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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 6

Due: Monday, 07.06.2020

Exercise 1 (4+2 Punkte)

Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion and $t > 0$.

(i) For $n \geq 1$ and $k \in \{0, 1, \dots, 2^n\}$ let $t_k^{(n)} := 2^{-n}tk$. Prove that almost surely

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \big(B_{t_k^{(n)}} - B_{t_{k-1}^{(n)}}\big)^2 = t.$$

(ii) Let $\mathcal{P} = \{\pi = (t_0, \dots, t_{n_\pi}) : n_\pi \geq 1, 0 \leq t_0 \leq \dots \leq t_{n_\pi}\}$ be the set of all partitions of the interval $[0, t]$. Show that almost surely

$$\sup_{\pi\in\mathcal{P}} \sum_{k=1}^{n_\pi} |B_{t_k} - B_{t_{k-1}}| = \infty,$$

i.e. almost surely the sample paths of Brownian motion are of unbounded variation.

Hint: Assuming the converse, use part (i) and continuity of Brownian motion to arrive at a contradiction.

Exercise 2 (5 Punkte)

Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion. Let $f\colon [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function with continuous partial derivative f_s in the first component and with two continuous partial derivatives in the second component, with second partial derivative denoted f_{xx}. Moreover, suppose that there is a bounded continuous function $g\colon [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$g(s, x) = f_s(s, x) + \frac{1}{2} f_{xx}(s, x).$$

Then

$$f(t, B_t) - f(0, B_0) - \int_0^t g(s, B_s)\,\mathrm{d}s$$

is a martingale.

Exercise 3 (3+6 Punkte)

Let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion.

(i) For $t \geq 0$, use Itô's formula to compute the following stochastic integrals:

(a) $\int_0^t \mathrm{d}B_s$.

(b) $\int_0^t B_s \,\mathrm{d}B_s$.

(c) $\int_0^t B_s^2 \,\mathrm{d}B_s$.

(ii) Let $f\colon [0,\infty) \times \Omega \longrightarrow \mathbb{R}$ be progressively measurable and assume that $\int_0^t f(s,\omega)^2 \,\mathrm{d}s < \infty$ almost surely for all $t \geq 0$. Consider the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = \exp\Big(\int_0^t f(s,\omega)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s,\omega)^2\,\mathrm{d}s\Big).$$

Use Itô's Formula to show that

(a) $Z_t = 1 + \int_0^t Z_s f(s,\omega)\,\mathrm{d}B_s$.

(b) The process $Y_t := 1/Z_t$ satisfies $Y_t = 1 + \int_0^t Y_s f(s,\omega)^2\,\mathrm{d}s - \int_0^t Y_s f(s,\omega)\,\mathrm{d}B_s$.