Large N limit and $1/N$ expansion of the observables for $O(N)$ linear sigma model via SPDE

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References: arXiv preprint

Joint Work with Hao Shen and Xiangchan Zhu

Introduction

$O(N)$ linear sigma model:

$$\nu^N = \frac{1}{C_N} \exp\left(- \int_{\mathbb{T}^d} \frac{1}{2} \sum_{j=1}^{N} |\nabla \Phi_j|^2 + \frac{m}{2} \sum_{j=1}^{N} \Phi_j^2 + \frac{1}{4N} \left(\sum_{j=1}^{N} \Phi_j^2 \right)^2 \mathrm{d}x \right) \mathcal{D}\Phi,$$

where $\Phi = (\Phi_1, \ldots, \Phi_N)$ is the (vector-valued) field.

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Stochastic quantization on $\mathbb{T}^d$, $d = 2, 3$:

$$\mathcal{L}\Phi_i = -\frac{1}{N} \sum_{j=1}^N \Phi_j^2 \Phi_i + \sqrt{2}\xi_i,$$

$\mathcal{L} = \partial_t - \Delta + m$; $(\xi_i)_{i=1}^N$: independent space-time white noises.

- The dynamical linear sigma model

$$\mathcal{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N} :\Phi_j^2\Phi_i: + \sqrt{2}\xi_i, \quad \Phi_i(0) = \phi_i$$

Limiting equation and convergence of the dynamics when $d = 2$

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Theorem [Shen, Smith, Zhu, Z. 20]

Suppose that $d = 2$ and (ψ_i, ψ_j) are independent and have the same law and for $p > 1$ $\mathbf{E}\|\phi_i - \psi_i\|_{C^{-\kappa}}^p \to 0$, as $N \to \infty$.

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- Mean field limit/ Propagation of chaos

Convergence of invariant measure (field)

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For $d = 2, 3$

- $\nu^{N,i}$ form a tight set of probability measures on $C^{-\frac{1}{2}-\kappa}$ for $\kappa > 0$.

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- $\nu^{N,i}$ form a tight set of probability measures on $C^{-\frac{1}{2}-\kappa}$ for $\kappa > 0$.
- For $m \geq m_0$, $\nu^{N,i}$ converges to ν; and ν_k^N converges to $\nu \times \cdots \times \nu$, as $N \to \infty$. Furthermore, $\mathbb{W}_2(\nu^{N,i}, \nu) \lesssim N^{-\frac{1}{2}}$.

Large N limit of Observables

$O(N)$ invariant Observables

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Since $\Phi_i \to Z_i$ with $Z_i \sim \mathcal{N}(0,(m-\Delta)^{-1})$, it is natural to ask

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} :\Phi_i^2: := \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} :Z_i^2: =^d \mathcal{Z} \sim \mathcal{N}(0,2C^2)?$$

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Questions:

Large N limit of the observables?

Large N limit of Observables

Set $:(\mathbf{\Phi}^2)^n: \; = :\Big(\sum_{i=1}^N \Phi_i^2\Big)^n: .$ Fix $d=2$ and $m \geq m_0$.

Theorem. Large N limit of Observables

- $(\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2:)_N$ converge in law in $H^{-\kappa}$ for any $\kappa > 0$ to a mean zero Gaussian field $\mathcal{Q}$ with covariance $G(x-y)$ determined by

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- For $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m, m \in \mathbb{N}$, as $N \to \infty$,

$$\left\{\left(\frac{1}{N^{n_1/2}} :(\mathbf{\Phi}^2)^{n_1}: , \dots , \frac{1}{N^{n_m/2}} :(\mathbf{\Phi}^2)^{n_m}: \right)\right\}_N$$

converge jointly in law to $(:\mathcal{Q}^{n_1}:_C, \dots, :\mathcal{Q}^{n_m}:_C)$ in $(H^{-\kappa})^m$ for $\kappa > 0$ with

$$:\mathcal{Q}^n:_C = \lim_{\varepsilon\to 0}(2C_\varepsilon^2(0))^{n/2} H_n((2C_\varepsilon^2(0))^{-1/2}\mathcal{Q}_\varepsilon) \qquad n \in \mathbb{N}.$$

Step 1. Uniform estimates from stochastic quantization

Stochastic quantization on $\mathbb{T}^2$:

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Uniform estimates from SPDEs

For $\ell \geq 0$, $n \in \mathbb{N}$, $\kappa > 0$

$$\mathbb{E}\Big(\Big\| :(\mathbf{\Phi}^2)^n: \Big\|_{H^{-\kappa}}^{\ell}\Big) \lesssim N^{\frac{n\ell}{2}},$$
$$\mathbb{E}\Big(\Big\| :\Phi_1(\mathbf{\Phi}^2)^n: \Big\|_{H^{-\kappa}}^{\ell}\Big) \lesssim N^{\frac{n\ell}{2}},$$

Formally for $\Phi_i = Z_i + Y_i$

$$\mathbf{\Phi}^2 = \sum_{i=1}^{N}(Y_i^2 + Y_iZ_i + :Z_i^2:) \sim \sqrt{N}.$$

Step 2. Dyson–Schwinger equations

Dyson–Schwinger equations (IBP):

$$\mathbb{E}\Big(\frac{\delta F(\Phi)}{\delta\Phi_1(x)}\Big) = \mathbb{E}\Big((m-\Delta)\Phi_1(x)F(\Phi)\Big) + \frac{1}{N}\mathbb{E}\Big(F(\Phi) :\Phi_1\mathbf{\Phi}^2(x): \Big).$$

$\Leftrightarrow$

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Choosing $F(\Phi) = \Phi_1(y_1) :\boldsymbol{\Phi}^2: (y_2)$ and $x = y_1$

$$2C(y_1-y_2)\mathbb{E}\Big(\Phi_1(y_1)\Phi_1(y_2)\Big) = \mathbb{E}\Big(:\Phi_1^2: (y_1) :\boldsymbol{\Phi}^2: (y_2)\Big)$$
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$\Rightarrow$ For

$$G(y_1,y_2) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\Big(:\mathbf{\Phi}^2: (y_1) :\mathbf{\Phi}^2: (y_2)\Big),$$
$$G + C^2 * G = 2C^2.$$

Step 3. Explicit solutions for recursive relation

$$f_k(y_1, \dots, y_k) = \lim_{N\to\infty} \frac{1}{N^{k/2}} \mathbb{E}\Big(\prod_{i=1}^{k} :\mathbf{\Phi}^2: (y_i) \Big).$$

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$\Rightarrow f_2=G$ and

$$f_k(y_1,\dots,y_k)=\sum_\pi\prod_{j=1}^{k/2} f_2(y_{\pi(2j-1)},y_{\pi(2j)}),\quad k\in 2\mathbb{N},$$

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$$\Rightarrow\quad \lim_{N\to\infty}\frac{1}{N^{1/2}}:\mathbf{\Phi}^2:\ \text{is Gaussian.}$$

General recursive relation

Set for $\mathbf{n} = (n_1, \dots, n_k)$,

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$$f_{\mathbf{n},k}(y_1, \dots, y_k) + \int C^2(y_1 - z) f_{\hat{\mathbf{n}},k+1}(z, y_1, \dots, y_k)\mathrm{d}z$$
$$= \sum_{j=2}^k 2n_j C^2(y_1 - y_j) f_{\widetilde{\mathbf{n}}_j,k}(y_1, \dots, y_k),$$

with $\hat{\mathbf{n}} = (1, n_1 - 1, n_2, \dots, n_k)$ and $\widetilde{\mathbf{n}}_j = (n_1 - 1, n_2, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_k)$.

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$$f_{\mathbf{n},k}(y_1, \dots, y_k) + \int C^2(y_1 - z) f_{\hat{\mathbf{n}},k+1}(z, y_1, \dots, y_k) \mathrm{d}z$$
$$= \sum_{j=2}^k 2n_j C^2(y_1 - y_j) f_{\widetilde{\mathbf{n}}_j,k}(y_1, \dots, y_k),$$

with $\hat{\mathbf{n}} = (1, n_1 - 1, n_2, \dots, n_k)$ and $\widetilde{\mathbf{n}}_j = (n_1 - 1, n_2, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_k)$.

$\Rightarrow$

$$\lim_{N\to\infty} \frac{1}{N^{n/2}} :(\mathbf{\Phi}^2)^n: \; = :\mathcal{Q}^n:_C.$$

$1/N$ expansion

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Set

$$f_k^N(y_1, \dots, y_k) = \frac{1}{N^{k/2}} \mathbb{E}\Big(\prod_{i=1}^{k} :\mathbf{\Phi}^2: (y_i) \Big).$$

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Theorem (1/N expansion)

For $p\geq 0$

$$f_k^N=\sum_{n=0}^p\frac{1}{N^n}F_n^{k,1}+\frac{1}{N^{p+1}}R_{p+1}^{k,1},\quad k\in 2\mathbb{N},$$

and

$$f_k^N=\sum_{n=0}^p\frac{1}{N^{n+1/2}}F_n^{k,2}+\frac{1}{N^{p+3/2}}R_{p+1}^{k,2},\quad k\in 2\mathbb{N}-1,$$

where $F_n^{k,1},F_n^{k,2}$ only depend on the Green's function of Gaussian free field and

$$\|R_{p+1}^{k,1}\|_{H^{-\kappa}}+\|R_{p+1}^{k,2}\|_{H^{-\kappa}}\lesssim 1,$$

with the proportional constant independent of N.

Graph notations

$$\int C(x-z)\mathbb{E}\Big(\frac{\delta F(\Phi)}{\delta \Phi_1(z)}\Big)\mathrm{d}z = \mathbb{E}\Big(\Phi_1(x)F(\Phi)\Big)+\frac{1}{N}\int C(x-z)\mathbb{E}\Big(F(\Phi)\, :\Phi_1\mathbf{\Phi}^2: (z)\Big)\mathrm{d}z.$$

We denote C by a line, and single / double / triple wavy lines represent Φ_1, $\frac{1}{\sqrt{N}}\mathbf{\Phi}^2$ and $\frac{1}{\sqrt{N}}\Phi_1\mathbf{\Phi}^2$.

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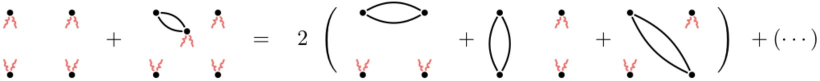

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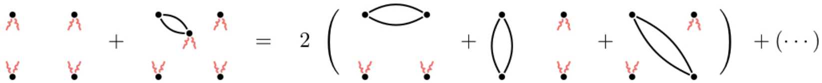
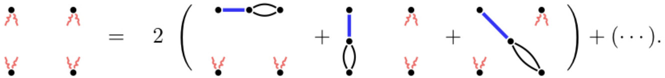


Denote $K = (I + C^2 *)^{-1}$ by a blue line:



Idea of Proof

1. Two types of IBP:

$$\frac{1}{N^{-k/2}}(I+C^2*)\mathbb{E}\Big(\prod_{i=1}^{k} :\mathbf{\Phi}^2:(y_i)\Big)=\frac{2}{N^{-(k-2)/2}}C^2\mathbb{E}\Big(\prod_{i=1}^{k-2} :\mathbf{\Phi}^2:(y_i)\Big)+O(\frac{1}{\sqrt{N}}),$$

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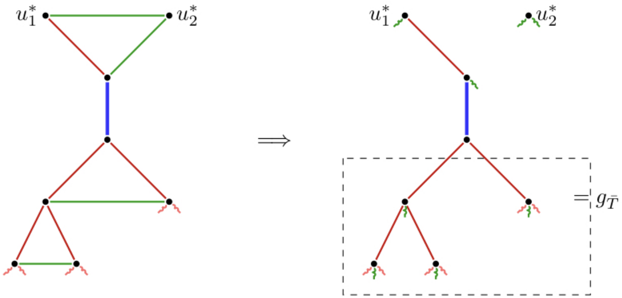

$$C(x-y) = \mathbb{E}(Z(x)Z(y)).$$

Next order SPDEs

Next order SPDEs

Stochastic quantization of $O(N)$ model ν^N

$$\mathcal{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N}\Phi_j^2\Phi_i + \sqrt{2}\xi_i,$$

Stochastic quantization of GFF ν

$$\mathcal{L}Z_i = \sqrt{2}\xi_i.$$

Theorem (Next order SPDEs)

In the stationary setting, $\sqrt{N}(\Phi_i - Z_i)$ converges to the stationary solution of

$$\mathcal{L}u_i = \mathcal{P}_i,$$

where $\{\mathcal{P}_1, \dots, \mathcal{P}_k\}$ is stationary process with the time marginal distribution

$$\{X_1\mathcal{Q}, \dots, X_k\mathcal{Q}\}.$$

Here $X_i, i = 1, \dots, k$, and $\mathcal{Q}$ are independent, $X_i =^d Z_i$ and $\mathcal{Q}$ is the large N limit of $\frac{1}{\sqrt{N}}$ $:\mathbf{\Phi}^2:$.

- How about $d = 3$? Tightness of $\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2:$ is known in [Shen, Zhu, Z. 21]
- Large N problem for other models?

Thank you !