Prof. Dr. Chiranjib Mukherjee
Konstantin Recke

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Gaussian Free Field and Liouville Quantum Gravity

Exercise Sheet 10

Due: Monday, 12.07.2021

Exercise 1 (15 Punkte)

Let $d \in \mathbb{N}$. Let $\mathbb{P}_0$ be the law of Simple Random Walk $X = (X_n)$ on $\mathbb{Z}^d$ started in 0, viewed as a measure on the path space

$$\Omega_{\text{traj}} = \{\omega = (\omega_i)_{i=0}^{\infty} : \omega_i \in \mathbb{Z}^d \text{ for all } i \in \mathbb{N}_0\}$$

which is equipped with the Borel σ-field. Moreover, let

$$\{\eta(i,x) : i \in \mathbb{N}_0, x \in \mathbb{Z}^d\}$$

be iid standard Gaussians defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In the following, expectations taken with respect to $\mathbb{P}_0$ will be denoted $\mathbb{E}_0$ and expectations with respect to $\mathbf{P}$ will be denoted $\mathbf{E}$.

For $n \geq 0$ and $\omega \in \Omega_{\text{traj}}$ define

$$H_n(\omega) \coloneqq \sum_{i=0}^{n} \eta(i, \omega_i)$$

and note that this defines a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$.

Now consider a parameter $\gamma > 0$. For every $n \geq 0$ we define a random measure $\mu_{n,\gamma}$ on the path space Ω_{traj} by setting

$$\mu_{n,\gamma}(\mathrm{d}\omega) = \frac{1}{Z_{n,\gamma}} e^{\gamma H_n(\omega)} \mathbb{P}_0(\mathrm{d}\omega),$$

where $Z_{n,\gamma}$ is a normalizing constant, which makes $\mu_{n,\gamma}$ a probability measure. In other words $Z_{n,\gamma}$ is a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ given by

$$Z_{n,\gamma} = \mathbb{E}_0\big[e^{\gamma H_n(X)}\big].$$

(i) Show that the sequence $(\overline{Z_{n,\gamma}})_{n=0}^{\infty}$, defined by

$$\overline{Z_{n,\gamma}} = \frac{Z_{n,\gamma}}{\mathbf{E}[Z_{n,\gamma}]},$$

is a martingale.

(ii) Deduce that the martingale $(\overline{Z_{n,\gamma}})$ converges $\mathbf{P}$-almost surely to a limit $\overline{Z_\infty}$ and show that the limit is either $\mathbf{P}$-almost surely positive or $\mathbf{P}$-almost surely zero.

(iii) Now let $d \geq 3$.

(a) Prove that there exists a constant $\gamma_{L^2} > 0$ such that the martingale $(\overline{Z_{n,\gamma}})$ is L^2-bounded for every $\gamma \in (0, \gamma_{L^2})$.

(b) For $\gamma \in (0, \gamma_{L^2})$, deduce that the convergence in part (ii) happens also in $L^2(\mathbf{P})$ and that $\overline{Z_\infty}$ is $\mathbf{P}$-almost surely positive.

In the following let D be a proper simply connected domain. Moreover let $(B_t)_{t\geq 0}$ be Brownian motion in the complex plane.

Exercise 2 (5 Punkte)

Fix $z \in D$. Prove that

$$\log R(z, D) = \mathbb{E}_z\big[\log |B_T - z|\big],$$

where $T := \inf\{t > 0 : B_t \notin D\}$ and $R(z, D)$ is the conformal radius of D from z.

Hint: Let g be a conformal map sending D to $\mathbb{D}$ and z to 0. Let $\phi(w) = g(w)/(w - z)$ for $w \neq z$ and $\phi(z) = g'(z)$. Now consider $\log |\phi|$.

Exercise 3 (5 Punkte)

Assume that the domain D is bounded. Let h be a Gaussian Free Field on D and let U be a subdomain of D. For $z \in U$, consider the *harmonic measure* ρ_z on ∂U as seen from z, in other words, with $T := \inf\{t > 0 : B_t \notin U\}$,

$$\rho(A) = \mathbb{P}_z\big[B_T \in A\big]$$

for every $A \subset \partial U$ measurable.

(i) Show that $\rho \in \mathcal{M}_0$.

(ii) Show that

$$\mathrm{Var}(h, \rho) = \log \frac{R(z, D)}{R(z, U)}.$$

What can you say about the (not necessarily bounded) proper simply connected case?