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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 4

Due: Monday, 17.05.2020

Exercise 1 (3+3 Punkte)

(a) Let Ω be a domain. Suppose there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Ω consisting of open discs $U_i \subset \mathbb{C}$ such that the first cohomology group $H^1(\mathcal{U}, \mathbb{C})$ defined in the lecture is trivial, i.e. suppose that

$$H^1(\mathcal{U}, \mathbb{C}) = \{0\}.$$

Show that every harmonic function $u \colon \Omega \longrightarrow \mathbb{R}$, $u \in C^2(\Omega)$, has a harmonic conjugate.

(b) Consider the function

$$u \colon \mathbb{D} \setminus \{0\} \longrightarrow \mathbb{R}, \ z \mapsto \log |z|.$$

Prove that u does not have a harmonic conjugate.

Exercise 2 (3+3 Punkte)

(a) Let γ be a parameterization of the unit circle of the upper half plane $\mathbb{H}$. Show that

$$\Big| \int_\gamma \frac{1}{2 + z^2} \, \mathrm{d}z \Big| \leq \pi.$$

(b) Suppose that $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic and satisfies

$$|f(z)| \leq 999 \, (1 + |z|)^{13} \quad \text{for every } z \in \mathbb{C}.$$

Show that f is a polynomial.

Hint: You may use Cauchy's inequalities.

Exercise 3 (The Mean-Value Property) (4+4 Punkte)

Let $n \geq 1$ and Ω be an open subset of $\mathbb{R}^n$.

(a) Let $u\colon \Omega \longrightarrow \mathbb{R}$ be a harmonic function, i.e. it is twice continuously differentiable and $\Delta u = 0$. Prove that for every $x \in \mathbb{R}^n$ and every $r > 0$ such that

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\} \subset \Omega$$

we have that

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\sigma(\mathrm{d}y) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,\mathrm{d}y, \tag{1}$$

where ω_n is the area of the n-dimensional unit sphere and σ is the $(n-1)$-dimensional surface measure on $\partial B_r(x)$.

(b) Conversely, let u be any locally integrable function on Ω, i.e.

$$\int_K |u(x)|\,\mathrm{d}x < \infty \quad \text{for every compact } K \subset \Omega.$$

Show that if u satisfies (1) for every ball in Ω, then it is infinitely differentiable and satisfies $\Delta u = 0$.