y \rangle_{\mathcal{H}} - \frac{\lambda}{2} \|p\|_{\mathcal{H}}^2 \right) \quad \text{subject to} \quad \Phi^* p \in \partial \mathrm{TV}(0). \tag{$\mathcal{D}_\lambda(y)$}$$

Proof. Alternatively, $\mathcal{P}_\lambda(y)$ can be reformulated as:

$$\inf_{u \in L^2(\mathbb{R}^2)} \left(\mathrm{TV}(u) + \frac{1}{2\lambda} \|\Phi u - y\|_{\mathcal{H}}^2 \right).$$

Our first objective is to establish the convexity of the functional $g(z) := \frac{1}{2\lambda} \|z - y\|_{\mathcal{H}}^2$. This is demonstrated by considering the following relations, which hold for any $\mu \in [0,1]$ and

$v, w \in \mathcal{H}$:

$$\begin{aligned} g(\mu v + (1-\mu)w) &= \frac{1}{2\lambda}\|(\mu v + (1-\mu)w) - y\|_{\mathcal{H}}^2 \\ &= \frac{1}{2\lambda}\|\mu v + (1-\mu)w - y\|_{\mathcal{H}}^2 \\ &= \frac{1}{2\lambda}\|\mu v + (1-\mu)w - (\mu + 1 - \mu)y\|_{\mathcal{H}}^2 \\ &= \frac{1}{2\lambda}\|\mu(v-y) + (1-\mu)(w-y)\|_{\mathcal{H}}^2 \\ &\le \frac{1}{2\lambda}(\mu\|(v-y)\|_{\mathcal{H}} + (1-\mu)\|(w-y)\|_{\mathcal{H}})^2 \\ &\overset{*}{\le} \frac{1}{2\lambda}(\mu\|v-y\|_{\mathcal{H}}^2 + (1-\mu)\|w-y\|_{\mathcal{H}}^2) \\ &= \mu(\frac{1}{2\lambda}\|v-y\|_{\mathcal{H}}^2) + (1-\mu)(\frac{1}{2\lambda}\|w-y\|_{\mathcal{H}}^2) \\ &= \mu g(v) + (1-\mu)g(w) \end{aligned}$$

where the inequality marked with $(*)$ follows from the fact that $x^2 : \mathbb{R} \longmapsto \mathbb{R}$ is a convex function. Thus, once again utilizing theorem 4.2, we obtain the following dual representation of $\mathcal{P}_\lambda(y)$:

$$\begin{aligned} &\sup_{p^* \in \mathcal{H}^*} (-\mathrm{TV}^*(\Phi^* p^*) - g^*(-p^*)) \\ =&\sup_{p \in \mathcal{H}} \left(-\underbrace{\sup\{\langle \Phi^* p, u\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(u)) \mid u \in L^2(\mathbb{R}^2)\}}_{a} - \underbrace{\sup\{\langle -p, z\rangle_{\mathcal{H}} - g(z) \mid z \in \mathcal{H}\}}_{b} \right). \end{aligned}$$

We can now observe that the function present in b exhibits both continuity and concavity. Consequently, it follows that:

$$\sup\{\langle -p, z\rangle_{\mathcal{H}} - g(z) \mid z \in \mathcal{H}\}) = \max\{\langle -p, z\rangle_{\mathcal{H}} - g(z) \mid z \in \mathcal{H}\}).$$

Therefore, we seek a $z \in \mathcal{H}$ that satisfies the following equation:

$$\begin{aligned} \frac{d}{dz}(\langle -p, z\rangle_{\mathcal{H}} - g(z)) = 0 \\ \frac{d}{dz}(\langle -p, z\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\|z-y\|_{\mathcal{H}}^2) = 0. \end{aligned} \tag{12}$$

Given $f(x) := \langle k, x\rangle_{\mathcal{H}}$ for any $h \in \mathcal{H}$, it follows:

$$\begin{aligned} Df(x)(h) &= \lim_{t\to 0} \frac{\langle k, x + th\rangle_{\mathcal{H}} - \langle k, x\rangle_{\mathcal{H}}}{t} \\ &= \lim_{t\to 0} \frac{\langle k, x\rangle_{\mathcal{H}} + t\langle k, h\rangle_{\mathcal{H}} - \langle k, x\rangle_{\mathcal{H}}}{t} \\ &= \lim_{t\to 0} \frac{t\langle k, h\rangle_{\mathcal{H}}}{t} \\ &= \langle k, h\rangle_{\mathcal{H}}. \end{aligned}$$

This leads to the following for (12):

$$
\begin{aligned}
&\frac{d}{dz}\left(\langle -p, z\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\|z-y\|_{\mathcal{H}}^2\right) = 0\\
\iff &\frac{d}{dz}\langle -p, z\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\frac{d}{dz}\langle z-y, z-y\rangle_{\mathcal{H}} = 0\\
\iff &\langle -p, \cdot\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\frac{d}{dz}\left(\langle z, z\rangle_{\mathcal{H}} - 2\langle z, y\rangle_{\mathcal{H}} + \langle y, y\rangle_{\mathcal{H}}\right) = 0\\
\iff &\langle -p, \cdot\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\left(2\langle z, \cdot\rangle_{\mathcal{H}} - 2\langle y, \cdot\rangle_{\mathcal{H}}\right) = 0\\
\iff &\langle -p, \cdot\rangle_{\mathcal{H}} - \frac{1}{\lambda}\langle z-y, \cdot\rangle_{\mathcal{H}} = 0\\
\iff &\langle -p - \frac{1}{\lambda}(z-y), \cdot\rangle_{\mathcal{H}} = 0.
\end{aligned}
$$

Therefore, it must hold for all $h \in \mathcal{H}$ (including for $h := -p - \frac{1}{\lambda}(z-y)$):

$$
\begin{aligned}
&\langle -p - \frac{1}{\lambda}(z-y), h\rangle_{\mathcal{H}} = 0\\
\overset{*}{\iff}\quad &-p - \frac{1}{\lambda}(z-y) = 0\\
\iff\quad &z = y - \lambda p,
\end{aligned}
$$

where $(*)$ follows from the definiteness of the inner product. Thus, for the expression b in its entirety, we obtain:

$$
\begin{aligned}
\max\left\{\langle -p, z\rangle_{\mathcal{H}} - g(z) \mid z \in \mathcal{H}\right\} &= \langle -p, y - \lambda p\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\|y - \lambda p - y\|_{\mathcal{H}}^2\\
&= \langle -p, y - \lambda p\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\|y - \lambda p - y\|_{\mathcal{H}}^2\\
&= \langle -p, y - \lambda p\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\|-\lambda p\|_{\mathcal{H}}^2\\
&= \langle -p, y\rangle_{\mathcal{H}} - \lambda\langle -p, p\rangle_{\mathcal{H}} - \frac{1}{2\lambda}\lambda^2\|p\|_{\mathcal{H}}^2\\
&= \langle -p, y\rangle_{\mathcal{H}} + \lambda\|p\|_{\mathcal{H}}^2 - \frac{\lambda}{2}\|p\|_{\mathcal{H}}^2\\
&= \langle -p, y\rangle_{\mathcal{H}} + \frac{\lambda}{2}\|p\|_{\mathcal{H}}^2.
\end{aligned}
$$

As established earlier in the proof of 4.2, the same applies here with $a = \iota_D(\Phi^* p)$. Hence, the dual representation of $\mathcal{P}_\lambda(y)$ is obtained as:

$$
\begin{aligned}
&\sup_{p\in\mathcal{H}}\Big(-\sup\{\langle\Phi^*p,u\rangle_{L^2(\mathbb{R}^2)}-\mathrm{TV}(u))\mid u\in L^2(\mathbb{R}^2)\}-\sup\{\langle -p,z\rangle_{\mathcal{H}}-g(z)\mid z\in\mathcal{H}\}\Big)\\
=&\sup_{p\in\mathcal{H}}\left(-\iota_D(\Phi^*p)-(\langle -p,y\rangle_{\mathcal{H}}+\frac{\lambda}{2}\|p\|_{\mathcal{H}}^2)\right)\\
=&\sup_{p\in\mathcal{H}}\left(\langle p,y\rangle_{\mathcal{H}}-\frac{\lambda}{2}\|p\|_{\mathcal{H}}^2-\iota_D(\Phi^*p)\right)\\
=&\sup_{p\in\mathcal{H}}\left(\langle p,y\rangle_{\mathcal{H}}-\frac{\lambda}{2}\|p\|_{\mathcal{H}}^2\right)\quad\text{subject to}\quad \Phi^*p\in\partial\mathrm{TV}(0).
\end{aligned}
$$

Strong duality: let us take $v\in L^2(\mathbb{R}^2)$, for instance $v=0$. This implies that $\mathrm{TV}(v)=0<\infty$ and $\frac{1}{2\lambda}\|\Phi v-y\|_{\mathcal{H}}^2<\infty$. Additionally, the functional $u\mapsto\frac{1}{2\lambda}\|u-y\|_{\mathcal{H}}^2$ is continuous at Φv. Consequently, the strong duality of $\mathcal{D}_\lambda(y)$ in relation to $\mathcal{P}_\lambda(y)$ is proven, leading to the conclusion that $\mathcal{D}_\lambda(y)=\mathcal{P}_\lambda(y)$. $\square$

With the strong duality of both problems established, we can now state the following condition for a solution p of $\mathcal{D}_\lambda(y)$ with respect to every solution u of $\mathcal{P}_\lambda(y)$. It is given that:

$$\Phi u=y-\lambda p\quad\text{and}\quad\Phi^*p\in\partial\mathrm{TV}(u).\tag{13}$$

Considering a p that solves $\mathcal{D}_\lambda(y)$, it follows from previous derivations that for all u solving $\mathcal{D}_\lambda(y)$, $\Phi u=y-\lambda p$. Thus, we have:

$$
\begin{aligned}
&\langle p,y\rangle_{\mathcal{H}}-\frac{\lambda}{2}\langle p,p\rangle_{\mathcal{H}}\\
=&\langle p,\Phi u+\lambda p\rangle_{\mathcal{H}}-\frac{\lambda}{2}\langle p,p\rangle_{\mathcal{H}}\\
=&\langle p,\Phi u\rangle_{\mathcal{H}}+\frac{\lambda}{2}\langle p,p\rangle_{\mathcal{H}}\\
=&\langle\Phi^*p,u\rangle_{L^2(\mathbb{R}^2)}+\frac{1}{2\lambda}\langle -\lambda p,-\lambda p\rangle_{\mathcal{H}}\\
=&\langle\Phi^*p,u\rangle_{L^2(\mathbb{R}^2)}+\frac{1}{2\lambda}\|\Phi u-y\|_{\mathcal{H}}^2.
\end{aligned}
$$

Therefore, the equality of the primal and dual problem formulations can only be given if $\langle\Phi^*p,u\rangle_{L^2(\mathbb{R}^2)}=\mathrm{TV}(u)$, which is the case only if $\Phi^*p\in\partial\mathrm{TV}(u)$.

Conversely, if (13) holds, then it follows that u and p solve $\mathcal{P}_\lambda(y)$ and $\mathcal{D}_\lambda(y)$. However, the uniqueness of u is not necessarily guaranteed in this case. Nonetheless, for any solutions u_1,u_2 of $\mathcal{P}_\lambda(y)$ for which (13) is valid, the following equations hold:

$$\Phi u_1=y-\lambda p=\Phi u_2,$$

and

$$\mathrm{TV}(u_1)=\langle\Phi^*p,u_1\rangle=\langle p,\Phi u_1\rangle=\langle p,\Phi u_2\rangle=\langle\Phi^*p,u_2\rangle=\mathrm{TV}(u_2).$$

4.4 Dual certificates

To achieve a more compact notation for (11) and (13), we will introduce in the following definition the concept of the 'dual certificate', whose existence ensures the optimality of solutions to $\mathcal{P}_0(y_0)$ and $\mathcal{P}_\lambda(y)$.

Definition 4.3 (Dual Certificate): We define $\eta \in L^2(\mathbb{R}^2)$ as a dual certificate for u with respect to $\mathcal{P}_0(y_0)$ if:

- $\eta = \Phi^* p$ and $\eta \in \partial \mathrm{TV}(u)$,

and with respect to $\mathcal{P}_\lambda(y)$ if:

- $\eta = -\Phi^*(\Phi u - y)/\lambda$ and $\eta \in \partial \mathrm{TV}(u)$.

As previously described, in the latter case, there can be several dual certificates. In this context, we are interested in the one with the minimal norm, given by the next definition.

Definition 4.4 (Minimal norm dual certificate): Given a solution to $(P_0(y_0))$, the dual certificate with the least norm, denoted by η_0, is defined as

$$\eta_0 = \Phi^* p_0, \quad \text{where } p_0 = \operatorname{argmin} \|p\|_{\mathcal{H}} \quad \text{subject to } \ p \text{ solving } \mathcal{D}_0(y_0).$$

Utilizing the definitions and results previously mentioned, we now approach the question posed in section 2.3 more closely. With the next proposition, we gain insight into the behavior of $\eta_{\lambda,w} = \Phi^* p_{\lambda,w}$, where $p_{\lambda,w}$ is the solution to $\mathcal{D}_\lambda(y)$, in the scenario where the noise w tend to zero.

Proposition 4.5: [14, Prop. 3] Assuming the existence of a solution to the problem $\mathcal{D}_0(y_0)$, the sequence $p_{\lambda,0}$ converges strongly to p_0 in norm as the regularization parameter λ tends to zero, i.e.,

$$\lim_{\lambda \to 0} \|p_{\lambda,0} - p_0\|_{\mathcal{H}} = 0.$$

Moreover, it is evident that the subsequent transformation can be applied to $\mathcal{D}_\lambda(y)$:

$$\begin{aligned}
&\sup_{p \in \mathcal{H}} \left(\langle p, y \rangle_{\mathcal{H}} - \frac{\lambda}{2} \|p\|_{\mathcal{H}}^2 \right) \text{s.t.}\, \Phi^* p \in \partial \mathrm{TV}(0) \qquad \Big| \cdot \left(-\frac{2}{\lambda}\right) < 0 \\
= &\operatorname*{argmin}_{\substack{p \in \mathcal{H}, \\ \Phi^* p \in \partial \mathrm{TV}(0)}} \left(\|p\|_{\mathcal{H}}^2 - \frac{2}{\lambda} \langle p, y \rangle_{\mathcal{H}} \right) \\
= &\operatorname*{argmin}_{\substack{p \in \mathcal{H}, \\ \Phi^* p \in \partial \mathrm{TV}(0)}} \left(\|p\|_{\mathcal{H}}^2 - \frac{2}{\lambda} \langle p, y \rangle_{\mathcal{H}} + \|\frac{y}{\lambda}\|_{\mathcal{H}}^2 \right) \\
= &\operatorname*{argmin}_{\substack{p \in \mathcal{H}, \\ \Phi^* p \in \partial \mathrm{TV}(0)}} \left(\|p - \frac{y}{\lambda}\|_{\mathcal{H}}^2 \right) \\
= &\operatorname*{argmin}_{\substack{p \in \mathcal{H}, \\ \Phi^* p \in \partial \mathrm{TV}(0)}} \left(\|p - \frac{y}{\lambda}\|_{\mathcal{H}} \right) =: P\left(\frac{y}{\lambda}\right).
\end{aligned}$$

Consequently, a solution $p_{\lambda,w}$ to $\mathcal{D}_\lambda(y)$ is identified as the projection P of $\frac{y_0+w}{\lambda}$ onto the closed convex set $\{p \in \mathcal{H} \mid \Phi^* p \in \partial \mathrm{TV}(0)\}$. Given the nonexpansiveness $(*)$ of the projection operation, it follows that for every $(\lambda, w) \in \mathbb{R}_+^* \times \mathcal{H}$:

$$\|p_{\lambda,w} - p_{\lambda,0}\|_{\mathcal{H}} \leq \| = \|P\left(\frac{y_0+w}{\lambda}\right) - P\left(\frac{y_0}{\lambda}\right)\|_{\mathcal{H}} \overset{*}{\leq} \|\frac{y_0+w}{\lambda} - \frac{y_0}{\lambda}\|_{\mathcal{H}} = \frac{\|w\|_{\mathcal{H}}}{\lambda}, \tag{14}$$

which implies that

$$\|\eta_{\lambda,w} - \eta_{\lambda,0}\|_{L^2(\mathbb{R}^2)} = \|\Phi^*(p_{\lambda,w} - p_{\lambda,0})\|_{L^2(\mathbb{R}^2)} \leq \frac{\|\Phi^*\|\|w\|_{\mathcal{H}}}{\lambda}. \tag{15}$$

With these estimates, if $\lambda \to 0$ and $\|w\|_{\mathcal{H}}/\lambda \to 0$, then the dual certificate $\eta_{\lambda,w}$ converges strongly in $L^2(\mathbb{R}^2)$ to the minimal norm dual certificate η_0.

$$\Big(\|\eta_{\lambda,w} - \eta_0\|_{L^2(\mathbb{R}^2)} = \|\eta_{\lambda,w} - \eta_{\lambda,0} + \eta_{\lambda,0} - \eta_0\|_{L^2(\mathbb{R}^2)} \leq \|\eta_{\lambda,w} - \eta_{\lambda,0}\|_{L^2(\mathbb{R}^2)} + \|\eta_{\lambda,0} - \eta_0\|_{L^2(\mathbb{R}^2)}\Big)$$

5 Exposed faces of $\{TV \leq 1\}$

In this section, we will examine certain exposed faces of the convex set $\{\mathrm{TV} \leq 1\} := \{u \in L^2(\mathbb{R}^2) \mid \mathrm{TV}(u) \leq 1\}$. Let us first define this concept.

Definition 5.1 (Exposed face): [19, p. 162] A subset $B \subseteq A$ of a convex set A is called an *exposed face* if there exists a linear functional h and a scalar α such that B is the set of all points in A where h attains its maximum value α:

$$B = \{x \in A \mid h(x) = \alpha\} = \underset{x \in A}{\operatorname{argmax}}\ h(x).$$

Specifically, we will explore the faces of $\{\mathrm{TV} \leq 1\}$ that are exposed by the functional $\langle \eta, \cdot \rangle_{L^2(\mathbb{R}^2)}$, where η is a dual certificate (see 4.4). In analyzing these sets, we will discover an interesting property for the elements of such exposed faces, which brings us closer to answering our question from Section 2.3.

5.1 Subgradients and exposed faces

To make a meaningful choice of a dual certificate that exposes a face of $\{\mathrm{TV} \leq 1\}$, let us first examine the subdifferential of the Fenchel conjugate of TV for any $\eta \in \partial \mathrm{TV}(0)$. It follows from (3.23) and (4.1) that:

$$\partial \mathrm{TV}^*(\eta) = \Big\{u \in L^2(\mathbb{R}^2) : \mathrm{TV}^*(v) - \mathrm{TV}^*(\eta) \geq \langle u, v - \eta \rangle_{L^2(\mathbb{R}^2)}, \forall v \in L^2(\mathbb{R}^2)\Big\}.$$

With $\eta \in \partial\mathrm{TV}(0)$ (i.e., $\langle \eta, x\rangle_{L^2(\mathbb{R}^2)} \leq \mathrm{TV}(x),\ \forall x \in L^2(\mathbb{R}^2)$), this simplifies to:

$$\sup_{x\in L^2(\mathbb{R}^2)} \Big(\langle v, x\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(x)\Big) - \underbrace{\sup_{x\in L^2(\mathbb{R}^2)} \Big(\langle \eta, x\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(x)\Big)}_{=0} \geq \langle u, v-\eta\rangle_{L^2(\mathbb{R}^2)}$$

$$\iff \sup_{x\in L^2(\mathbb{R}^2)} \Big(\langle v, x\rangle_{L^2(\mathbb{R}^2)} - \mathrm{TV}(x)\Big) \geq \langle v, u\rangle_{L^2(\mathbb{R}^2)} - \langle u, \eta\rangle_{L^2(\mathbb{R}^2)},\ \forall v \in L^2(\mathbb{R}^2).$$

This relationship holds true if and only if $\eta \in \partial\mathrm{TV}(u)$, thus leading us to

$$\partial\mathrm{TV}^*(\eta) = \Big\{u \in L^2(\mathbb{R}^2) : \eta \in \partial\mathrm{TV}(u)\Big\} = \underset{u\in L^2(\mathbb{R}^2)}{\operatorname{argmax}} \Big(\int_{\mathbb{R}^2} \eta u - \mathrm{TV}(u)\Big). \tag{16}$$

We now establish a connection between this set and the faces exposed by η. We define $\mathcal{F}_\eta$ as the exposed face of the set $\{\mathrm{TV} \leq 1\}$, according to definition (5.1), using the linear function $\langle \eta, \cdot\rangle_{L^2(\mathbb{R}^2)}$. Thus we derive

$$\mathcal{F}_\eta = \underset{u\in\{\mathrm{TV}\leq 1\}}{\operatorname{argmax}} \int_{\mathbb{R}^2} \eta u.$$

From this definition, it can be seen that:

$$\mathcal{F}_0 = \{\mathrm{TV} \leq 1\} \text{ and } \mathcal{F}_{t\eta} = \mathcal{F}_\eta \text{ for } t > 0, \tag{17}$$

since in the latter case, the maximum is attained for the same $u \in \{TV \leq 1\}$ under positive scaling.

By introducing the so-called G-norm [12, p. 275]:

$$\forall \eta \in L^2(\mathbb{R}^2), \quad \|\eta\|_G := \sup_{u\in\{TV\leq 1\}} \int_{\mathbb{R}^2} \eta u,$$

we can demonstrate the relationship between subgradients and exposed faces of the total variation unit ball. Consider the following cases with $0 \neq \eta \in L^2(\mathbb{R}^2)$ and $v := u/\mathrm{TV}(u) \in \{\mathrm{TV} \leq 1\}$, where $0 \neq u \in \partial\mathrm{TV}^*(\eta)$:

1. If $\|\eta\|_G > 1$:
   there exists a non-zero $v \in \{TV \leq 1\}$ such that:

   $$\int_{\mathbb{R}^2} \eta v > 1 \iff \int_{\mathbb{R}^2} \eta\Big(\frac{\lambda u}{\mathrm{TV}(u)}\Big) > 1 \iff \mathrm{TV}(u) < \int_{\mathbb{R}^2} \eta\lambda u < \int_{\mathbb{R}^2} \eta u,$$

   implying that $\eta \notin \partial\mathrm{TV}(0)$, hence $\eta \notin \partial\mathrm{TV}(w)$ for all $w \in L^2(\mathbb{R}^2)$ and thus $\partial\mathrm{TV}^*(\eta) = \emptyset$.

2. If $\|\eta\|_G \leq 1$ for $v = 0$:

it follows that:

$$\int_{\mathbb{R}^2} \eta v = 0 \leq \mathrm{TV}(u) \text{ for all } u \in L^2(\mathbb{R}^2).$$

Therefore, $\eta \in \partial\mathrm{TV}(0)$ and hence $0 \in \partial\mathrm{TV}^*(\eta)$.

3. If $\|\eta\|_G \leq 1$ with $v \neq 0$: define $v' := v/\lambda \neq 0$, then for all such v':

$$\begin{aligned} \int_{\mathbb{R}^2} \eta v' \leq 1 &\iff \int_{\mathbb{R}^2} \eta\Big(\frac{u}{\mathrm{TV}(u)}\Big) \leq 1 \\ &\iff \int_{\mathbb{R}^2} \eta u \leq \mathrm{TV}(u). \end{aligned} \tag{18}$$

Because of $0 \neq u \in \partial\mathrm{TV}^*(\eta)$, this already implies equality in (18), hence

$$\int_{\mathbb{R}^2} \eta\Big(\frac{u}{\mathrm{TV}(u)}\Big) = 1, \tag{19}$$

thus $\|\eta\|_G = 1$ and $u/\mathrm{TV}(u) \in \mathcal{F}_\eta$.

In summary, with (17) we derive the expression:

$$\partial\mathrm{TV}^*(\eta) = \begin{cases} \emptyset & \text{if } \|\eta\|_G > 1, \\ \{0\} \cup (\bigcup_{t>0} t\mathcal{F}_\eta) & \text{if } \|\eta\|_G = 1, \\ \{0\} & \text{if } \|\eta\|_G < 1. \end{cases} \tag{20}$$

Based on this definition, we can now observe that our subsequent analysis will focus on the set $\mathrm{TV}^*(\eta)$ for $\|\eta\|_G = 1$, as this is the sole case where it is equivalent to analyzing the exposed faces of $\{\mathrm{TV} \leq 1\}$ by an $\eta \neq 0$.

5.2 Analysis of the extreme points for a specific face

Given the face $\mathcal{F}$ associated with an η possessing attributes as outlined in the previous section (also including $\eta \in \partial\mathrm{TV}(0)$), and further assuming that $\eta \in C^1(\mathbb{R}^2)$, this section is dedicated to investigating the extreme points (refer to definition 3.21) of $\mathcal{F}$. Specifically the main objective of this section is to prove the following proposition:

Proposition 5.2: For any extreme point u of $\mathcal{F}$, there exists a unique pair (s, E), where E is a C^3 simply connected open set and $s \in \{-1, 1\}$, such that $u = s\chi_E/P(E)$. For distinct extremal points u_1 and u_2 of $\mathcal{F}$, with associated pairs (s_i, E_i), it holds that $\partial E_1 \cap \partial E_2 = \emptyset$.

In order to maintain a logical separation in this chapter, it is necessary to refer to results from section 6 for some proofs. These references are clearly marked to allow quick navigation to these results if desired. Let us first examine a result from [2].

Proposition 5.3: [2, Prop. 8] Consider the convex set $\{\mathrm{TV} \leqslant 1\}$ in $L^2(\mathbb{R}^2)$ defined by

$$\{\mathrm{TV} \leqslant 1\} := \{u \mid \mathrm{TV}(u) \leq 1\}.$$

The extreme points of this set are represented by functions of the form $\pm\chi_E/P(E)$, where E is a simple set satisfying $0 < |E| < +\infty$.

This implies that the extreme points of $\mathcal{F}$ also have the same structure. Inspired by this result, we define the following set:

$$\mathcal{E} := \mathcal{E}^+ \cup \mathcal{E}^- \cup \{\emptyset, \mathbb{R}^2\}, \tag{21}$$

where the subsets $\mathcal{E}^+, \mathcal{E}^-$ are defined as:

$$\begin{aligned}
\mathcal{E}^+ &:= \{E \subset \mathbb{R}^2 \mid |E| < \infty, 0 < P(E) < \infty, \frac{\chi_E}{P(E)} \in \mathcal{F}\}, \\
\mathcal{E}^- &:= \{E \subset \mathbb{R}^2 \mid |E^c| < \infty, 0 < P(E^c) < \infty, \frac{-\chi_{E^c}}{P(E^c)} \in \mathcal{F}\}.
\end{aligned}$$

Subsequently, we will explore and illustrate the properties of this set.

Analysis of $\mathcal{E}$

First, we will examine the properties associated with intersections and unions of elements within the set $\mathcal{E}$. To achieve this, let us prove the following proposition.

Proposition 5.4: Let $E, F \in \mathcal{E}$. Then both $E \cap F \in \mathcal{E}$ and $E \cup F \in \mathcal{E}$.

Proof. Suppose $E \in \mathcal{E}^+$ and $F \in \mathcal{E}^+$. Due to $(*)$ from [2, Prop. 1] and the fact that $\chi_{E\cup F} = \chi_E + \chi_F - \chi_{E\cap F}$ $(**)$, we obtain:

$$\begin{aligned}
P(E\cap F) + P(E\cup F) \overset{*}{\leq} P(E) + P(F) &= \int_E \eta + \int_F \eta \\
&= \int_{\mathbb{R}^2} \eta(\chi_E + \chi_F) \\
&\overset{**}{=} \int_{\mathbb{R}^2} \eta(\chi_{E\cup F} + \chi_{E\cap F}) \\
&= \int_{E\cap F} \eta + \int_{E\cup F} \eta.
\end{aligned}$$

Consequently, we deduce, knowing $\eta \in \partial\mathrm{TV}(0)$:

$$\left(\underbrace{P(E\cap F) - \int_{E\cap F} \eta}_{\geq 0}\right) + \left(\underbrace{P(E\cup F) - \int_{E\cup F} \eta}_{\geq 0}\right) \leq 0.$$

Thus, both terms are already equal to 0, and we conclude that if $E \cap F \neq \emptyset$:

$$\int_{\mathbb{R}^2} \eta \Big(\frac{\chi_{E\cap F}}{\mathrm{TV}(\chi_{E\cap F})} \Big) = 1 \text{ and } \int_{\mathbb{R}^2} \eta \Big(\frac{\chi_{E\cup F}}{\mathrm{TV}(\chi_{E\cup F})} \Big) = 1.$$

Therefore, in accordance to equation (19), it holds that $E \cap F \in \mathcal{E}^+$ and $E \cup F \in \mathcal{E}^+$ (unless $E \cup F = \mathbb{R}^2$). The same argument also applies for $E \in \mathcal{E}^-$ and $F \in \mathcal{E}^-$.

Let us now assume that $E \in \mathcal{E}^+$ and $F \in \mathcal{E}^-$ (or vice versa). Consider for $x \in \mathbb{R}^2$:

$$(\chi_E - \chi_{F^c})(x) = \begin{cases} 1 & \text{if } x \in E \wedge x \in F, \\ -1 & \text{if } x \in E^c \wedge x \in F^c, \\ 0 & \text{otherwise.} \end{cases}$$

It thus follows that

$$\chi_E - \chi_{F^c} = \chi_{E\cap F} - \chi_{E^c \cap F^c} \overset{*}{=} \chi_{E\cap F} - \chi_{(E\cup F)^c}, \tag{22}$$

where $(*)$ follows from the De Morgan's laws.

Then, with the property of the perimeter $P(A) = P(A^c)$, we obtain:

$$\begin{aligned} P(E\cap F) + P((E\cup F)^c) = P(E\cap F) + P(E\cup F) &\leq P(E) + P(F) \\ &= P(E) + P(F^c) \\ &= \int_E \eta - \int_{F^c} \eta \\ &= \int_{E\cap F} \eta - \int_{(E\cup F)^c} \eta. \end{aligned}$$

Thus, we again obtain:

$$\Bigg(\underbrace{P(E\cap F) - \int_{E\cap F} \eta}_{\geq 0} \Bigg) + \Bigg(\underbrace{P((E\cup F)^c) - \Big(- \int_{(E\cup F)^c} \eta \Big)}_{\geq 0} \Bigg) \leq 0.$$

By similar reasoning as above, we find $E \cap F \in \mathcal{E}^+$ (unless $E \cap F = \emptyset$) and $E \cup F \in \mathcal{E}^-$ (unless $E \cup F = \mathbb{R}^2$).

□

To deepen the study of $\mathcal{E}$, we will focus on the intersection over the boundaries of two sets of $\mathcal{E}$. As we will see in Section 6, for a $\eta \in C^1(\mathbb{R}^2)$, it is established that elements of $\mathcal{E}$ possess a C^3 representation (refer to 3.14). Consequently, these intersections can be characterized using their outward unit normals (refer to 2), as delineated in the forthcoming proposition.

Proposition 5.5: Suppose $E, F \in \mathcal{E}$. It follows:

$$\partial E \cap \partial F = \{\nu_E = \nu_F\} \cup \{\nu_E = -\nu_F\}$$

where

$$\{\nu_E = -\nu_F\} := \{x \in \mathbb{R}^2 \mid \nu_E(x) = -\nu_F(x)\}$$
$$\{\nu_E = \nu_F\} := \{x \in \mathbb{R}^2 \mid \nu_E(x) = \nu_F(x)\}.$$

Furthermore, $\{\nu_E = -\nu_F\}$ and $\{\nu_E = \nu_F\}$ are simultaneously open and closed in ∂E and ∂F.

The proof of this proposition is supported by the following two lemmas.

Lemma 5.6: Suppose E and F are two C^1-class sets. If the intersection $E \cap F$ and the union $E \cup F$ either belong to class C^1 or are such that $E \cap F$ is the empty set and $E \cup F$ is the entire plane $\mathbb{R}^2$, it follows that

$$\partial E \cap \partial F = \{\nu_E = \nu_F\} \cup \{\nu_E = -\nu_F\}. \tag{23}$$

Moreover, the set

$$\{\nu_E = -\nu_F\} \text{ is simultaneously open and closed in both } \partial E \text{ and } \partial F. \tag{24}$$

Proof. From definition (3.14), it follows that the densities θ_E, θ_F, $\theta_{E\cap F}$, and $\theta_{E\cup F}$ are well-defined on $\mathbb{R}^2$ and their values are restricted to $\{0, 1/2, 1\}$. Additionally, the following relation holds:

$$|E \cup F| = |E| + |F| - |E \cap F|,$$

leading to the equation:

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} + \frac{|F \cap B(x,r)|}{|B(x,r)|} = \frac{|(E \cap F) \cap B(x,r)|}{|B(x,r)|} + \frac{|(E \cup F) \cap B(x,r)|}{|B(x,r)|}.$$

As $r \to 0^+$, this implies:

$$\theta_E + \theta_F = \theta_{E\cap F} + \theta_{E\cup F}. \tag{25}$$

Furthermore, for all $x \in \partial E \cap \partial F$, it is established that $\theta_E(x) = \theta_F(x) = 1/2$, which, in conjunction with (25), yields:

$$\theta_{E\cap F}(x) + \theta_{E\cup F}(x) = 1.$$

Since $\theta_{E\cap F} \leq \theta_{E\cup F}$ evidently holds, we can deduce that $(\theta_{E\cap F}(x), \theta_{E\cup F}(x))$ must be either

(0,1) or (1/2,1/2).

Additionally, employing [16, Thm. 15.5], we can demonstrate the following convergence:

$$\left|\frac{(E\cap F)-x}{r}\cap B(0,1)\right| = \left|\left(\frac{E-x}{r}\cap B(0,1)\right)\cap\left(\frac{F-x}{r}\cap B(0,1)\right)\right| \xrightarrow[r\to 0^+]{} \left|B_{\nu_E(x)}\cap B_{\nu_F(x)}\right|,$$

where B_ν is the set $\{x\in B(0,1)\mid x\cdot\nu\le 0\}$. Consequently, we find that:

$$\begin{aligned}\theta_{E\cap F}(x) = \lim_{r\to 0^+}\frac{|(E\cap F)\cap B(x,r)|}{|B(x,r)|} &= \lim_{r\to 0^+}\frac{|(E\cap F-x)\cap B(0,r)|}{\pi r^2}\\ &= \lim_{r\to 0^+} r^2\frac{|(\frac{E\cap F-x}{r}\cap B(0,1)|}{\pi r^2}\\ &= \lim_{r\to 0^+}\frac{|(\frac{E\cap F-x}{r}\cap B(0,1)|}{\pi}\\ &= \frac{|B_{\nu_E(x)}\cap B_{\nu_F(x)}|}{\pi}.\end{aligned} \tag{26}$$

This leads to the conclusion that if $\theta_{E\cap F}(x)=0$, then $|B_{\nu_E(x)}\cap B_{\nu_F(x)}|=0$. Therefore, the intersection $B_{\nu_E(x)}\cap B_{\nu_F(x)}$ is Lebesgue negligible in $\mathbb{R}^2$. In light of the definition of B_ν, this situation is possible only if $\nu_E(x)$ and $\nu_F(x)$ are oriented in opposite directions (as illustrated in the figure 15 below). Consequently, we deduce that $\nu_E(x)=-\nu_F(x)$.

Similarly, if $\theta_{E\cap F}(x)=1/2$, then $|B_{\nu_E(x)}\cap B_{\nu_F(x)}|=\pi/2$, indicating that $B_{\nu_E(x)}\cap B_{\nu_F(x)}$ exactly corresponds to one half of the circle $B(0,1)$. This can only be true if $B_{\nu_E(x)}=B_{\nu_F(x)}$ and thus $\nu_E(x)=\nu_F(x)$ (like before see figure 15 below).
Next, our objective is to demonstrate that the set $\{\nu_E=-\nu_F\}$ is both open and closed within ∂E (the same applies to ∂F).

(i) **Closed:** Since E and F are of class C^1, it follows from definition (2) that ν_E and ν_F are continuous $(*)$. Consider a sequence $(x_n)_{n\in\mathbb{N}}\subseteq\{\nu_E=-\nu_F\}$ converging to some $x\in\mathbb{R}^2$. $(\nu_E(x_n)+\nu_F(x_n))_{n\in\mathbb{N}}$ is then a null sequence. Consequently, we have:

$$\begin{aligned}0=\lim_{n\to\infty}\nu_E(x_n)+\nu_F(x_n) &= \lim_{n\to\infty}\nu_E(x_n)+\lim_{n\to\infty}\nu_F(x_n)\\ &\overset{*}{=}\nu_E(\lim_{n\to\infty}x_n)+\nu_F(\lim_{n\to\infty}x_n)\\ &=\nu_E(x)+\nu_F(x).\end{aligned}$$

Therefore, x also belongs to the set $\{\nu_E=-\nu_F\}$.

(ii) **Open:** Through Equation (26) we know that:

$$\begin{aligned}\{\nu_E=-\nu_F\} &= \left\{x\in\mathbb{R}^2\mid\theta_{E\cup F}(x)=1\right\}\cap\left\{x\in\mathbb{R}^2\mid\theta_{E\cap F}(x)=0\right\}\\ &=(E\cup F)\cap\left(\mathbb{R}^2\setminus\overline{E\cap F}\right).\end{aligned}$$

Since either $E \cap F$ and $E \cup F$ are trivial or correspond to an open set of class C^1, the openness of the two latter sets in the equation also follows, yielding our result.

$\square$

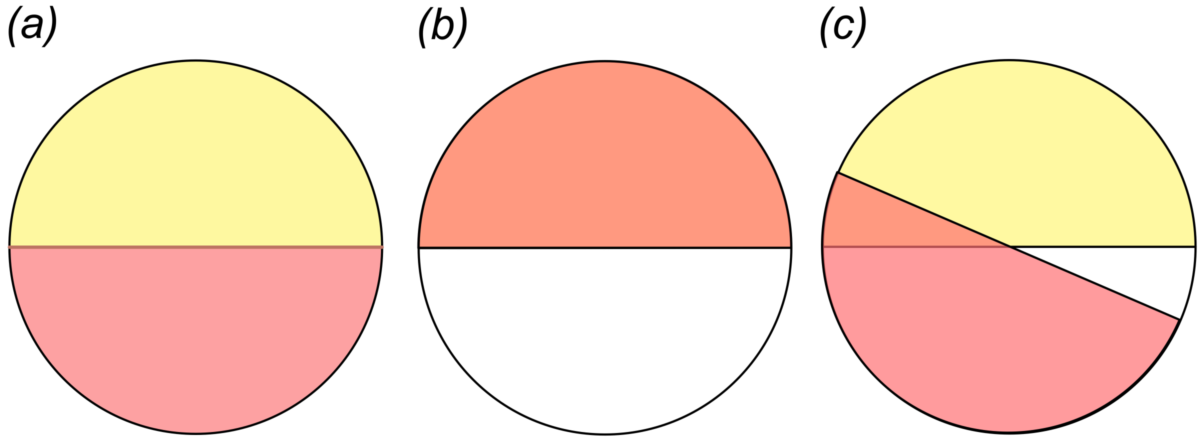

Figure 15: Illustration of the sets within $B(0,1)$: The set $B^-_{\nu_E(x)}$ is depicted with a yellow area, and $B^-_{\nu_F(x)}$ is depicted with a red area, while their intersection is shown in orange. Subfigure (a) represents the case where $\nu_E(x) = -\nu_F(x)$, subfigure (b) corresponds to the case $\nu_E(x) = \nu_F(x)$, and subfigure (c) illustrates the scenario where neither of the previous conditions holds. In this last case, it is noted that the intersection of $B^-_{\nu_E(x)}$ and $B^-_{\nu_F(x)}$ always possesses a positive Lebesgue measure.

In the next lemma, we demonstrate the final step to obtain the result referred to proposition (5.5). It can be observed that the property to be shown, namely that $\{\nu_F = \nu_E\}$ is both open and closed, does not generally hold for the assumptions made in the previous lemma (refer to figure 16 below for more details).

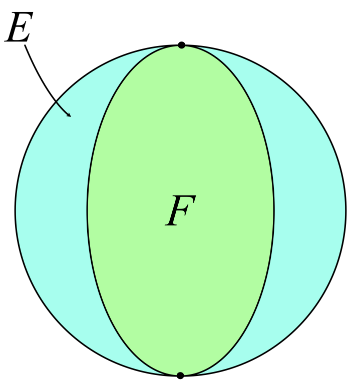

Figure 16: Demonstration of two C^1 class sets: Set E represented as a blue circle in $\mathbb{R}^2$, and set F depicted in green. The marked locations are the only points where $\nu_E = \nu_F$. Consequently, the set $\{\nu_E = \nu_F\}$ is not open in ∂E, ∂F.

Lemma 5.7: For $E, F \in \mathcal{E}$, the set $\{\nu_E = \nu_F\}$ is both open and closed in ∂E and ∂F.

Proof. As initially outlined in this section, the sets E, F within $\mathcal{E}$ are assumed to be of class C^3. Consequently, paralleling our earlier proof, the closedness of the set $\{\nu_E = \nu_F\}$ can be established through the continuity of ν_E and ν_F.
To demonstrate the openness of the set, we consider the following cases.

- For $E, F \in \mathcal{E}^+$:
  For a given $x \in \{\nu_E = \nu_F\}$, we define $\nu := \nu_E(x) = \nu_F(x)$. Since E and F are of class C^3, there exists an open square $C(x, r, \nu)$ where both ∂E and ∂F match with functions from $C^3([-r, r])$ (see 3.1). We will later show that for such functions the following holds (refer to (6.6)):

$$\frac{u''(z)}{(1+u'(z)^2)^{3/2}} = H(z, u(z)) \text{ with } H(z,t) = \eta(x + R_\nu(z,t)), \tag{27}$$

  where $z \in [-r, r]$ and $u(0) = u'(0) = 0$. Thus, we are dealing with a second-order ODE. By introducing $v_1(x) := u(x)$ and $v_2(x) := u'(x)$, we can convert (27) into the following first-order ODE system:

$$\begin{aligned} v_1' &= u', \\ v_2' &= H(t, v_1)(1 + v_2^2)^{3/2}. \end{aligned}$$

  Let us now define the mapping:

$$f(t, v) := \begin{pmatrix} v_2 \\ H(t, v_1)(1 + v_2^2)^{3/2} \end{pmatrix}.$$

  Then, for all $t \in [-r, r]$, it holds that:

$$\begin{aligned} \|f(t, y_1) - f(t, y_2)\| &= \left\| \begin{pmatrix} y_{1,2} \\ H(t, y_{1,1})(1 + y_{1,2}^2)^{3/2} \end{pmatrix} - \begin{pmatrix} y_{2,2} \\ H(t, y_{2,1})(1 + y_{2,2}^2)^{3/2} \end{pmatrix} \right\| \\ &= |y_{1,2} - y_{2,2}| + \left| H(t, y_{1,1})(1 + y_{1,2}^2)^{3/2} - H(t, y_{2,1})(1 + y_{2,2}^2)^{3/2} \right| \\ &\leq |y_{1,2} - y_{2,2}| + |H(t, y_{1,1}) - H(t, y_{2,1})| \cdot \left| (1 + y_{1,2}^2)^{3/2} \right| \\ &\quad + |H(t, y_{2,1})| \cdot \left| (1 + y_{1,2}^2)^{3/2} - (1 + y_{2,2}^2)^{3/2} \right| \\ &\overset{*}{\leq} |y_{1,2} - y_{2,2}| + L_1 M_2 |y_{1,1} - y_{2,1}| + L_2 M_1 |y_{1,2} - y_{2,2}| \\ &= (L_2 M_1 + 1)|y_{1,2} - y_{2,2}| + L_1 M_2 |y_{1,1} - y_{2,1}| \\ &\overset{**}{\leq} \mathbf{L} \left(|y_{1,2} - y_{2,2}| + |y_{1,1} - y_{2,1}| \right) \\ &= \mathbf{L} \|y_1 - y_2\|. \end{aligned}$$

  The marked estimates can be justified as follows:

  (∗) (i) $|H(t, y_{1,1}) - H(t, y_{2,1})|$:
  Since H, y_1, and y_2 are continuous, without loss of generality, H is Lips-

chitz continuous on the compact set $[-r,r]\times([y_{1,1}(-r),y_{1,1}(r)]\cup[y_{2,1}(-r),y_{2,1}(r)])$ with a Lipschitz constant L_1.

(ii) $|H(t,y_{2,1})|$:
H attains its maximum value M_1, without loss of generality, on $[-r,r]\times[y_{2,1}(-r),y_{2,1}(r)]$.

(iii) $\left|(1+y_{1,2}^2)^{3/2}-(1+y_{2,2}^2)^{3/2}\right|$:
The function $(1+x^2)^{3/2}$ is also continuous and thus attains its maximum, without loss of generality, on $[y_{1,2}(-r),y_{1,2}(r)]\cup[y_{2,2}(-r),y_{2,2}(r)]$ and hence is Lipschitz continuous with a constant L_2.

(iv) $\left|(1+y_{1,2}^2)^{3/2}\right|$:
The function $(1+x^2)^{3/2}$ attains its maximum value M_2, without loss of generality, on $[y_{1,2}(-r),y_{1,2}(r)]$.

(**) $\mathbf{L}:=\max(L_2M_1+1,L_1M_2)$.

Hence, $f(t,v)$ is locally Lipschitz continuous in the second variable, and by Picard-Lindelöf's theorem, there exists a unique solution u on the interval $(-r,r)$. Therefore, both aforementioned functions must coincide on this interval. Consequently, the same property also applies to the outward unit normals on $C(x,r,\nu)$, because:

$$\nu_E(x+R_v(z,u(z)))=\frac{1}{\sqrt{1+(u'(z))^2}}\begin{pmatrix}-u'(z)\\1\end{pmatrix}=\nu_F(x+R_v(z,u(z))).$$

Thus, it follows that:

$$\{\nu_E=\nu_F\}\cap C(x,r,v)=\underbrace{\partial E\cap C(x,r,v)}_{\text{open in }\partial E}=\underbrace{\partial F\cap C(x,r,v)}_{\text{open in }\partial F}.$$

Therefore, $\{\nu_E=\nu_F\}$ is open in ∂E and ∂F.

- $E,\ F\in\mathcal{E}^-$:
The complements $E^c,\ F^c$ are thus of class C^3, i.e., we apply the same argument as before with the modification in (27) of $H=-\eta$. Consequently, we deduce that $\{\nu_{E^c}=\nu_{F^c}\}=\{-\nu_E=-\nu_F\}=\{\nu_E=\nu_F\}$ is open in $\partial E^c,\ \partial F^c$ and therefore also in $\partial E,\ \partial F$.

- $E\in\mathcal{E}^+,\ F\in\mathcal{E}^+$ (or vice versa): Let us now define $\nu:=\nu_E(x)=\nu_F(x)=-\nu_{F^c}(x)$. As before, there exists $r>0$ such that ∂E can locally be described on $C(x,r,\nu)$ by a C^3 function u solving (27) with $u(0)=0$ and $u'(0)=0$. Similarly, ∂F^c can be locally represented on $C(x,r,-\nu)$ for some $r>0$ by a function v, satisfying the following equation:

$$\frac{u''(z)}{(1+u'(z)^2)^{3/2}}=G(z,u(z))\text{ with }H(z,t)=-\eta(x+R_{-\nu}(z,t)),\qquad(28)$$

with $v(0)=0$, $v'(0)=0$. Due to symmetry, $C(x,r,-\nu)=C(x,r,\nu)$ and additionally $R_{-\nu}(z,t))=R_\nu(z,-t))$ for all $t\in(-r,r)$. This means that ∂F can locally be described on $C(x,r,\nu)$ by an $\tilde{u}=-v$ which solves (27) and satisfies $\tilde{u}(0)=0$, $\tilde{u}'(0)=0$. Thus, again by the Picard-Lindelöf theorem, u and $\tilde{u}$ must be identical on $(-r,r)$ and therefore $\{\nu_E=\nu_F\}$ must be open in ∂E and ∂F.

□

Proof of proposition 5.2

Equipped with proposition 5.5, we now finish this section and proceed to prove (5.2).

Proposition 5.2: For any extreme point u of $\mathcal{F}$, there exists a unique pair (s,E), where E is a C^3 simply connected open set and $s\in\{-1,1\}$, such that $u=s\chi_E/P(E)$. For distinct extremal points u_1 and u_2 of $\mathcal{F}$, with associated pairs (s_i,E_i), it holds that $\partial E_1\cap\partial E_2=\emptyset$.

Proof. Let x be an extremal point of $\mathcal{F}$. Since $\mathcal{F}$ is a subset of $\{TV\le 1\}$, x is also an extremal point of $\{TV\le 1\}$. Following proposition 5.3, it implies that $u=s\chi_E/P(E)$ for a simple set E from $\mathbb{R}^2$ with $0<|E|<\infty$. Thus, by lemma 6.6, there exists a C^3 representative for E. As E is simple, with [2, Thm. 7] we can establish that E is the interior of a rectifiable Jordan curve (a closed loop in $\mathbb{R}^2$). Then we can conclude with the Jordan-Schoenflies theorem, that E is homeomorphic to $\mathrm{int}(B(0,1))\subseteq\mathbb{R}^2$ and consequently simply connected. Now, let us demonstrate the second part of the proposition. Suppose we have two distinct extremal points of $\mathcal{F}$, u_1 and u_2, corresponding to simple sets E_1 and E_2 such that:

$$u_1=s_1\chi_{E_1}/P(E_1)\neq s_2\chi_{E_2}/P(E_2)=u_2. \tag{29}$$

We observe that $E_1\neq E_2$ must hold because otherwise, it would imply $s_1\neq s_2$ which is equivalent to $s_1=-s_2$. Since $\mathcal{F}$ is convex, it also follows that $\lambda u_1+(1-\lambda)u_2\in\mathcal{F}$ for $\lambda\in[0,1]$. Specifically, for $\lambda=\frac{1}{2}$, we have:

$$0=\frac{1}{2}s_1\chi_{E_1}/P(E_1)+\frac{1}{2}s_2\chi_{E_2}/P(E_2)=:u\in\mathcal{F}=\underset{v\in\{TV\le 1\}}{\operatorname{argmax}}\int_{\mathbb{R}^2}\eta v.$$

Consequently, for u, the following relation is established:

$$\int_{\mathbb{R}^2}\eta u=0\neq 1=\|\eta\|_G=\max_{v\in\{TV\le 1\}}\int_{\mathbb{R}^2}\eta v,$$

which would imply that $u\notin\mathcal{F}$. Therefore, this leads to a contradiction.

We deduce that both E_i must be elements of $\mathcal{E}$, since for $s_i=-1$, $E_i\in\mathcal{E}^-$, and for $s_i=1$, $E_i\in\mathcal{E}^+$. Using Proposition (5.5), we observe that $\partial E_1\cap\partial E_2$ is both open and closed in ∂E_1 and ∂E_2. Since ∂E_1 and ∂E_2 are closed and bounded, openness can only occur if $\partial E_1\cap\partial E_2=\emptyset$ or $\partial E_1=\partial E_2$. However, in the latter case, according to the Jordan curve theorem, it would imply $E_1 = E_2$, which contradicts our earlier assertion. Thus, we conclude that $\partial E_1 \cap \partial E_2 = \emptyset$.

$\square$

5.3 Structure of finite-dimensional exposed faces

With the insights from 5.2, in this section, we will establish that k-dimensional exposed faces $\mathcal{F}$ exhibit the following structure:

$$\mathcal{F} = \left\{ \sum_{i=0}^{k} \lambda_i u_i \mid \lambda_i \geq 0,\ \sum_{i=0}^{k} \lambda_i = 1 \right\}, \tag{30}$$

where the u_i denote the extreme points of $\mathcal{F}$. Consequently, $\mathcal{F}$ can be described as a k-simplex.

To support this, we first present the following corollary to Proposition 5.2.

Corollary 5.8: Each family of pairwise distinct extreme points of $\mathcal{F}$ is linearly independent.

Proof. Let I be an index set. Proposition 5.2 shows that for a family of extreme points $\{u_i\}_{i\in I}$ of $\mathcal{F}$, there exist corresponding simple sets $(E_i)_{i\in I}$ and $s_i \in \{-1,1\}$ such that:

$$\{u_i\}_{i\in I} = \left\{ \frac{s_i \chi_{E_i}}{P(E_i)} \right\}_{i\in I} \tag{31}$$

and for each $E_i,\ E_j$ with $i \neq j$ it holds that $\partial E_i \cap \partial E_j = \emptyset$.

Then, for an arbitrary family $(\lambda_i)_{i\in I}$ with $\lambda_i \in \mathbb{R}$ (assuming, in the event that the cardinality of I is infinite, there exists an index $i \in I$ such that for all $j \geq i$, $\lambda_j = 0$), and with $\sum_{i\in I} \lambda_i u_i = 0$, it also follows that $\sum_{i\in I} \lambda_i D u_i = 0$.

Furthermore, due to the constancy of χ_{E_i} both within $\text{int}(E_i)$ and $\mathbb{R}^2 \setminus \overline{E}$, we have that $\text{supp}(D(u_i)) = \partial E_i$. Consequently, for any $i,j \in I$ with $i \neq j$, we obtain

$$\emptyset = \partial E_i \cap \partial E_j = \text{supp}(D(u_i)) \cap \text{supp}(D(u_j)). \tag{32}$$

This implies $\lambda_i = 0$ for all $i \in I$, thereby demonstrating the linear independence of the u_i. $\square$

With this property of the extreme points of a face $\mathcal{F}$, we can now demonstrate equality to a simplex in the following theorem:

Theorem 5.9: If a face $\mathcal{F}$ is of dimension k, then it has exactly $k+1$ extreme points, and is therefore a k-simplex.

Remark 5.10: Since the concept of dimension is meaningful only in the context of vector spaces, we will briefly explain what it signifies in this context.

A set M is characterized as being of dimension k if the dimension of its affine hull, $\text{Aff}(M)$, is equivalently k. The construction of $\text{Aff}(M)$ is as follows: An arbitrary point $m_0 \in M$ is selected. Subsequently, the linear hull, denoted as H, of the set $\{m - p \mid m, p \in M\}$ is determined. Thereupon, $\text{Aff}(M)$ is expressed as $m_0 + H$. It is noteworthy that the dimension of $\text{Aff}(M)$ is congruent with the dimension of H.

Proof of 5.9. Consider $u_1, \ldots, u_m$ as pairwise distinct extremal points of $\mathcal{F}$. By Corollary 5.8, the linear independence of $u_1, \ldots, u_m$ is established. Consequently, this also holds for $u_2 - u_1, \ldots, u_m - u_1$, since for all $\lambda \in \mathbb{R}^m$ with

$$\begin{aligned}
&\sum_{i=1}^{m} \lambda_i u_i = 0 \\
\Leftrightarrow &\sum_{i=2}^{m} \lambda_i u_i + \lambda_1 u_1 = 0 \quad \mid \text{define } \lambda_1 := -\sum_{i=2}^{m} \lambda_i \\
\Leftrightarrow &\sum_{i=2}^{m} \lambda_i u_i - \sum_{i=2}^{m} \lambda_i u_1 = 0 \\
\Leftrightarrow &\sum_{i=2}^{m} \lambda_i (u_i - u_1) = 0
\end{aligned}$$

it follows that $\lambda = 0$. We can now discern that $\{u_2 - u_1, \ldots, u_m - u_1\} \subseteq \{u - p \mid u, p \in \mathcal{F}\}$. This implies that

$$\dim(\text{span}\{u_2 - u_1, \ldots, u_m - u_1\}) \leq \dim(\text{Aff}(\mathcal{F})).$$

Since $\dim(\text{Aff}(\mathcal{F})) = \dim \mathcal{F} = k$, it follows that $m \leq k+1$. Now suppose that $m < k+1$ (without loss of generality, let $m = k$). From the properties of the extrema of $\mathcal{F}$ it follows that for all x in $\mathcal{F}$:

$$x = \sum_{i=1}^{k} (\lambda_i u_i) \quad \text{with} \quad \sum_{i=1}^{k} (\lambda_i) = 1. \tag{33}$$

Since $\mathcal{F}$ is convex, we already have $\mathcal{F} = \text{conv}(\mathcal{F})$. By Carathéodory's theorem, it follows that $\dim(\mathcal{F}) = k - 1$, which contradicts the assumption. $\square$

Let us now consolidate our insights from this section. Concerning an extreme point of $\mathcal{F}$, Proposition 5.2 demonstrates the existence of a pair (s, E) satisfying $u = \frac{s\chi_E}{P(E)}$, where E is a simply connected set in the class C^3, and s takes values in $\{-1, 1\}$. Invoking Theorem 5.9 and Carathéodory's theorem, we deduce a representation for all elements of $\mathcal{F}$ through its extreme points. Specifically:

$$u = \sum_{i=1}^{d+1} \lambda_i \frac{s_i \chi_{E_i}}{P(E_i)}, \quad \text{subject to} \quad \sum_{i=1}^{d+1} \lambda_i = 1.$$

Define $a_i := \frac{\lambda_i s_i}{P(E_i)}$ and select $I \subseteq \{1, \ldots, d+1\}$ such that $a_i \neq 0$ for all $i \in I$. This leads to the expression:

$$u = \sum_{i \in I} a_i \chi_{E_i}. \tag{34}$$

The uniqueness of this formulation is guaranteed by the corollary 5.8. With this knowledge, let us revisit our problem statement. For a solution p to the problem $\mathcal{D}_0(y_0)$, we obtain a dual certificate $\eta = \Phi^* p$, which gives a face of the set $\mathcal{F} \subseteq \{\mathrm{TV}(x) \leq \min(\mathcal{P}_0(y_0))\}$ with dimension d. Then the solutions of the problem $\mathcal{P}_0(y_0)$ can be represented as in (34). Furthermore, if the operator defined by

$$\Phi_{\mathcal{F}} : \mathbb{R}^{d+1} \to \mathcal{H}, \quad a \mapsto \Phi\left(\sum_{i=1}^{d+1} a_i \chi_{E_i}\right), \tag{35}$$

is injective, then the uniqueness of the solution follows.

This means that for solutions of $\mathcal{P}_0(y_0)$ under these assumptions, we naturally obtain a k-sparse representation as described in section 2.3. In our case, this refers to the notation of k-simple functions as described in the following definition.

Definition 5.11 (k-simple functions): Let $k \in \mathbb{N}^*$. We say that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is *k-simple* if there exists a collection $\{E_i\}_{i \leq i \leq k}$ of simple sets of class C^1 with positive finite measure such that $E_i \cap E_j = \varnothing$ for every $i \neq j$, and a vector $a \in \mathbb{R}^k$ such that

$$u = \sum_{i=1}^{k} a_i \chi_{E_i}. \tag{36}$$

6 The prescribed curvature problem

In this section, we will transform the subdifferential property from (11) into a problem formulation (the so called prescribed curvature problem) that involves the sets from the previously demonstrated decomposition. After introducing this new problem formulation, we will analyze its behavior for for a varying dual certificate. This allows us to establish important theses that we need to demonstrate the main result of this work in Section 7.

6.1 Properties and behavior of the level sets of solutions

We begin by demonstrating some properties for sets from the decomposition in (34). For this, we consider an $\eta \in \partial\mathrm{TV}(a\chi_E)$ with positive a. Therefore, the following equation must be satisfied:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \eta a \chi_E = \mathrm{TV}(a\chi_E) \\
\iff &\int_{\mathbb{R}^2} \eta a \chi_E = |a|\mathrm{TV}(\chi_E) \\
\iff &\int_{\mathbb{R}^2} \eta \chi_E = \mathrm{TV}(\chi_E) \\
\iff &\int_E \eta = P(E).
\end{aligned}$$

With the results from [6] and [14], it then follows.

Lemma 6.1: [6, 14, p. 25ff, Lem. 5] Consider $(\eta_n)_{n\geq 0} \subset \partial TV(0)$ converging strongly in $L^2(\mathbb{R}^2)$ to η_∞. Define the set

$$\mathcal{E} := \left\{ E \subseteq \mathbb{R}^2 \;\middle|\; 0 < |E| < +\infty, \exists n \in \mathbb{N} \cup \{\infty\} \text{ such that } P(E) = \left| \int_E \eta_n \right| \right\}.$$

Then, every $E \in \mathcal{E}$ satisfies the following properties:

1. $\inf\limits_{E\in\mathcal{E}} P(E) > 0$ and $\sup\limits_{E\in\mathcal{E}} P(E) < \infty$,
2. $\inf\limits_{E\in\mathcal{E}} |E| > 0$ and $\sup\limits_{E\in\mathcal{E}} |E| < \infty$,
3. There exists a $R > 0$ such that $E \subseteq B(0,R)$,
4. There exists $r_0 > 0$ and $C \in (0, 1/2)$, such that for every $r \in (0, r_0]$:

$$\forall x \in \partial E,\; C \leq \frac{|E \cap B(x,r)|}{|B(x,r)|} \leq 1 - C.$$

The next proposition provides insight into the behavior of these sets for decreasing noise w and regularization parameter λ. It is important to note that the noise diminishes faster than the regularization parameter. If this is not the case, we actually observe the behavior described in Section 2.2.2, where there is an overfitting to the noise.

Proposition 6.2: [14, Thm. 2] Let us consider that there exists a solution to $\mathcal{D}_0(y_0)$. If $\lambda_n \to 0$ and the following condition

$$\frac{\|w_n\|_H}{\lambda_n} \leq \frac{1}{4c_2\|\Phi^*\|}$$

is fulfilled, then for each $n \in \mathbb{N}$, with u_n as a solution to $P_{\lambda_n}(y_0 + w_n)$, the support sets $(\mathrm{supp}(u_n))_{n\geq 0}$ are bounded. In addition, by considering a subsequence (without changing the notation), u_n converges strictly in $BV(\mathbb{R}^2)$ to u_*, a solution to $\mathcal{P}_0(y_0)$ (see 3.19). Furthermore for almost every $t \in \mathbb{R}$, the following holds:

$$|U_n^{(t)} \Delta U_*^{(t)}| \to 0 \quad \text{and} \quad \mathcal{H}(\partial U_n^{(t)}, \partial U_*^{(t)}) \to 0,$$

where $\mathcal{H}$ denotes the Hausdorff-metric (see 3.12).

6.2 An alternative problem formulation

We now transform the subgradient property (11), which indicates the optimality of a solution, into a problem formulation that involves the aforementioned sets. Let us consider the following proposition for this purpose.

Proposition 6.3: [6, Prop. 3] Let u be an element of $L^2(\mathbb{R}^2)$ for which $\mathrm{TV}(u) < \infty$, and let η belong to $L^2(\mathbb{R}^2)$. It is then established that the following statements are equivalent:

(i) $\eta \in \partial \mathrm{TV}(u)$.

(ii) $\eta \in \partial \mathrm{TV}(0)$ and for the level sets of u, the following holds:

$$\forall t > 0, \quad P(U(t)) = \int_{U(t)} \eta,$$
$$\forall t < 0, \quad P(U(t)) = -\int_{U(t)} \eta.$$

(iii) For the level sets of u, we have:

$$\forall t > 0, \quad U(t) \in \operatorname*{Argmin}_{E \subseteq \mathbb{R}^2, |E| < \infty} \left(P(E) - \int_E \eta \right),$$
$$\forall t < 0, \quad U(t) \in \operatorname*{Argmin}_{E \subseteq \mathbb{R}^2, |E| < \infty} \left(P(E) + \int_E \eta \right).$$

We can now see from (iii) that u is a solution of $\mathcal{P}_0(y_0)$ with p being a solution of $\mathcal{D}_0(y_0)$ $(\Phi^* p = \eta)$ if and only if the level sets of u solve the following problem:

$$\inf_{E \subset \mathbb{R}^2, |E| < \infty} J(E) := P(E) - \int_E \eta. \tag{$\mathcal{PC}(\eta)$}$$

This problem is called the prescribed curvature problem for a specified $\eta \in L^2(\mathbb{R}^2)$. The naming of the problem originates from the fact that for a sufficiently regular η, every boundary of a solution to $\mathcal{PC}(\eta)$ exhibits a curvature that corresponds to η (in the distributional sense). The existence of such solutions is established for any $\eta \in \partial \mathrm{TV}(0)$. This is because, for such an η, the following holds true for all $u \in L^2(\mathbb{R}^2)$ (including for $u := \pm \chi_E$ where $|E| < \infty$):

$$\mathrm{TV}(\chi_E) = P(E) \geq \int_{\mathbb{R}^2} \eta \pm \chi_E = \pm \int_E \eta$$
$$\iff P(E) \pm \int_E \eta \geq 0.$$

In this scenario, J is non-negative and equals 0 when $E = \emptyset$.
We will now demonstrate several properties of this problem setting.

Lemma 6.4 (Boundedness): [6, Lem. 4] For each solution E of $\mathcal{PC}(\eta)$, there exists a radius $R > 0$ such that:

$$E \subseteq B(0,R)$$

where $B(0,R)$ describes a ball in $\mathbb{R}^2$ centered at 0 with radius R.

Lemma 6.5: [3, Def. 4.7.3, Thm. 4.7.4] Assuming additionally that $\eta \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, and therefore in $L^\infty(B(0,R))$, it follows that every solution of $\mathcal{PC}(\eta)$ is a strong quasi-minimizer (see definiton 3.27) of the perimeter. This implies that such a solution possesses a $C^{1,1}$ representative.

Lemma 6.6 (Regularity of solutions): For $\eta \in C^0(\mathbb{R}^2) \cap L^\infty_{\text{loc}}(\mathbb{R}^2)$, consider a solution E of $\mathcal{PC}(\eta)$. In this case, ∂E can be locally represented by a function $u \in C^1$ (see 6.5 above), and the signed curvature H_E of E (refer to 3) aligns with η in the classical sense, that is:

$$H_E(x,u(x)) = \left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \eta(x,u(x)), \tag{37}$$

which implies, that $u \in C^{k+2,\alpha}$ if $\eta \in C^{k,\alpha}(\mathbb{R}^2)$.

Proof. With lemma 6.6 we derive a local description $u \in C^1([-r,r])$ of the set E (see definiton 3.1 and figure 11). We observe that the perimeter of E, restricted to $C(x,r,\nu)$, is effectively described by the length of the graph of u over the interval $(-r,r)$. Therefore, we have:

$$P(E;C(x,r,\nu)) = \mathcal{H}^1(\partial E)\llcorner_{C(x,r,\nu)} = \mathcal{H}^1(\text{graph}(u)) = \int_{-r}^{r} \sqrt{1+u'(x)^2}dx$$

where the final equation can be elucidated with reference to Figure 17 below, especially in the case where the interval $[a,b]$ becomes infinitesimally small.

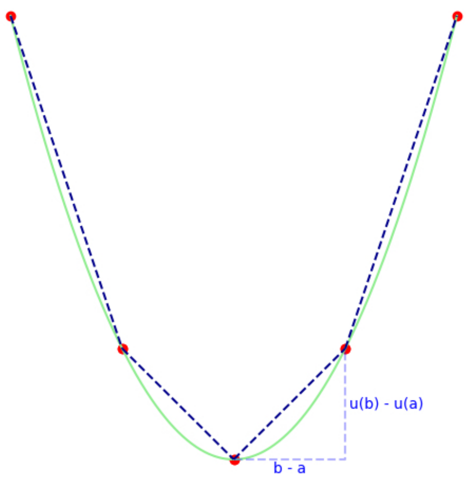

Figure 17: The figure depicts a continuous function in light green, complemented by four dashed blue secants, positioned at uniformly distributed support points.

This given, the problem stated in $\mathcal{PC}(\eta)$ simplifies for a set $E \subseteq \mathbb{R}^2$ when locally considered on $C(x,r,\nu)$ to:

$$\min_{u\in C^1([-r,r])}\left\{\int_{-r}^{r}\sqrt{1+u'(x)^2}\,dx-\int_{-r}^{r}\int_{-\infty}^{u(x)}\eta(x,y)\,dy\,dx\right\}=:E(u).$$

Therefore, for a test function $\varphi\in C_0^\infty([-r,r])$ satisfying $\varphi(-r)=\varphi(r)=0$ and $t\in\mathbb{R}$, it must hold that:

$$\begin{aligned}
&\frac{d}{dt}E(u+t\varphi)\Big|_{t=0}=0\\
\Longleftrightarrow\ &\frac{d}{dt}\Big(\int_{-r}^{r}\sqrt{1+(u'(x)+t\varphi(x))^2}dx-\int_{-r}^{r}\int_{-\infty}^{u(x)+t\varphi(x)}\eta(x,y)dydx\Big)\Big|_{t=0}=0\\
\Longleftrightarrow\ &\Big(\int_{-r}^{r}\frac{d}{dt}\sqrt{1+(u'(x)+t\varphi(x))^2}dx-\int_{-r}^{r}\frac{d}{dt}\int_{-\infty}^{u(x)+t\varphi(x)}\eta(x,y)dydx\Big)\Big|_{t=0}=0\\
\Longleftrightarrow\ &\Big(\int_{-r}^{r}\frac{(u'(x)+t\varphi(x))\varphi'(x)}{\sqrt{1+(u'(x)+t\varphi(x))^2}}dx-\int_{-r}^{r}\varphi(x)\eta(x,u(x)+t\varphi(x))dx\Big)\Big|_{t=0}=0\\
\Longleftrightarrow\ &\int_{-r}^{r}\frac{u'(x)\varphi'(x)}{\sqrt{1+u'(x)^2}}dx-\int_{-r}^{r}\varphi(x)\eta(x,u(x))dx=0\\
\overset{*}{\Longleftrightarrow}\ &\int_{-r}^{r}\varphi(x)\Big(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\Big)'dx-\int_{-r}^{r}\varphi(x)\eta(x,u(x))dx=0\\
\Longleftrightarrow\ &\int_{-r}^{r}\varphi(x)\left(\Big(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\Big)'-\eta(x,u(x))\right)dx=0\\
\overset{**}{\Longleftrightarrow}\ &\Big(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\Big)'-\eta(x,u(x))=0\\
\Longleftrightarrow\ &H_E(x,u(x))=\eta(x,u(x)).
\end{aligned}$$

Where $(*)$ results from partial integration with φ having 0 boundary values, and $(**)$ follows from the fundamental lemma of calculus of variations (see [15, Lem. 1.1.1]). □

6.3 Analysis of the problem

As derived in Section 5.3, we obtain k-simple functions as solutions to $\mathcal{P}_\lambda(y)$ $(y=y_0+w)$. The simple sets appearing in the decomposition of these functions, representing the level sets according to theorem 3.25, thus form solutions to the prescribed curvature problem with respect to $\eta_{\lambda,w}$. Since we have already established in Section 4.4 that $\eta_{\lambda,w}$ converges to a minimal norm certificate η_0 as $w,\lambda\to 0$, we will next investigate how the solutions to the prescribed curvature problem behave as $\eta_{\lambda,w}$ approaches η_0.
Specifically, we aim to determine for two sufficiently close functionals η and $\tilde{\eta}$ whether:

(i) Are solutions of $\mathcal{PC}(\eta)$ are close to the solutions of $\mathcal{PC}(\tilde{\eta})$?

(ii) How many solutions of $\mathcal{PC}(\tilde{\eta})$ exist within a small neighborhood of $\mathcal{PC}(\eta)$?

6.3.1 Convergence:

In the following, we will address Question (i) using the ensuing proposition. We will show in the next proposition that for two functionals sufficiently close to each other with respect to the C^1 and L^2 norms, the solutions of the prescribed curvature problem for one function can be represented as C^2 deformations (see 3.2) of the solutions of the other function. The proof of this proposition is found in Appendix A in order to maintain the continuity of the text.

Proposition 6.7: Let $\eta_0 \in \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$.
For every $\epsilon > 0$ there exists $r > 0\ \forall \eta \in \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ with $\|\eta - \eta_0\|_{L^2(\mathbb{R}^2)} + \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \leq r$, the following is true: each non-empty solution F of $\mathcal{PC}(\eta)$ can be characterized as a C^2-normal deformation of size at most ϵ of a non-empty solution E of $\mathcal{PC}(\eta_0)$, that is, using the notation of Proposition 3.6, $F = E_\varphi$ with $\|\varphi\|_{C^2(\partial E)} \leq \epsilon$.

6.3.2 Stability

To address question (ii), it is necessary to delve deeper into the subject. We examine the functional J in the neighborhood of a solution E to $\mathcal{PC}(\eta)$ with respect to C^2 deformations φ. Analogous to [8], we consider the second derivative of the functional defined below, involving J and the aforementioned deformations φ. Intuitively, this can be envisioned as demonstrating that J is, in a sense, twice continuously differentiable. We will then use the differentiability to determine a unique minimum of the function and to investigate the surrounding sets. This then aids in addressing question (ii).

Structure of shape derivatives

For further analysis we introduce the mapping denoted by j_E, which associates a normal deformation φ with the functional J evaluated on the deformed set E_φ, as described in section 3.2:

$$\begin{aligned} j_E \colon C^1(\partial E) &\to \mathbb{R} \\ \varphi &\mapsto J(E_\varphi). \end{aligned}$$

With the notation introduced above, we now present the following proposition:

Proposition 6.8: [13, p. 243f, p. 251]
Let $\eta \in C^1(\mathbb{R}^2)$, then the functional j_E is twice Fréchet differentiable at 0. For any function ψ in $C^1(\partial E)$, the following expressions hold:

$$\begin{aligned} j_E'(0)(\psi) &= \int_{\partial E} (H - \eta)\psi \, d\mathcal{H}^1, \\ j_E''(0)(\psi, \psi) &= \int_{\partial E} \left(|\nabla_\tau \psi|^2 - \left(H + \frac{\partial \eta}{\partial \nu} \right) \psi^2 \right) d\mathcal{H}^1, \end{aligned}$$

where H is the curvature of E and $\nabla_\tau \psi := \nabla\psi - (\nabla\psi \cdot \nu)\nu$ is the tangential gradient of ψ with respect to E.

We can see from the definitions of j'_E and j''_E that both functionals are well-defined even for weaker requirements on ψ. Therefore, we will consider j'_E over $L^1(\partial E)$ and j''_E over $H^1(\partial E)$ in the following.

To address question (ii), we will henceforth impose constraints on the potential solutions of $\mathcal{PC}(\eta)$. Subsequently, we shall examine the convergence behavior and continuity of j''_E. The proofs for these can be found in the Appendix A.

Definition 6.9 (Strict Stability): In accordance to [8, p. 3012], a non-trivial open set E, which solves $\mathcal{PC}(\eta)$, is said to exhibit strict stability if $j''_E(0)$ is coercive in $H^1(\partial E)$, meaning the ensuing criterion is met:

$$\exists \alpha > 0,\ \forall \psi \in H^1(\partial E), \quad j''_E(0)(\psi, \psi) \geq \alpha \|\psi\|^2_{H^1(\partial E)}.$$

To ensure the strict stability of E, let us consider what must hold for the functional j''_E. Again we consider a η of $\partial \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ and E, which is regarded as a solution of $\mathcal{PC}(\eta)$. With the lemma 6.6 we obtain that H_E is equivalent to η on the boundary set ∂E. So we derive the following expression for $j''_E(0)$: For every function ψ within $H^1(\partial E)$:

$$j''_E(0)(\psi, \psi) = \int_{\partial E} \left[|\nabla_{\tau_E} \psi|^2 - \left(H_E^2 + \frac{\partial \eta}{\partial \nu_E} \right) \psi^2 \right] \mathrm{d}H^1.$$

It can now be directly seen that the set E is strictly stable if and only if the term $(H_E^2 + \frac{\partial \eta}{\partial \nu_E})$ is negative. This leads us to the following proposition.

Proposition 6.10: If the supremum

$$\sup_{x \in \partial E} \left[H_E(x)^2 + \frac{\partial \eta}{\partial \nu_E}(x) \right] < 0 \tag{38}$$

then $j''_E(0)$ is coercive.

For the following propositions, let X be a vector space over $\mathbb{R}$, and let $\mathcal{Q}(X)$ be the space of all quadratic forms over X, where the norm for a $q \in \mathcal{Q}(X)$ is given by:

$$\|q\|_{\mathcal{Q}(X)} := \sup_{x \in X \setminus \{0\}} \frac{|q(x,x)|}{\|x\|^2_X}.$$

Proposition 6.11: If $\eta \in C^1(\mathbb{R}^2)$, the mapping

$$\begin{aligned} j''_E \colon C^2(\partial E) &\to \mathcal{Q}(H^1(\partial E)) \\ \varphi &\mapsto j''_E(\varphi) \end{aligned}$$

is continuous at 0.

Proposition 6.12: Let $\eta_0 \in C^1(\mathbb{R}^2)$. There exists $\epsilon > 0$ such that

$$\lim_{\|\eta-\eta_0\|_{C^1(\mathbb{R}^2)}\to 0} \sup_{\|\varphi\|_{C^2(\partial E)}\le\epsilon} \left\| j''_E(\varphi) - j''_{0,E}(\varphi) \right\|_{Q(H^1(\partial E))} = 0,$$

where j''_E and $j''_{0,E}$ describe the functionals with respect to η and η_0.

With these properties of the functional j_E and under the assumption that the set E is a strictly stable solution of $\mathcal{PC}(\eta_0)$, we can now answer question (ii). It follows that there exists at most one φ close to 0 such that E_φ is a solution of $\mathcal{PC}(\eta)$, provided that $\|\eta-\eta_0\|_{C^1(\mathbb{R}^2)}$ is sufficiently small. Precisely, we obtain the following proposition:

Proposition 6.13: Consider η_0 belonging to $\partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ and E a strictly stable solution to $\mathcal{PC}(\eta_0)$.
Then there exists $\epsilon > 0$ and $r > 0$ such that for every $\eta \in \partial\mathrm{TV}(0)$ with $\|\eta-\eta_0\|_{C^1(\mathbb{R}^2)} \le r$ there is at most one $\varphi \in C^2(\partial E)$ such that $\|\varphi\|_{C^2(\partial E)} \le \epsilon$ and E_φ solves $\mathcal{PC}(\eta)$.

Proof. Given that E is a strictly stable solution of $\mathcal{PC}(\eta_0)$, we have:

$$\begin{aligned} &\exists\alpha_0 > 0,\ \forall\psi \in H^1(\partial E), \quad j''_{0,E}(0)(\psi,\psi) \ge \alpha_0\|\psi\|^2_{H^1(\partial E)} \\ &\qquad\qquad\qquad\qquad \iff |j''_{0,E}(0)(\psi,\psi)| \ge \alpha_0\|\psi\|^2_{H^1(\partial E)}. \end{aligned}$$

As this holds for all $\psi \in H^1(\partial E)$, it also applies to the supremum over all $H^1(\partial E)\backslash\{0\}$, yielding:

$$\|j''_{0,E}(0)\|_{\mathcal{Q}(H^1(\partial E))} \ge \alpha_0.$$

Furthermore, due to the continuity of $j''_{0,E}$ at 0, we can select an $\epsilon_1 > 0$ such that for all φ in $C^2(\partial E)$ with $\|\varphi\|_{C^2(\partial E)} \le \epsilon_1$, it holds:

$$\|j''_{0,E}(0) - j''_{0,E}(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} \le \frac{1}{3}\alpha_0.$$

Similarly, employing Proposition 6.12, we choose $\epsilon_2 > 0$ and $r > 0$ such that for all $\eta \in C^1(\mathbb{R}^2)$ with $\|\eta-\eta_0\|_{C^1(\mathbb{R}^2)} \le r$ and $\|\varphi\|_{C^2(\partial E)} \le \epsilon_2$, the following holds:

$$\left\| j''_E(\varphi) - j''_{0,E}(\varphi) \right\|_{Q(H^1(\partial E))} \le \frac{1}{3}\alpha_0.$$

Collectively, for a φ with $\|\varphi\|_{H^1(\partial E)} \leq \min(\epsilon_1, \epsilon_2) =: \epsilon$ and $\alpha := \frac{1}{3}\alpha_0 > 0$, we get:

$$\begin{aligned}
\alpha_0 &\leq \|j''_{0,E}(0)\|_{\mathcal{Q}(H^1(\partial E))} \\
&= \|j''_{0,E}(0) + j''_{0,E}(\varphi) - j''_{0,E}(\varphi) + j''_E(\varphi) - j''_E(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} \\
&\leq \|j''_{0,E}(0) - j''_{0,E}(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} + \|j''_{0,E}(\varphi) - j''_E(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} + \|j''_E(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} \\
&\leq \frac{1}{3}\alpha_0 + \frac{1}{3}\alpha_0 + \|j''_E(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} \\
\Longleftrightarrow \alpha &\leq \|j''_E(\varphi)\|_{\mathcal{Q}(H^1(\partial E))} \\
\Longleftrightarrow \alpha &\leq \sup_{\psi \in H^1(\partial E)\setminus\{0\}} \frac{|j''_E(\varphi)(\psi,\psi)|}{\|\psi\|^2_{H^1(\partial E)}}.
\end{aligned}$$

Therefore, we can deduce that $j''_E(\varphi)$ is coercive and thus positive definite. Consequently, j_E is strictly convex on the set $\{\varphi \in C^2(\mathbb{R}^2) \mid \|\varphi\| \leq \epsilon\} =: A$ $(*)$. Suppose there exist $\varphi, \psi \in A$ with $\varphi \neq \psi$ such that E_φ and E_ψ are solutions of $\mathcal{PC}(\eta)$. Then we have $\frac{1}{2}(\varphi+\psi) \in A$ and it follows:

$$\begin{aligned}
J(E_{\frac{1}{2}(\varphi+\psi)}) = j_E(\frac{1}{2}(\varphi+\psi)) &\overset{*}{<} \frac{1}{2}j_E(\varphi) + \frac{1}{2}j_E(\psi) \\
&= \frac{1}{2}J(E_\varphi) + \frac{1}{2}J(E_\psi) \\
&= J(E_\varphi) = J(E_\psi),
\end{aligned}$$

which contradicts the minimality of E_φ, E_ψ, thereby implying $\varphi = \psi$ and leading to the asserted result. □

By synthesizing the findings from propositions 6.7 and 6.13, it has been established that, assuming η approximates η_0 adequately in both $C^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ norms, any solution to $\mathcal{PC}(\eta)$ is contained within a neighborhood (characterized by C^2-normal deformations) of a solution to $\mathcal{PC}(\eta_0)$. Furthermore, given a strict stability condition, each such neighborhood is guaranteed to have no more than a single solution to $\mathcal{PC}(\eta)$.
Transferring these results back to our original problem formulation from Section 2, we can conclude, using the results from this section, that if we have a suitable original image (for example, the top view of the unit circle in $\mathbb{R}^2$, i.e., $u = a\chi_{B(0,1)}$), then solutions to the problem with noise can be represented as continuous deformations of solutions to the problem without noise, provided the respective dual certificates are sufficiently close to each other. We will next generalize this result to the previously mentioned k-simple functions.

7 Exact support recovery

Now we come to the final chapter of this thesis. Here we will first extend the results of the previous chapter to the faces defined in section 5, and then define a 'non-degenerate source condition' under which we can prove the main thesis of this work, theorem 7.3.

Finally, we will use a simple example to show that, under our conditions, there are indeed solutions to $\mathcal{P}_0(y_0)$.

7.1 Stability of $\mathcal{F} \subseteq \{TV \leq 1\}$

As in the previous section, we again choose $\eta, \eta_n \in \partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ as functionals which expose the faces $\mathcal{F}$ and $\mathcal{F}_n$ of $\{\mathrm{TV} \leq 1\}$. As seen in proposition 5.2, extreme points of such faces have the form $s\chi_E/P(E)$, where $s \in \{-1,1\}$ and E is a simply connected open set of class C^3. As defined earlier for solutions of $(\mathcal{PC}(\eta))$, we now extend the concept of strict stability to these extreme points.

Definition 7.1 (Strictly stable extremepoints): Let $\mathrm{extr}(\mathcal{F})$ be the set of all extreme points of $\mathcal{F}$. We call an extreme point $s\chi_E/P(E) \in \mathrm{extr}(\mathcal{F})$ strictly stable if:

(i) $s = 1$ and E is a strictly stable solution to $\mathcal{PC}(\eta)$ (see (6.9)),

(ii) $s = -1$ and E is a strictly stable solution to $\mathcal{PC}(-\eta)$.

To obtain stability result, Theorem 7.3, in this section, we will first demonstrate the following Lemma.

Lemma 7.2: Consider the sequence $(\eta_n)_{n\in\mathbb{N}\setminus\{0\}}$ in $\partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$, such that $\|\eta_n - \eta_0\|_{L^2(\mathbb{R}^2)} \to 0$ and $\|\eta_n - \eta_0\|_{C^1(\mathbb{R}^2)} \to 0$, ensuring that the set $\mathcal{F}_0$ is finite-dimensional and contains only strictly stable extreme points. Moreover, suppose there exists an infinite subset of $\mathbb{N}\setminus\{0\}$, where each element n corresponds to $\mathcal{F}_n$ having at least m distinct extreme points, represented as $(s_{n,i}\chi_{E_{n,i}}/P(E_{n,i}))$ for $i = 1,\dots,m$.

Then, there exist $(s_i)_{1\leq i\leq m}$ and pairwise distinct sets $(E_i)_{1\leq i\leq m}$, with $s_i\chi_{E_i}/P(E_i) \in \mathrm{extr}(\mathcal{F}_0)$ for all $i \in \{1,\dots,m\}$. Furthermore, after extracting a (not relabeled) subsequence, the following holds:

$$\forall n \in \mathbb{N}\setminus\{0\},\ \forall i \in \{1,\dots,m\}, \quad \begin{cases} s_{n,i} = s_i, \\ E_{n,i} = (E_i)_{\varphi_{n,i}}, \quad \text{with} \quad \lim\limits_{n\to\infty} \|\varphi_{n,i}\|_{C^2(\partial E_i)} = 0. \end{cases} \tag{39}$$

In particular, $m \leq \mathrm{card}(\mathrm{extr}(F_0))$.

Proof. For each i in $\{1,\dots,m\}$, since $s_{n,i}$ converges to either -1 or 1, there exist infinitely many n such that $s_{n,i} = 1$ or $s_{n,i} = -1$. This means that we can select a subsequence (without relabeling) such that $s_{n,i} = s_i$ for all $n \in \mathbb{N}\setminus\{0\}$ and $i \in \{1,\dots,m\}$.
By applying Proposition 6.7, we obtain for a sufficiently large $n \in \mathbb{N}\setminus\{0\}$, and thus by selecting a suitable subsequence (again not relabeled), for each $i \in \{1,\dots,m\}$, $E_{n,i}$ can be expressed as a C^2 deformation of E_i, denoted by $E_{n,i} = (E_i)_{\varphi_{n,i}}$. We choose the subsequence such that $\|\varphi_{n,i}\| < \epsilon/n$. Consequently, $\|\varphi_{n,i}\| \to 0$, and thus the sequence $(E_{i,n})_{n\in\mathbb{N}}$ converges in C^3 to a solution E_i of $\mathcal{PC}(s_i\eta_0)$ (see proposition 3.7), thereby demonstrating

(39). Furthermore, since $E_{n,i}$ is simple and diffeomorphic to E_i (for sufficiently large n), it follows that E_i is also simple, and thus $s_i\chi_{E_i}/P(E_i)$ is an extreme point of $\mathcal{F}_0$ for all $i \in \{1,\dots,m\}$.

The final step of the proof is to show that the sets $(E_i)_{1\le i\le m}$ are pairwise distinct. For this purpose, assume there exist $i,j \in \{1,\dots,m\}$ with $i \neq j$ and $E_i = E_j$. Since $s_i\chi_{E_i}/P(E_i)$ and $s_j\chi_{E_j}/P(E_j)$ are in $\mathcal{F}_0$, it follows that

$$\int_{\mathbb{R}^2} \frac{s_i\chi_{E_i}}{\mathrm{TV}(\chi_{E_i})}\eta_0 = 1 = \int_{\mathbb{R}^2} \frac{s_j\chi_{E_j}}{\mathrm{TV}(\chi_{E_j})}\eta_0,$$

and thus

$$s_i = \int_{E_i} \frac{\eta_0}{P(E_i)} = \int_{E_j} \frac{\eta_0}{P(E_j)} = s_j.$$

However, this would imply that two different extreme points $s_{n,i}\chi_{E_{n,i}}/P(E_{n,i})$ and $s_{n,j}\chi_{E_{n,j}}/P(E_{n,j})$ of $\mathcal{F}_n$ exist that converge to $s_i\chi_{E_i}/P(E_i) = s_j\chi_{E_j}/P(E_j)$ with $E_{n,i} \neq E_{n,j}$ for all $n \in \mathbb{N}\setminus\{0\}$. Consequently, for sufficiently large n_0, Proposition 6.7 ensures that for all $n \ge n_0$, there exist distinct functions $\psi_n \neq \phi_n$ in $C^2(\partial E_i)$ such that $E_{n,i} = (E_i)_{\phi_n}$ and $E_{n,j} = (E_i)_{\psi_n}$. Therefore, $(E_i)_{\psi_n}$ and $(E_i)_{\phi_n}$ represent two distinct solutions to $\mathcal{PC}(\eta_n)$ for all $n \ge n_0$, which contradicts the strict stability of E_i as stated in Proposition 6.13. □

Thus, we can now derive:

Theorem 7.3: Let $\eta_0 \in \partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ be such that $\mathcal{F}_0$ has finite dimension, with all its extreme points strictly stable. Then for every $\epsilon > 0$, there exists $r > 0$ such that, for every $\eta \in \partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ with

$$\|\eta - \eta_0\|_{L^2(\mathbb{R}^2)} + \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \le r,$$

there exists an injective mapping $\theta\colon \mathrm{extr}(\mathcal{F}) \to \mathrm{extr}(\mathcal{F}_0)$ such that, for every $u = s\chi_F/P(F)$ in $\mathrm{extr}(\mathcal{F})$, we have $\theta(u) = s\chi_E/P(E)$ with

$$F = E_\varphi \quad \text{and} \quad \|\varphi\|_{C^2(\partial E)} < \epsilon.$$

In particular, $\dim(F) \le \dim(F_0)$.

Proof. By contradiction, let us consider the existence of a certain $\epsilon > 0$ and a sequence $(\eta_n)_{n\in\mathbb{N}^*}$ in $\partial\mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$, which converges in $L^2(\mathbb{R}^2)$ and $C^1(\mathbb{R}^2)$ to η_0, but fails to maintain the property in question for all $n \in \mathbb{N}\setminus\{0\}$.

Define m to be the maximal number of extreme points attainable by any subsequence $(\mathcal{F}_n)_{n\in\mathbb{N}\setminus\{0\}}$:

$$m := \limsup_{n\to\infty} \Big(\mathrm{card}(\mathrm{extr}(\mathcal{F}_n))\Big).$$

Lemma 7.2 guarantees that $m \le \mathrm{card}(\mathrm{extr}(\mathcal{F}_0))$, and by passing to a subsequence if necessary, we can find an injective mapping $\theta_n : \mathrm{extr}(\mathcal{F}_n) \to \mathrm{extr}(\mathcal{F}_0)$. This mapping ensures that for every $u = s1_F/P(F)$ in $\mathrm{extr}(\mathcal{F}_n)$, we have $\theta_n(u) = s1_E/P(E)$ where $F = E_{\varphi_u}$, and the following condition holds:

$$\lim_{n\to\infty}\left(\max_{u\in\mathrm{extr}(\mathcal{F}_n)} \|\varphi_u\|_{C^2(\partial E)}\right) = 0. \tag{40}$$

Consequently, for sufficiently large n, $\|\varphi_u\|_{C^2(\partial E)} \leq \epsilon$ applies to all deformations φ_u. This means that the conclusion in Theorem 7.3 is valid for our arbitrarily chosen $\epsilon > 0$. This constitutes a contradiction to the assumption.

□

7.2 Main result

To demonstrate the main result of this work, we need to impose some restrictions on possible solutions u_0 of $\mathcal{P}_0(y_0)$. This will be outlined in the following as a non-degenerate source condition, which is defined by:

Definition 7.4 (Non-degenerate source condition): Let $u_0 = \sum_{i=1}^N a_i\chi_{E_i}$ be a N-simple function. We assert that u_0 fulfills the non-degenerate source condition if

1. the source condition $\mathrm{Im}\Phi^* \cap \partial\mathrm{TV}(u_0) \neq \emptyset$ is satisfied,
2. for each $i \in \{1,\ldots,N\}$, the set E_i constitutes a strictly stable solution to $\mathcal{PC}(\mathrm{sign}(a_i)\eta_0)$,
3. for any simple set $E \subset \mathbb{R}^2$ with $|E\Delta E_i| > 0$ for all $i \in \{1,\ldots,N\}$, it holds that $|\int_E \eta_0| < P(E)$.

Under these conditions, η_0 is considered non-degenerate.

The first point includes the source condition mentioned earlier (11). With these constraints in place, we can finally prove the main thesis of this work, which is as follows:

Theorem 7.5: Assume that $u_0 = \sum_{i=1}^N a_i\chi_{E_i}$ is a N-simple function satisfying the non-degenerate source condition, and that $\Phi_{\mathcal{F}_0}$ is injective (see (35)). Then there exist constants α, $\lambda_0 \in \mathbb{R}_+^*$ such that, for every $(\lambda, w) \in \mathbb{R}_+ \times \mathcal{H}$ with $\lambda \leq \lambda_0$ and $\|w\|_{\mathcal{H}}/\lambda \leq \alpha$, every solution $u_{\lambda,w}$ of $(\mathcal{P}_\lambda(y))$ is such that

$$u_{\lambda,w} = \sum_{i=1}^N a_i^{\lambda,w}\chi_{E_i^{\lambda,w}}, \tag{41}$$

with

$$\forall i \in \{1,\ldots,N\},\ \begin{cases} \mathrm{sign}(a_i^{\lambda,w}) = \mathrm{sign}(a_i), \\ \quad E_i^{\lambda,w} = (E_i)_{\varphi_i^{\lambda,w}} \quad \text{with } \varphi_i^{\lambda,w} \in C^2(\partial E_i). \end{cases} \tag{42}$$

Moreover,

$$\lim_{\substack{(\lambda,w)\to(0,0)\\0<\lambda\leq\lambda_0\\\|w\|_{\mathcal{H}}\leq\alpha\lambda}} a_i^{\lambda,w} = a_i \quad \text{and} \quad \lim_{\substack{(\lambda,w)\to(0,0)\\0<\lambda\leq\lambda_0\\\|w\|_{\mathcal{H}}\leq\alpha\lambda}} \left\|\varphi_i^{\lambda,w}\right\|_{C^2(\partial E_i)} = 0. \tag{43}$$

Proof. Define the open δ-neighborhood of the set $A \subseteq \mathbb{R}^2$ by the equation

$$A^\delta := \bigcup_{x\in A} B(x,\delta).$$

Now, thanks to proposition 5.2, we can choose a $\delta > 0$ sufficiently small such that

$$(\partial E_i)^\delta \cap (\partial E_j)^\delta = \emptyset \tag{44}$$

holds. Furthermore, we select an $\epsilon > 0$ such that $\epsilon < |a_i|P(E_i)$ for all $i \in \{1,\ldots,N\}$ and for a $\varphi \in C^2(\partial E_i)$ with $\|\varphi\|_{C^2(\partial E_i)} < \epsilon$, the following condition is satisfied:

$$(Id + \varphi\nu_{E_i})(E_i) \subset (\partial E_i)^\delta.$$

Finally, we select a $r > 0$ such that the assumptions in theorem 7.3 are satisfied. Due to the injectivity of $\Phi_{\mathcal{F}}$, we obtain u_0 as the unique solution to problem $\mathcal{P}_0(y_0)$. By applying Proposition 6.2), we ascertain that $u_{\lambda,w}$ converges strictly in $BV(\mathbb{R}^2)$ to u_0 as $\lambda \to 0$ and $\frac{\|w\|_{\mathcal{H}}}{\lambda} \to 0$. That is, employing Definition 3.19, we achieve weak* convergence of $|Du_{\lambda,w}|$ to $|Du_0|$.

Moreover, due to

$$\operatorname{supp}(|Du_0|) = \partial E_i \text{ for all } i \in \{1,\ldots,N\}$$

it follows that

$$\langle |Du_{\lambda,w}|, \chi_{((\partial E_i)^\delta)^c}\rangle_{L^2(\mathbb{R}^2)} \to \langle |Du_0|, \chi_{((\partial E_i)^\delta)^c}\rangle_{L^2(\mathbb{R}^2)} = 0.$$

Consequently, there exist $\alpha > 0$ and $\lambda_0 > 0$ such that for all (λ, w) in $\mathbb{R}_+^* \times \mathcal{H}$ with $\lambda \leq \lambda_0$ and $\frac{\|w\|_{\mathcal{H}}}{\lambda} \leq \alpha$, and for all i in $\{1,\ldots,N\}$, it holds that

$$\Big||Du_{\lambda,w}|((\partial E_i)^\delta) - |Du_0|((\partial E_i)^\delta)\Big| \leq \epsilon. \tag{45}$$

Furthermore, using proposition (4.5), we can find an $\epsilon_\lambda > 0$ such that

$$\begin{aligned}
\|\eta_{\lambda,w} - \eta_0\|_{L^2(\mathbb{R}^2)} &= \|\Phi^*(p_{\lambda,w} - p_0)\|_{L^2(\mathbb{R}^2)} \\
&= \|\Phi^*(p_{\lambda,w} - p_{\lambda,0} + p_{\lambda,0} - p_0)\|_{L^2(\mathbb{R}^2)} \\
&\leq \|\Phi^*\|_{C^0(\mathbb{R}^2)} \Big(\|p_{\lambda,w} - p_{\lambda,0}\|_{\mathcal{H}} + \|p_{\lambda,0} - p_0\|_{\mathcal{H}}\Big) \\
&\overset{*}{\leq} \|\Phi^*\|_{C^0(\mathbb{R}^2)} \left(\frac{\|w\|_{\mathcal{H}}}{\lambda} + \epsilon_\lambda\right)
\end{aligned}$$

and

$$\begin{aligned}\|\eta_{\lambda,w}-\eta_0\|_{C^1(\mathbb{R}^2)} &= \|\Phi^*(p_{\lambda,w}-p_0)\|_{C^1(\mathbb{R}^2)}\\ &= \|\Phi^*\|_{C^1(\mathbb{R}^2)}\|p_{\lambda,w}-p_0\|_{\mathcal{H}}\\ &\overset{**}{\leq} \|\Phi^*\|_{C^0(\mathbb{R}^2)}\|p_{\lambda,w}-p_0\|_{\mathcal{H}}\\ &\leq \|\Phi^*\|_{C^0(\mathbb{R}^2)}\left(\frac{\|w\|_{\mathcal{H}}}{\lambda}+\epsilon_\lambda\right).\end{aligned}$$

Where $(**)$ follows from the linearity and continuity of Φ^*. In this case $D\Phi^* = \Phi^*$ holds.

Thus, we adjust the choice of α and λ_0 (also including ϵ_λ) such that

$$\left(\|\Phi^*\|_{C^0(\mathbb{R}^2)}\alpha+\|\Phi^*\|_{C^0(\mathbb{R}^2)}\epsilon_\lambda\right)\leq \frac{1}{2}r$$

which yields:

$$\|\eta_{\lambda,w}-\eta_0\|_{L^2(\mathbb{R}^2)}+\|\eta_{\lambda,w}-\eta_0\|_{C^1(\mathbb{R}^2)}\leq \frac{1}{2}r+\frac{1}{2}r=r.$$

We now choose a pair (λ,w) with $\lambda<\lambda_0$ and obtain for the face $\mathcal{F}_{\lambda,w}$, exposed by $\eta_{\lambda,w}$, the following extreme points:

$$\operatorname{extr}(\mathcal{F}_{\lambda,w})=\left\{s_i^{\lambda,w}\frac{\chi_{E_i^{\lambda,w}}}{P(E_i^{\lambda,w})}\right\}_{1\leq i\leq N_{\lambda,w}}.$$

Thus, using theorem 7.3, we can find an injective mapping $\theta_{\lambda_w}:\{1,...,N_{\lambda,w}\}\to\{1,...,N\}$ such that:

$$\forall i\in\{1,\dots,N_{\lambda,w}\},\quad \begin{cases} s_i^{\lambda,w}=s_{\theta_{\lambda,w}(i)},\\ E_i^{\lambda,w}=(E_{\theta_{\lambda,w}(i)})_{\varphi_i^{\lambda,w}} \quad \text{with } \varphi_i^{\lambda,w}\in C^2(\partial E_i).\end{cases} \tag{46}$$

In the next step, we show that $\theta_{\lambda,w}$ is also surjective, i.e., $N_{\lambda,w}=N$.

With (45) for all $i\in\{1,..N\}$:

$$\begin{aligned}|Du_{\lambda,w}|((\partial E_i)^\delta) &\geq |Du_0|((\partial E_i)^\delta)-\epsilon\\ &= \mathrm{TV}\left(u\big|_{(\partial E_i)^\delta}\right)-\epsilon\\ &\overset{*}{=} \mathrm{TV}\left(a_i\chi_{E_i}\right)-\epsilon\\ &= |a_i|\mathrm{TV}\left(\chi_{E_i}\right)-\epsilon\\ &= |a_i|P(E_i)-\epsilon>0\end{aligned}$$

where $(*)$ follows from (44).

We can thus see that $\operatorname{supp}(|Du_{\lambda,w}|)\cap((\partial E_i)^\delta)\neq\emptyset$ $(\forall i\in\{1,..,N\})$. Therefore, it is established that:

$$\operatorname{supp}(|Du_{\lambda,w}|) \subseteq \bigcup_{i=1}^{N_{\lambda,w}} \partial E_i^{\lambda,w} \subseteq \bigcup_{i=1}^{N_{\lambda,w}} (\partial E_{\theta_{\lambda,w}(i)})^{\delta}.$$

Since the sets in the latter union are all disjoint, this establishes the surjectivity of $\theta_{\lambda,w}$.

Thus, upon reordering, the relationship

$$\forall i \in \{1,\ldots,N\}, \quad s_i^{\lambda,w} = s_i \quad \text{and} \quad E_i^{\lambda,w} = (E_i)_{\varphi_i^{\lambda,w}}$$

is maintained. As per proposition 6.7, the norm $\left\|\varphi_i^{\lambda,w}\right\|_{C^2(\partial E_i)}$ diminishes as the pair $(\lambda, \|w\|_{\mathcal{H}}/\lambda) \to (0,0)$. Additionally, $s_i^{\lambda,w} = s_i$ infers identical signs for $a_i^{\lambda,w}$ and a_i. In conclusion, we observe that

$$\left||Du_{\lambda,w}|\left((\partial E_i)^{\delta}\right) - |Du_0|\left((\partial E_i)^{\delta}\right)\right| = \left|a_i^{\lambda,w} P(E_i^{\lambda,w}) - a_i P(E_i)\right|$$

tends to zero as a result of the weak-$*$ convergence when $(\lambda, \|w\|_{\mathcal{H}}/\lambda)$ approaches $(0,0)$. Considering the convergence $P(E_i^{\lambda,w})$ towards $P(E_i)$, it follows that

$$a_i^{\lambda,w} \longrightarrow a_i.$$

$\square$

With this theorem, we are now in a position to answer the question from Section 2.3. Let us take such a u_0 which possesses the properties from the assumptions of Theorem 7.3 and is k-simple. Then, for a noise w small enough and a small parameter λ, it holds that solutions of $\mathcal{P}_\lambda(y)$ are not only k-simple but also converge to u_0 for $w \to 0$ for a well chosen λ.
Next, we will verify whether such a required u_0 can actually exist through a simple example.

7.3 Numerical verification of the non-degenerate source condition

To demonstrate that the solution set under our constraints of the non-degenerate source condition is not empty. For this, we investigate the existence of a dual certificate. This is achieved by defining what we call a dual pre-certificate, which represents a 'genuine' dual certificate under numerically verifiable properties. This will be defined in the following section.

The figures used in this section can be generated using the Jupyter notebook that was included with the paper on which this thesis is based (see [9, p. 28]).

Definition of a pre-certificate

Let $N \in \mathbb{N}\setminus\{0\}$ and let u be an N-simple function ($u = \sum_{i=1}^N a_i \chi_{E_i}$, $a_i \in \mathbb{R}^*$). As explained in Chapter 6, for such a function, a dual certificate $\eta = \Phi^* p$ exists if and only if all level sets of u (E_i) are solutions to $\mathcal{PC}(\eta)$, thus fulfilling the property:

$$\begin{aligned} &\forall i \in \{1,\ldots,N\}, \quad E_i \in \operatorname*{Argmin}_{E\in\mathbb{R}^2, |E|<+\infty} \left(P(E) - \operatorname{sign}(a_i)\int_E \Phi^* p\right) \\ \iff &\forall i \in \{1,\ldots,N\}, \quad \int_{E_i} \Phi^* p = \operatorname{sign}(a_i) P(E_i). \end{aligned} \tag{i}$$

Furthermore, as we are looking for a $\Phi^* p \in C^1(\mathbb{R}^2)$ (otherwise we do not necessarily obtain the k-simple representation of u, see section 5), due to lemma 6.6, it must hold that:

$$\forall i \in \{1,\ldots,N\}, \quad \Phi^* p|_{\partial E_i} = \operatorname{sign}(a_i) H_{E_i}. \tag{ii}$$

Therefore, similarly to [11, p. 20ff.], we can define a pre-certificate in such a way that it solves (i) and (ii) while having the smallest norm. Thus, we arrive at the following definition:

Definition 7.6 (Vanishing Derivatives Pre-certificate): The vanishing derivatives pre-certificate for an N-simple function $u = \sum_{i=1}^N a_i \chi_{E_i}$ is defined as the function $\eta_v = \Phi^* p_v$, where p_v is the unique solution of (i) and (ii), that is:

$$\min_{p\in\mathcal{H}} \|p\|_{\mathcal{H}}^2 \quad \text{s.t.} \quad \forall i \in \{1,\ldots,N\}, \quad \begin{aligned} &\int_{E_i} \Phi^* p = \operatorname{sign}(a_i) P(E_i), \\ &\Phi^* p|_{\partial E_i} = \operatorname{sign}(a_i) H_{E_i}. \end{aligned} \tag{47}$$

It can be seen that the set generated by both properties (i) and (ii) is weakly closed. Therefore, the minimum exists if this set is not empty. Since every dual certificate (i.e., $\eta \in \partial \mathrm{TV}(u)$) already satisfies (i) and (ii), a solution for (47) also exists (In this case, we say that (47) is feasible), and we can state the following proposition.

Proposition 7.7: If equation (47) is feasible and $\eta_v \in \partial TV(0)$, then qualifies as the dual certificate of minimal norm. In accordance with Definition 4.4, we have $\eta_v = \eta_0$.

Deconvolution of radial simple functions

In the following, we will demonstrate how we can numerically verify the existence of a $\eta_v = \Phi^* p_v$ that represents a dual certificate. For simplicity, we will consider the noise-free case where $\mathcal{H} = L^2(\mathbb{R}^2)$, $\Phi = h \star \cdot$ represents the convolution operation with a Gaussian kernel h of variance σ, and each $E_i = B(0, R_i)$ for i in $\{1,\ldots,N\}$ with a strictly increasing sequence of radii $0 < R_1 < \ldots < R_N$ (thus for an unknown image $u_0 = \sum_{i=1}^N a_i \chi_{B(0,R_i)}$). To this end, we introduce the following mappings:

$$\Phi_E \colon \mathbb{R}^N \to \mathcal{H}, \qquad \Phi'_E \colon \mathbb{R}^N \to \mathcal{H}, \qquad \Gamma_E \colon \mathbb{R}^{2N} \to \mathcal{H}.$$
$$a \mapsto \sum_{i=1}^N a_i\, h \star \chi_{E_i} \qquad b \mapsto \sum_{i=1}^N b_i\, h \star (\mathcal{H}^1 \llcorner_{\partial E_i}) \qquad (a,b) \mapsto \Phi_E a + \Phi'_E b$$

With these mappings, we can now discretize our problem given by (47), as stated in the following proposition.

Lemma 7.8: If (47) is feasible, p_v is radial (i.e. there exists a function $\tilde{p_v}: \mathbb{R}_+ \to \mathbb{R}$ such that $\tilde{p_v}(x) = p_v(\|x\|)$ for almost every $x \in \mathbb{R}^2$) and is the unique solution of

$$\min_{p \in \mathcal{H}} \|p\|^2_{\mathcal{H}} \quad \text{s.t.} \quad \Gamma^*_E p = \begin{pmatrix} (\mathrm{sign}(a_i) P(E_i))_{1 \leq i \leq N} \\ (\mathrm{sign}(a_i) 2\pi)_{1 \leq i \leq N} \end{pmatrix}. \tag{48}$$

Proof. First, we introduce the radialization $\tilde{p}$ of any function $p \in L^2(\mathbb{R}^2)$, defined as follows:

$$\tilde{p}(x) = \frac{1}{2\pi} \int_{S^1} p(\|x\| e) d\mathcal{H}^1(e) \quad \text{for a.e. } x \in \mathbb{R}^2. \tag{49}$$

Moreover, $\tilde{p}$ can also be represented using a rotation matrix R_e as follows:

$$\tilde{p} = \frac{1}{2\pi} \int_{S^1} (p \circ R_e)\, d\mathcal{H}^1(e) \quad \text{for all } p \in L^2(\mathbb{R}^2). \tag{50}$$

Furthermore, for all $e \in S^1$ and $x \in \mathbb{R}^2$, the following holds:

$$\begin{aligned} h \star (p \circ R_e)(x) &= \int_{\mathbb{R}^2} h(t)\, p \circ R_e(x-t) dt \\ &= \int_{\mathbb{R}^2} h(t)\, p(R_e(x-t)) dt \quad \Big| \text{ define } t := R_e^{-1}(t') \ (t' \in \mathbb{R}^2) \\ &= \int_{\mathbb{R}^2} h(R_e^{-1}(t'))\, p(R_e(x - R_e^{-1}(t'))) d\left(R_e^{-1}(t')\right) \\ &\overset{*}{=} \int_{\mathbb{R}^2} h(t')\, p(R_e(x) - t') dt' \quad \Big| \text{ define } \bar{t} := R_e(x) - t' \\ &= \int_{\mathbb{R}^2} h(R_e(x) - \bar{t})\, p(\bar{t}) d\,(R_e(x) - \bar{t}) \\ &\overset{**}{=} \int_{\mathbb{R}^2} h(R_e(x) - \bar{t})\, p(\bar{t}) d\bar{t} \\ &= p \star h(R_e(x)) \\ &= h \star p(R_e(x)), \end{aligned} \tag{51}$$

where $(*)$ follows from the radial nature of the Gaussian kernel h and the fact that $\det(R_e) = 1$, and $(**)$ follows from $\det(R_e) - t' = 1$.
Thus, we will show in the following that for a function p that satisfies the optimality condition from (i) and (ii), the same holds true for its radial version $\tilde{p}$.

(i): Here we obtain:

$$\begin{aligned}
\int_{E_i} \Phi^* \tilde{p}(x)dx &= \int_{E_i} h \star \tilde{p}(x)dx \\
&= \int_{E_i} \left(h \star \frac{1}{2\pi} \int_{S^1} (p \circ R_e) \, d\mathcal{H}^1(e) \right)(x)dx \\
&= \int_{E_i} \int_{\mathbb{R}^2} \left(h(t) \frac{1}{2\pi} \int_{S^1} (p \circ R_e) \, d\mathcal{H}^1(e)(x-t) \, dt \right) dx \\
&\overset{*}{=} \frac{1}{2\pi} \int_{S^1} \int_{E_i} \left(\int_{\mathbb{R}^2} h(t) \, (p \circ R_e)) \, (x-t)dt \right) dx \, d\mathcal{H}^1(e) \\
&= \frac{1}{2\pi} \int_{S^1} \int_{E_i} h \star (p \circ R_e)(x)dx \, d\mathcal{H}^1(e) \\
&\overset{(51)}{=} \frac{1}{2\pi} \int_{S^1} \left(\int_{E_i} h \star p(R_e(x))dx \right) d\mathcal{H}^1(e) \\
&\overset{**}{=} \int_{E_i} h \star p(x)dx \\
&= \int_{E_i} \Phi^* p(x)dx \\
&= \operatorname{sign}(a_i) P(E_i)
\end{aligned}$$

where $(*)$ follows from the linearity of the integral, Fubini's theorem [1, Thm. 1.74], and the radial nature of h. $(**)$ follows from the fact that by defining the E_i as spheres in $\mathbb{R}^2$, $\int_{E_i} h \star p(x)dx$ is already radial.

(ii): Here we receive:

$$\begin{aligned}
\Phi^* \tilde{p}(x)\Big|_{x \in \partial E_i} &= h \star \tilde{p}(x)\Big|_{x \in \partial E_i} \\
&= h \star \left(\frac{1}{2\pi} \int_{S^1} (p \circ R_e) d\mathcal{H}^1(e) \right)(x)\Big|_{x \in \partial E_i} \\
&= \int_{\mathbb{R}^2} \left(h(t) \frac{1}{2\pi} \int_{S^1} (p \circ R_e) \, d\mathcal{H}^1(e)(x-t) \right) dt\Big|_{x \in \partial E_i} \\
&= \frac{1}{2\pi} \int_{S^1} \left(\int_{\mathbb{R}^2} h(t)(p \circ R_e)(x-t) \, dt \right) d\mathcal{H}^1(e)\Big|_{x \in \partial E_i} \\
&= \frac{1}{2\pi} \int_{S^1} h \star p(R_e(x)) \, d\mathcal{H}^1(e)\Big|_{x \in \partial E_i} \\
&= h \star p(x)\Big|_{x \in \partial E_i} \\
&= \Phi^* p(x)\Big|_{x \in \partial E_i} \\
&= \operatorname{sign}(a_i) H_{E_i} = \operatorname{sign}(a_i) \frac{1}{R_i}.
\end{aligned}$$

Given the additional knowledge that $\|\tilde{p}\|_{L^2(\mathbb{R}^2)} \leq \|p\|_{L^2(\mathbb{R}^2)}$, and considering the uniqueness of the solution to (47), it can be deduced that if a solution p_v exists for (47), then necessarily p_v is equal to its radial counterpart $\tilde{p_v}$.

To demonstrate the final part of the proposition, the following holds true for such a function p with $a,\ b\ \in \mathbb{R}^N$:

$$\begin{aligned}
\left\langle \Gamma_E^* p, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_{\mathbb{R}^{2N}} &= \left\langle p, \Gamma_E \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} p \left(\Phi_E a + \Phi_E' b \right)(x) dx \\
&= \int_{\mathbb{R}^2} p \left(\sum_{i=1}^N a_i \, h \star \chi_{E_i} + \sum_{i=1}^N b_i \, h \star (\mathcal{H}^1 \llcorner \partial E_i) \right)(x) dx \\
&= a \cdot \left(\int_{\mathbb{R}^2} p \, h \star \chi_{E_i}(x) dx \right) + b \cdot \left(\int_{\mathbb{R}^2} p \, h \star (\mathcal{H}^1 \llcorner \partial E_i)(x) dx \right) \\
&\overset{*}{=} a \cdot \left(\int_{E_i} p \star h(x) dx \right) + b \cdot \left(\int_{\mathbb{R}^2} p \star h \; d\mathcal{H}^1 \llcorner \partial E_i \right) \\
&= a \cdot \left(\int_{E_i} \Phi^* p(x) dx \right) + b \cdot \left(\int_{\mathbb{R}^2} \Phi^* p \, d\mathcal{H}^1 \llcorner \partial E_i \right) \\
&= a \cdot (\operatorname{sign}(a_i) P(E_i))_{1 \le i \le N} + b \cdot \left(\int_{\mathbb{R}^2} \operatorname{sign}(a_i) \frac{1}{R_i} \, d\mathcal{H}^1 \llcorner \partial E_i \right)_{1 \le i \le N} \\
&= a \cdot (\operatorname{sign}(a_i) P(E_i))_{1 \le i \le N} + b \cdot \left(\operatorname{sign}(a_i) \frac{1}{R_i} 2\pi R_i \right)_{1 \le i \le N} \\
&= a \cdot (\operatorname{sign}(a_i) P(E_i))_{1 \le i \le N} + b \cdot (\operatorname{sign}(a_i) 2\pi)_{1 \le i \le N} \\
&= \left\langle \begin{pmatrix} \operatorname{sign}(a_i) P(E_i)_{1 \le i \le N} \\ \operatorname{sign}(a_i) 2\pi_{1 \le i \le N} \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_{\mathbb{R}^{2N}}
\end{aligned}$$

where $(*)$ follows from the fact that convolution with h is self-adjoint.

□

Since we are interested in obtaining an $\eta_v = \Phi^* p_v$ for our further analysis, we need to find a way to invert the mapping Γ_E^*. As this is not generally possible, we will introduce a more general concept of invertibility, the so-called Moore-Penrose pseudo-inverse $(\Gamma_E^*)^+ : \mathbb{R}^N \mapsto \mathbb{R}^N$ (see, for example, [4, p. 40ff.]). How to determine such an inverse will be shown in the proof of the following proposition.

Proposition 7.9: The operator Γ_E is injective. Furthermore, given that (48) is feasible, it follows that the vanishing derivatives pre-certificate η_v can be expressed as

$$\eta_v = \Phi^* (\Gamma_E^*)^+ \begin{pmatrix} (\operatorname{sign}(a_i) P(E_i))_{1 \le i \le N} \\ (\operatorname{sign}(a_i) 2\pi)_{1 \le i \le N} \end{pmatrix},$$

where $(\Gamma_E^*)^+$ is defined as $(\Gamma_E^*)^+ = \Gamma_E (\Gamma_E^* \Gamma_E)^{-1}$.

Proof. The first step is to demonstrate the injectivity of Γ_E. Since Γ_E is a linear operator, we aim to show that $\ker(\Gamma_E) = \{0\}$. Let us assume that for some $(a, b) \in \mathbb{R}^{2N}$, we have $\Gamma_E(a, b) = \Phi_E a + \Phi_E' b = h * (\sum_{i=1}^N a_i \chi_{E_i} + b_i \mathcal{H}^1 \llcorner \partial E_i) = 0$. Given that the convolution with h is injective (this follows from the fact that a convolution can be represented as multiplication by its Fourier transform, and this representation is injective), we obtain

$$\sum_{i=1}^{N} a_i \chi_{E_i} + b_i \mathcal{H}^1 \llcorner_{\partial E_i} = 0.$$

This implies that for all test functions $\varphi \in C_c^\infty(\mathbb{R}^2)$, it must also hold that:

$$\int_{\mathbb{R}^2} \varphi \left(\sum_{i=1}^{N} a_i \chi_{E_i} + b_i \mathcal{H}^1 \llcorner_{\partial E_i} \right) dx = 0.$$

Now choose $\operatorname{supp}(\varphi) = \operatorname{int}(E_N \setminus E_{N-1}) = \operatorname{int}(B(0,R_N) \setminus B(0,R_{N-1}))$ and we derive:

$$\int_{\mathbb{R}^2} \varphi \left(\sum_{i=1}^{N} a_i \chi_{E_i} + b_i \mathcal{H}^1 \llcorner_{\partial E_i} \right) dx = \int_{\mathbb{R}^2} \varphi (a_N \chi_{E_N}) dx = 0,$$

which can only be true if $a_N = 0$. Thus, iteratively applying this argument, we can show that $a_1 = \ldots = a_N = 0$. Moreover, since for $i, j \in \{1, \ldots, N\}$, we have $\operatorname{supp}(\mathcal{H}^1 \llcorner_{\partial E_i}) \cap \operatorname{supp}(\mathcal{H}^1 \llcorner_{\partial E_j}) = \emptyset$, it follows that $b_1 = \ldots = b_N = 0$.
The injectivity of Γ_E consequently implies the surjectivity of Γ_E^* since otherwise $(\operatorname{Im}(\Gamma_E^*))^\perp \neq \emptyset$. Therefore, a $0 \neq x \in \mathbb{R}^{2N}$ exists such that for all $y \in L^2(\mathbb{R}^2)$, it holds that

$$\langle \Gamma_E^*(y), x \rangle_{\mathbb{R}^{2N}} = 0 \iff \langle y, \Gamma_E(x) \rangle_{L^2(\mathbb{R}^2)} = 0.$$

Therefore, $\Gamma_E(x) = 0$, which means $x \in \operatorname{Ker}(\Gamma_E)$, contradicting our assumption.

Consequently, the matrix $\Gamma_E^* \Gamma_E$ is invertible, and we can demonstrate the existence of an unique Moore-Penrose inverse for Γ_E^*, denoted by $(\Gamma_E^*)^+ = \Gamma_E(\Gamma_E^* \Gamma_E)^{-1}$. Next, we will verify that the properties for such an inverse, as defined in [4, p. 40ff.], are satisfied by $(\Gamma_E^*)^+$:

(i)

$$\Gamma_E^* (\Gamma_E^*)^+ \Gamma_E^* = \Gamma_E^* (\Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) \Gamma_E^* = (\Gamma_E^* \Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) \Gamma_E^* = I \, \Gamma_E^* = \Gamma_E^*$$

(ii)

$$\begin{aligned}
(\Gamma_E^*)^+ \Gamma_E^* (\Gamma_E^*)^+ &= (\Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) \Gamma_E^* (\Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) \\
&= (\Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) (\Gamma_E^* \Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) \\
&= (\Gamma_E (\Gamma_E^* \Gamma_E)^{-1}) = (\Gamma_E^*)^+
\end{aligned}$$

(iii)

$$
\begin{aligned}
(\Gamma_E^*(\Gamma_E^*)^+)^* &= (\Gamma_E^*(\Gamma_E(\Gamma_E^*\Gamma_E)^{-1}))^* \\
&= (\Gamma_E^*\Gamma_E(\Gamma_E^*\Gamma_E)^{-1})^* \\
&= (I)^* = I \\
&= \Gamma_E^*\Gamma_E(\Gamma_E^*\Gamma_E)^{-1} = \Gamma_E^*(\Gamma_E^*)^+
\end{aligned}
$$

(iv)

$$
\begin{aligned}
((\Gamma_E^*)^+\Gamma_E^*)^* &= ((\Gamma_E(\Gamma_E^*\Gamma_E)^{-1})\Gamma_E^*)^* \\
&= (\Gamma_E^*)^*((\Gamma_E^*\Gamma_E)^{-1})^*\Gamma_E^* \\
&\overset{*}{=} \Gamma_E((\Gamma_E^*\Gamma_E)^{-1})\Gamma_E^* \\
&= (\Gamma_E^*)^+\Gamma_E^*
\end{aligned}
$$

where $(*)$ indicates that since $\Gamma_E^*\Gamma_E$ is self-adjoint, its inverse also shares this property. Thus, we obtain the desired p_v in Equation (48) via:

$$
p_v = (\Gamma_E^*)^+ \begin{pmatrix} (\mathrm{sign}(a_i)P(E_i))_{1\leq i\leq N} \\ (\mathrm{sign}(a_i)2\pi)_{1\leq i\leq N} \end{pmatrix}.
$$

And by applying Φ^* to both sides, we obtain the desired result. $\square$

With this proposition, we have shown that $(a,b)\in\mathbb{R}^{2N}$ exist

$$
\left(\text{specifically: } \begin{pmatrix} a \\ b \end{pmatrix} = (\Gamma_E^*\Gamma_E)^{-1} \begin{pmatrix} (\mathrm{sign}(a_i)P(E_i))_{1\leq i\leq N} \\ (\mathrm{sign}(a_i)2\pi)_{1\leq i\leq N} \end{pmatrix} \right)
$$

such that:

$$
\begin{aligned}
\eta_v &= \Phi^*\left(\sum_{i=1}^N a_i\, h\star\chi_{E_i} + \sum_{i=1}^N b_i\, h\star(\mathcal{H}^1\llcorner_{\partial E_i})\right) \\
&= \Phi^*\left(\sum_{i=1}^N a_i\, h\star\chi_{E_i}\right) + \Phi^*\left(\sum_{i=1}^N b_i\, h\star(\mathcal{H}^1\llcorner_{\partial E_i})\right) \\
&= \sum_{i=1}^N a_i\,\Phi^*\left(h\star\chi_{E_i}\right) + \sum_{i=1}^N b_i\,\Phi^*\left(h\star(\mathcal{H}^1\llcorner_{\partial E_i})\right).
\end{aligned}
$$

This means such a pre-certificate η_v is composed of the functions $\Phi^*(h\star\chi_E)$ and $\Phi^*(h\star\mathcal{H}^1\llcorner_{\partial E})$. For a more intuitive understanding, one may refer to Figure 18 below to view the graphs of both functions for the case $E = B(0,1)$.

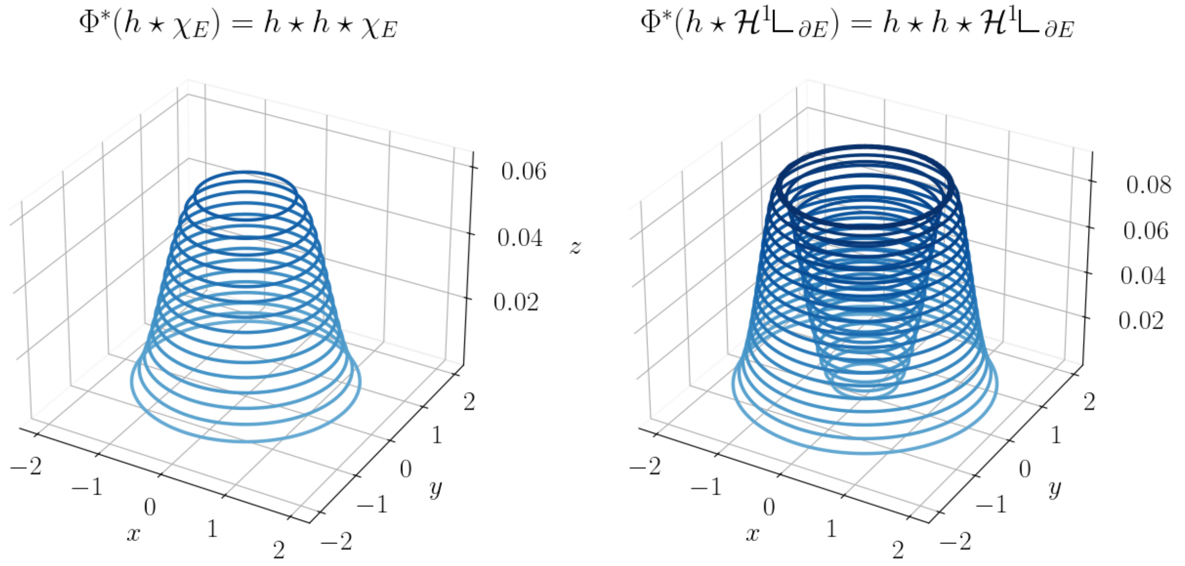

Figure 18: Graphical representations of $\Phi^*(h * \chi_{E_i})$ and $\Phi^*\left(h * (\mathcal{H}^1 \llcorner_{\partial E_i})\right)$ for $E = B(0,1)$ using a Gaussian kernel $h = \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)$ with a variance of $\sigma = 0.2$.

Validating η_v as an appropriate dual certificate

We have now established a method to determine a vanishing pre-certificate η_v in our specific case. However, to proceed to the minimal norm dual certificate as per proposition 7.7, it is essential to demonstrate that $\eta_v \in \partial \mathrm{TV}(0)$. We will elaborate on this in the following.

Referring to the set C given in (7), to establish this claim, it is necessary to identify a vector $z \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\|z\|_\infty \leq 1$ with $\operatorname{div} z = \eta_v$. Given that p_v is radial, the convolution with a Gaussian kernel, resulting in η_v, retains this radiality. Thus, our task is to find a suitable radial vector field z.

In polar coordinates, the divergence of such a z is described by:

$$\operatorname{div} z = \frac{1}{r}\left(\frac{\partial(r z_r)}{\partial r} + \frac{\partial z_\theta}{\partial \theta}\right) = \frac{1}{r}\left(\frac{\partial(r z_r)}{\partial r}\right).$$

This is due to the fact that, in a radial function, there are no changes along the angle θ.

From this, we deduce that $\eta_v = \operatorname{div} z$ is achieved if and only if for all $x \in \mathbb{R}^2$ with $||x||_{\mathbb{R}^2} =: r > 0$:

$$\begin{aligned} \eta_v(r) &= \frac{1}{r}\left(\frac{\partial(r z_r)}{\partial r}\right) \\ \iff r\eta_v(r) &= \frac{\partial}{\partial r}(r z_r)(r) \\ \iff z_r(r) &= \frac{1}{r}\int_0^r \eta_v(s) s \, ds. \end{aligned}$$

Therefore, by demonstrating that the mapping f_v, defined by

$$
\begin{aligned}
f_v : \mathbb{R}_+ &\to \mathbb{R} \\
r &\mapsto \frac{1}{r}\int_0^r \eta_v(s) s\, ds
\end{aligned} \tag{52}
$$

satisfies $||f_v||_\infty \le 1$, we can consequently show that $||z||_\infty \le 1$, and therefore $\eta_v \in \partial \mathrm{TV}(0)$.

Remark 7.10: It can be further demonstrated that the assumption of radial symmetry for z is not a restriction but a natural consequence stemming from the fact that η_v is radial. For clarity, we will denote the radial versions of η_v and z by $\tilde{\eta_v}$ and $\tilde{z}$, respectively:

If there exists a $z \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\eta_v = \mathrm{div} z$ and $\|z\| \le 1$, then we deduce that:

$$
\begin{aligned}
\tilde{\eta_v}(r) = \frac{1}{2\pi}\int_0^{2\pi} \eta_v(r,\theta) d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{div} z\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r}\left(\frac{\partial(r z_r)}{\partial r} + \frac{\partial z_\theta}{\partial \theta}\right) d\theta \\
&= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\int_0^{2\pi} \frac{z_r(r,\theta)}{2\pi} d\theta\right) + \underbrace{\frac{1}{r}\int_0^{2\pi} \frac{1}{2\pi}\frac{\partial z_\theta(r,\theta)}{\partial \theta} d\theta}_{=0 \to (*)} \\
&= -\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{z}_r) = \mathrm{div}\tilde{z}.
\end{aligned}
$$

Where $(*)$ follows from $z_\theta(r,0) = z_\theta(r,2\pi)$ for all $r > 0$.

Numerical verification of the non-degenerate source condition

As we bring our analysis to a close, our objective is to demonstrate that for the function $u = \sum_{i=1}^N a_i \chi_{B(0,R_i)}$, assuming the non-degenerate source condition holds, we can construct a minimal norm dual certificate. This enables the application of the central finding of our study (see Theorem 7.5), rendering u a solution to $\mathcal{P}_0(y_0)$.

We shall now address the three essential features of the non-degenerate source condition and tailor them to fit our specific problem scenario (see definition 7.4):

1. The condition $\mathrm{Im}(\Phi^*) \cap \partial \mathrm{TV}(u) \neq \emptyset$ holds true if the following two properties are satisfied:

   (i) $\eta_v \in \partial \mathrm{TV}(0)$, which, as previously demonstrated, occurs precisely when $\|f_v\|_\infty \le 1$.

   (ii) For all $i \in \{1, \ldots, N\}$ with $\mathrm{sign}(a_i) = 1$, the following is true:

$$
\int_{E_i} \eta_v = P(E_i) = 2\pi R_i.
$$

Hence, we have:

$$\begin{aligned}
P(E_i) &= \int_{E_i} \eta_v = \int_{B(0,R_i)} \eta_v(r,\theta) r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{R_i} \eta_v(r,\theta) r\,dr\,d\theta \\
&= 2\pi R_i \cdot \frac{1}{R_i}\int_0^{R_i} \eta_v(r) r\,dr \\
&= 2\pi R_i \cdot f_v(R_i).
\end{aligned}$$

In this case, it follows that $f_v(R_i) = 1$. Analogously, for $\operatorname{sign}(a_i) = -1$, it follows that $f_v(R_i) = -1$.

2. For all $i \in \{1, \dots, N\}$, E_i is a strictly stable solution to $\mathcal{PC}(\operatorname{sign}(a_i)\eta_v)$:
As described in Section 6.3, this is true if and only if for all $i \in \{1, \dots, N\}$:

$$-\operatorname{sign}(a_i) \sup_{x \in \partial E_i} \left[H_{E_i}^2(x) + \frac{\partial \eta_v}{\partial \nu_{E_i}}(x) \right] > 0.$$

As mentioned earlier, H_{E_i} is a constant equal to $1/R_i$, and since η_v is radial, $\frac{\partial \eta_v}{\partial \nu_{E_i}}$ is constant on ∂E_i. Thus, demonstrating that:

For all $i \in \{1, \dots, N\}$

$$-\operatorname{sign}(a_i) \left[\frac{1}{R_i^2} + \frac{\partial \eta_v}{\partial r}(R_i) \right] > 0 \tag{53}$$

is sufficient. To simplify the demonstration of this property, we compute the following:

$$\begin{aligned}
f_v'(r) &= \frac{d}{dr}\left(\frac{1}{r}\int_0^r \eta_v(s) s\,ds \right) \\
&= -\frac{1}{r^2}\int_0^r \eta_v(s) s\,ds + \frac{1}{r}\eta_v(r) r \\
&= -\frac{1}{r^2}\int_0^r \eta_v(s) s\,ds + \eta_v(r)
\end{aligned}$$

and

$$\begin{aligned}
f_v''(r) &= \frac{d}{dr}\left(-\frac{1}{r^2}\int_0^r \eta_v(s) s\,ds + \eta_v(r) \right) \\
&= \frac{d}{dr}\left(-\frac{1}{r^2}\int_0^r \eta_v(s) s\,ds \right) + \frac{\partial \eta_v}{\partial r}(r) \\
&= \frac{2}{r^3}\int_0^r \eta_v(s) s\,ds - \frac{1}{r}\eta_v(r) + \frac{\partial \eta_v}{\partial r}(r).
\end{aligned}$$

Under the assumption that $\eta_v = \Phi^* p_v$ solves (47), then for all $i \in \{1, \dots, N\}$, f_v''

simplifies at R_i to:

$$\begin{aligned}
f_v''(R_i) &= \frac{2}{R_i^3}\int_0^{R_i} \eta_v(s)s\,ds - \frac{1}{R_i}\eta_v(R_i) + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \frac{2}{R_i^3}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{R_i} \eta_v(s)s\,ds\,d\theta - \frac{1}{R_i}\eta_v(R_i) + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \frac{2}{R_i^3}\frac{1}{2\pi}\int_{E_i}\eta_v - \frac{1}{R_i}\eta_v(R_i) + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \frac{2}{R_i^3}\frac{1}{2\pi}\operatorname{sign}(a_i)P(E_i) - \frac{1}{R_i}\operatorname{sign}(a_i)H_{E_i} + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \frac{2}{R_i^3}\frac{1}{2\pi}\operatorname{sign}(a_i)2\pi R_i - \frac{1}{R_i}\operatorname{sign}(a_i)\frac{1}{R_i} + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \operatorname{sign}(a_i)\frac{2}{R_i^2} - \operatorname{sign}(a_i)\frac{1}{R_i^2} + \operatorname{sign}(a_i)\frac{\partial \eta_v}{\partial r}(R_i)\\
&= \operatorname{sign}(a_i)\left(\frac{1}{R_i^2} + \frac{\partial \eta_v}{\partial r}(R_i)\right).
\end{aligned}$$

Thus (53) can be directly verified by inspecting the graph of f_v.

3. For every simple set $E \subseteq \mathbb{R}^2$ such that $|E\Delta E_i| > 0$ for all $i \in \{1,\ldots,N\}$, we have $|\int_E \eta_0| < P(E)$:
   This condition is also satisfied if 1. (i) and (ii) hold.

Therefore, we need to verify the following conditions:

$$\begin{aligned}
&\forall R \in \mathbb{R}_+ \setminus \{R_1,\ldots,R_N\}, \quad |f_v(R)| < 1,\\
&\forall i \in \{1,\ldots,N\}, \quad f_v(R_i) = \operatorname{sign}(a_i) \text{ and } \operatorname{sign}(a_i)f_v''(R_i) < 0.
\end{aligned} \tag{54}$$

Results for the case $N = 1$

We begin by examining a simple scenario where $u = a_1\chi_{B(0,1)}$ and $a_1 > 0$. In Figure 19, we present the graph of the function f_v across various variances σ for the convolution with the Gaussian kernel. We investigate at which point in increasing σ, the previously derived properties, mandated by the non-degenerate source condition, cease to apply. Therefore, when considering the examples of the Gaussian filter in Section 2, we will demonstrate up to what degree of blurring in the image to be reconstructed the non-degenerate source condition holds, thus making theorem 7.3 applicable. It is observable that for $r > 2$ and $\sigma > 1$, the non-degenerate source condition is violated since $f_v(r) < -1$. Moreover, given that $\operatorname{sign}(a_1)f_v''(1) < 0$ even for $\sigma > 1$ (see figure 21), we infer the likely existence of a threshold $\sigma_0 > 0.75$, where the non-degenerate source condition holds for all $\sigma \leq \sigma_0$. Contrary to expectations, $f_v''(1)$ decreases for increasing variance, even though $\eta_v \notin \partial\mathrm{TV}(u)$ has been established well before this point.

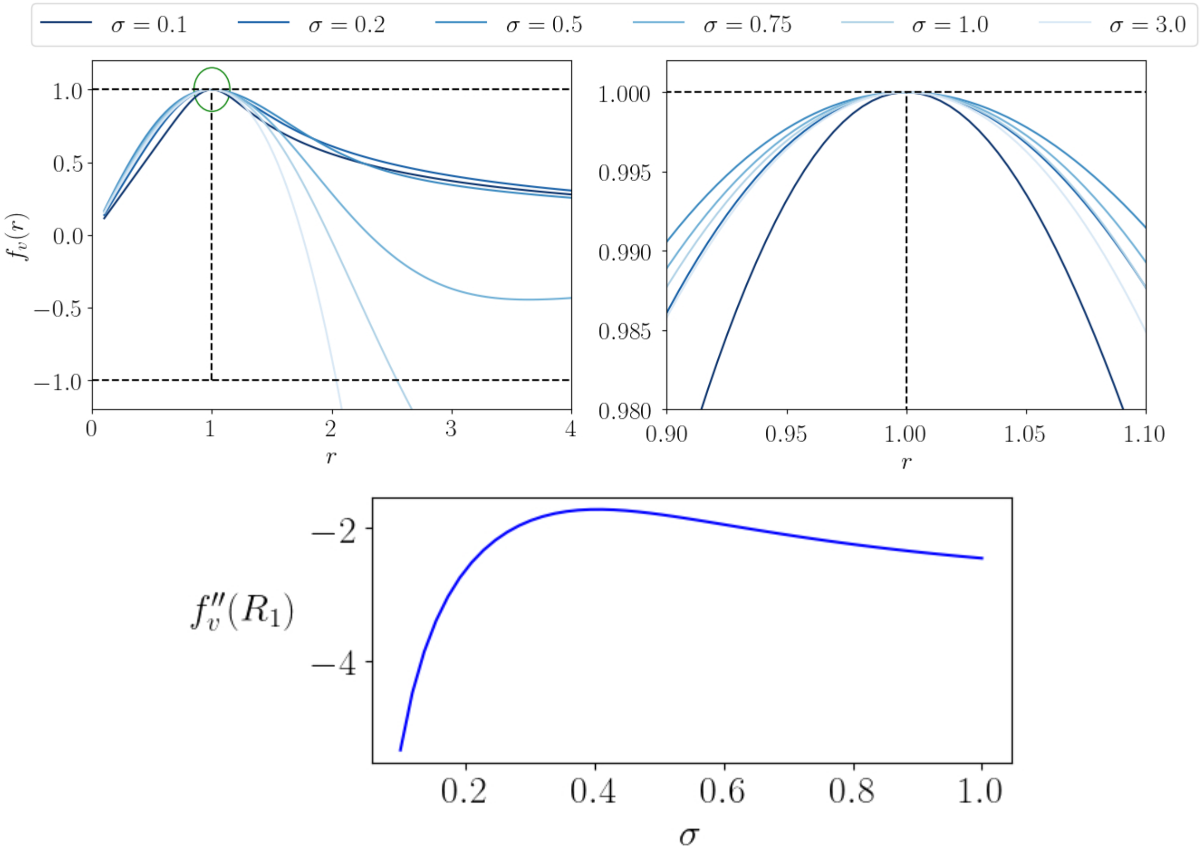

Figure 19: Graph of f_v, for $N=1$, $R_1=1$, and $\text{sign}(a_1)=1$ (top left: overall view, top right: detailed view near $r=1$) and $\sigma \mapsto f_v''(R_1)$ bottom.Graph of f_v with $N=1$, $R_1=1$, and $\text{sign}(a_1)=1$ (Top left: Global view, Top right: Close-up near $r=1$), and the plot of $\sigma \mapsto f_v''(R_1)$ at the bottom.

Results for the case $N \geq 2$

Expanding our analysis, we consider $u_0 = a_1\chi_{B(0,R_1)} + a_2\chi_{B(0,R_2)}$, focusing on scenarios where R_1 approaches R_2, with a fixed Gaussian kernel variance of 0.2. figure 20 reveals that when $\text{sign}(a_1) \neq \text{sign}(a_2)$, a certain minimum distance (in this case, ≥ 0.5) must be maintained between R_1 and R_2 to ensure $|f_v(r)| \leq 1$ for all $r>0$ (note that this distance increases with higher variance). The curvature of f_v at R_1 and R_2 ($\text{sign}(a_i)f_v''(R_i)<0$) indicates that u_0 satisfies the non-degenerate source condition under these circumstances.

In contrast, for $\text{sign}(a_1) = \text{sign}(a_2)$ (see Figure 21), a different behavior emerges. Here, R_1 and R_2 can be arbitrarily close while still maintaining $|f_v(r)|<1$. However, it is crucial to consider that as $R_2 \to R_1$, f_v increasingly approaches a saddle point within the interval $[R_1, R_2]$, leading to $\text{sign}(a_i)f_v''(R_1) = 0$. This convergence implies that u_0 no longer fulfills the non-degenerate source condition.

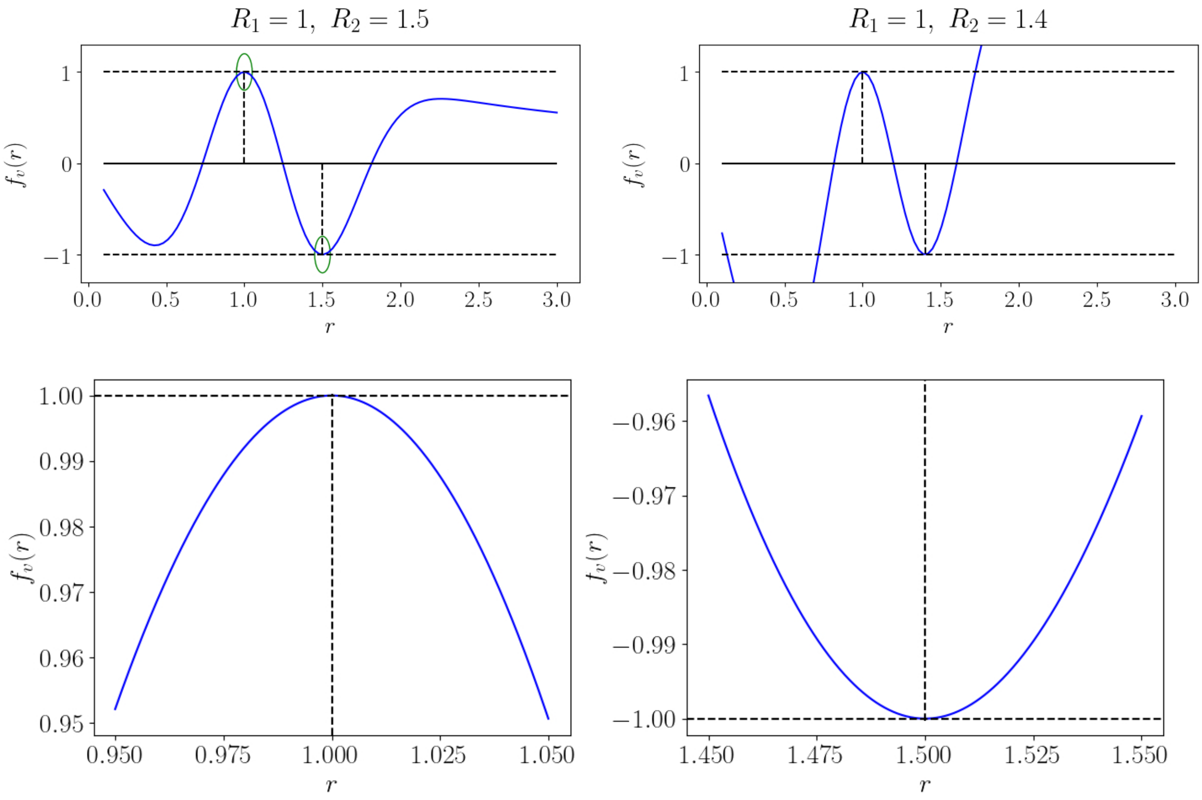

Figure 20: Graphs of f_v for variance $\sigma = 0.2$, $\text{sign}(a_1) = -\text{sign}(a_2)$, and $R_1 = 1$, $R_2 = 1.5$ (top left), $R_2 = 1.4$ (top right). The lower two graphs provide a zoomed-in view of the extreme points R_1 and R_2 from the top left graph.

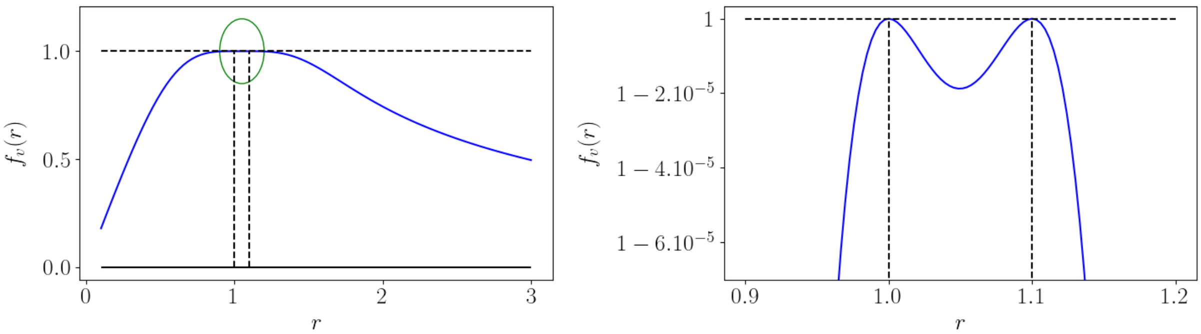

Figure 21: Graph of f_v for $\sigma = 0.2$, $N = 2$, $\text{sign}(a_1) = \text{sign}(a_2)$, $R_1 = 1$, and $R_2 = 1.1$ (Left: Overall view, Right: Zoomed-in around $r = 1$).

7.4 Conclusion

In this work, we have provided an overview of the types of disturbances that can occur during the acquisition and processing of images and how these images can be qualitatively improved through post-processing algorithms. In particular, we have delved into one algorithm, the TV regularization, and illustrated that this method is best suited for reconstructing piecewise constant images. Subsequently, we extended our discussion to the findings presented in [9]. In this part of our work, we provided a detailed explanation of the principal elements crucial for this analysis. Our focus was on the reconstruction of piecewise constant images, specifically those composed of simple shapes, using TV regularization. The results from this analysis revealed that in the reconstruction of images affected by continuous (linear) distortions (blurr) and noise, assuming a non-degenerate source condition, the reconstructed image retains the original image's structure. This includes preserving the same number of shapes, which can also be described as continuous deformations of the initial forms. Finally, we illustrated the conditions under which the non-degenerate source condition is applicable in a noise-free scenario (deconvolution case) for a radial image.

References

[1] Ambrosio, L., Fusco, N., and Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. Oxford University Press, Oxford, New York.

[2] Ambrosio, L., Caselles, V., Masnou, S., and Morel, J.-M. (2001). Connected components of sets of finite perimeter and applications to image processing. *Journal of the European Mathematical Society*, 3(1):39–92.

[3] Ambrosio L. (2010). *Corso introduttivo alla teoria geometrica della misura e alle superfici minime*. Scuola Normale Superiore.

[4] Ben-Israel, A., and Greville, T. N. E. (2003). *Generalized Inverses: Theory and Applications*. Second Edition. Springer-Verlag, New York. Editors-in-Chief: Jonathan Borwein, Peter Borwein.

[5] Borwein, J. M. and Zhu, Q. J. (2004). *Techniques of Variational Analysis*. Springer, Berlin Heidelberg New York.

[6] Chambolle A., V. Duval, G. Peyré, and C. Poon (2016). *Geometric properties of solutions to the total variation denoising problem*. Inverse Problems, 33(1):015002.

[7] Condat, L. (2017). *Discrete Total Variation: New Definition and Minimization*. SIAM J. Imaging Sciences, 10(3):1258–1290.

[8] Dambrine, M. and Lamboley, J. (2019). *Stability in shape optimization with second variation*. Journal of Differential Equations, 267(5):3009–3045.

[9] De Castro, Y., Duval, V., and Petit, R. (2023). *Exact recovery of the support of piecewise constant images via total variation regularization*.

[10] Delfour, M. C. and Zolesio, J.-P. (2011). *Shapes and Geometries: Metrics, Analysis, Differential Calculus, and Optimization, Second Edition*. SIAM.

[11] Duval, V. and G. Peyré (2015). *Exact Support Recovery for Sparse Spikes Deconvolution*. Foundations of Computational Mathematics, 15(5):1315–1355.

[12] Haddad, A. (2007). Texture Separation BV–G and BV–L1 Models. *Multiscale Modeling & Simulation*, 6(1):273–286. Publisher: Society for Industrial and Applied Mathematics.

[13] Henrot, A. and Pierre, M. (2018). *Shape Variation and Optimization: A Geometrical Analysis*. Number 28 in Tracts in Mathematics. European Mathematical Society.

[14] Iglesias, J. A., Mercier, G., and Scherzer, O. (2018). *A note on convergence of solutions of total variation regularized linear inverse problems*. Inverse Problems, 34(5):055011.

[15] Jost, J., & Li-Jost, X. (1998). *Calculus of Variations*. Cambridge University Press. ISBN 0-521-64203-5.

[16] Maggi, F. (2012). *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge.

[17] The MathWorks, Inc. *Wavelet Toolbox version: R2023b*. Available at: https://www.mathworks.com. Accessed: December 27, 2023.

[18] Ovchinnikov, S. (2018). *Functional Analysis: An Introductory Course*. Universitext. Springer International Publishing. ISBN 978-3-319-91511-1. https://doi.org/10.1007/978-3-319-91512-8.

[19] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press.

[20] Rudin, L. I., Osher, S., and Fatemi, E. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1):259–268, 1992.

[21] University of Southern California Signal and Image Processing Institute. *USC-SIPI Image Database*. Available at: https://sipi.usc.edu/database/database.php. Accessed on January 10, 2024.

A Appendix

Proof of proposition 6.7

Propositon 6.7
Let $\eta_0 \in \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$. For every $\epsilon > 0$ there exists $r > 0$ $\forall \eta \in \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$ with $\|\eta - \eta_0\|_{L^2(\mathbb{R}^2)} + \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \leq r$, the following is true: each non-empty solution F

of $\mathcal{PC}(\eta)$ can be characterized as a C^2-normal deformation of size at most ϵ of a non-empty solution E of $\mathcal{PC}(\eta_0)$, that is, using the notation of Proposition 3.6, $F = E_\varphi$ with $\|\varphi\|_{C^2(\partial E)} \leq \epsilon$.

Proof. By Contradiction: Suppose there exist sequences $(\eta_n)_{n\in\mathbb{N}\backslash\{0\}}$ and $(F_n)_{n\in\mathbb{N}\backslash\{0\}}$ such that:

- $\forall n \in \mathbb{N}\backslash\{0\}, \eta_n \in \mathrm{TV}(0) \cap C^1(\mathbb{R}^2)$.
- $\|\eta_n - \eta_0\|_{C^1(\mathbb{R}^2)} \to 0$ and $\|\eta_n - \eta_0\|_{L^2(\mathbb{R}^2)} \to 0$.
- For each $n \in \mathbb{N}\backslash\{0\}$, $F_n \neq \emptyset$ is a solution of $\mathcal{PC}(\eta_n)$ and for all C^2 deformations φ, there exists ϵ with $\|\varphi\|_{C^2(\mathbb{R}^2)} \leq \epsilon$ such that $F_n \neq E_\varphi$ where E is a solution of $\mathcal{PC}(\eta_0)$.

Thus, $P(F_n) = \int_{F_n} \eta_n$ holds, allowing the application of lemma 6.1(3.), which yields $R > 0$ with $F_n \subseteq B(0,R)$ for all $n \in \mathbb{N}\backslash\{0\}$. Consequently, due to the minimality of F_n and $\eta_n \in \partial\mathrm{TV}(u)$, for all $C \subseteq \mathbb{R}^2$ with $F_n \triangle C \subset\subset B(0,r)$ and $r < R$:

$$\begin{aligned}
P(F_n, B(0,r)) - \int_{F_n} \eta_n &\leq P(C, B(0,r)) - \int_C \eta_n \\
\iff P(F_n, B(0,r)) &\leq P(C, B(0,r)) - \int_C \eta_n + \int_{F_n} \eta_n \\
&= P(C, B(0,r)) + \int_{\mathbb{R}^2} \eta_n \,(\chi_{F_n} - \chi_C) \\
&\leq P(C, B(0,r)) + \|\eta_n\|_{C^0(\mathbb{R}^2)} \int_{\mathbb{R}^2} (\chi_{F_n} - \chi_C) \\
&= P(C, B(0,r)) + \|\eta_n\|_{C^0(\mathbb{R}^2)} |C \triangle F_n|.
\end{aligned}$$

Define $\Lambda := \sup\left\{\|\eta_n\|_{C^0(\mathbb{R}^2)} \mid n \in \mathbb{N}\backslash\{0\}\right\}$, showing that for all $n \in \mathbb{N}\backslash\{0\}$, F_n is a (Λ, r)-perimeter minimizer according to definition 3.27. By choosing $0 < r < R$ such that $\Lambda r < 1$, we apply [16, prop. 21.14] to obtain a (Λ, r)-perimeter minimizer E satisfying the stated conditions:

- $\mathcal{H}(\partial F_n, \partial E) \to 0$ (see definition 3.12)
- $|D\chi_{F_n}| \xrightarrow{*} |D\chi_E|$.

The latter implies $\int_{\mathbb{R}^2} (\chi_E - \chi_{F_n}) \to 0$, hence $|E \triangle F_n| \to 0$. Given that $\inf_{n\in\mathbb{N}\backslash\{0\}} |F_n| > 0$, lemma 6.1 confirms E as a non-empty solution of $\mathcal{PC}(n_0)$.

Moreover, the convergence of ∂F_n to ∂E ensures an open neighborhood around ∂E and an $n_0 \in \mathbb{N}\backslash\{0\}$, such that for any $n \geq n_0$, ∂F_n lies within this neighborhood. Specifically, we have:

$$\forall r > 0\, \exists n_0 \in \mathbb{N}\backslash\{0\} : \forall n \geq n_0 :\ \partial F_n \subset \bigcup_{x\in\partial E} C(x, r, \nu_E(x)),$$

where $C(x, r, \nu_E(x))$ is defined as in (1).

Utilizing [3, 4.7.4], similar reasoning as in the proof of [16, Thm. 26.6], and [16, Thm. 26.3], for every $x \in \partial E$ there exists $r > 0$, $n_0 \in \mathbb{N}\backslash\{0\}$, $u_x \in C^{1,1}([-r,r])$ and a uniformly bounded sequence $(u_{x,n})_{n\geq n_0}$. These functions satisfy that within $C(x, r, \nu_E(x))$, the set E is the hypograph of u_x, and for all $n \geq n_0$, the set F_n is the hypograph of $u_{x,n}$ (see figure 11), with $\|u_{x,n} - u_x\|_{C^1([-r,r])} \to 0$.

Our next objective is to demonstrate that the F_n converge to E in C^3 according to definition (3.3). To achieve this, we need to verify condition (iii) of 3.3, namely that:

$$\sup_{x\in\partial E} \|u_{x,n} - u_x\|_{C^3([-r,r])} \to 0.$$

Lemma 6.6 informs us that for all $x \in \partial E$, u_x and $u_{x,n}$ (for $n \geq n_0$) satisfy:

$$\begin{aligned} \frac{u_x''(z)}{(1+u_x'(z)^2)^{3/2}} &= H(z, u_x(z)), \quad \text{with} \quad H(z,t) = \eta_0(x + R_{\nu_E(x)}(z,t)), \\ \frac{u_{x,n}''(z)}{(1+u_{x,n}'(z)^2)^{3/2}} &= H_n(z, u_{x,n}(z)), \quad \text{with} \quad H_n(z,t) = \eta_n(x + R_{\nu_E(x)}(z,t)). \end{aligned} \tag{55}$$

Thus, both u_x and $u_{x,n}$ are in $C^3([-r,r])$. The remaining steps to show are:

(1.) $\sup_{x\in\partial E} \|u''_{x,n} - u''_x\|_{C^0([-r,r])} \to 0$:
For a given $x \in \partial E$ and $z \in (-r,r)$, we derive from (55):

$$\begin{aligned} |u''_{x,n}(z) - u''_x(z)| &= \Big|H_n(z,u_{x,n}(z))(1+u'_{x,n}(z)^2)^{3/2} - H(z,u_x(z))(1+u'_x(z)^2)^{3/2}\Big| \\ &\leq \Big|H_n(z,u_{x,n}(z))(1+u'_{x,n}(z)^2)^{3/2} - H(z,u_{x,n}(z))(1+u'_{x,n}(z)^2)^{3/2}\Big| \\ &\quad + \Big|H(z,u_{x,n}(z))(1+u'_{x,n}(z)^2)^{3/2} - H(z,u_x(z))(1+u'_{x,n}(z)^2)^{3/2}\Big| \\ &\quad + \Big|H(z,u_x(z))(1+u'_{x,n}(z)^2)^{3/2} - H(z,u_x(z))(1+u'_x(z)^2)^{3/2}\Big| \\ &\leq \Big|(1+u'_{x,n}(z)^2)^{3/2}\Big|\Big|H_n(z,u_{x,n}(z)) - H(z,u_{x,n}(z))\Big| \\ &\quad + \Big|(1+u'_{x,n}(z)^2)^{3/2}\Big|\Big|H(z,u_{x,n}(z)) - H(z,u_x(z))\Big| \\ &\quad + \Big|H(z,u_x(z))\Big|\Big|(1+u'_{x,n}(z)^2)^{3/2} - (1+u'_x(z)^2)^{3/2}\Big| \\ &\leq \underbrace{\Big|(1+u'_{x,n}(z)^2)^{3/2}\Big|}_{<\infty}\left(\underbrace{\Big\|H_n - H\Big\|_{C^0(R^2)}}_{\to 0} + \underbrace{\Big|H(z,u_{x,n}(z)) - H(z,u_x(z))\Big|}_{\to 0}\right) \\ &\quad + \underbrace{\Big\|H\Big\|_{C^0(R^2)}}_{<\infty}\underbrace{\Big|(1+u'_{x,n}(z)^2)^{3/2} - (1+u'_x(z)^2)^{3/2}\Big|}_{\to 0} \\ &\longrightarrow 0. \end{aligned}$$

Furthermore, since ∂E is compact, the supremum is attained, and thus the assertion follows.

(2.) $\sup_{x\in\partial E} \|u^{(3)}_{x,n} - u^{(3)}_x\|_{C^0([-r,r])} \to 0$:
Let's first determine $u^{(3)}_{x,n}$ and $u^{(3)}_x$ for an arbitrary $x \in \partial E$ and $z \in (-r,r)$. It follows from (55):

$$
\begin{aligned}
u_x^{(3)}(z) = \frac{d}{dz}u_x''(z) &= \frac{d}{dz}\Big(H(z,u_x(z))(1+u_x'(z)^2)^{3/2}\Big)\\
&= \frac{d}{dz}\left(H(z,u_x(z))\right)(1+u_x'(z)^2)^{3/2} + \frac{d}{dz}\Big((1+u_x'(z)^2)^{3/2}\Big)H(z,u_x(z))\\
&= \Big(\partial_1 H(z,u_x(z)) + \partial_2 H(z,u_x(z))u_x'(z)\Big)(1+u_x'(z)^2)^{3/2}\\
&\quad + \Big(\frac{3}{2}(1+u_x'(z)^2)^{1/2}2u_x'(z)u_x''(z)\Big)H(z,u_x(z))\\
&= \Big(\partial_1 H(z,u_x(z)) + \partial_2 H(z,u_x(z))u_x'(z)\Big)(1+u_x'(z)^2)^{3/2}\\
&\quad + 3H(z,u_x(z))u_x'(z)u_x''(z)(1+u_x'(z)^2)^{1/2}.
\end{aligned}
$$

Similarly, we have:

$$
\begin{aligned}
u_{x,n}^{(3)}(z) = &\Big(\partial_1 H_n(z,u_{x,n}(z)) + \partial_2 H(z,u_{x,n}(z))u_{x,n}'(z)\Big)(1+u_{x,n}'(z)^2)^{3/2}\\
&+ 3H_n(z,u_x(z))u_{x,n}'(z)u_{x,n}''(z)(1+u_{x,n}'(z)^2)^{1/2}.
\end{aligned}
$$

Now, we can make the following estimations:

(i)

$$
\begin{aligned}
&|\partial_1 H(z,u_x(z))(1+u_x'(z)^2)^{3/2} - \partial_1 H_n(z,u_{x,n}(z))(1+u_{x,n}'(z)^2)^{3/2}|\\
&\le |\partial_1 H(z,u_x(z))(1+u_x'(z)^2)^{3/2} - \partial_1 H(z,u_x(z))(1+u_{x,n}'(z)^2)^{3/2}|\\
&\quad + |\partial_1 H(z,u_x(z))(1+u_{x,n}'(z)^2)^{3/2} - \partial_1 H_n(z,u_x(z))(1+u_{x,n}'(z)^2)^{3/2}|\\
&\quad + |\partial_1 H_n(z,u_x(z))(1+u_{x,n}'(z)^2)^{3/2} - \partial_1 H_n(z,u_{x,n}(z))(1+u_{x,n}'(z)^2)^{3/2}|\\
&\le \underbrace{\|\partial_1 H\|_{C^0(\mathbb{R}^2)}}_{<\infty}\underbrace{|(1+u_x'(z)^2)^{3/2} - (1+u_{x,n}'(z)^2)^{3/2}|}_{\to 0}\\
&\quad + \left(\underbrace{\|\partial_1 H - \partial_1 H_n\|_{C^0(\mathbb{R}^2)}}_{\to 0} + \underbrace{|\partial_1 H_n(z,u_x(z)) - \partial_1 H_n(z,u_{x,n}(z))|}_{\to 0}\right)\underbrace{|(1+u_{x,n}'(z)^2)^{3/2}|}_{<\infty}\\
&\longrightarrow 0,
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&|\partial_2 H(z,u_x(z))u_x'(z)(1+u_x'(z)^2)^{3/2} - \partial_2 H(z,u_{x,n}(z))u_{x,n}'(z)(1+u_{x,n}'(z)^2)^{3/2}|\\
&\le \underbrace{\|\partial_2 H\|_{C^0(\mathbb{R}^2)}}_{<\infty}\underbrace{|u_x'(z)(1+u_x'(z)^2)^{3/2} - u_{x,n}'(z)(1+u_{x,n}'(z)^2)^{3/2}|}_{\to 0}\\
&\quad + \left(\underbrace{\|\partial_2 H - \partial_2 H_n\|_{C^0(\mathbb{R}^2)}}_{\to 0} + \underbrace{|\partial_2 H_n(z,u_x(z)) - \partial_2 H_n(z,u_{x,n}(z))|}_{\to 0}\right)\underbrace{|u_{x,n}'(z)(1+u_{x,n}'(z)^2)^{3/2}|}_{<\infty}\\
&\longrightarrow 0,
\end{aligned}
$$

(iii)

$$\begin{aligned}
&|3H(z,u_x(z))u_x'(z)u_x''(z)(1+u_x'(z)^2)^{1/2} - 3H_n(z,u_x(z))u_{x,n}'(z)u_{x,n}''(z)(1+u_{x,n}'(z)^2)^{1/2}| \\
&\leq \underbrace{3\|H\|_{C^0(\mathbb{R}^2)}}_{<\infty}\underbrace{|u_x'(z)u_x''(z)(1+u_x'(z)^2)^{1/2} - u_{x,n}'(z)u_{x,n}''(z)(1+u_{x,n}'(z)^2)^{1/2}|}_{\to 0} \\
&\quad + \left(\underbrace{3\|H-H_n\|_{C^0(\mathbb{R}^2)}}_{\to 0} + \underbrace{3|H_n(z,u_x(z)) - H_n(z,u_{x,n}(z))|}_{\to 0}\right)\underbrace{|u_{x,n}'(z)u_{x,n}''(z)(1+u_{x,n}'(z)^2)^{1/2}|}_{<\infty} \\
&\longrightarrow 0.
\end{aligned}$$

Therefore, we obtain:

$$|u_x^{(3)}(z) - u_{x,n}^{(3)}(z)| \longrightarrow 0.$$

And, as before, this also holds for the supremum over ∂E.

Hence, we have demonstrated that $(F_n)_{n\in\mathbb{N}\setminus\{0\}}$ (by choosing an appropriate subsequence) converges to E in C^3 as per definition 3.3. Therefore, we can apply proposition 3.7 and obtain, for sufficiently large n, $\tilde{\varphi}_n \in C^2(\partial E)$ with $F_n = E_{\tilde{\varphi}_n}$. By choosing n large enough such that additionally $\|\tilde{\varphi}_n\| \leq \epsilon$ holds, we arrive at a contradiction to our initial assumption, that no such C^2-deformation exists.

□

Proofs of section 6.3.2

To demonstrate propositions 6.12 and 6.13, we will first establish two auxiliary lemmas. For a bounded set E, we henceforth define $f_\varphi = \mathrm{Id} + \xi_\varphi$ (where ξ_φ is as described in 3.5). As noted in proposition 3.6, f_φ is a C^1 diffeomorphism provided $\|\varphi\|_{C^1(\partial E)}$ is sufficiently small. The inverse of this function will be denoted as g_φ.
Additionally, we define the vectorfield $\tau = \nu^\perp$ and $\tau_\varphi = \nu_\varphi^\perp$, which are obtained by rotating ν and ν_φ by an angle of $\pi/2$, respectively, thus describing a tangential vector field to ∂E and ∂E_φ, correspondingly.

Lemma A.1: Assume E is a bounded set of class C^2. Consider any φ near 0 within $C^1(\partial E)$, and any ψ within $H^1(\partial E)$. It follows that:

$$j_E''(\varphi)(\psi,\psi) = j_{E_\varphi}''(0)(\xi_\varphi \circ g_\varphi \cdot \nu_\varphi, \xi_\varphi \circ g_\varphi \cdot \nu_\varphi) + j_{E_\varphi}'(0)(Z_{\varphi,\psi})$$

where ν_φ denotes the unit outward normal to E_φ, and

$$Z_{\varphi,\psi} = B_\varphi((\xi_\varphi \circ g_\varphi)_{\tau_\varphi}, (\xi_\varphi \circ g_\varphi)_{\tau_\varphi}) - 2(\nabla_{\tau_\varphi}(\xi_\varphi \circ g_\varphi \cdot \nu_\varphi)) \cdot (\xi_\varphi \circ g_\varphi)_{\tau_\varphi}.$$

Furthermore, it holds that:

- The expression $\zeta_{\tau_\varphi} = \zeta - (\zeta \cdot \nu_\varphi)\nu_\varphi$ defines the tangential component of ζ on E_φ,
- The term $\nabla_{\tau_\varphi}\zeta = (D(\zeta)\tau_\varphi)\tau_\varphi$ denotes the gradient projected onto the tangent plane of E_φ.
- Lastly, B_φ denotes the second fundamental form of E_φ, a symmetric bilinear form given by:
  $B_\varphi(a,b) = \langle D(\nu_\varphi)a, b\rangle$, for vectors a, b that are tangential to ∂E_φ.

Proof. To establish the stated proposition, we introduce the operator J_E, defined as follows:

$$\begin{aligned}\mathcal{J}_E \colon C_b^1(\mathbb{R}^2, \mathbb{R}^2) &\to \mathbb{R}\\ \xi &\mapsto J((\mathrm{Id} + \xi)(E)).\end{aligned}$$

Let ν represent the exterior unit normal to the set E, and let B denote the second fundamental form associated with it. Furthermore, we define ξ_τ and $\nabla_\tau \xi$ to be the tangential component and the tangential gradient of ξ, respectively, in relation to E.

Invoking the structural theorem (refer to [13, Thm. 5.9.2] or [8, Thm. 2.1]), we deduce for any ξ that is sufficiently smooth:

$$\begin{aligned}\mathcal{J}_E'(0).\xi &= j_E'(0) \cdot (\xi_\tau \cdot \nu),\\ \mathcal{J}_E''(0).(\xi, \xi) &= j_E''(0) \cdot (\xi_\tau \cdot \nu, \xi_\tau \cdot \nu) + j_E'(0) \cdot (Z_\xi),\end{aligned}$$

where

$$Z_\xi := B(\xi_\tau, \xi_\tau) - 2\nabla_\tau(\xi \cdot \nu) \cdot \xi_\tau.$$

Moreover, for a set of class C^2, with $t \neq 0$ in $\mathbb{R}$, and for any vector fields ξ and ζ, and defining $F := (\mathrm{id} + \xi)$, we have:

$$\mathcal{J}_E(\xi + t\zeta) = J((\mathrm{id} + \xi + t\zeta)E) = J((\mathrm{Id} + t\zeta \circ (\mathrm{id} + \xi)^{-1})F) = \mathcal{J}_F(t\zeta \circ (\mathrm{id} + \xi)^{-1}),$$

Hence, by differentiation, we can demonstrate the following equality:

$$\begin{aligned}\mathcal{J}_E'(\xi)(\zeta) &= \frac{d}{dt}\left[J((\mathrm{Id} + \xi + t\zeta)E)\right]\Big|_{t=0}\\ &= \frac{d}{dt}\left[J((\mathrm{Id} + t\zeta \circ (\mathrm{id} + \xi)^{-1})F)\right]\Big|_{t=0} = \mathcal{J}_F'(0)(\zeta \circ (\mathrm{id} + \xi)^{-1}),\end{aligned}$$

and finally,

$$\begin{aligned}
\mathcal{J}_E''(\xi)(\zeta,\zeta) &= \frac{d}{dt}\left[\frac{d}{dt}\left(J((Id+\xi+t\zeta+s\zeta)F)\right)\right]\Big|_{s=t=0} \\
&= \frac{d}{dt}\left[\frac{d}{dt}\Big(J((Id+t\zeta\circ(Id+\xi)^{-1}+s\zeta\circ(Id+\xi)^{-1})F)\Big)\right]\Big|_{t=0} \\
&= \mathcal{J}_F''(0)(\zeta\circ(Id+\xi)^{-1},\zeta\circ(Id+\xi)^{-1}).
\end{aligned}$$

Now, selecting $\xi=\xi_\varphi=\nu\varphi$ and $\zeta=\xi_\psi=\psi\nu$, we arrive at:

$$j_E''(\varphi)(\psi,\psi)=\mathcal{J}_E''(\xi_\varphi)(\xi_\psi,\xi_\psi)=\mathcal{J}_{E_\varphi}''(0)(\xi_\varphi\circ g_\varphi,\xi_\psi\circ g_\varphi),$$

whereby, through the application of the structure theorem, we acquire the statement to be demonstrated. □

Lemma A.2: Let E be a bounded C^2 set. If $\|\varphi\|_{C^1(\partial E)}\to 0$ we have:

$$\begin{aligned}
&\text{(i)} && \|f_\varphi-\mathrm{Id}\|_{C^1(\partial E)}\to 0,\\
&\text{(ii)} && \|g_\varphi-\mathrm{Id}\|_{C^1(\partial E_\varphi)}\to 0,\\
&\text{(iii)} && \|\nu_\varphi\circ f_\varphi-\nu\|_{C^0(\partial E)}\to 0,\\
&\text{(iv)} && \|\mathrm{Jac}_\tau f_\varphi-1\|_{C^0(\partial E)}\to 0. \text{ where } \mathrm{Jac}_\tau f_\varphi=\|Df_\varphi\tau\|.
\end{aligned}$$

If $\|\varphi\|_{C^2(\partial E)}\to 0$ then we also have:

$$\begin{aligned}
&\text{(v)} && \|H_\varphi\circ f_\varphi-H\|_{C^0(\partial E)}\to 0,\\
&\text{(vi)} && \|B_\varphi\circ f_\varphi-B\|_{C^0(\partial E)}\to 0.
\end{aligned}$$

Moreover, the following holds:

$$\begin{aligned}
&\text{(a)} && \lim_{\|\varphi\|_{C^1(\partial E)}\to 0}\ \sup_{\psi\in L^2(\partial E)\setminus\{0\}}\frac{\|(\xi_\varphi\circ g_\varphi)_\tau\|_{L^2(\partial E_\varphi)}}{\|\psi\|_{L^2(\partial E)}}=0,\\
&\text{(b)} && \lim_{\|\varphi\|_{C^1(\partial E)}\to 0}\ \sup_{\psi\in H^1(\partial E)\setminus\{0\}}\frac{\|\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\|_{L^2(\partial E_\varphi)}-\|\nabla_\tau\psi\|_{L^2(\partial E)}}{\|\psi\|_{H^1(\partial E)}}=0,\\
&\text{(c)} && \lim_{\|\varphi\|_{C^2(\partial E)}\to 0}\ \sup_{\psi\in H^1(\partial E)\setminus\{0\}}\frac{\|Z_{\psi,\varphi}\|_{L^1(\partial E_\varphi)}}{\|\psi\|^2_{H^1(\partial E)}}=0.
\end{aligned}$$

Proof. (i) Invoking lemma 3.5, there exists a constant $C>0$ such that:

$$\begin{aligned}
\|f_\varphi-\mathrm{Id}\|_{C^1(\partial E)} &= \|(\mathrm{Id}+\xi_\varphi)-\mathrm{Id}\|_{C^1(\partial E)}\\
&= \|\xi_\varphi\|_{C^1(\partial E)}\\
&\le C\|\varphi\|_{C^1(\partial E)}\to 0.
\end{aligned}$$

(ii) It holds that:

$$\begin{aligned}
&\|g_\varphi - \mathrm{Id}\|_{C^1(\partial E)} \\
&= \|(\mathrm{Id} - f_\varphi)\circ g_\varphi\|_{C^1(\partial E)} \\
&= \max\Big(\|(\mathrm{Id} - f_\varphi)\circ g_\varphi\|_{C^0(\partial E)}, \|\nabla[(\mathrm{Id} - f_\varphi)\circ g_\varphi]\|_{C^0(\partial E)}\Big) \\
&\le \max\Big(\|\mathrm{Id} - f_\varphi\|_{C^0(\partial E)}\|g_\varphi\|_{C^0(\partial E)}, \|\nabla(\mathrm{Id} - f_\varphi)\|_{C^0(\partial E)}\|\nabla g_\varphi\|_{C^0(\partial E)}\Big) \\
&\ \|\mathrm{Id} - f_\varphi\|_{C^1(\partial E)} \underbrace{\|g_\varphi\|_{C^1(\partial E)}}_{<\infty\ (*)} \to 0
\end{aligned}$$

where $(*)$ follows from the continuity of $g_\varphi, \nabla g_\varphi$ and the compactness of ∂E.

(iii) Since every unit vector w can be represented as follows:

$$w = (w\cdot\nu)\nu + (w\cdot\nu^\perp)\nu^\perp = (w\cdot\nu)\nu \pm \sqrt{1-(w\cdot\nu)^2}\nu^\perp$$

we obtain for all $x\in\partial E$:

$$\begin{aligned}
&|(\nu_\phi\circ f_\phi - \nu)(x)| \\
&= |((\nu_\varphi\circ f_\varphi\cdot\nu)\nu)(x) - \nu(x) \pm (\sqrt{1-(\nu_\varphi\circ f_\varphi\cdot\nu)^2}\nu^\perp)(x)| \\
&\le |((\nu_\varphi\circ f_\varphi\cdot\nu - 1)\nu)(x)| + |(\sqrt{1-(\nu_\varphi\circ f_\varphi\cdot\nu)^2}\nu^\perp)(x)| \\
&= |(\nu_\varphi\circ f_\varphi\cdot\nu - 1)(x)| + (\sqrt{1-(\nu_\varphi\circ f_\varphi\cdot\nu)^2})(x) \\
&= |(\nu_\varphi\circ f_\varphi\cdot\nu - 1)(x)| + \left(\sqrt{(\nu_\varphi\circ f_\varphi\cdot\nu - 1)(\nu_\varphi\circ f_\varphi\cdot\nu + 1)}\right)(x) \\
&= |(\nu_\varphi\circ f_\varphi\cdot\nu - 1)(x)| + \left(\sqrt{(\nu_\varphi\circ f_\varphi\cdot\nu - 1)(\nu_\varphi\circ f_\varphi\cdot\nu - 1 + 2)}\right)(x) \\
&\le ||\nu_\varphi\circ f_\varphi\cdot\nu - 1||_{C^0(\partial E)} + \sqrt{||\nu_\varphi\circ f_\varphi\cdot\nu - 1||_{C^0(\partial E)}(||\nu_\varphi\circ f_\varphi\cdot\nu - 1||_{C^0(\partial E)} + 2)}, \\
&\to 0,
\end{aligned}$$

where $||\nu_\varphi\circ f_\varphi\cdot\nu - 1||_{C^0(\partial E)} \to 0$ is obtained via [8, Lem. 4.7].

(iv) / (v) / (vi): These three statements are obtained directly by applying [8, Lem. 4.7].

(a) To achieve the convergence described in (a), we first consider the parametrization of ∂E by $\rho : [0, \mathcal{H}^1(\partial E)] \to \partial E$, then we have:

$$\begin{aligned}
\left\|(\xi_\psi \circ g_\psi)_{\tau_\psi}\right\|^2_{L^2(\partial E_\varphi)} &= \int_{\partial E_\varphi} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi}\, d\mathcal{H}^1 \\
&= \int_{f_\varphi \circ \rho([0,\mathcal{H}^1(\partial E)])} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi}\, d\mathcal{H}^1 \\
&= \int_0^{\mathcal{H}^1(\partial E)} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \circ \rho(s) |D(f_\varphi \circ \rho(s))|\, ds \\
&= \int_0^{\mathcal{H}^1(\partial E)} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \circ \rho(s) |Df_\varphi \circ \rho(s)\rho'(s)|\, ds \\
&= \int_0^{\mathcal{H}^1(\partial E)} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \circ \rho(s) |Df_\varphi(\rho(s))\tau||\rho'(s)|\, ds \\
&= \int_0^{\mathcal{H}^1(\partial E)} (\xi_\varphi \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \circ \rho(s) \mathrm{Jac}_\tau f_\varphi \circ \rho(s) |\rho'(s)|\, ds \\
&= \int_{\partial E} ((\nu\psi) \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \mathrm{Jac}_\tau f_\varphi\, d\mathcal{H}^1 \\
&= \int_{\partial E} (\nu \circ g_\psi)^2_{\tau_\psi} \circ f_\varphi \mathrm{Jac}_\tau f_\varphi \psi^2\, d\mathcal{H}^1 \\
&\leq ||(\nu \circ g_\psi)_{\tau_\psi} \circ f_\varphi||^2_{C^0(\partial E)} ||\mathrm{Jac}_\tau f_\varphi||_{C^0(\partial E)} ||\psi||^2_{L^2(\partial E)} \\
&\leq ||\Big((\nu \circ g_\psi) - (\nu \circ g_\psi \cdot \nu_\varphi)\nu_\varphi\Big) \circ f_\varphi||^2_{C^0(\partial E)} ||\mathrm{Jac}_\tau f_\varphi||_{C^0(\partial E)} ||\psi||^2_{L^2(\partial E)} \\
&\leq \underbrace{||\nu - (\nu \cdot \nu_\varphi \circ f_\varphi)\nu_\varphi \circ f_\varphi||^2_{C^0(\partial E)}}_{\to 0\ (*)} \underbrace{||\mathrm{Jac}_\tau f_\varphi||_{C^0(\partial E)}}_{\to 1\ (iv)} ||\psi||^2_{L^2(\partial E)}
\end{aligned}$$

Where $(*)$ follows from the convergence of $\|\nu \cdot \nu_\varphi \circ f_\varphi\| \to 1$ according to [8, Lem. 4.7], (iii), and from the fact that the product of uniformly convergent functions is also uniformly convergent. Since the choice of ψ was arbitrary, we thus obtain (a).

(b) During the proof of statement (b), it has unfortunately come to light that there is an incompleteness in the argumentation of the underlying paper [9]. Nonetheless, we will assume for the subsequent proofs that this statement is correct and will present a counterexample to a claim made in this proof in a subsequent remark.

We first demonstrate:

$$
\begin{aligned}
&\nabla_{\tau_\varphi}(\ \xi_\varphi \circ g_\varphi \cdot \nu_\varphi) = \nabla_{\tau_\varphi}(\psi \circ g_\varphi \nu \circ g_\varphi \cdot \nu_\varphi)\\
&= (D(\psi \circ g_\varphi \nu \circ g_\varphi \cdot \nu_\varphi)\tau_\varphi)\tau_\varphi\\
&= [\nu \circ g_\varphi \cdot \nu_\varphi D(\psi \circ g_\varphi) + \psi \circ g_\varphi D(\nu \circ g_\varphi \cdot \nu_\varphi)\tau_\varphi]\tau_\varphi\\
&= [\nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi) + \psi \circ g_\varphi (D(\nu \circ g_\varphi)^T \nu_\varphi + D(\nu_\varphi)^T \nu \circ g_\varphi)^T \tau_\varphi]\tau_\varphi\\
&= [\nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi) + \psi \circ g_\varphi ((D(\nu) \circ g_\varphi D(g_\varphi))^T \nu_\varphi + D(\nu_\varphi)^T \nu \circ g_\varphi)^T \tau_\varphi]\tau_\varphi\\
&= [\nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi) + \psi \circ g_\varphi (\tau_\varphi \cdot ((D(\nu) \circ g_\varphi D(g_\varphi))^T \nu_\varphi + D(\nu_\varphi)^T \nu \circ g_\varphi))]\tau_\varphi\\
&= [\nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi)\tau_\varphi + \psi \circ g_\varphi ((D(\nu) \circ g_\varphi D(g_\varphi))^T \nu_\varphi \cdot \tau_\varphi + D(\nu_\varphi)^T \nu \circ g_\varphi \cdot \tau_\varphi)]\tau_\varphi\\
&= [\underbrace{\nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi)\tau_\varphi}_{a} + \underbrace{\psi \circ g_\varphi (\tau_\varphi^T (D(\nu) \circ g_\varphi D(g_\varphi))^T \nu_\varphi + \tau_\varphi^T D(\nu_\varphi)^T \nu \circ g_\varphi)}_{b}]\tau_\varphi
\end{aligned}
$$

Upon further transforming term a and using the fact that $Dg_\varphi \tau_\varphi$ is a multiple of τ, we get:

$$
\begin{aligned}
&= \nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi D(g_\varphi)\tau_\varphi\\
&= \nu \circ g_\varphi \cdot \nu_\varphi D(\psi) \circ g_\varphi |D(g_\varphi)\tau_\varphi|\tau\\
&= \nu \circ g_\varphi \cdot \nu_\varphi |D(g_\varphi)\tau_\varphi|(D(\psi) \circ g_\varphi \tau\tau) \cdot \tau\\
&= \nu \circ g_\varphi \cdot \nu_\varphi \nabla_\tau(\psi) \circ g_\varphi \cdot D(g_\varphi)\tau_\varphi\\
&= \underbrace{\nu \circ g_\varphi \cdot \nu_\varphi D(g_\varphi)\tau_\varphi}_{=: c_\varphi^2} \cdot \nabla_\tau \psi \circ g_\varphi.
\end{aligned}
$$

For the term b we encounter the incompleteness mentioned above. Ideally, for the following proof structure, we should define c_φ^1 as follows:

$$
\begin{aligned}
&\psi \circ g_\varphi ((D(\nu) \circ g_\varphi D(g_\varphi)\tau_\varphi)^T \nu_\varphi + (D(\nu_\varphi)\tau_\varphi)^T \nu \circ g_\varphi)\\
&= \psi \circ g_\varphi \underbrace{(\nu_\varphi \cdot (D(\nu) \circ g_\varphi D(g_\varphi)\tau_\varphi) + \nu \circ g_\varphi \cdot (D(\nu_\varphi)\tau_\varphi))}_{=: c_\varphi^1}.
\end{aligned}
\tag{56}
$$

In the original paper this term was specified as follows (in the further course of the proof we will use this term):

$$
c_\varphi^1 := \tau \circ g_\varphi \cdot \nu_\varphi (Dg_\varphi \tau_\varphi) \cdot \tau \circ g_\varphi + \tau_\varphi \cdot \nu \circ g_\varphi \tag{57}
$$

Hence, we obtain the overall representation:

$$
\nabla_{\tau_\varphi}(\ \xi_\varphi \circ g_\varphi \cdot \nu_\varphi) = \left[c_\varphi^1 \psi \circ g_\varphi + c_\varphi^2 \cdot \nabla_\tau \psi \circ g_\varphi\right] \tau_\varphi. \tag{58}
$$

Now, for all $x \in \partial E$, the following estimate can be made:

$$
\begin{aligned}
&|(\nabla_{\tau_\varphi}(\xi_\psi \circ g_\varphi \cdot \nu_\varphi) \circ f_\varphi \sqrt{\mathrm{Jac}_\tau f_\varphi} - \nabla_\tau \psi)(x)| \\
=&|([c_\varphi^1 \psi \circ g_\varphi + c_\varphi^2 \cdot \nabla_\tau \psi \circ g_\varphi]\tau_\varphi) \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \nabla_\tau \psi(x)| \\
=&|[c_\varphi^1 \circ f_\varphi(x)\psi \circ g_\varphi \circ f_\varphi(x) \\
&\quad + c_\varphi^2 \circ f_\varphi(x) \cdot \nabla_\tau \psi \circ g_\varphi \circ f_\varphi(x)]\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \nabla_\tau \psi(x)| \\
=&|[c_\varphi^1 \circ f_\varphi(x)\psi(x) + c_\varphi^2 \circ f_\varphi(x) \cdot \nabla_\tau \psi(x)]\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \nabla_\tau \psi(x)| \\
=&|[c_\varphi^1 \circ f_\varphi(x)\psi(x)\tau_\varphi \circ f_\varphi(x) + c_\varphi^2 \circ f_\varphi(x) \cdot \nabla_\tau \psi(x)\tau_\varphi \circ f_\varphi(x)]\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \nabla_\tau \psi(x)| \\
=&|c_\varphi^1 \circ f_\varphi(x)\psi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) \\
&+ c_\varphi^2 \circ f_\varphi(x) \cdot \nabla_\tau \psi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \nabla_\tau \psi(x)| \\
=&|c_\varphi^1 \circ f_\varphi(x)\psi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) \\
&+ \tau_\varphi \circ f_\varphi(x)(c_\varphi^2 \circ f_\varphi(x))^T \sqrt{\mathrm{Jac}_\tau f_\varphi}(x)\nabla_\tau \psi(x) - \nabla_\tau \psi(x)| \\
=&|c_\varphi^1 \circ f_\varphi(x)\psi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) \\
&+ \tau_\varphi \circ f_\varphi(x)(c_\varphi^2 \circ f_\varphi(x))^T \sqrt{\mathrm{Jac}_\tau f_\varphi}(x) \pm |\nabla_\tau \psi(x)|\tau(x) - \pm|\nabla_\tau \psi(x)|\tau(x)| \\
\leq&|c_\varphi^1 \circ f_\varphi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x)||\psi(x)| \\
&+ |\tau_\varphi \circ f_\varphi(x)(c_\varphi^2)^T \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x)\tau(x) - \tau(x)||\nabla_\tau \psi(x)| \\
=&|c_\varphi^1 \circ f_\varphi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x)||\psi(x)| \\
&+ |(c_\varphi^2 \circ f_\varphi(x) \cdot \tau(x))\tau_\varphi \circ f_\varphi(x) \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \tau(x)||\nabla_\tau \psi(x)| \\
\leq& c_\varphi(x)(|\psi(x)| + |\nabla_\tau \psi(x)|),
\end{aligned}
$$

where

$$
\begin{aligned}
c_\varphi(x) :=&|c_\varphi^1 \circ f_\varphi(x)\tau_\varphi \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x)| \\
&+ |(c_\varphi^2 \circ f_\varphi(x) \cdot \tau(x))\tau_\varphi \circ f_\varphi(x) \circ f_\varphi(x)\sqrt{\mathrm{Jac}_\tau f_\varphi}(x) - \tau(x)|.
\end{aligned}
$$

Next, we show that the following holds true:

$$
\lim_{\|\varphi\|_{C^1(\partial E)}} \|c_\varphi\|_{C^0 \partial E} \to 0. \tag{59}
$$

Defining $R_{\pi/2}$ as the rotation matrix by the angle $\pi/2$, and with (iii) for $\|\varphi\|_{C^1(\partial E)} \to 0$, we get:

$$
\begin{aligned}
\|\tau_\varphi \circ f_\varphi - \tau\|_{C^0 \partial E} &= \|R_{\pi/2}(\nu_\varphi) \circ f_\varphi - R_{\pi/2}(\nu)\|_{C^0 \partial E} \\
&\leq \|R_{\pi/2}\|\|\nu_\varphi \circ f_\varphi - \nu\|_{C^0 \partial E} \to 0.
\end{aligned}
$$

Therefore, we have:

$$\begin{aligned}\|c^1_\varphi \circ f_\varphi\|_{C^0\partial E} &= \|(\tau\circ g_\varphi \cdot \nu_\varphi (Dg_\varphi \tau_\varphi)\cdot \tau\circ g_\varphi + \tau_\varphi\cdot\nu\circ g_\varphi)\circ f_\varphi\|_{C^0\partial E} \\ &= \|\underbrace{\underbrace{\tau\cdot\underbrace{\nu_\varphi\circ f_\varphi}_{\to\nu}}_{\to 0}\underbrace{(\underbrace{Dg_\varphi}_{\to Id(ii)}\ \tau_\varphi\circ f_\varphi)\cdot\tau}_{<\infty}}_{\to 0} + \underbrace{\underbrace{\tau_\varphi\circ f_\varphi}_{\to\tau}\cdot\nu}_{\to 0}\|_{C^0\partial E}\to 0.\end{aligned}$$

Since the product of uniformly convergent functions is again uniformly convergent, we obtain the following convergence for the first term of c_φ:

$$\lim_{\|\varphi\|_{C^1(\partial E)}} \|\underbrace{c^1_\varphi\circ f_\varphi}_{\to 0}\underbrace{\tau_\varphi\circ f_\varphi}_{\to\tau}\underbrace{\sqrt{\mathrm{Jac}_\tau f_\varphi}}_{\to 1}\|_{C^0\partial E}\to 0.$$

For the second term, we obtain:

$$\begin{aligned}&\lim_{\|\varphi\|_{C^1(\partial E)}\to 0}\|c^2_\varphi\circ f_\varphi\cdot\tau\ \sqrt{\mathrm{Jac}_\tau f_\varphi}\tau_\varphi\circ f_\varphi-\tau\| \\ &\lim_{\|\varphi\|_{C^1(\partial E)}\to 0}\|(\nu\circ g_\varphi\cdot\nu_\varphi D(g_\varphi)\tau_\varphi)\circ f_\varphi\cdot\tau\ \sqrt{\mathrm{Jac}_\tau f_\varphi}\tau_\varphi\circ f_\varphi-\tau\| \\ &\lim_{\|\varphi\|_{C^1(\partial E)}\to 0}\|\underbrace{\underbrace{\nu\cdot\nu_\varphi\circ f_\varphi}_{\to 1}\underbrace{D(g_\varphi)}_{\to I}\underbrace{\tau_\varphi\circ f_\varphi\cdot\tau}_{\to 1}\ \underbrace{\sqrt{\mathrm{Jac}_\tau f_\varphi}}_{\to 1}\underbrace{\tau_\varphi\circ f_\varphi}_{\to\tau}}_{\to\tau}-\tau\|\to 0.\end{aligned}$$

Again, this is due to the product of uniformly convergent functions being uniformly convergent.

Thus, we have demonstrated the convergence of (59) and can finally make the following estimate:

$$
\begin{aligned}
&\|\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\|_{L^2(\partial E_\varphi)}-\|\nabla_\tau\psi\|_{L^2(\partial E)}\\
&=\left(\int_{f_\varphi(\partial E)}(\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi))^2 d\mathcal{H}^1\right)^{1/2}-\|\nabla_\tau\psi\|_{L^2(\partial E)}\\
&=\left(\int_{\partial E}(\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi))^2\circ f_\varphi \mathrm{Jac}_\tau f_\varphi d\mathcal{H}^1\right)^{1/2}-\|\nabla_\tau\psi\|_{L^2(\partial E)}\\
&=\left(\int_{\partial E}(\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\circ f_\varphi\sqrt{\mathrm{Jac}_\tau f_\varphi})^2 d\mathcal{H}^1\right)^{1/2}-\|\nabla_\tau\psi\|_{L^2(\partial E)}\\
&=\left\|\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\circ f_\varphi\sqrt{\mathrm{Jac}_\tau f_\varphi}\right\|_{L^2(\partial E)}-\|\nabla_\tau\psi\|_{L^2(\partial E)}\\
&\leq\left\|\nabla_\tau(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\circ f_\varphi\sqrt{\mathrm{Jac}_\tau f_\varphi}-\nabla_\tau\psi\right\|_{L^2(\partial E)}\\
&\leq\|c_\varphi(|\psi|+|\nabla_\tau\psi|)\|_{L^2(\partial E)}\\
&\leq\|c_\varphi\|_{C^0(\partial E)}\||\psi|+|\nabla_\tau\psi|\|_{L^2(\partial E)}\\
&=\|c_\varphi\|_{C^0(\partial E)}(\int_{\partial E}(|\psi|+|\nabla_\tau\psi|)^2\, dH^1)^{\frac{1}{2}}\\
&\leq\|c_\varphi\|_{C^0(\partial E)}\int_{\partial E}(2|\psi|^2+2|\nabla_\tau\psi|^2\, dH^1)^{\frac{1}{2}}\\
&=\|c_\varphi\|_{C^0(\partial E)}\sqrt{2}\,\|\psi\|_{H^1(\partial E)}\,.
\end{aligned}
$$

As a result, for $\|\varphi\|_{C^1(\partial E)}\to 0$, we achieve outcome (b).

(c) It follows that:

$$
\begin{aligned}
\frac{\|Z_{\varphi,\psi}\|_{L^1(\partial E_\varphi)}}{\|\psi\|^2_{H^1(\partial E)}}&=\frac{\|B_\varphi((\xi_\varphi\circ g_\varphi)_{\tau_\varphi},(\xi_\varphi\circ g_\varphi)_{\tau_\varphi})-2(\nabla_{\tau_\varphi}(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi))\cdot(\xi_\varphi\circ g_\varphi)_{\tau_\varphi}\|_{L^1(\partial E_\varphi)}}{\|\psi\|^2_{H^1(\partial E)}}\\
&\leq\frac{\|B_\varphi\|_{C^0(\partial E_\varphi)}\|(\xi_\varphi\circ g_\varphi)_{\tau_\varphi}\|^2_{L^2(\partial E_\varphi)}}{\|\psi\|^2_{H^1(\partial E)}}\\
&\quad-2\frac{\|\nabla_{\tau_\varphi}(\xi_\varphi\circ g_\varphi\cdot\nu_\varphi)\|_{L^2(\partial E_\varphi)}\|(\xi_\varphi\circ g_\varphi)_{\tau_\varphi}\|_{L^2(\partial E_\varphi)}}{\|\psi\|^2_{H^1(\partial E)}}.
\end{aligned}
$$

Therefore, since B_φ is continuous and linear, and considering the fact that $\|\psi\|_{H^1(\partial E)}\geq\|\psi\|_{L^2(\partial E)}$, along with the previously demonstrated convergences (a) and (b), we obtain the result (c).

□

Remark A.3: As previously mentioned, we will now demonstrate that the proof of Lemma A.2 is incomplete. Let us consider both definitions (56) and (57) of c^1_φ using a specific example. Our objective is to show that the two expressions do not coincide at at least one point for a certain example.

To address this matter, we examine the set E of class C^3, where the boundary ∂E is such that locally it represents a straight line. Therefore, we may assume (modulo translation and rotation) that there exists an $r > 0$ such that: $\partial E \cap C((1,0),r,\nu) = \{x \in C((1,0),r,\nu) \mid x_2 = 0\} =: \partial E_{loc}$, with $\nu(x) = (0,1)$ and thus $\tau(x) = (1,0)$.

Now, let us define $\varphi(x) := x_1^2/2$. This gives us a local representation of the set ∂E_φ through $f_\varphi(\partial E_{loc}) = \partial E_{\varphi,loc}$, where $f_\varphi(x) = \mathrm{Id}(x) + \varphi(x)\nu(x)$ and its inverse $g_\varphi(x) = \mathrm{Id}(x) - \varphi(x)\nu(x)$.

It can then be seen that $\partial E_{\varphi,loc}$ can be described by the function $u(z) = z^2/2$. Consequently, we obtain with (2):

$$\nu_\varphi(z,u(z)) = \begin{pmatrix} \frac{-z}{\sqrt{1+z^2}} \\ \frac{1}{\sqrt{1+z^2}} \end{pmatrix} \text{ and } \tau_\varphi(z,u(z)) = \begin{pmatrix} \frac{-z}{\sqrt{1+z^2}} \\ \frac{-1}{\sqrt{1+z^2}} \end{pmatrix}.$$

We now proceed to analyze for $(a,b) \in \partial E_{\varphi,loc}$:

$$D(g_\varphi(a,b)) = D\left(\begin{pmatrix} a \\ b - a^2/2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix},$$

$$D(\nu_\varphi(a,b)) = D\left(\begin{pmatrix} \frac{-a}{\sqrt{1+a^2}} \\ \frac{1}{\sqrt{1+a^2}} \end{pmatrix}\right) = \begin{pmatrix} \frac{-1}{(1+a^2)^{3/2}} & 0 \\ \frac{-a}{(1+a^2)^{3/2}} & 0 \end{pmatrix}.$$

And for $(a,b) \in \partial E_{loc}$

$$D(\nu(a,b)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Noting that ν is constant on this set.

Next, we compare the expressions given by (56) and (57) at the point $f_\varphi(1,0) = (1,1/2) \in \partial E_{\varphi,loc}$. Let:

$$\begin{aligned} c_\varphi^1 &= \nu_\varphi \cdot (D(\nu) \circ g_\varphi D(g_\varphi)\tau_\varphi) + \nu \circ g_\varphi \cdot (D(\nu_\varphi)\tau_\varphi) \\ \tilde{c_\varphi^1} &= \tau \circ g_\varphi \cdot \nu_\varphi (D g_\varphi \tau_\varphi) \cdot \tau \circ g_\varphi + \tau_\varphi \cdot \nu \circ g_\varphi. \end{aligned}$$

Then we obtain:

$$\begin{aligned} c_\varphi^1(f_\varphi(1,0)) =& \nu_\varphi(1,1/2) \cdot (D(\nu)(1,0)\, D(g_\varphi)(1,1/2)\tau_\varphi(1,1/2)) \\ &+ \nu(1,0) \cdot (D(\nu_\varphi)(1,1/2)\tau_\varphi(1,1/2)) \\ =& \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{-1}{(2)^{3/2}} & 0 \\ \frac{-1}{(2)^{3/2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \\ =& \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ =& \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned}\tilde{c^1_\varphi}(f_\varphi(1,0)) &= \tau(1,0)\cdot\nu_\varphi(1,1/2)(Dg_\varphi(1,1/2)\tau_\varphi(1,1/2))\cdot\tau(1,0)+\tau_\varphi(1,1/2)\cdot\nu(1,0)\\ &= \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\left(\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\\ &= \frac{-1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right)+\frac{-1}{\sqrt{2}}=\frac{1}{2}-\frac{1}{\sqrt{2}}.\end{aligned}$$

It is evident from the continuity of c^1_φ and $\tilde{c^1_\varphi}$ that both functions do not coincide in an open neighborhood around $f_\varphi(1,0)$. Therefore, both expressions cannot be the same, rendering the proof of Statement A.2 (b) incomplete, as our derived term for c^1_φ (56) may not necessarily have the desired convergence properties for (59) to hold. Nonetheless, we will continue to assume that (b) is valid for the remaining proofs.

As the next step, to demonstrate Proposition 6.11, we will decompose the functional j''_E into two parts and show the continuity at 0 for each part separately.

Proposition A.4: Suppose E is a bounded C^2 domain and $p_E : \varphi \mapsto P(E_\varphi)$, then the function

$$p''_E : C^2(\partial E) \to Q(H^1(\partial E)), \quad \varphi \mapsto p''_E(\varphi)$$

is continuous at 0.

Proof. Utilizing Lemma A.1 and considering every $\varphi \in C^2(\partial E)$ near 0 and $\psi \in H^1(\partial E)$, we obtain the following:

$$\begin{aligned}\left(p''_E(\varphi)(\psi,\psi)\right)-p''_E(0)(\psi,\psi) &= \left(p''_{E_\varphi}(0)((\xi_\varphi\circ g_\varphi)\cdot\nu_\varphi,(\xi_\varphi\circ g_\varphi)\cdot\nu_\varphi)+p'_{E_\varphi}(0)(Z_{\varphi,\psi})\right)-p''_E(0)(\psi,\psi)\\ &= \underbrace{p''_{E_\varphi}(0)((\xi_\varphi\circ g_\varphi)\cdot\nu_\varphi,(\xi_\varphi\circ g_\varphi)\cdot\nu_\varphi)-p''_E(0)(\psi,\psi)}_{=:A}+p'_{E_\varphi}(0)(Z_{\varphi,\psi}).\end{aligned}$$

Upon examining the definition of j''_E, we obtain the following expression for A:

$$A = \|\nabla_{\tau_\varphi}((\xi_\varphi\circ g_\varphi)\cdot\nu_\varphi)\|^2_{L^2(\partial E_\varphi)}-\|\nabla_\tau\psi\|^2_{L^2(\partial E)}.$$

Moreover, since it holds that

$$|p'_{E_\varphi}(0)(Z_{\varphi,\psi})| \le \underbrace{\|H_\varphi\|_{C^0(\partial E_\varphi)}}_{<\infty\to A.2(v)}\|Z_{\varphi,\psi}\|_{L^2(\partial E_\varphi)},$$

with lemma A.2 (b), (c), we conclude that the term $p''_E(\varphi)(\psi,\psi)-p''_E(0)(\psi,\psi)$ becomes arbitrarily small as $\|\varphi\|_{C^2(\partial E)}\to 0$, thereby demonstrating continuity at 0. □

Proposition A.5: If E is a bounded C^2 domain, $\eta \in C^1(\mathbb{R}^2)$ and $g_E : \varphi \mapsto \int_{E_\varphi}\eta$, then the map

$$g''_E : C^2(\partial E) \to Q(H^1(\partial E)), \quad \varphi \mapsto g''_E(\varphi)$$

is continuous at 0.

Proof. Similarly to the previous proof, as $\|\varphi\|_{C^2(\partial E)}$ approaches zero, we derive:

$$\begin{aligned}
A &:= g''_{E_\varphi}(0)((\xi_\varphi \circ g_\varphi)\cdot\nu_\varphi, (\xi_\varphi \circ g_\varphi)\cdot\nu_\varphi) - g''_E(0)(\psi,\psi) \\
&= \int_{\partial E_\varphi} \left[H_\varphi \eta + \frac{\partial \eta}{\partial \nu_\varphi}\right] ((\psi\nu)\circ g_\varphi \cdot \nu_\varphi)^2 \, d\mathcal{H}^1 - \int_{\partial E} \left[H\eta + \frac{\partial \eta}{\partial \nu}\right] (\psi)^2 \, d\mathcal{H}^1 \\
&= \int_{\partial E} \left(\left[H_\varphi \eta + \frac{\partial \eta}{\partial \nu_\varphi}\right] (\nu \circ g_\varphi \cdot \nu_\varphi)^2 \right) \circ f_\varphi \mathrm{Jac}_\tau f_\varphi \psi^2 d\mathcal{H}^1 - \int_{\partial E} \left[H\eta + \frac{\partial \eta}{\partial \nu}\right] (\psi)^2 \, d\mathcal{H}^1 \\
&\leq \underbrace{\left\| \underbrace{\left[H_\varphi \eta + \frac{\partial \eta}{\partial \nu_\varphi}\right] \circ f_\varphi}_{\to \left[H\eta + \frac{\partial \eta}{\partial \nu}\right]} \Big(\underbrace{\nu \cdot \nu_\varphi \circ f_\varphi}_{\to 1} \Big)^2 \underbrace{\mathrm{Jac}_\tau f_\varphi}_{\to 1} - \left[H\eta + \frac{\partial \eta}{\partial \nu}\right] \right\|_{C^0(\partial E)}}_{\to 0} \|\psi\|_{L^2(\partial E)}.
\end{aligned}$$

Moreover, given that

$$\left| g'_{E_\varphi}(0).(Z_{\varphi,\psi}) \right| \leq \underbrace{\|\eta\|_{C^0(\partial E_\varphi)}}_{<\infty} \|Z_{\varphi,\psi}\|_{L^1(\partial E_\varphi)}$$

and invoking Lemma A.2 (c), we infer that:

$$\begin{aligned}
&\lim_{\|\varphi\|_{C^2(\partial E)} \to 0} \sup_{\psi \in H^1(\partial E)\setminus\{0\}} \frac{|g''_E(0)(\psi,\psi) - g''_E(\varphi)(\psi,\psi)|}{\|\psi\|_{H^1(\partial E)}} \\
&\leq \lim_{\|\varphi\|_{C^2(\partial E)} \to 0} \sup_{\psi \in H^1(\partial E)\setminus\{0\}} \left(\frac{|A|}{\|\psi\|_{L^2(\partial E)}} + \frac{\left| g'_{E_\varphi}(0).(Z_{\varphi,\psi}) \right|}{\|\psi\|_{H^1(\partial E)}} \right) \\
&= 0.
\end{aligned}$$

This confirms the continuity at zero. Since the composition of continuous functions is continuous, by A.4 and A.5, the continuity of j''_E at zero is thus established. $\square$

Now we proceed to the final proof of Proposition 6.12:

Proof. One can discern that the inequality $(j_E - j_{0,E})''(\varphi).(\psi,\psi) \leq c^1_\varphi + c^2_\varphi$ holds with

$$\begin{aligned}
c^1_\varphi &:= \left| \int_{\partial E_\varphi} \left(H_\varphi(\eta - \eta_0) + \frac{\partial(\eta - \eta_0)}{\partial \nu_\varphi} \right) \Big((\xi_\psi \circ g_\varphi)\cdot \nu_\varphi \Big)^2 d\mathcal{H}^1 \right| \\
&= \left| \int_{\partial E} \left(H_\varphi(\eta - \eta_0) + \frac{\partial(\eta - \eta_0)}{\partial \nu_\varphi} \right) \circ f_\varphi \Big((\xi_\psi \circ g_\varphi)\cdot \nu_\varphi \Big)^2 \circ f_\varphi \mathrm{Jac}_\tau f_\varphi d\mathcal{H}^1 \right| \\
&= \left| \int_{\partial E} \left(H_\varphi(\eta - \eta_0) + \frac{\partial(\eta - \eta_0)}{\partial \nu_\varphi} \right) \circ f_\varphi \left((\nu \circ g_\varphi)\cdot \nu_\varphi \right)^2 \circ f_\varphi \mathrm{Jac}_\tau f_\varphi \psi^2 d\mathcal{H}^1 \right| \\
&\leq \Big(\|H_\varphi\|_{C^0(\partial E_\varphi)} \|\eta - \eta_0\|_{C^0(\mathbb{R}^2)} + \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \Big) \|f_\varphi\|_{C^0(\partial E)} \|\nu \cdot \nu_\varphi \circ f_\varphi\|_{C^0(\partial E)} \|\psi\|^2_{L^2(\partial E)} \\
&\leq \Big(\|H_\varphi\|_{C^0(\partial E_\varphi)} + 1 \Big) \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \|f_\varphi\|_{C^0(\partial E)} \|\nu \cdot \nu_\varphi \circ f_\varphi\|_{C^0(\partial E)} \|\psi\|^2_{L^2(\partial E)},
\end{aligned}$$

and

$$\begin{aligned} c_\varphi^2 &:= \left| \int_{\partial E_\varphi} (\eta - \eta_0) Z_{\varphi,\psi} d\mathcal{H}^1 \right| \\ &\leq \|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \|Z_{\varphi,\psi}\|_{L^1(\partial E_\varphi)}, \end{aligned}$$

where, by invoking lemma A.2 and [8, Lem. 4.7], we ascertain the boundedness of expressions such as $\|H_\varphi\|_{C^0(\partial E_\varphi)}$,
$\|f_\varphi\|_{C^0(\partial E)}$, $\|Z_{\varphi,\psi}\|_{L^1(\partial E_\varphi)}/\|\psi\|_{H^1(\partial E)}$, $\|\nu \cdot \nu_\varphi \circ f_\varphi\|_{C^0(\partial E)}$ for sufficiently small $\varphi \in C^2(\partial E)$ (there exists some $\epsilon > 0$ such that this holds for all φ with $\|\varphi\| \leq \epsilon$). Consequently, for all $\psi \in H^1(\partial E)\backslash\{0\}$, we have:

$$\begin{aligned} \frac{|(j_E - j_{0,E})''(\varphi).(\psi,\psi)|}{\|\psi\|_{H^1(\partial E)}^2} &\leq \frac{c_\varphi^1 + c_\varphi^2}{\|\psi\|_{H^1(\partial E)}^2} \\ &\leq \frac{c_\varphi^1}{\|\psi\|_{L^2(\partial E)}^2} + \frac{c_\varphi^2}{\|\psi\|_{H^1(\partial E)}^2} \to 0 \end{aligned}$$

as $\|\eta - \eta_0\|_{C^1(\mathbb{R}^2)} \to 0$. This demonstrates the statement to be proven. □