Prof. Dr. Chiranjib Mukherjee
Konstantin Recke

Sommersemester 2021
3. Mai 2021

Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 3

Due: Monday, 10.05.2020

Exercise 1 (Reflection Principle) (5+1 Punkte)

Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion started in 0.

(i) For $t \geq 0$ and $a > 0$, use the strong Markov property to compute

$$\mathbb{P}\Big[\sup_{0\leq s\leq t} B_s \geq a\Big] = 2\,\mathbb{P}[B_t \geq a] = \frac{\sqrt{2}}{\sqrt{\pi t}}\int_a^\infty \exp\Big(-\frac{y^2}{2t}\Big)\,\mathrm{d}y$$

(ii) Deduce a similar bound for $\mathbb{P}\big[\sup_{0\leq s\leq t}|B_s| \geq a\big]$ in terms of $\mathbb{P}^{B_t}$.

Exercise 2 (6 Punkte)

Let $p_t(\cdot,\cdot)$ denote the transition probability function of two-dimensional Brownian motion. Show that, as $z \to w$, $z \neq w$

$$\pi\int_0^1 p_t(z,w)\,\mathrm{d}t = -\log\big(|z-w|\big) + O(1).$$

Exercise 3 (Complex Analysis) (1+3+4 Punkte)

(i) Let $z \in \mathbb{C}$ with $|z| = 1$. Simplify the expression

$$\Big|\frac{z-w}{1-\overline{z}w}\Big|.$$

(ii) Is the function

$$f\colon \mathbb{C} \longrightarrow \mathbb{C},\ f(x,y) := \big(xe^x\cos(y) - e^x y\sin(y)\big) + i\big(e^x y\cos(y) + xe^x\sin(y)\big)$$

holomorphic?

Hint: You may use that f is holomorphic, if its partial derivatives exist, are continuous and satisfy the Cauchy-Riemann equations.

(iii) Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $u\colon \Omega \longrightarrow \mathbb{R}$ be a function. A function $v\colon \Omega \longrightarrow \mathbb{R}$ is called a *harmonic conjugate* of u, if the complex function $u + iv$ is holomorphic.

(a) Show that for any two harmonic conjugates v_1, v_2 of u, the function $v_1 - v_2$ is constant.

Hint: You may use the Cauchy-Riemann equations.

(b) Consider $\Omega = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that every harmonic function $u \in C^2(\mathbb{D})$ admits a harmonic conjugate.

Hint: You may try to define the function

$$v(x, y) := \int_0^y \partial_1 u(0, t)\,\mathrm{d}t - \int_0^x \partial_2 u(t, y)\,\mathrm{d}t,$$

where ∂_1, respectively ∂_2, denotes the partial derivative with respect to the first, respectively second, coordinate.